MASK OPTIMIZATION DURING ANALYSIS

Triangular mask

Location

The Sum of distances squared (assuming an upright equilateral triangle with edge length a) is:

$$L^2 = (x_1)^2 + (y_1 - a\frac{2\sqrt{0.75}}{3})^2 + (x_2 - \frac{a}{2})^2 + \\ + (y_2 + a\frac{\sqrt{0.75}}{3})^2 + (x_3 + \frac{a}{2})^2 + (y_3 + a\frac{\sqrt{0.75}}{3})^2.$$

Shifting the triangle by ϵ_x yields:

$$(x_1+\epsilon_x)^2+(y_1-a\frac{2\sqrt{0.75}}{3})^2+(x_2+\epsilon_x-\frac{a}{2})^2+(y_2+a\frac{\sqrt{0.75}}{3})^2+(x_3+\epsilon_x+\frac{a}{2})^2+(y_3+a\frac{\sqrt{0.75}}{3})^2.$$

Taking the derivative with respect to ϵ_x and equating to 0 yields:

$$0 = 2(x_1 + \epsilon_x) + 2(x_2 + \epsilon_x - \frac{a}{2}) + 2(x_3 + \epsilon_x - \frac{a}{2}) \rightarrow \epsilon_x = -\frac{x_1 + x_2 + x_3}{3}.$$

Thus, the triangle's x coordinate should be located in the center of mass of the x coordinates. The same computation can be done for the y coordinate.

Rotation

Starting with a triangle centered optimally and aligned as before, we now rotate it by an angle $\tilde{\theta}$. Let us denote $l = a\frac{2\sqrt{0.75}}{3}$.

$$L^2 = (x_1 - lcos(\theta + \frac{\pi}{2}))^2 + (y_1 - lsin(\theta + \frac{\pi}{2}))^2 + (x_2 - lcos(\theta + \frac{\pi}{2} + \frac{4\pi}{3}))^2 + \\ +(y_2 - lsin(\theta + \frac{\pi}{2} + \frac{4\pi}{3}))^2 + (x_3 - lcos(\theta + \frac{\pi}{2} + \frac{2\pi}{3})))^2 + (y_3 - lsin(\theta + \frac{\pi}{2} + \frac{2\pi}{3}))^2.$$

Taking the derivative with respect to θ yields:

$$\theta = tan^{-1}(\frac{l(2x1 - x2 - x3) + \sqrt{3}(y3 - y2)}{\sqrt{3}l(x3 - x2) - 2y1 + y2 + y3})$$

Rectangular mask

Location

The Sum of distances squared is (assuming that the long axis of the rectangle is of length $2a$ and is aligned with the x axis and the other axis is of length $2b$):

$$L^2 = (x_1+a)^2+(y_1-b)^2+(x_2-a)^2+(y_2-b)^2+(x_3-a)^2+(y_3+b)^2+(x_4+a)^2+(y_4+b)^2.$$

Shifting the rectangle by ϵ_x yields:

$$L^2 = (x_1+\epsilon_x+a)^2+(y_1-b)^2+(x_2+\epsilon_x-a)^2+(y_2-b)^2+(x_3+\epsilon_x-a)^2+(y_3+b)^2+(x_4+\epsilon_x+a)^2+(y_4+b)^2.$$

Taking the derivative with respect to ϵ_x and equating to 0 yields:

$$0 = 2(x_1+\epsilon_x+a)+2(x_2+\epsilon_x-a)+2(x_3+\epsilon_x-a)+2(x_4+\epsilon_x+a) \rightarrow \epsilon_x = -\frac{x_1+x_2+x_3+x_4}{4}.$$

Thus, the rectangle's x coordinate should be located in the center of mass of the x coordinates. The same computation can be done for the y coordinate.

Rotation

Starting with a rectangle centered optimally and aligned as before, we now rotate it by an angle $\tilde{\theta}$. Let us denote $l = \sqrt{a^2+b^2}$ and $\theta = \tilde{\theta} + tan^{-1}(\frac{b}{a})$ due to the angle of the diagonal of the rectangle.

$$L^2 = (x_1 - lcos(\theta + 2tan^{-1}(\frac{a}{b})))^2 + (y_1 - lsin(\theta + 2tan^{-1}(\frac{a}{b})))^2 + (x_2 - lcos(\theta))^2 + (y_2 - lsin(\theta))^2 +$$
$$+(x_3 + lcos(\theta + 2tan^{-1}(\frac{a}{b})))^2 + (y_3 + lsin(\theta + 2tan^{-1}(\frac{a}{b})))^2 + (x_4 + lcos(\theta))^2 + (y_4 + lsin(\theta))^2.$$

Since the every θ term appears with $\pm$ sign, the location computation holds. Taking the derivative with respect to θ yields:

$$\begin{aligned}
0 &= (x_1 - lcos(\theta + 2tan^{-1}(\frac{a}{b})))sin(\theta + 2tan^{-1}(\frac{a}{b})) - (y_1 - lsin(\theta + 2tan^{-1}(\frac{a}{b})))cos(\theta + 2tan^{-1}(\frac{a}{b})) + \\
&+ (x_2 - lcos(\theta))(sin(\theta)) + (y_2 - lsin(\theta))(-cos(\theta)) + (x_4 + lcos(\theta))(-sin(\theta)) + (y_4 + lsin(\theta))(cos(\theta)) \\
&- (x_3 + lcos(\theta + 2tan^{-1}(\frac{a}{b})))sin(\theta + 2tan^{-1}(\frac{a}{b})) + (y_3 + lsin(\theta + 2tan^{-1}(\frac{a}{b})))cos(\theta + 2tan^{-1}(\frac{a}{b})) = \\
&= (x_1 - x_3)sin(\theta + 2tan^{-1}(\frac{a}{b})) - (y_1 - y_3)cos(\theta + 2tan^{-1}(\frac{a}{b})) + (x_2 - x_4)sin(\theta) - (y_2 - y_4)cos(\theta).
\end{aligned}$$

The solution for this equation reads (up to π):

$$\theta = cos^{-1}(\pm\frac{C + Acos(2tan^{-1}(\frac{a}{b})) + Bsin(2tan^{-1}(\frac{a}{b})}{\sqrt{A^2+B^2+C^2+D^2+2(AC+BD)cos(2tan^{-1}(\frac{a}{b}))+2(BC-AD)sin(2tan^{-1}(\frac{a}{b}))}}) =$$

$$= cos^{-1}(\pm\frac{C + A\frac{1-(\frac{a}{b})^2}{1+(\frac{a}{b})^2} + B\frac{2\frac{a}{b}}{1+(\frac{a}{b})^2}}{\sqrt{A^2+B^2+C^2+D^2+2(AC+BD)\frac{1-(\frac{a}{b})^2}{1+(\frac{a}{b})^2}+2(BC-AD)\frac{2\frac{a}{b}}{1+(\frac{a}{b})^2}}})$$

where $A = x_1 - x_3$, $B = y_1 - y_3$, $C = x_2 - x_4$ and $D = y_2 - y_4$. Last, the rotation is $\theta - tan^{-1}(\frac{b}{a})$.

Hexagonal mask

Location

The Sum of distances squared is (assuming an hexagon with parameters $[a, b, c]$ where a is the edge length along the long axis, b is the width of the hexagon along it's narrow dimension and $2c + a$ is the length of the long diagonal of the hexagon):

$$L^2 = (x_1 + (\frac{a}{2} + c))^2 + (y_1)^2 + (x_2 + \frac{a}{2})^2 + (y_2 - \frac{b}{2})^2 + (x_3 - \frac{a}{2})^2 + (y_3 - \frac{b}{2})^2 +$$
$$+ (x_4 - (\frac{a}{2} + c))^2 + (y_4)^2 + (x_5 - \frac{a}{2})^2 + (y_5 + \frac{b}{2})^2 + (x_6 + \frac{a}{2})^2 + (y_6 + \frac{b}{2})^2.$$

Adding ϵ_x to the x terms, and taking the partial derivative with respect to it yields:

$$0 = (x_1 + (\frac{a}{2} + c) + \epsilon_x) + (x_2 + \frac{a}{2} + \epsilon_x) + (x_3 - \frac{a}{2} + \epsilon_x) +$$
$$+ (x_4 - (\frac{a}{2} + c) + \epsilon_x) + (x_5 - \frac{a}{2} + \epsilon_x) + (x_6 + \frac{a}{2} + \epsilon_x).$$

So the localization is as expected in the center of mass.

Rotation

Starting with an hexagon centered optimally and aligned as before, we now rotate it by an angle θ.

$$L^2 = (x_1 + cos(\theta)(\frac{a}{2} + c))^2 + (y_1 + sin(\theta)(\frac{a}{2} + c))^2 + (x_4 - cos(\theta)(\frac{a}{2} + c))^2 + (y_4 - sin(\theta)(\frac{a}{2} + c))^2 +$$
$$+ (x_2 + sqrt(a^2 + b^2)cos(\theta - tan^{-1}(\frac{b}{a})))^2 + (y_2 + sqrt(a^2 + b^2)sin(\theta - tan^{-1}(\frac{b}{a})))^2 +$$
$$+ (x_3 - sqrt(a^2 + b^2)cos(\theta + tan^{-1}(\frac{b}{a})))^2 + (y_3 - sqrt(a^2 + b^2)sin(\theta + tan^{-1}(\frac{b}{a})))^2 +$$
$$+ (x_5 - sqrt(a^2 + b^2)cos(\theta - tan^{-1}(\frac{b}{a})))^2 + (y_5 - sqrt(a^2 + b^2)sin(\theta - tan^{-1}(\frac{b}{a})))^2 +$$
$$+ (x_6 + sqrt(a^2 + b^2)cos(\theta + tan^{-1}(\frac{b}{a})))^2 + (y_6 + sqrt(a^2 + b^2)sin(\theta + tan^{-1}(\frac{b}{a})))^2.$$

By taking the derivative with respect to θ and inverting the resulting relation one gets:

$$\theta = \tan^{-1}\left(\frac{-\frac{(a+2c)(\mathrm{y1}-\mathrm{y4})}{2\sqrt{a^2+b^2}} - \frac{b(\mathrm{x2}+\mathrm{x3}-\mathrm{x5}-\mathrm{x6})}{a\sqrt{\frac{b^2}{a^2}+1}} - \frac{\mathrm{y2}-\mathrm{y3}-\mathrm{y5}+\mathrm{y6}}{\sqrt{\frac{b^2}{a^2}+1}}}{\frac{(a+2c)(\mathrm{x1}-\mathrm{x4})}{2\sqrt{a^2+b^2}} + \frac{\mathrm{x2}-\mathrm{x3}-\mathrm{x5}+\mathrm{x6}}{\sqrt{\frac{b^2}{a^2}+1}} + \frac{b(-\mathrm{y2}-\mathrm{y3}+\mathrm{y5}+\mathrm{y6})}{a\sqrt{\frac{b^2}{a^2}+1}}}\right).$$