

# FDU 高等线性代数 Homework 01

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## Problem 1

设  $n$  为给定的正整数, 求  $n$  阶矩阵  $A$  的所有特征值和特征向量.

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$$

- Insight:

实际上, 我们观察到  $A$  是一个 **Frobenius 友型**,

故其特征多项式可以一眼看出来:  $\det(\lambda I - A) = \lambda^n - 1$ .

因此其特征值为  $\lambda_k = \omega^k$ , 对应的特征向量为  $[1, \omega^k, \dots, \omega^{(n-2)k}, \omega^{(n-1)k}]^T$ ,

其中  $\omega = \exp(2\pi i/n)$ ,  $k = 0, 1, \dots, n-1$ .

一般的 **Frobenius 友型**形如:

$$A := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix} \text{ or } \begin{bmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 & -\alpha_0 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & & & & -\alpha_0 \\ 1 & 0 & & & -\alpha_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -\alpha_{n-2} \\ & & & 1 & -\alpha_{n-1} \end{bmatrix}$$

可以证明其极小多项式  $m_A(t)$  和特征多项式  $p_A(t)$  均为  $t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1t + a_0$ .

特别地, 取第一种形式, 可以证明:

若  $\lambda$  为  $A$  的特征值 (即满足  $\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0$ ),

则  $[1, \lambda, \dots, \lambda^{n-2}, \lambda^{n-1}]^T$  为对应的特征向量:

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} \lambda$$

### Solution:

(我们这里给出最基础的做法, 不使用 Frobenius 友型的结论)

方阵  $A$  的特征多项式为:

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \\ -1 & & & & \lambda \end{vmatrix}_n \\ &= \lambda \begin{vmatrix} \lambda & -1 & & & \\ & \ddots & \ddots & & \\ & & \lambda & -1 & \\ & & & \lambda & \end{vmatrix}_{n-1} + (-1)^{n+1} \cdot (-1) \begin{vmatrix} -1 & & & & \\ \lambda & -1 & & & \\ & \ddots & \ddots & & \\ & & \lambda & -1 & \end{vmatrix}_{n-1} \\ &= \lambda \cdot \lambda^{n-1} + 1 \cdot (-1)^{n-1} \\ &= \lambda^n - 1 \end{aligned}$$

令  $\det(\lambda I - A) = \lambda^n - 1 = 0$ , 可解得  $n$  个根为:

$$\lambda_k = \sqrt[n]{1} \exp \left\{ i \left( \frac{0 + 2k\pi}{n} \right) \right\} = \left( \exp \left( \frac{2\pi i}{n} \right) \right)^k \quad (k = 0, 1, \dots, n-1)$$

若记  $\omega = \exp(2\pi i/n)$ , 则我们可以将方阵  $A$  的  $n$  个特征值写为:

$$\lambda_k = \omega^k \quad (k = 0, 1, \dots, n-1)$$

求解  $\lambda_k$  对应的特征向量就是要求解方程组:

$$(\lambda_k I_n - A)x = \begin{bmatrix} \lambda_k & -1 & & & \\ & \lambda_k & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_k & -1 \\ -1 & & & & \lambda_k \end{bmatrix} x = 0_n$$

$$\Updownarrow$$

$$\begin{cases} x_2 = \lambda_k x_1 \\ x_3 = \lambda_k x_2 \\ \vdots \\ x_n = \lambda_k x_{n-1} \\ x_1 = \lambda_k x_n \end{cases}$$

我们可以取  $\lambda_k = \omega^k$  ( $k = 0, 1, \dots, n-1$ ) 对应的特征向量  $x^{(k)}$  为:

$$x^{(k)} = \begin{bmatrix} 1 \\ \lambda_k \\ \vdots \\ \lambda_k^{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ \omega^k \\ \vdots \\ \omega^{(n-1)k} \end{bmatrix}$$

## Problem 2

试证明复数  $z_1, z_2, z_3$  满足  $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$  的充要条件是

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

- **Lemma 1:**

设  $\omega = \exp(2\pi i/n)$ , 则我们有:

$$\begin{aligned} \sum_{k=0}^{n-1} \omega^k &= 1 + \omega + \dots + \omega^{n-1} \\ &= \frac{1 - \omega^n}{1 - \omega} \quad (\text{note that } \omega^n = \exp\left(\frac{2\pi i}{n} \cdot n\right) = \exp(2\pi i) = 1) \\ &= \frac{1 - 1}{1 - \omega} \\ &= 0 \end{aligned}$$

- **Lemma 2 (两个非零复数的乘积也不是零):**

若  $z_1 z_2 = 0$ , 则  $z_1$  和  $z_2$  至少有一个是零.

- 当  $z_1 = 0$  时, 结论成立.
- 当  $z_1 \neq 0$  时, 可知逆元  $z_1^{-1}$  存在, 我们有  $z_2 = z_2(z_1 z_1^{-1}) = z_1^{-1}(z_1 z_2) = z_1^{-1} \cdot 0 = 0$ .

**Solution:**

若  $z_1, z_2, z_3$  中有任意两个是相等的,

则可根据  $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| = 0$  推出  $z_1 = z_2 = z_3 = 0$ ,

进而有  $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$

此时命题成立.

下证  $z_1, z_2, z_3$  互不相同命题成立.

- **① 必要性:**

若  $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$ , 则  $z_1, z_2, z_3$  三点确定了一个正三角形.

记  $\omega = \exp(2\pi i/3)$ , 则我们有:

$$\begin{cases} z_2 - z_3 = (z_2 - z_1)\omega \\ z_1 - z_3 = (z_2 - z_1)\omega^2 \end{cases}$$

于是有:

$$\begin{aligned} z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 &= \frac{1}{2}(z_2 - z_1)^2 + \frac{1}{2}(z_2 - z_3)^2 + \frac{1}{2}(z_1 - z_3)^2 \\ &= \frac{1}{2}(z_2 - z_1)^2(1 + \omega^2 + \omega^4) \quad (\text{note that } \omega^3 = 1) \\ &= \frac{1}{2}(z_2 - z_1)^2(1 + \omega^2 + \omega) \quad (\text{utilize Lemma 1}) \\ &= \frac{1}{2}(z_2 - z_1)^2 \cdot 0 \\ &= 0 \end{aligned}$$

• ② 充分性:

根据 Lemma 1 我们有  $\omega^2 + \omega + 1 = 0$ , 即有  $\omega^2 + \omega = -1$ .

若  $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$ , 则我们有:

$$\begin{aligned} 0 &= z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 \quad (\text{note that } \omega^3 = 1 \text{ and } \omega^2 + \omega = -1) \\ &= z_1^2 + \omega^3 z_2^2 + \omega^3 z_3^2 + (\omega^2 + \omega)z_1z_2 + (\omega^2 + \omega)z_2z_3 + (\omega^2 + \omega)z_3z_1 \\ &= (z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) \end{aligned}$$

根据 Lemma 2 可知  $z_1 + \omega z_2 + \omega^2 z_3$  和  $z_1 + \omega^2 z_2 + \omega z_3$  至少有一个是零.

不失一般性, 设  $z_1 + \omega z_2 + \omega^2 z_3 = 0$ , 则左右同乘  $(\omega^2 - \omega)$  可得:

$$\begin{aligned} 0 &= (\omega^2 - \omega)(z_1 + \omega z_2 + \omega^2 z_3) \\ &= (\omega^2 - \omega)z_1 + (\omega^3 - \omega^2)z_2 + (\omega^4 - \omega^3)z_3 \\ &= (\omega^2 - \omega)z_1 + (1 - \omega^2)z_2 + (\omega - 1)z_3 \\ &= (\omega - 1)(z_3 - z_1) + (\omega^2 - 1)(z_1 - z_2) \\ &= (\omega - 1)((z_3 - z_1) + (\omega + 1)(z_1 - z_2)) \\ &= (\omega - 1)((z_3 - z_2) + \omega(z_1 - z_2)) \end{aligned}$$

由于  $\omega - 1 = \exp(2\pi i/3) - 1 \neq 0$ , 故根据 Lemma 2 可知:

$$\begin{cases} (z_3 - z_1) + (\omega + 1)(z_1 - z_2) = 0 \\ (z_3 - z_2) + \omega(z_1 - z_2) = 0 \end{cases}$$

注意到:

$$\begin{cases} |\omega + 1| = |-\frac{1}{2} + \frac{\sqrt{3}}{2}i + 1| = 1 \\ |\omega| = |\exp(\frac{2\pi i}{3})| = 1 \end{cases}$$

于是我们有:

$$\begin{aligned} |z_3 - z_1| &= |-(\omega + 1)(z_2 - z_1)| \\ &= |\omega + 1| \cdot |z_2 - z_1| \\ &= 1 \cdot |z_2 - z_1| \\ &= |z_2 - z_1| \\ \hline |z_3 - z_2| &= |-\omega(z_2 - z_1)| \\ &= |\omega| \cdot |z_2 - z_1| \\ &= 1 \cdot |z_2 - z_1| \\ &= |z_2 - z_1| \end{aligned}$$

因此  $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$ .

综上所述, 命题得证.

事实上, 下列命题是等价的:

- $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$
- $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = \frac{1}{2}[(z_2 - z_1)^2 + (z_2 - z_3)^2 + (z_1 - z_3)^2] = 0$
- $z_1 + \omega z_2 + \omega^2 z_3 = 0$  (其中  $\omega = \exp(2\pi i/3)$ )

(2025 补充习题)

已知  $z_1 = 1 - i$ ,  $z_2 = 2 + 3i$ , 试求复数  $z_3$  使得  $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$ .

**Solution:**

根据  $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$  可知  $z_1, z_2, z_3$  在复平面中构成正三角形的顶点.

因此令  $z_2 - z_1$  顺/逆时针旋转  $\pi/3$  便可得到  $z_3 - z_1$ .

设  $z_3 = \alpha + \beta i$ , 令  $z_3 - z_1 = \exp(\pm \pi i/3)(z_2 - z_1)$ , 则我们有:

$$\begin{aligned} z_3 &= z_1 + \exp(\pm \pi i/3)(z_2 - z_1) \\ &= 1 - i + \frac{1}{2}(1 \pm \sqrt{3}i)(2 + 3i - 1 + i) \\ &= \frac{1}{2}(3 \mp 4\sqrt{3} + (2 \pm \sqrt{3})i) \end{aligned}$$

因此  $z_3$  有两个解:

- 一个是  $\frac{1}{2}(3 - 4\sqrt{3} + (2 + \sqrt{3})i)$
- 一个是  $\frac{1}{2}(3 + 4\sqrt{3} + (2 - \sqrt{3})i)$

## Problem 3

给定的正整数  $m, n$ , 记  $\omega = \exp(2\pi i/m)$ .

试证明对任何  $A, B \in \mathbb{C}^{n \times n}$  都有:

$$A^m + B^m = \frac{1}{m} \sum_{k=0}^{m-1} (A + \omega^k B)^m$$

- **Lemma (两个非零复数的乘积也不是零):**

若  $z_1 z_2 = 0$ , 则  $z_1$  和  $z_2$  至少有一个是零.

- 当  $z_1 = 0$  时, 结论成立.
- 当  $z_1 \neq 0$  时, 可知逆元  $z_1^{-1}$  存在, 我们有  $z_2 = z_2(z_1 z_1^{-1}) = z_1^{-1}(z_1 z_2) = z_1^{-1} \cdot 0 = 0$ .

**Solution:**

注意到  $A, B$  不一定是可交换的 (即  $AB \neq BA$ ).

因此  $(A + B)^m$  不能简单展开为  $\sum_{j=0}^m \binom{m}{j} A^{m-j} B^j$ .

我们记  $(A + B)^m$  的展开式中  $\binom{m}{j}$  个由  $m - j$  个  $A$  和  $j$  个  $B$  构成的项之和为  $\text{term}(A, B, j)$ .

显然我们有  $\text{term}(A, \omega^k B, j) = \omega^{jk} \text{term}(A, B, j)$  成立.

因为有限求和是可以交换次序的, 所以我们有:

$$\begin{aligned} \frac{1}{m} \sum_{k=0}^{m-1} (A + \omega^k B)^m &= \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \sum_{j=0}^m \text{term}(A, \omega^k B, j) \right\} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \sum_{j=0}^m \omega^{jk} \text{term}(A, B, j) \right\} \\ &= \frac{1}{m} \sum_{j=0}^m \left\{ \left( \sum_{k=0}^{m-1} \omega^{jk} \right) \text{term}(A, B, j) \right\} \end{aligned} \tag{3.1}$$

注意到  $\omega^m = 1$  (即  $\omega$  是 1 的一个  $m$  次复根).

- 当  $j$  为  $m$  的整数倍 (即  $j$  为 0 或  $m$ ) 时, 我们有:

$$\sum_{k=0}^{m-1} \omega^{jk} = \sum_{k=0}^{m-1} 1 = m$$

- 当  $j$  不为  $m$  的整数倍 (即  $j = 1, \dots, m-1$ ) 时, 我们有:

$$\begin{aligned} \omega^j \sum_{k=0}^{m-1} \omega^{jk} &= \omega^j (1 + \omega^j + \dots + \omega^{(m-2)j} + \omega^{(m-1)j}) \\ &= \omega^j + \omega^{2j} + \dots + \omega^{(m-1)j} + \omega^{mj} \quad (\text{note that } \omega^m = 1) \\ &= \omega^j + \omega^{2j} + \dots + \omega^{(m-1)j} + 1 \\ &= \sum_{k=0}^{m-1} \omega^{jk} \end{aligned}$$

于是有  $(\omega^j - 1) \sum_{k=0}^{m-1} \omega^{jk} = 0$  成立.

由于  $\omega^j - 1 = \exp\left(\frac{2j\pi i}{m}\right) - 1 \neq 0$ , 故根据 **Lemma** 可知  $\sum_{k=0}^{m-1} \omega^{jk} = 0$ .

综上所述, 我们有:

$$\sum_{k=0}^{m-1} \omega^{jk} = \begin{cases} m, & \text{if } j = 0, m \\ 0, & \text{if } j = 1, \dots, m-1 \end{cases}$$

将上述结果代入 (3.1) 式中我们有:

$$\begin{aligned} \frac{1}{m} \sum_{k=0}^{m-1} (A + \omega^k B)^m &= \frac{1}{m} \sum_{j=0}^m \left\{ \left( \sum_{k=0}^{m-1} \omega^{jk} \right) \text{term}(A, B, j) \right\} \\ &= \frac{1}{m} (mA^m + mB^m) \\ &= A^m + B^m \end{aligned}$$

命题得证.

## Problem 4

以下内容均来自 **Complex Variables and Applications (9th Edition J. Brown, R. Churchill) Chapter 6**

### (Complex Variables and Applications 第 74 节)

若函数在简单闭围道  $C$  的内部除了有限多个奇点以外处处解析, 则这些奇点必定是孤立奇点.

特殊地, 有理函数 (即两个多项式函数的商) 的奇点总是孤立奇点, 因为分母中的多项式函数仅有有限个零点.

### (Complex Variables and Applications 第 75 节)

若  $z_0$  是函数  $f$  的孤立奇点, 则存在正数  $R > 0$  使得  $f$  在  $0 < |z - z_0| < R$  中的任意一点  $z$  处解析

因此函数  $f$  关于  $z_0$  的 Laurent 级数展开式为:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \\ a_n &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (n = 0, 1, \dots) \\ b_n &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \quad (n = 1, 2, \dots) \end{aligned}$$

其中  $C$  为  $0 < |z - z_0| < R$  中任意围绕  $z_0$  的简单正向闭围道.

特别地,  $b_1$  的表达式为  $b_1 = \frac{1}{2\pi i} \oint_C f(z) dz$

我们称其为函数  $f$  在孤立奇点  $z_0$  处的 **留数** (residue), 记为  $\text{Res}f(z)$

于是我们有:

$$\oint_C f(z) dz = 2\pi i \cdot \text{Res}f(z)$$

### (Cauchy 留数定理, Complex Variables and Applications 第 76 节)

设  $C$  为正向简单闭围道.

若函数  $f$  在  $C$  及其内部除了有限多个奇点  $z_k$  ( $k = 1, \dots, n$ ) 以外处处解析 (自然是孤立奇点),

则我们有:

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}f(z)$$

即  $f$  沿  $C$  的积分值  $\oint_C f(z) dz$  为其内部有限个奇点处的留数之和的  $2\pi i$  倍.

下面的定理仅仅涉及一个留数, 故运用起来有时比 Cauchy 留数定理更加方便:

### (Complex Variables and Applications 第 77 节 定理)

若函数  $f$  在有限平面上除了有限多个奇点以外处处解析, 且这些奇点落在一条正向简单闭围道  $C$  的内部,

则我们有:

$$\oint_C f(z) dz = 2\pi i \cdot \text{Res}_{z=0} \left\{ \frac{1}{z^2} f\left(\frac{1}{z}\right) \right\}$$

计算复积分:

$$\oint_{|z|=4} \frac{1}{z^2 - 3z + 2} dz$$

记围道  $C$  为  $|z| = 4$  确定的正向圆周 (即逆时针方向).

注意到多项式函数  $p(z) = z^2 - 3z + 2$  在整个复平面都是解析的, 且仅有  $z = 1, 2$  两个零点.

因此  $f(z) = 1/p(z)$  在复平面上仅有  $z = 1, 2$  两个孤立奇点, 且都落在围道  $C$  的内部.

#### Solution 1:

最直接的做法是计算  $f(z)$  的留数.

- 计算  $f(z)$  在  $z = 1$  附近的 Laurent 级数展开:

$$\begin{aligned} f(z) &= \frac{1}{z^2 - 3z + 2} \quad (\text{suppose that } |z - 1| < 1) \\ &= \frac{1}{(z - 1)(z - 2)} \quad (\text{denote } w := z - 1) \\ &= \frac{1}{w(w - 1)} \quad (\text{note that } \frac{1}{1 - w} = \sum_{n=0}^{\infty} w^n, \text{ whenever } |w| < 1) \\ &= -\frac{1}{w} \sum_{n=0}^{\infty} w^n \\ &= -\sum_{n=0}^{\infty} w^{n-1} \end{aligned}$$

其  $1/w$  项的系数为  $-1$ , 故  $f(z)$  在  $z = 1$  处的留数  $\operatorname{Res}_{z=1} f(z) = -1$ .

事实上, 由于  $z = 1$  是  $f(z)$  的简单奇点, 我们可通过极限计算对应的留数:

$$\operatorname{Res}_{z=1} f(z) = \lim_{z \rightarrow 1} (z - 1)f(z) = \lim_{z \rightarrow 1} \frac{1}{z - 2} = \frac{1}{1 - 2} = -1$$

- 计算  $f(z)$  在  $z = 2$  附近的 Laurent 级数展开:

$$\begin{aligned} f(z) &= \frac{1}{z^2 - 3z + 2} \quad (\text{suppose that } |z - 2| < 1) \\ &= \frac{1}{(z - 1)(z - 2)} \quad (\text{denote } w := z - 2) \\ &= \frac{1}{(w + 1)w} \quad (\text{note that } \frac{1}{1 + w} = \sum_{n=0}^{\infty} (-w)^n, \text{ whenever } |w| < 1) \\ &= \frac{1}{w} \sum_{n=0}^{\infty} (-w)^n \\ &= -\sum_{n=0}^{\infty} (-w)^{n-1} \end{aligned}$$

其  $1/w$  项的系数为  $1$ , 故  $f(z)$  在  $z = 2$  处的留数  $\operatorname{Res}_{z=2} f(z) = 1$ .

事实上, 由于  $z = 2$  是  $f(z)$  的简单奇点, 我们可通过极限计算对应的留数:

$$\operatorname{Res}_{z=2} f(z) = \lim_{z \rightarrow 2} (z - 2)f(z) = \lim_{z \rightarrow 2} \frac{1}{z - 1} = \frac{1}{2 - 1} = 1$$

根据 Cauchy 留数定理可知:

$$\begin{aligned} \oint_{|z|=4} f(z) dz &= 2\pi i \cdot \left( \operatorname{Res}_{z=1} f(z) + \operatorname{Res}_{z=2} f(z) \right) \\ &= 2\pi i \cdot (-1 + 1) \\ &= 0 \end{aligned}$$

#### Solution 2:

另一种方法是计算  $\frac{1}{z^2} f\left(\frac{1}{z}\right)$  在  $z = 0$  处的留数.

定义  $g(z) := \frac{1}{z^2} f\left(\frac{1}{z}\right)$ , 则我们有:

$$\begin{aligned}
g(z) &= \frac{1}{z^2} f\left(\frac{1}{z}\right) \\
&= \frac{1}{z^2} \cdot \frac{z^2}{1 - 3z + 2z^2} \\
&= \frac{1}{(1 - 2z)(1 - z)}
\end{aligned}$$

根据 **Lemma** 可知  $g(z)$  的 Laurent 级数中  $1/z$  项的系数即为所求留数.  
注意到  $g(z)$  在  $z = 0$  处是解析的, 其 Laurent 级数展开即为 Taylor 级数展开,  
因此没有  $1/z$  项, 于是有:

$$\operatorname{Res}_{z=0} \left\{ \frac{1}{z^2} f\left(\frac{1}{z}\right) \right\} = 0$$

因此  $f$  在  $C$  上的积分为:

$$\oint_C f(z) dz = \oint_C \frac{1}{z^2 - 3z + 2} dz = 2\pi i \cdot \operatorname{Res}_{z=0} \left\{ \frac{1}{z^2} f\left(\frac{1}{z}\right) \right\} = 2\pi i \cdot 0 = 0$$

## Problem 5

试证明任何复方阵都可以在复数域上相似上三角化,  
即对于任意复方阵  $A \in \mathbb{C}^{n \times n}$  总存在非奇异阵  $P$  使得  $P^{-1}AP$  为上三角矩阵.

**Solution:**

当  $n = 1$  时, 命题显然成立.

当  $n \geq 2$  时, 假设对于所有维数小于  $n$  的复方阵, 上述命题都成立.

下面对  $n$  维复方阵证明该命题.

设  $(\lambda_1, x_1)$  是  $A \in \mathbb{C}^{n \times n}$  的一个特征对, 即满足  $Ax_1 = x_1\lambda_1$ .

将  $x_1$  扩充为  $\mathbb{C}^n$  的一组基  $x_1, v_2, \dots, v_n$ ,

定义非奇异阵  $P_1 := [x_1, v_2, \dots, v_n] = [x_1, V]$ , 则我们有:

$$\begin{aligned}
AP_1 &= A[x_1, V] \\
&= [Ax_1, AV] \\
&= [x_1\lambda_1, AV] \\
&= [x_1, V][\lambda_1 e_1, P_1^{-1}AV] \quad (\text{denote } P_1^{-1}AV = \begin{bmatrix} * \\ A_2 \end{bmatrix} \in \mathbb{C}^{n \times (n-1)}) \\
&= [x_1, V] \begin{bmatrix} \lambda_1 & * \\ & A_2 \end{bmatrix} \\
&= P_1 \begin{bmatrix} \lambda_1 & * \\ & A_2 \end{bmatrix}
\end{aligned}$$

根据归纳假设可知, 存在非奇异阵  $\tilde{P}_2 \in \mathbb{C}^{(n-1) \times (n-1)}$  使得  $T_2 := \tilde{P}_2^{-1}A_2\tilde{P}_2$  为上三角阵.

定义  $P_2 := 1 \oplus \tilde{P}_2$  和  $P = P_1P_2$  可知:

$$\begin{aligned}
P^{-1}AP &= P_2^{-1}P_1^{-1}AP_1P_2 \\
&= \begin{bmatrix} 1 & \\ & P_2^{-1} \end{bmatrix} \begin{bmatrix} \lambda_1 & * \\ & A_2 \end{bmatrix} \begin{bmatrix} 1 & \\ & P_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & * \\ & P_2^{-1}A_2P_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & * \\ & T_2 \end{bmatrix}
\end{aligned}$$

因此  $T := P^{-1}AP$  为上三角阵.

根据数学归纳法, 命题得证.

## Problem 6 (optional)

设  $n$  为正整数.

已知  $n$  次系数多项式  $f(z) = \sum_{k=0}^n a_k z^k$  的系数满足  $a_0 > \dots > a_n > 0$

证明:  $f(z)$  的所有复根都在单位圆外.

- **Hint:** 考察  $g(z) = (1-z)f(z)$

**Solution:**

当  $|z| \leq 1$  时, 我们有:

$$\begin{aligned}
 |g(z)| &= |(1-z)f(z)| \\
 &= \left| (1-z) \sum_{k=0}^n a_k z^k \right| \\
 &= \left| a_0 + \sum_{k=1}^n (a_k - a_{k-1})z^k - a_n z^{n+1} \right| \quad (\text{triangle inequality}) \\
 &\geq |a_0| - \left| \sum_{k=1}^n (a_k - a_{k-1})z^k - a_n z^{n+1} \right| \quad (\text{triangle inequality and } |z_1 z_2| = |z_1| |z_2| \text{ for all } z_1, z_2 \in \mathbb{C}) \\
 &\geq |a_0| - \sum_{k=1}^n |a_k - a_{k-1}| |z|^k - |a_n| |z|^{n+1} \quad (\text{note that } a_0 > \dots > a_n > 0) \\
 &= a_0 - \sum_{k=1}^n (a_{k-1} - a_k) |z|^k - a_n |z|^{n+1} \quad (\text{note that } |z| \leq 1) \\
 &\geq a_0 - \sum_{k=1}^n (a_{k-1} - a_k) \cdot 1 - a_n \cdot 1 \\
 &= a_0 - (a_0 - a_n) - a_n \\
 &= 0
 \end{aligned}$$

上述三个不等号同时取等的充要条件是:

- ①  $\sum_{k=1}^n (a_k - a_{k-1})z^k - a_n z^{n+1}$  与  $a_0$  反方向 (即与 1 反方向)  
(注意  $a_0$  是正实数, 而  $a_k - a_{k-1} < 0$  ( $k = 1, \dots, n$ ))
- ②  $z, z^2, \dots, z^{n+1}$  同方向
- ③  $|z| = 1$

容易验证这样的  $z$  只能是  $z = \exp(2m\pi i) = 1$  ( $m \in \mathbb{Z}$ ).

因此当  $|z| \leq 1$  且  $z \neq 1$  时, 我们都有  $|g(z)| > 0$  成立, 表明这样的  $z$  不是  $g(z)$  的根.

于是  $g(z)$  的根要么是  $z = 1$ , 要么满足  $|z| > 1$ .

注意到  $g(z) = (1-z)f(z)$  的复根除了额外的 1 以外, 其余复根都与  $f(z)$  的相同.

而根据  $f(1) = \sum_{k=0}^n a_k > 0$  可知  $z = 1$  不是  $f(z)$  的根.

因此  $f(z)$  的所有根都满足  $|z| > 1$ , 即都落在单位圆周  $|z| = 1$  的外部.

## Problem 7 (optional)

证明下面的函数不是解析函数, 但在复平面上处处满足 Cauchy-Riemann 方程:

$$f(z) = \begin{cases} \exp(-z^{-4}), & z \neq 0, \\ 0, & z = 0. \end{cases}$$

- (可导的必要条件, Complex Variables and Applications 第 21 节)

设函数  $f(z) = u(x, y) + iv(x, y)$  在点  $z_0 = (x_0, y_0)$  处可导,

则  $u, v$  在  $(x_0, y_0)$  处可偏导, 且其一阶偏导数满足 **Cauchy-Riemann 方程**:

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{cases}$$

此时导数  $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$ .

(极坐标形式)

设函数  $f(z) = u(\rho, \theta) + iv(\rho, \theta)$  在点  $z_0 = \rho_0 e^{i\theta_0}$  处可导,

则  $u, v$  在  $(\rho_0, \theta_0)$  处可偏导, 且其一阶偏导数满足 **Cauchy-Riemann 方程**:

$$\begin{cases} \rho u_\rho(\rho_0, \theta_0) = v_\theta(\rho_0, \theta_0) \\ u_\theta(\rho_0, \theta_0) = -\rho v_\rho(\rho_0, \theta_0) \end{cases}$$

此时导数  $f'(z_0) = e^{-i\theta_0}(u_\rho(\rho_0, \theta_0) + iv_\rho(\rho_0, \theta_0))$ .

**Solution:**

当  $z = 0$  时:



- 计算  $u_x(0, 0)$  和  $v_x(0, 0)$ :  
(取  $z = h$ , 从实轴方向逼近 0)

$$u_x(0, 0) = \lim_{h \rightarrow 0} \frac{u(h, 0) - u(0, 0)}{h - 0} = \lim_{h \rightarrow 0} \frac{\operatorname{Re}\{\exp(-h^{-4})\} - 0}{h - 0} = \lim_{h \rightarrow 0} \frac{\exp(-h^{-4}) - 0}{h - 0} = 0$$

$$v_x(0, 0) = \lim_{h \rightarrow 0} \frac{v(h, 0) - v(0, 0)}{h - 0} = \lim_{h \rightarrow 0} \frac{\operatorname{Im}\{\exp(-h^{-4})\} - 0}{h - 0} = \lim_{h \rightarrow 0} \frac{0 - 0}{h - 0} = 0$$

- 计算  $u_y(0, 0)$  和  $v_y(0, 0)$ :  
(取  $z = ih$ , 从虚轴方向逼近 0)

$$u_y(0, 0) = \lim_{h \rightarrow 0} \frac{u(0, h) - u(0, 0)}{h - 0} = \lim_{h \rightarrow 0} \frac{\operatorname{Re}\{\exp(-(ih)^{-4})\} - 0}{h - 0} = \lim_{h \rightarrow 0} \frac{\exp(-h^{-4}) - 0}{h - 0} = 0$$

$$v_y(0, 0) = \lim_{h \rightarrow 0} \frac{v(0, h) - v(0, 0)}{h - 0} = \lim_{h \rightarrow 0} \frac{\operatorname{Im}\{\exp(-(ih)^{-4})\} - 0}{h - 0} = \lim_{h \rightarrow 0} \frac{0 - 0}{h - 0} = 0$$

因此我们有:

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) = 0 \\ u_y(x_0, y_0) = -v_x(x_0, y_0) = 0 \end{cases}$$

即  $f$  在  $z = 0$  处满足 Cauchy-Riemann 方程.

当  $z \neq 0$  时, 我们有:

$$\begin{aligned} f(z) &= \exp(-z^{-4}) \\ &= \exp(-\rho^{-4}e^{-4i\theta}) \\ &= \exp(-\rho^{-4}(\cos(-4\theta) + i\sin(-4\theta))) \\ &= \exp(-\rho^{-4}\cos(-4\theta)) \cdot \exp(i \cdot (-\rho^{-4}\sin(-4\theta))) \\ &= \exp(-\rho^{-4}\cos(-4\theta)) \cdot (\cos(-\rho^{-4}\sin(-4\theta)) + i\sin(-\rho^{-4}\sin(-4\theta))) \\ &= u(\rho, \theta) + iv(\rho, \theta) \end{aligned}$$

其中我们记:

$$\begin{aligned} u(\rho, \theta) &= \exp(-\rho^{-4}\cos(-4\theta)) \cos(-\rho^{-4}\sin(-4\theta)) = g(\rho, \theta) \cos(h(\rho, \theta)) \\ v(\rho, \theta) &= \exp(-\rho^{-4}\cos(-4\theta)) \sin(-\rho^{-4}\sin(-4\theta)) = g(\rho, \theta) \sin(h(\rho, \theta)) \end{aligned}$$

where  $\begin{cases} g(\rho, \theta) = \exp(-\rho^{-4}\cos(-4\theta)) \\ h(\rho, \theta) = -\rho^{-4}\sin(-4\theta) \end{cases} \Rightarrow \begin{cases} g_\rho(\rho, \theta) = 4\rho^{-5}\cos(-4\theta)g(\rho, \theta) \\ g_\theta(\rho, \theta) = -4\rho^{-4}\sin(-4\theta)g(\rho, \theta) \\ h_\rho(\rho, \theta) = 4\rho^{-5}\sin(-4\theta) \\ h_\theta(\rho, \theta) = 4\rho^{-4}\cos(-4\theta) \end{cases}$

经计算可得:

$$\begin{aligned} u_\rho(\rho, \theta) &= g_\rho(\rho, \theta) \cos(h(\rho, \theta)) + g(\rho, \theta)[- \sin(h(\rho, \theta))h_\rho(\rho, \theta)] \\ &= 4\rho^{-5}\cos(-4\theta)g(\rho, \theta) \cos(h(\rho, \theta)) - g(\rho, \theta) \sin(h(\rho, \theta))4\rho^{-5}\sin(-4\theta) \\ &= 4\rho^{-5}g(\rho, \theta) \cos(h(\rho, \theta) - 4\theta) \\ \hline u_\theta(\rho, \theta) &= g_\theta(\rho, \theta) \cos(h(\rho, \theta)) + g(\rho, \theta)[- \sin(h(\rho, \theta))h_\theta(\rho, \theta)] \\ &= -4\rho^{-4}\sin(-4\theta)g(\rho, \theta) \cos(h(\rho, \theta)) - g(\rho, \theta) \sin(h(\rho, \theta))4\rho^{-4}\cos(-4\theta) \\ &= -4\rho^{-4}g(\rho, \theta) \sin(h(\rho, \theta) - 4\theta) \\ \hline v_\rho(\rho, \theta) &= g_\rho(\rho, \theta) \sin(h(\rho, \theta)) + g(\rho, \theta) \cos(h(\rho, \theta))h_\rho(\rho, \theta) \\ &= 4\rho^{-5}\cos(-4\theta)g(\rho, \theta) \sin(h(\rho, \theta)) + g(\rho, \theta) \cos(h(\rho, \theta))4\rho^{-5}\sin(-4\theta) \\ &= 4\rho^{-5}g(\rho, \theta) \sin(h(\rho, \theta) - 4\theta) \\ \hline v_\theta(\rho, \theta) &= g_\theta(\rho, \theta) \sin(h(\rho, \theta)) + g(\rho, \theta) \cos(h(\rho, \theta))h_\theta(\rho, \theta) \\ &= -4\rho^{-4}\sin(-4\theta)g(\rho, \theta) \sin(h(\rho, \theta)) + g(\rho, \theta) \cos(h(\rho, \theta))4\rho^{-4}\cos(-4\theta) \\ &= 4\rho^{-4}g(\rho, \theta) \cos(h(\rho, \theta) - 4\theta) \end{aligned}$$

因此对于任意  $\rho > 0$  和  $\theta \in \mathbb{R}$  我们都有:

$$\begin{cases} \rho u_\rho(\rho, \theta) = v_\theta(\rho, \theta) \\ u_\theta(\rho, \theta) = -\rho v_\rho(\rho, \theta) \end{cases}$$

因此  $f$  在任意  $z \neq 0$  处都满足 Cauchy-Riemann 方程.

综上所述,  $f$  在复平面上处处满足 Cauchy-Riemann 方程.

解析函数要求在定义域内处处可导, 但  $f(z)$  在  $z = 0$  处不可导.

我们考虑  $|f(z)|$  在  $z \rightarrow 0$  时的极限行为:

$$\begin{aligned} |f(z)| &= |\exp(-z^{-4})| \\ &= |\exp(-\rho^{-4}e^{-4i\theta})| \\ &= |\exp(-\rho^{-4}\cos(-4\theta)) \exp(-\rho^{-4}\sin(-4\theta)i)| \\ &= \exp(-\rho^{-4}\cos(4\theta)) \end{aligned}$$

- 若  $\cos(4\theta) > 0$ , 则当  $\rho \rightarrow 0_+$  时有  $|f(z)| \rightarrow 0$ .
- 若  $\cos(4\theta) = 0$ , 则  $|f(z)| = 1$  ( $\forall \rho \geq 0$ ).
- 若  $\cos(4\theta) < 0$ , 则当  $\rho \rightarrow 0_+$  时有  $|f(z)| \rightarrow \infty$ .

因此  $f$  在  $z = 0$  处不连续, 更谈不上在  $z = 0$  处复可微了.

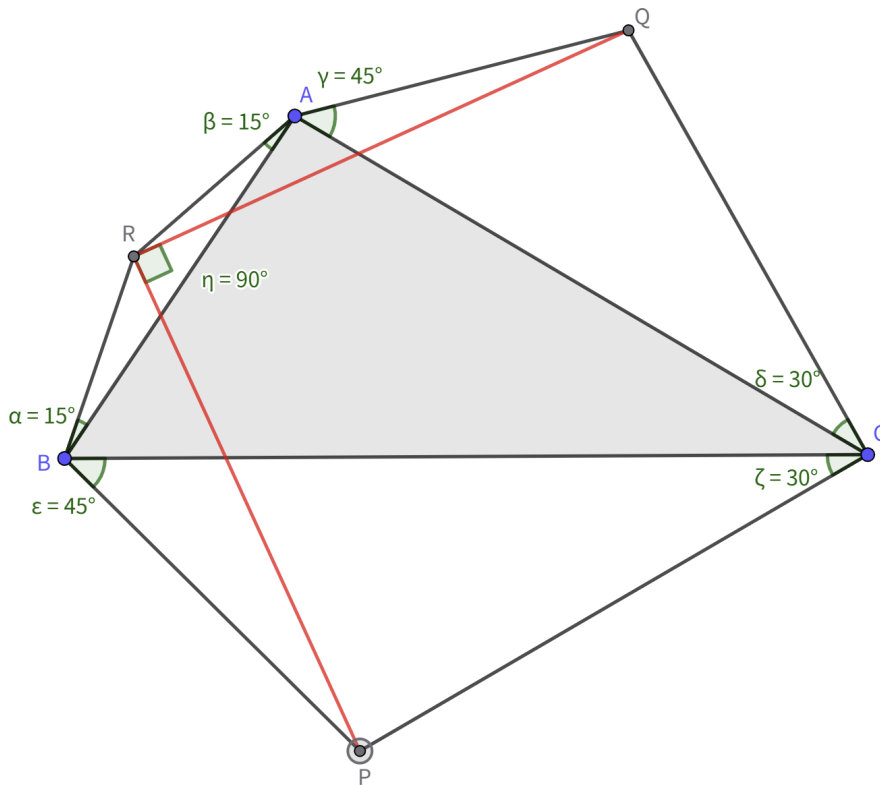
故  $f$  不是解析函数.

## Problem 8 (optional)

对于 Euclid 平面  $\mathbb{R}^2$  内的任意三角形  $\triangle ABC$

向外作  $\angle ABR, \angle BCP, \angle CAQ$  使得  $\begin{cases} \angle CBP = \angle CAQ = 45^\circ \\ \angle BCP = \angle ACQ = 30^\circ \\ \angle ABR = \angle BAR = 15^\circ \end{cases}$

试利用复数证明  $\angle QRP = 90^\circ$  且  $|QR| = |RP|$ .



**Solution:**

记  $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OR}, \overrightarrow{OP}, \overrightarrow{OQ}$  的复数表示为  $a, b, c, z_1, z_2, z_3$ .

根据  $1 - 2(\sin(\frac{\pi}{12}))^2 = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$  可解得  $\sin(\frac{\pi}{12}) = \frac{\sqrt{6}-\sqrt{2}}{4}$ , 进而有  $\cos(\frac{\pi}{12}) = \frac{\sqrt{6}+\sqrt{2}}{4}$ .

根据正弦定理可知:

$$\begin{aligned} \frac{|BR|}{|BA|} &= \frac{|AR|}{|AB|} = \frac{\sin(\frac{\pi}{12})}{\sin(\frac{5\pi}{6})} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{1}{2}} = \frac{\sqrt{6}-\sqrt{2}}{2} \\ \frac{|BP|}{|BC|} &= \frac{|AQ|}{|AC|} = \frac{\sin(\frac{\pi}{6})}{\sin(\frac{7\pi}{12})} = \frac{\frac{1}{2}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{\sqrt{6}-\sqrt{2}}{2} \end{aligned}$$

记  $\omega = \exp(\pi i/12)$ , 则我们有:

$$\begin{cases} z_1 - b = \overrightarrow{BR} = \frac{\sqrt{6}-\sqrt{2}}{2} \overrightarrow{\omega BA} = \frac{\sqrt{6}-\sqrt{2}}{2} \omega(a-b) \\ z_1 - a = \overrightarrow{AR} = \frac{\sqrt{6}-\sqrt{2}}{2} \overrightarrow{\bar{\omega} AB} = \frac{\sqrt{6}-\sqrt{2}}{2} \bar{\omega}(b-a) \\ z_2 - b = \overrightarrow{BP} = \frac{\sqrt{6}-\sqrt{2}}{2} \bar{\omega}^3 \overrightarrow{BC} = \frac{\sqrt{6}-\sqrt{2}}{2} \bar{\omega}^3(c-b) \\ z_3 - a = \overrightarrow{AQ} = \frac{\sqrt{6}-\sqrt{2}}{2} \omega^3 \overrightarrow{AC} = \frac{\sqrt{6}-\sqrt{2}}{2} \omega^3(c-a) \end{cases} \quad \text{where} \quad \begin{cases} \omega = \cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i \\ \bar{\omega} = \cos\left(-\frac{\pi}{12}\right) + i \sin\left(-\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}i \\ \bar{\omega}^3 = \cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \\ \omega^3 = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \end{cases}$$

要证明 " $\angle QRP = 90^\circ$  且  $|QR| = |RP|$ ", 即要证  $\overrightarrow{RQ} = \exp(i\pi/2)\overrightarrow{RP}$ ,  
也就等价于证明  $z_3 - z_1 = i(z_2 - z_1)$ .

$$\begin{aligned} z_3 - z_1 - i(z_2 - z_1) &= ((z_3 - a) - (z_1 - a)) - i((z_2 - b) - (z_1 - b)) \\ &= \left( \frac{\sqrt{6}-\sqrt{2}}{2} \omega^3(c-a) - \frac{\sqrt{6}-\sqrt{2}}{2} \bar{\omega}(b-a) \right) - i \left( \frac{\sqrt{6}-\sqrt{2}}{2} \bar{\omega}^3(c-b) - \frac{\sqrt{6}-\sqrt{2}}{2} \omega(a-b) \right) \\ &= \frac{\sqrt{6}-\sqrt{2}}{2} (\omega^3(c-a) - \bar{\omega}(b-a) - i\bar{\omega}^3(c-b) + i\omega(a-b)) \end{aligned}$$

因此要证明  $z_3 - z_1 - i(z_2 - z_1) = 0$ , 等价于证明  $\omega^3(c-a) - \bar{\omega}(b-a) - i\bar{\omega}^3(c-b) + i\omega(a-b) = 0$ ,  
也就等价于证明  $a, b, c$  项的系数分别为 0:

- $a$  的系数为:

$$\begin{aligned} -\omega^3 + \bar{\omega} + i\omega &= -\left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right) + \left( \frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}i \right) + i \left( \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i \right) \\ &= \left( -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4} \right) + i \left( -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{6}+\sqrt{2}}{4} \right) \\ &= 0 \end{aligned}$$

- $b$  的系数为:

$$\begin{aligned} -\bar{\omega} + i\bar{\omega}^3 - i\omega &= -\left( \frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}i \right) + i \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right) - i \left( \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i \right) \\ &= \left( -\frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{2}}{2} + \frac{\sqrt{6}-\sqrt{2}}{4} \right) + i \left( \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{2}}{2} - \frac{\sqrt{6}+\sqrt{2}}{4} \right) \\ &= 0 \end{aligned}$$

- $c$  的系数为:

$$\begin{aligned} \omega^3 - i\bar{\omega}^3 &= \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right) - i \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right) \\ &= \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) + i \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \\ &= 0 \end{aligned}$$

命题得证.

## Problem 9

记  $z = x + iy$  ( $x, y \in \mathbb{R}$ ).

关于复变量  $z$  的函数  $f$  可以视为关于独立实变量  $x, y$  的二元函数,

而  $x, y$  与  $z, \bar{z}$  可以互相线性表示, 因此函数  $f$  从形式上可视为关于独立变量  $z, \bar{z}$  的二元函数.

试证明在此意义下 Cauchy-Riemann 方程可表示为  $\partial f / \partial \bar{z} = 0$ .

**Solution:**

注意到  $x, y$  与  $z, \bar{z}$  可以互相线性表示:

$$\begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(z + \bar{z}) \\ y = \frac{1}{2i}(z - \bar{z}) = -\frac{i}{2}(z - \bar{z}) \end{cases}$$

于是我们有:

$$\begin{aligned}
df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\
&= \frac{\partial f}{\partial x} \left( \frac{\partial x}{\partial z} dz + \frac{\partial x}{\partial \bar{z}} d\bar{z} \right) + \frac{\partial f}{\partial y} \left( \frac{\partial y}{\partial z} dz + \frac{\partial y}{\partial \bar{z}} d\bar{z} \right) \\
&= \frac{\partial f}{\partial x} \left( \frac{1}{2} dz + \frac{1}{2} d\bar{z} \right) + \frac{\partial f}{\partial y} \left( -\frac{i}{2} dz + \frac{i}{2} d\bar{z} \right) \\
&= \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) dz + \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) d\bar{z} \\
&= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}
\end{aligned}$$

因此我们有:

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \\
\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)
\end{aligned}$$

记  $f(x, y) = u(x, y) + iv(x, y)$ , 则我们有:

$$\begin{aligned}
\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \\
&= \frac{1}{2} (u_x + iv_x + i(u_y + iv_y)) \\
&= \frac{1}{2} (u_x - v_y + i(v_x + u_y))
\end{aligned}$$

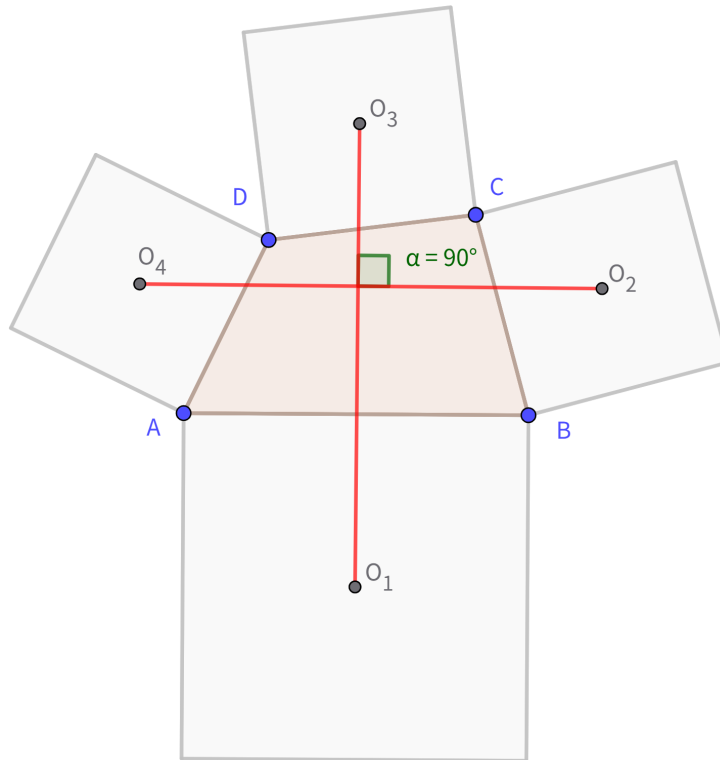
因此 Cauchy-Riemann 方程  $\begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$  就等价于  $\partial f / \partial \bar{z} = 0$ .

## Problem 10 (optional)

利用复数证明 **Van Aubel 定理**:

在 Euclid 平面内, 由凸四边形  $ABCD$  的各边分别向外作正方形,

其中心依次记为  $O_1, O_2, O_3, O_4$ , 那么  $O_1O_3 \perp O_2O_4$ , 且  $|O_1O_3| = |O_2O_4|$ .



**Solution:**

设  $A, B, C, D, O_1, O_2, O_3, O_4$  对应的复数为  $z_1, z_2, z_3, z_4, v_1, v_2, v_3, v_4$ ,

记  $\omega = \exp(\pi i/4)$ , 则我们有:

$$\begin{aligned}
v_1 - z_2 &= \frac{\omega}{\sqrt{2}}(z_1 - z_2) \\
v_2 - z_3 &= \frac{\omega}{\sqrt{2}}(z_2 - z_3) \\
v_3 - z_4 &= \frac{\omega}{\sqrt{2}}(z_3 - z_4) \\
v_4 - z_1 &= \frac{\omega}{\sqrt{2}}(z_4 - z_1)
\end{aligned}$$

因此我们有:

$$\begin{aligned}
v_3 - v_1 &= \left( z_4 + \frac{\omega}{\sqrt{2}}(z_3 - z_4) \right) - \left( z_2 + \frac{\omega}{\sqrt{2}}(z_1 - z_2) \right) \\
&= \frac{1}{2}((-z_1 - z_2 + z_3 + z_4) + i(-z_1 + z_2 + z_3 - z_4)) \\
v_4 - v_2 &= \left( z_1 + \frac{\omega}{\sqrt{2}}(z_4 - z_1) \right) - \left( z_3 + \frac{\omega}{\sqrt{2}}(z_2 - z_3) \right) \\
&= \frac{1}{2}((z_1 - z_2 - z_3 + z_4) + i(-z_1 - z_2 + z_3 + z_4))
\end{aligned}$$

于是我们有  $v_4 - v_2 = i(v_3 - v_1)$  成立.

注意到  $i = \exp(\pi i/2)$ , 故  $O_2O_4$  是由  $O_1O_3$  顺时针旋转  $90^\circ$  得到的,

即有  $O_1O_3 \perp O_2O_4$  和  $|O_1O_3| = |O_2O_4|$  成立.

## Problem 11 (optional)

定义:

$$\mathbb{H} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

试证明  $\mathbb{H}$  对矩阵加法和乘法封闭, 并建立这两种运算所满足的运算律.

- **Insight:**

作为 4 维实代数,  $\mathbb{H}$  与 Hamilton 的四元数代数同构.

换言之, 它就是四元数代数  $\{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$  的一个  $2 \times 2$  复矩阵表示,

其中  $i, j, k$  满足  $i^2 = j^2 = k^2 = ijk = -1$ .

根据邵老师课上的内容我们知道, 四元数代数亏损了乘法交换律, 因此不是数域.

**Solution:**

定义:

$$M(\alpha, \beta) := \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \quad (\forall \alpha, \beta \in \mathbb{C})$$

- 对矩阵加法封闭.

对于任意  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ , 我们都有:

$$\begin{aligned}
M(\alpha_1, \beta_1) + M(\alpha_2, \beta_2) &= \begin{bmatrix} \alpha_1 & \beta_1 \\ -\bar{\beta}_1 & \bar{\alpha}_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & \beta_2 \\ -\bar{\beta}_2 & \bar{\alpha}_2 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 + \alpha_2 & \beta_1 + \beta_2 \\ -\overline{\beta_1 + \beta_2} & \overline{\alpha_1 + \alpha_2} \end{bmatrix} \\
&= M(\alpha_1 + \alpha_2, \beta_1 + \beta_2) \in \mathbb{H}
\end{aligned}$$

- 对矩阵乘法封闭.

对于任意  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ , 我们都有:

$$\begin{aligned}
M(\alpha_1, \beta_1) \cdot M(\alpha_2, \beta_2) &= \begin{bmatrix} \alpha_1 & \beta_1 \\ -\bar{\beta}_1 & \bar{\alpha}_1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_2 & \beta_2 \\ -\bar{\beta}_2 & \bar{\alpha}_2 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1\alpha_2 - \beta_1\bar{\beta}_2 & \alpha_1\beta_2 + \beta_1\bar{\alpha}_2 \\ -\bar{\beta}_1\alpha_2 - \bar{\alpha}_1\bar{\beta}_2 & -\bar{\beta}_1\beta_2 + \bar{\alpha}_1\bar{\alpha}_2 \end{bmatrix} \\
&= M(\alpha_1\alpha_2 - \beta_1\bar{\beta}_2, \alpha_1\beta_2 + \beta_1\bar{\alpha}_2) \in \mathbb{H}
\end{aligned}$$

根据矩阵加法和乘法的性质可知,  $\mathbb{H}$  几乎满足所有的运算律, 唯独不满足乘法交换律.

- **(矩阵加法)** 对于任意  $M, M_1, M_2, M_3 \in \mathbb{H}$  我们有:
    - 封闭 (closed):  $M_1 + M_2 \in \mathbb{H}$
    - 可结合 (associative):  $(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$
    - 可交换 (commutative):  $M_1 + M_2 = M_2 + M_1$
    - 单位元 (identity):  $0_{2 \times 2} \in \mathbb{H}$  such that  $M + 0_{2 \times 2} = M$
    - 逆元 (inverse):  $\exists (-M) \in \mathbb{H}$  such that  $M + (-M) = 0_{2 \times 2}$
  - **(矩阵乘法)** 对于任意  $M, M_1, M_2, M_3 \in \mathbb{H}$  我们有:
    - 封闭:  $M_1 \cdot M_2 \in \mathbb{H}$
    - 可结合:  $(M_1 \cdot M_2) \cdot M_3 = M_1 \cdot (M_2 \cdot M_3)$
    - 单位元:  $I_2 \in \mathbb{H}$  such that  $M \cdot I_2 = M$
    - 逆元: 若  $\det(M) = |\alpha|^2 + |\beta|^2 \neq 0$ ,  
 则  $\exists M^{-1} = \frac{1}{|\alpha|^2 + |\beta|^2} \begin{bmatrix} \bar{\alpha} & -\bar{\beta} \\ \bar{\beta} & \alpha \end{bmatrix} \in \mathbb{H}$  such that  $M \cdot M^{-1} = I_2$
- 通常不可交换:  $M_1 \cdot M_2$  通常不等于  $M_2 \cdot M_1$ .

## Problem 12

给定正整数  $m, n$ , 设  $A \in \mathbb{C}^{m \times m}$ ,  $B \in \mathbb{C}^{n \times n}$ ,  $X \in \mathbb{C}^{m \times n}$ .  
 试证明  $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$ .

**Solution:**

我们首先证明此命题的一个简化版本.

设  $a \in \mathbb{C}^m$ ,  $b \in \mathbb{C}^n$ ,  $X = [x_1, x_2, \dots, x_n] \in \mathbb{C}^{m \times n}$ , 则  $a^T X b = (b^T \otimes a^T)\text{vec}(X)$ .

$$\begin{aligned}
 (b^T \otimes a^T)\text{vec}(X) &= [b_1 a^T \quad b_2 a^T \quad \cdots \quad b_n a^T] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
 &= \sum_{j=1}^n b_j a^T x_j \\
 &= a^T [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
 &= a^T X b
 \end{aligned}$$

现将  $A \in \mathbb{C}^{m \times m}$ ,  $B \in \mathbb{C}^{n \times n}$  记为:

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad B = [b_1 \quad b_2 \quad \cdots \quad b_n]$$

则我们有:

$$\begin{aligned}
(B^T \otimes A) \text{vec}(X) &= \begin{bmatrix} b_1^T \otimes a_1^T \\ b_1^T \otimes a_2^T \\ \vdots \\ b_1^T \otimes a_m^T \\ b_2^T \otimes a_1^T \\ b_2^T \otimes a_2^T \\ \vdots \\ b_2^T \otimes a_m^T \\ \vdots \\ b_n^T \otimes a_1^T \\ b_n^T \otimes a_2^T \\ \vdots \\ b_n^T \otimes a_m^T \end{bmatrix} \text{vec}(X) = \begin{bmatrix} (b_1^T \otimes a_1^T) \text{vec}(X) \\ (b_1^T \otimes a_2^T) \text{vec}(X) \\ \vdots \\ (b_1^T \otimes a_m^T) \text{vec}(X) \\ (b_2^T \otimes a_1^T) \text{vec}(X) \\ (b_2^T \otimes a_2^T) \text{vec}(X) \\ \vdots \\ (b_2^T \otimes a_m^T) \text{vec}(X) \\ \vdots \\ (b_n^T \otimes a_1^T) \text{vec}(X) \\ (b_n^T \otimes a_2^T) \text{vec}(X) \\ \vdots \\ (b_n^T \otimes a_m^T) \text{vec}(X) \end{bmatrix} = \begin{bmatrix} a_1^T X b_1 \\ a_2^T X b_1 \\ \vdots \\ a_m^T X b_1 \\ a_1^T X b_2 \\ a_2^T X b_2 \\ \vdots \\ a_m^T X b_2 \\ \vdots \\ a_1^T X b_n \\ a_2^T X b_n \\ \vdots \\ a_m^T X b_n \end{bmatrix} = \text{vec}(AXB)
\end{aligned}$$

**The End**