

FDU 高等线性代数 Homework 01

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Problem 1

设 n 为给定的正整数, 求 n 阶矩阵 A 的所有特征值和特征向量.

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$$

- Insight:

实际上, 我们观察到 A 是一个 **Frobenius 友型**,

故其特征多项式可以一眼看出来: $\det(\lambda I - A) = \lambda^n - 1$.

因此其特征值为 $\lambda_k = \omega^k$, 对应的特征向量为 $[1, \omega^k, \dots, \omega^{(n-2)k}, \omega^{(n-1)k}]^T$,

其中 $\omega = \exp(2\pi i/n)$, $k = 0, 1, \dots, n-1$.

一般的 **Frobenius 友型**形如:

$$A := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix} \text{ or } \begin{bmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 & -\alpha_0 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & & & & -\alpha_0 \\ 1 & 0 & & & -\alpha_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -\alpha_{n-2} \\ & & & 1 & -\alpha_{n-1} \end{bmatrix}$$

可以证明其极小多项式 $m_A(t)$ 和特征多项式 $p_A(t)$ 均为 $t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1t + a_0$.

特别地, 取第一种形式, 可以证明:

若 λ 为 A 的特征值 (即满足 $\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0$),

则 $[1, \lambda, \dots, \lambda^{n-2}, \lambda^{n-1}]^T$ 为对应的特征向量:

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} \lambda$$

Solution:

(我们这里给出最基础的做法, 不使用 Frobenius 友型的结论)

方阵 A 的特征多项式为:

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \\ -1 & & & & \lambda \end{vmatrix}_n \\ &= \lambda \begin{vmatrix} \lambda & -1 & & & \\ & \ddots & \ddots & & \\ & & \lambda & -1 & \\ & & & \lambda & \end{vmatrix}_{n-1} + (-1)^{n+1} \cdot (-1) \begin{vmatrix} -1 & & & & \\ \lambda & -1 & & & \\ & \ddots & \ddots & & \\ & & \lambda & -1 & \end{vmatrix}_{n-1} \\ &= \lambda \cdot \lambda^{n-1} + 1 \cdot (-1)^{n-1} \\ &= \lambda^n - 1 \end{aligned}$$

令 $\det(\lambda I - A) = \lambda^n - 1 = 0$, 可解得 n 个根为:

$$\lambda_k = \sqrt[n]{1} \exp \left\{ i \left(\frac{0 + 2k\pi}{n} \right) \right\} = \left(\exp \left(\frac{2\pi i}{n} \right) \right)^k \quad (k = 0, 1, \dots, n-1)$$

若记 $\omega = \exp(2\pi i/n)$, 则我们可以将方阵 A 的 n 个特征值写为:

$$\lambda_k = \omega^k \quad (k = 0, 1, \dots, n-1)$$

求解 λ_k 对应的特征向量就是要求解方程组:

$$(\lambda_k I_n - A)x = \begin{bmatrix} \lambda_k & -1 & & & \\ & \lambda_k & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_k & -1 \\ -1 & & & & \lambda_k \end{bmatrix} x = 0_n$$

$$\Updownarrow$$

$$\begin{cases} x_2 = \lambda_k x_1 \\ x_3 = \lambda_k x_2 \\ \vdots \\ x_n = \lambda_k x_{n-1} \\ x_1 = \lambda_k x_n \end{cases}$$

我们可以取 $\lambda_k = \omega^k$ ($k = 0, 1, \dots, n-1$) 对应的特征向量 $x^{(k)}$ 为:

$$x^{(k)} = \begin{bmatrix} 1 \\ \lambda_k \\ \vdots \\ \lambda_k^{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ \omega^k \\ \vdots \\ \omega^{(n-1)k} \end{bmatrix}$$

Problem 2

证明: 复数 z_1, z_2, z_3 满足 $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$ 的充要条件是

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

- **Lemma 1:**

设 $\omega = \exp(2\pi i/n)$, 则我们有:

$$\begin{aligned} \sum_{k=0}^n \omega^k &= 1 + \omega + \dots + \omega^{n-1} \\ &= \frac{1 - \omega^n}{1 - \omega} \quad (\text{note that } \omega^n = \exp\left(\frac{2\pi i}{n} \cdot n\right) = \exp(2\pi i) = 1) \\ &= \frac{1 - 1}{1 - \omega} \\ &= 0 \end{aligned}$$

- **Lemma 2 (两个非零复数的乘积也不是零):**

若 $z_1 z_2 = 0$, 则 z_1 和 z_2 至少有一个是零.

- 当 $z_1 = 0$ 时, 结论成立.
- 当 $z_1 \neq 0$ 时, 可知逆元 z_1^{-1} 存在, 我们有 $z_2 = z_2(z_1 z_1^{-1}) = z_1^{-1}(z_1 z_2) = z_1^{-1} \cdot 0 = 0$.

Solution:

若 z_1, z_2, z_3 中有任意两个是相等的,

则可根据 $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| = 0$ 推出 $z_1 = z_2 = z_3 = 0$,

进而有 $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$

此时命题成立.

下证 z_1, z_2, z_3 互不相同命题成立.

- **① 必要性:**

若 $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$, 则 z_1, z_2, z_3 三点确定了一个正三角形.

记 $\omega = \exp(2\pi i/3)$, 则我们有:

$$\begin{cases} z_2 - z_3 = (z_2 - z_1)\omega \\ z_1 - z_3 = (z_2 - z_1)\omega^2 \end{cases}$$

于是有:

$$\begin{aligned} z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 &= \frac{1}{2}(z_2 - z_1)^2 + \frac{1}{2}(z_2 - z_3)^2 + \frac{1}{2}(z_1 - z_3)^2 \\ &= \frac{1}{2}(z_2 - z_1)^2(1 + \omega^2 + \omega^4) \quad (\text{note that } \omega^3 = 1) \\ &= \frac{1}{2}(z_2 - z_1)^2(1 + \omega^2 + \omega) \quad (\text{utilize Lemma 1}) \\ &= \frac{1}{2}(z_2 - z_1)^2 \cdot 0 \\ &= 0 \end{aligned}$$

• ② 充分性:

根据 Lemma 1 我们有 $\omega^2 + \omega + 1 = 0$, 即有 $\omega^2 + \omega = -1$.

若 $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$, 则我们有:

$$\begin{aligned} 0 &= z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 \quad (\text{note that } \omega^3 = 1 \text{ and } \omega^2 + \omega = -1) \\ &= z_1^2 + \omega^3 z_2^2 + \omega^3 z_3^2 + (\omega^2 + \omega)z_1z_2 + (\omega^2 + \omega)z_2z_3 + (\omega^2 + \omega)z_3z_1 \\ &= (z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) \end{aligned}$$

根据 Lemma 2 可知 $z_1 + \omega z_2 + \omega^2 z_3$ 和 $z_1 + \omega^2 z_2 + \omega z_3$ 至少有一个是零.

不失一般性, 设 $z_1 + \omega z_2 + \omega^2 z_3 = 0$, 则左右同乘 $(\omega^2 - \omega)$ 可得:

$$\begin{aligned} 0 &= (\omega^2 - \omega)(z_1 + \omega z_2 + \omega^2 z_3) \\ &= (\omega^2 - \omega)z_1 + (\omega^3 - \omega^2)z_2 + (\omega^4 - \omega^3)z_3 \\ &= (\omega^2 - \omega)z_1 + (1 - \omega^2)z_2 + (\omega - 1)z_3 \\ &= (\omega - 1)(z_3 - z_1) + (\omega^2 - 1)(z_1 - z_2) \\ &= (\omega - 1)((z_3 - z_1) + (\omega + 1)(z_1 - z_2)) \\ &= (\omega - 1)((z_3 - z_2) + \omega(z_1 - z_2)) \end{aligned}$$

由于 $\omega - 1 = \exp(2\pi i/3) - 1 \neq 0$, 故根据 Lemma 2 可知:

$$\begin{cases} (z_3 - z_1) + (\omega + 1)(z_1 - z_2) = 0 \\ (z_3 - z_2) + \omega(z_1 - z_2) = 0 \end{cases}$$

注意到:

$$\begin{cases} |\omega + 1| = |-\frac{1}{2} + \frac{\sqrt{3}}{2}i + 1| = 1 \\ |\omega| = |\exp(\frac{2\pi i}{3})| = 1 \end{cases}$$

于是我们有:

$$\begin{aligned} |z_3 - z_1| &= |-(\omega + 1)(z_2 - z_1)| \\ &= |\omega + 1| \cdot |z_2 - z_1| \\ &= 1 \cdot |z_2 - z_1| \\ &= |z_2 - z_1| \\ \hline |z_3 - z_2| &= |-\omega(z_2 - z_1)| \\ &= |\omega| \cdot |z_2 - z_1| \\ &= 1 \cdot |z_2 - z_1| \\ &= |z_2 - z_1| \end{aligned}$$

因此 $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$.

综上所述, 命题得证.

事实上, 下列命题是等价的:

- $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$
- $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = \frac{1}{2}[(z_2 - z_1)^2 + (z_2 - z_3)^2 + (z_1 - z_3)^2] = 0$
- $z_1 + \omega z_2 + \omega^2 z_3 = 0$ (其中 $\omega = \exp(2\pi i/3)$)

(2025 补充习题)

已知 $z_1 = 1 - i$, $z_2 = 2 + 3i$, 试求复数 z_3 使得 $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$.

Solution:

根据 $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$ 可知 z_1, z_2, z_3 在复平面中构成正三角形的顶点.

因此令 $z_2 - z_1$ 顺/逆时针旋转 $\pi/3$ 便可得到 $z_3 - z_1$.

设 $z_3 = \alpha + \beta i$, 令 $z_3 - z_1 = \exp(\pm \pi i/3)(z_2 - z_1)$, 则我们有:

$$\begin{aligned} z_3 &= z_1 + \exp(\pm \pi i/3)(z_2 - z_1) \\ &= 1 - i + \frac{1}{2}(1 \pm \sqrt{3}i)(2 + 3i - 1 + i) \\ &= \frac{1}{2}(3 \mp 4\sqrt{3} + (2 \pm \sqrt{3})i) \end{aligned}$$

Problem 3

给定的正整数 m, n , 记 $\omega = \exp(2\pi i/m)$.

试证明对任何 $A, B \in \mathbb{C}^{n \times n}$ 都有:

$$A^m + B^m = \frac{1}{m} \sum_{k=0}^{m-1} (A + \omega^k B)^m$$

• **Lemma (两个非零复数的乘积也不是零):**

若 $z_1 z_2 = 0$, 则 z_1 和 z_2 至少有一个是零.

◦ 当 $z_1 = 0$ 时, 结论成立.

◦ 当 $z_1 \neq 0$ 时, 可知逆元 z_1^{-1} 存在, 我们有 $z_2 = z_2(z_1 z_1^{-1}) = z_1^{-1}(z_1 z_2) = z_1^{-1} \cdot 0 = 0$.

Solution:

注意到 A, B 不一定是可交换的 (即 $AB = BA$).

因此 $(A + B)^m$ 不能简单展开为 $\sum_{j=0}^m \binom{m}{j} A^{m-j} B^j$.

我们记 $(A + B)^m$ 的展开式中 $\binom{m}{j}$ 个由 $m-j$ 个 A 和 j 个 B 构成的项之和为 $\text{term}(A, B, j)$.

显然我们有 $\text{term}(A, \omega^k B, j) = \omega^{jk} \text{term}(A, B, j)$ 成立.

因为有限求和是可以交换次序的, 所以我们有:

$$\begin{aligned} \frac{1}{m} \sum_{k=0}^{m-1} (A + \omega^k B)^m &= \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \sum_{j=0}^m \text{term}(A, \omega^k B, j) \right\} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \sum_{j=0}^m \omega^{jk} \text{term}(A, B, j) \right\} \\ &= \frac{1}{m} \sum_{j=0}^m \left\{ \left(\sum_{k=0}^{m-1} \omega^{jk} \right) \text{term}(A, B, j) \right\} \end{aligned} \tag{3.1}$$

注意到 $\omega^m = 1$ (即 ω 是 1 的一个 m 次复根).

• 当 j 为 m 的整数倍 (即 j 为 0 或 m) 时, 我们有:

$$\sum_{k=0}^{m-1} \omega^{jk} = \sum_{k=0}^{m-1} 1 = m$$

• 当 j 不为 m 的整数倍 (即 $j = 1, \dots, m-1$) 时, 我们有:

$$\begin{aligned} \omega^j \sum_{k=0}^{m-1} \omega^{jk} &= \omega^j (1 + \omega^j + \dots + \omega^{(m-2)j} + \omega^{(m-1)j}) \\ &= \omega^j + \omega^{2j} + \dots + \omega^{(m-1)j} + \omega^{mj} \quad (\text{note that } \omega^m = 1) \\ &= \omega^j + \omega^{2j} + \dots + \omega^{(m-1)j} + 1 \\ &= \sum_{k=0}^{m-1} \omega^{jk} \end{aligned}$$

于是有 $(\omega^j - 1) \sum_{k=0}^{m-1} \omega^{jk} = 0$ 成立.

由于 $\omega^j - 1 = \exp(\frac{2j\pi i}{m}) - 1 \neq 0$, 故根据 **Lemma** 可知 $\sum_{k=0}^{m-1} \omega^{jk} = 0$.

综上所述, 我们有:

$$\sum_{k=0}^{m-1} \omega^{jk} = \begin{cases} m, & \text{if } j = 0, m \\ 0, & \text{if } j = 1, \dots, m-1 \end{cases}$$

将上述结果代入 (3.1) 式中我们有:

$$\begin{aligned} \frac{1}{m} \sum_{k=0}^{m-1} (A + \omega^k B)^m &= \frac{1}{m} \sum_{j=0}^m \left\{ \left(\sum_{k=0}^{m-1} \omega^{jk} \right) \text{term}(A, B, j) \right\} \\ &= \frac{1}{m} (mA^m + mB^m) \\ &= A^m + B^m \end{aligned}$$

命题得证.

Problem 4

以下内容均来自 **Complex Variables and Applications (9th Edition J. Brown, R. Churchill) Chapter 6**

(Complex Variables and Applications 第 74 节)

若函数在简单闭围道 C 的内部除了有限多个奇点以外处处解析, 则这些奇点必定是孤立奇点.

特殊地, 有理函数 (即两个多项式函数的商) 的奇点总是孤立奇点, 因为分母中的多项式函数仅有有限个零点.

(Complex Variables and Applications 第 75 节)

若 z_0 是函数 f 的孤立奇点, 则存在正数 $R > 0$ 使得 f 在 $0 < |z - z_0| < R$ 中的任意一点 z 处解析

因此函数 f 关于 z_0 的 Laurent 级数展开式为:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \\ a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (n = 0, 1, \dots) \\ b_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \quad (n = 1, 2, \dots) \end{aligned}$$

其中 C 为 $0 < |z - z_0| < R$ 中任意围绕 z_0 的简单正向闭围道.

特别地, b_1 的表达式为 $b_1 = \frac{1}{2\pi i} \int_C f(z) dz$

我们称其为函数 f 在孤立奇点 z_0 处的 **留数 (residue)**, 记为 $\text{Res}_{z=z_0} f(z)$

于是我们有:

$$\int_C f(z) dz = 2\pi i \cdot \text{Res}_{z=z_0} f(z)$$

(Cauchy 留数定理, Complex Variables and Applications 第 76 节)

设 C 为正向简单闭围道.

若函数 f 在 C 及其内部除了有限多个奇点 z_k ($k = 1, \dots, n$) 以外处处解析 (自然是孤立奇点),

则我们有:

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)$$

即 f 沿 C 的积分值 $\int_C f(z) dz$ 为其内部有限个奇点处的留数之和的 $2\pi i$ 倍.

下面的定理仅仅涉及一个留数, 故运用起来有时比 Cauchy 留数定理更加方便:

(Complex Variables and Applications 第 77 节 定理)

若函数 f 在有限平面上除了有限多个奇点以外处处解析, 且这些奇点落在一条正向简单闭围道 C 的内部,

则我们有:

$$\int_C f(z) dz = 2\pi i \cdot \text{Res}_{z=0} \left\{ \frac{1}{z^2} f\left(\frac{1}{z}\right) \right\}$$

计算复积分:

$$\int_{|z|=4} \frac{1}{z^2 - 3z + 2} dz$$

Solution:

记围道 C 为 $|z| = 4$ 确定的正向圆周 (即逆时针方向).

注意到多项式函数 $p(z) = z^2 - 3z + 2$ 在整个复平面都是解析的, 且仅有 $z = 1, 2$ 两个零点.

因此 $f(z) = 1/p(z)$ 在复平面上仅有 $z = 1, 2$ 两个孤立奇点, 且都落在围道 C 的内部.

下面我们计算 $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ 在 $z = 0$ 处的留数.

定义 $g(z) := \frac{1}{z^2} f\left(\frac{1}{z}\right)$, 则我们有:

$$\begin{aligned} g(z) &= \frac{1}{z^2} f\left(\frac{1}{z}\right) \\ &= \frac{1}{z^2} \cdot \frac{z^2}{1 - 3z + 2z^2} \\ &= \frac{1}{(1 - 2z)(1 - z)} \end{aligned}$$

根据 **Lemma** 可知 $g(z)$ 的 Laurent 级数中 $1/z$ 项的系数即为所求留数.

注意到 $g(z)$ 在 $z = 0$ 处是解析的, 其 Laurent 级数展开即为 Taylor 级数展开, 因此没有 $1/z$ 项, 于是有:

$$\operatorname{Res}_{z=0} \left\{ \frac{1}{z^2} f\left(\frac{1}{z}\right) \right\} = 0$$

因此 f 在 C 上的积分为:

$$\int_C f(z) dz = \int_C \frac{1}{z^2 - 3z + 2} dz = 2\pi i \cdot \operatorname{Res}_{z=0} \left\{ \frac{1}{z^2} f\left(\frac{1}{z}\right) \right\} = 2\pi i \cdot 0 = 0$$

Problem 5

试证明任何复方阵都可以在复数域上相似上三角化,

即对于任意复方阵 $A \in \mathbb{C}^{n \times n}$ 总存在非奇异阵 P 使得 $P^{-1}AP$ 为上三角矩阵.

Solution:

当 $n = 1$ 时, 命题显然成立.

当 $n \geq 2$ 时, 假设对于所有维数小于 n 的复方阵, 上述命题都成立.

下面对 n 维复方阵证明该命题.

设 (λ_1, x_1) 是 $A \in \mathbb{C}^{n \times n}$ 的一个特征对, 即满足 $A_1 x_1 = x_1 \lambda_1$.

将 x_1 扩充为 \mathbb{C}^n 的一组基 x_1, v_2, \dots, v_n ,

定义非奇异阵 $P_1 := [x_1, v_2, \dots, v_n] = [x_1, V]$, 则我们有:

$$\begin{aligned} AP_1 &= A[x_1, V] \\ &= [Ax_1, AV] \\ &= [x_1 \lambda_1, AV] \\ &= [x_1, V][\lambda_1 e_1, P_1^{-1} AV] \quad (\text{denote } P_1^{-1} AV = \begin{bmatrix} * \\ A_2 \end{bmatrix} \in \mathbb{C}^{n \times (n-1)}) \\ &= [x_1, V] \begin{bmatrix} \lambda_1 & * \\ & A_2 \end{bmatrix} \\ &= P_1 \begin{bmatrix} \lambda_1 & * \\ & A_2 \end{bmatrix} \end{aligned}$$

根据归纳假设可知, 存在非奇异阵 $\tilde{P}_2 \in \mathbb{C}^{(n-1) \times (n-1)}$ 使得 $T_2 := \tilde{P}_2^{-1} A_2 \tilde{P}_2$ 为上三角阵.

定义 $P_2 := 1 \oplus \tilde{P}_2$ 和 $P = P_1 P_2$ 可知:

$$\begin{aligned} P^{-1}AP &= P_2^{-1}P_1^{-1}AP_1P_2 \\ &= \begin{bmatrix} 1 & \\ & P_2^{-1} \end{bmatrix} \begin{bmatrix} \lambda_1 & * \\ & A_2 \end{bmatrix} \begin{bmatrix} 1 & \\ & P_2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & * \\ & P_2^{-1} A_2 P_2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & * \\ & T_2 \end{bmatrix} \end{aligned}$$

因此 $T := P^{-1}AP$ 为上三角阵.

根据数学归纳法, 命题得证.

Problem 6 (optional)

设 n 为正整数.

已知 n 次系数多项式 $f(z) = \sum_{k=0}^n a_k z^k$ 的系数满足 $a_0 > \cdots > a_n > 0$

证明: $f(z)$ 的所有复根都在单位圆外.

- Hint: 考察 $g(z) = (1-z)f(z)$

Solution:

当 $|z| \leq 1$ 时, 我们有:

$$\begin{aligned} |g(z)| &= |(1-z)f(z)| \\ &= \left| (1-z) \sum_{k=0}^n a_k z^k \right| \\ &= \left| a_0 + \sum_{k=1}^n (a_k - a_{k-1}) z^k - a_n z^{n+1} \right| \quad (\text{triangle inequality}) \\ &\geq |a_0| - \left| \sum_{k=1}^n (a_k - a_{k-1}) z^k - a_n z^{n+1} \right| \quad (\text{triangle inequality and } |z_1 z_2| = |z_1| |z_2| \text{ for all } z_1, z_2 \in \mathbb{C}) \\ &\geq |a_0| - \sum_{k=1}^n |a_k - a_{k-1}| |z|^k - |a_n| |z|^{n+1} \quad (\text{note that } a_0 > \cdots > a_n > 0) \\ &= a_0 - \sum_{k=1}^n (a_{k-1} - a_k) |z|^k - a_n |z|^{n+1} \quad (\text{note that } |z| \leq 1) \\ &\geq a_0 - \sum_{k=1}^n (a_{k-1} - a_k) \cdot 1 - a_n \cdot 1 \\ &= a_0 - (a_0 - a_n) - a_n \\ &= 0 \end{aligned}$$

上述三个不等号同时取等的充要条件是:

- ① $\sum_{k=1}^n (a_k - a_{k-1}) z^k - a_n z^{n+1}$ 与 a_0 反方向 (即与 1 反方向)
(注意 a_0 是正实数, 而 $a_k - a_{k-1} < 0$ ($k = 1, \dots, n$))
- ② z, z^2, \dots, z^{n+1} 同方向
- ③ $|z| = 1$

容易验证这样的 z 只能是 $z = e^{2m\pi i} = 1$ ($m \in \mathbb{Z}$).

因此当 $|z| \leq 1$ 且 $z \neq 1$ 时, 我们都有 $|g(z)| > 0$ 成立, 表明这样的 z 不是 $g(z)$ 的根.

于是 $g(z)$ 的根要么是 $z = 1$, 要么满足 $|z| > 1$.

注意到 $g(z) = (1-z)f(z)$ 的复根除了额外的 1 以外, 其余复根都与 $f(z)$ 的相同.

而根据 $f(1) = \sum_{k=0}^n a_k > 0$ 可知 $z = 1$ 不是 $f(z)$ 的根.

因此 $f(z)$ 的所有根都满足 $|z| > 1$, 即都落在单位圆周 $|z| = 1$ 的外部.

Problem 7 (optional)

证明下面的函数不是解析函数, 但在复平面上处处满足 Cauchy-Riemann 方程:

$$f(z) = \begin{cases} \exp(-z^{-4}), & z \neq 0, \\ 0, & z = 0. \end{cases}$$

- (极坐标下可导的充分条件, Complex Variables and Applications 第 24 节)

若函数 $f(z) = u(\rho, \theta) + iv(\rho, \theta)$ 在非零点 $z_0 = \rho_0 e^{i\theta_0}$ 的某个邻域内有定义, 且满足:

- 函数 u, v 在 $z_0 = \rho_0 e^{i\theta_0}$ 的该邻域内可偏导
- 函数 u, v 的一阶偏导数在 (ρ_0, θ_0) 处连续且满足极坐标形式的 Cauchy-Riemann 方程:

$$\begin{cases} \rho u_\rho(\rho_0, \theta_0) = v_\theta(\rho_0, \theta_0) \\ u_\theta(\rho_0, \theta_0) = -\rho v_\rho(\rho_0, \theta_0) \end{cases}$$

则 f 在 $z_0 = \rho_0 e^{i\theta_0}$ 处可导, 且导数 $f'(z_0) = e^{-i\theta} (u_\rho(\rho_0, \theta_0) + iv_\rho(\rho_0, \theta_0))$

Solution:

当 $z \neq 0$ 时, 我们有:

$$\begin{aligned}
 f(z) &= \exp(-z^{-4}) \\
 &= \exp(-\rho^{-4}e^{-4i\theta}) \\
 &= \exp(-\rho^{-4}(\cos(-4\theta) + i\sin(-4\theta))) \\
 &= \exp(-\rho^{-4}\cos(-4\theta)) \cdot \exp(i \cdot (-\rho^{-4}\sin(-4\theta))) \\
 &= \exp(-\rho^{-4}\cos(-4\theta)) \cdot (\cos(-\rho^{-4}\sin(-4\theta)) + i\sin(-\rho^{-4}\sin(-4\theta))) \\
 &= u(\rho, \theta) + iv(\rho, \theta)
 \end{aligned}$$

其中我们记:

$$\begin{aligned}
 u(\rho, \theta) &= \exp(-\rho^{-4}\cos(-4\theta)) \cos(-\rho^{-4}\sin(-4\theta)) = g(\rho, \theta) \cos(h(\rho, \theta)) \\
 v(\rho, \theta) &= \exp(-\rho^{-4}\cos(-4\theta)) \sin(-\rho^{-4}\sin(-4\theta)) = g(\rho, \theta) \sin(h(\rho, \theta)) \\
 \text{where } \begin{cases} g(\rho, \theta) = \exp(-\rho^{-4}\cos(-4\theta)) \\ h(\rho, \theta) = -\rho^{-4}\sin(-4\theta) \end{cases} &\Rightarrow \begin{cases} g_\rho(\rho, \theta) = 4\rho^{-5}\cos(-4\theta)g(\rho, \theta) \\ g_\theta(\rho, \theta) = -4\rho^{-4}\sin(-4\theta)g(\rho, \theta) \\ h_\rho(\rho, \theta) = 4\rho^{-5}\sin(-4\theta) \\ h_\theta(\rho, \theta) = 4\rho^{-4}\cos(-4\theta) \end{cases}
 \end{aligned}$$

经计算可得:

$$\begin{aligned}
 u_\rho(\rho, \theta) &= g_\rho(\rho, \theta) \cos(h(\rho, \theta)) + g(\rho, \theta)[- \sin(h(\rho, \theta))h_\rho(\rho, \theta)] \\
 &= 4\rho^{-5}\cos(-4\theta)g(\rho, \theta) \cos(h(\rho, \theta)) - g(\rho, \theta) \sin(h(\rho, \theta))4\rho^{-5}\sin(-4\theta) \\
 &= 4\rho^{-5}g(\rho, \theta) \cos(h(\rho, \theta) - 4\theta) \\
 \hline
 u_\theta(\rho, \theta) &= g_\theta(\rho, \theta) \cos(h(\rho, \theta)) + g(\rho, \theta)[- \sin(h(\rho, \theta))h_\theta(\rho, \theta)] \\
 &= -4\rho^{-4}\sin(-4\theta)g(\rho, \theta) \cos(h(\rho, \theta)) - g(\rho, \theta) \sin(h(\rho, \theta))4\rho^{-4}\cos(-4\theta) \\
 &= -4\rho^{-4}g(\rho, \theta) \sin(h(\rho, \theta) - 4\theta) \\
 \hline
 v_\rho(\rho, \theta) &= g_\rho(\rho, \theta) \sin(h(\rho, \theta)) + g(\rho, \theta) \cos(h(\rho, \theta))h_\rho(\rho, \theta) \\
 &= 4\rho^{-5}\cos(-4\theta)g(\rho, \theta) \sin(h(\rho, \theta)) + g(\rho, \theta) \cos(h(\rho, \theta))4\rho^{-5}\sin(-4\theta) \\
 &= 4\rho^{-5}g(\rho, \theta) \sin(h(\rho, \theta) - 4\theta) \\
 \hline
 v_\theta(\rho, \theta) &= g_\theta(\rho, \theta) \sin(h(\rho, \theta)) + g(\rho, \theta) \cos(h(\rho, \theta))h_\theta(\rho, \theta) \\
 &= -4\rho^{-4}\sin(-4\theta)g(\rho, \theta) \sin(h(\rho, \theta)) + g(\rho, \theta) \cos(h(\rho, \theta))4\rho^{-4}\cos(-4\theta) \\
 &= 4\rho^{-4}g(\rho, \theta) \cos(h(\rho, \theta) - 4\theta)
 \end{aligned}$$

因此对于任意 $\rho > 0$ 和 $\theta \in \mathbb{R}$ 我们都有:

$$\begin{cases} \rho u_\rho(\rho, \theta) = v_\theta(\rho, \theta) \\ u_\theta(\rho, \theta) = -\rho v_\rho(\rho, \theta) \end{cases}$$

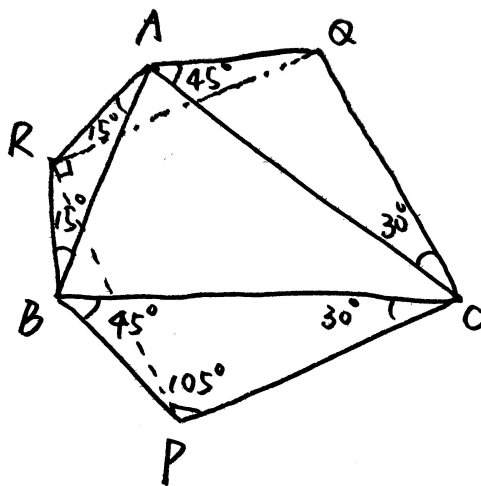
这表明 $f(z)$ 在复平面上处处满足 Cauchy-Riemann 方程.

Problem 8 (optional)

对于欧式平面 \mathbb{R}^2 内的任意三角形 $\triangle ABC$

向外作 $\angle ABR, \angle BCP, \angle CAQ$ 使得 $\begin{cases} \angle CBP = \angle CAQ = 45^\circ \\ \angle BCP = \angle ACQ = 30^\circ \\ \angle ABR = \angle BAR = 15^\circ \end{cases}$.

试利用复数证明 $\angle QRP = 90^\circ$ 且 $|QR| = |RP|$.



Solution:

记 $\vec{OA}, \vec{OB}, \vec{OC}, \vec{OR}, \vec{OP}, \vec{OQ}$ 的复数表示为 a, b, c, z_1, z_2, z_3

根据 $1 - 2\sin^2(\frac{\pi}{12}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ 可解得 $\sin(\frac{\pi}{12}) = \frac{\sqrt{6}-\sqrt{2}}{4}$, 进而有 $\cos(\frac{\pi}{12}) = \frac{\sqrt{6}+\sqrt{2}}{4}$

根据正弦定理可知:

$$\frac{|BR|}{|BA|} = \frac{|AR|}{|AB|} = \frac{\sin(\frac{\pi}{12})}{\sin(\frac{5\pi}{6})} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{1}{2}} = \frac{\sqrt{6}-\sqrt{2}}{2}$$

$$\frac{|BP|}{|BC|} = \frac{|AQ|}{|AC|} = \frac{\sin(\frac{\pi}{6})}{\sin(\frac{7\pi}{12})} = \frac{\frac{1}{2}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{\sqrt{6}-\sqrt{2}}{2}$$

若记 $\omega = e^{i\frac{\pi}{12}}$, 则我们有:

$$\begin{cases} z_1 - b = \vec{BR} = \frac{\sqrt{6}-\sqrt{2}}{2}\omega\vec{BA} = \frac{\sqrt{6}-\sqrt{2}}{2}\omega(a-b) \\ z_1 - a = \vec{AR} = \frac{\sqrt{6}-\sqrt{2}}{2}\bar{\omega}\vec{AB} = \frac{\sqrt{6}-\sqrt{2}}{2}\bar{\omega}(b-a) \\ z_2 - b = \vec{BP} = \frac{\sqrt{6}-\sqrt{2}}{2}\bar{\omega}^3\vec{BC} = \frac{\sqrt{6}-\sqrt{2}}{2}\bar{\omega}^3(c-b) \\ z_3 - a = \vec{AQ} = \frac{\sqrt{6}-\sqrt{2}}{2}\omega^3\vec{AC} = \frac{\sqrt{6}-\sqrt{2}}{2}\omega^3(c-a) \end{cases} \text{ where } \begin{cases} \omega = \cos(\frac{\pi}{12}) + i\sin(\frac{\pi}{12}) = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i \\ \bar{\omega} = \cos(-\frac{\pi}{12}) + i\sin(-\frac{\pi}{12}) = \frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}i \\ \bar{\omega}^3 = \cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \\ \omega^3 = \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \end{cases}$$

要证明 " $\angle QRP = 90^\circ$ 且 $|QR| = |RP|$ ", 即要证 $\vec{RQ} = e^{i\frac{\pi}{2}}\vec{RP}$

也就等价于证明 $z_3 - z_1 = i(z_2 - z_1)$

$$\begin{aligned} z_3 - z_1 - i(z_2 - z_1) &= [(z_3 - a) - (z_1 - a)] - i[(z_2 - b) - (z_1 - b)] \\ &= \left[\frac{\sqrt{6}-\sqrt{2}}{2}\omega^3(c-a) - \frac{\sqrt{6}-\sqrt{2}}{2}\bar{\omega}(b-a) \right] - i \left[\frac{\sqrt{6}-\sqrt{2}}{2}\bar{\omega}^3(c-b) - \frac{\sqrt{6}-\sqrt{2}}{2}\omega(a-b) \right] \\ &= \frac{\sqrt{6}-\sqrt{2}}{2} \{ \omega^3(c-a) - \bar{\omega}(b-a) - i\bar{\omega}^3(c-b) + i\omega(a-b) \} \end{aligned}$$

因此要证明 $z_3 - z_1 - i(z_2 - z_1) = 0$, 等价于证明 $\omega^3(c-a) - \bar{\omega}(b-a) - i\bar{\omega}^3(c-b) + i\omega(a-b) = 0$

也就等价于证明 a, b, c 项的系数分别为 0:

- a 的系数为:

$$\begin{aligned} -\omega^3 + \bar{\omega} + i\omega &= -\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) + \left(\frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}i\right) + i\left(\frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i\right) \\ &= \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}\right) + i\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{6}+\sqrt{2}}{4}\right) \\ &= 0 \end{aligned}$$

- b 的系数为:

$$\begin{aligned}
 -\bar{\omega} + i\bar{\omega}^3 - i\omega &= -\left(\frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}i\right) + i\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) - i\left(\frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i\right) \\
 &= \left(-\frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{2}}{2} + \frac{\sqrt{6}-\sqrt{2}}{4}\right) + i\left(\frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{2}}{2} - \frac{\sqrt{6}+\sqrt{2}}{4}\right) \\
 &= 0
 \end{aligned}$$

- c 的系数为:

$$\begin{aligned}
 \omega^3 - i\bar{\omega}^3 &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) - i\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \\
 &= \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \\
 &= 0
 \end{aligned}$$

命题得证.

The End