The Josephus problem – Our variation

$\dot{J}(n)/\omega(n) = 3$

n	$\omega(n)$
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5
11	7
12	9
13	11
14	13
15	15
16	1

if, $n = K$
then,
$\omega(n) = 1$

Ex: 1

$13 = 2^3 + (\text{Something})$
$= 2^3 + 5$
$= 2^K + l$
$= 2l + 1$

Ex: 2

$41 = 2^5 + 9$
$= 2l + 1$
$= 9 \times 2 + 1$
$= 19$

Ex: 3

$77 = 2^6 + 13$
$= 2l + 1$
$= 2 \times 13 + 1$
$= 27$

Ex: 4

$$77 = \begin{array}{ccccccc} 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}$$

$= (0\,0\,1\,1\,0\,1\,1)_2$
$= (27)_{10}$

Recurrence form:

Even : $\dot{J}(2n) = 2 \cdot \dot{J}(n) - 1$

Odd : $\dot{J}(2n+1) = 2 \cdot \dot{J}(n) + 1$

Q. Find $J(100)$ using recurrences:

Ans: $\dot{J}(100) = 2\,\dot{J}(50) - 1$
$= 2 \cdot \{2 \cdot \dot{J}(25) - 1\} - 1$
~~$= 2 \cdot 2\{2\,\dot{J}(12) + 1 - 1\} - 2$~~ 1
~~$= 8\{$~~
$= 4\,\dot{J}(25) - 2 - 1$
$= 4\{2\dot{J}(12) + 1\} - 3$
$= 8\,\dot{J}(12) + 4 - 3$
$= 8 \cdot \{2\,\dot{J}(6) - 1\} + 1$

$\dot{J}(100) = 16\,\dot{J}(6) - 8 + 1$
$= 16 \cdot \{2 \cdot \dot{J}(3) - 1\} - 7$
$= 32\,\dot{J}(3) - 16 - 7$
$= 32\{2 \cdot \dot{J}(1) + 1\} - 23$
$= 64\,\dot{J}(1) + 32 - 23$
$= 64 \times 1 + 9$

Q. Prove by induction:

Ans: $j(n) = 2l+1$

$\Rightarrow j(2^m + l) = 2l+1$

Basis: $0 \le l \le 2^m$

if $m=0$, $l=0$,

$j(2^m + l) = 2l+1$

$\Rightarrow j(2^0 + 0) = 2\times 0 + 1$

$\therefore j(1) = 1$ $\boxed{\text{proved}}$

Hypothesis: Assuming from 1 to $m-1$, then equation holds:

Even: $j(2n) = 2.j(n) - 1$ ---------- ①

$\Rightarrow 2n = 2^m + l$

$\Rightarrow n = \frac{2^m+l}{2} = \frac{2^m}{2} + \frac{l}{2} = 2^{-1}.2^m + \frac{l}{2}$

$\therefore n = 2^{m-1} + \frac{l}{2}$.

(i) $j(2n) = 2j(n) - 1$

$= 2j(2^{m-1} + l/2) - 1$

$= 2.(2\frac{l}{2} + 1) - 1$ $[\because j(2^m + l) = 2l+1]$

$= 2l + 2 - 1$

$= 2l + 1$

$j(2n) = 2j(n) - 1$ -----①

break $2n$ into the lower power of $2(2^m) + l$

$\therefore 2n = 2^m + l$

$\Rightarrow n = (2^m + l)/2$

$\Rightarrow n = \frac{2^m}{2} + \frac{l}{2}$

$\Rightarrow n = 2^{-1}.2^m + \frac{l}{2}$

$\therefore n = 2^{m-1} + \frac{l}{2}$

Odd: $\dot{J}(2n+1) = 2 \cdot \dot{J}(n) + 1$

$$\Rightarrow 2n+1 = 2^m + l \cdot$$

$$\Rightarrow n = \frac{2^m + l - 1}{2}$$

$$\therefore n = 2^{m-1} + \frac{l-1}{2} \cdot$$

(i) $\dot{J}(2n+1) = 2 \cdot \dot{J}(n) + 1$

$$\Rightarrow \dot{J}(2n+1) = 2 \cdot \dot{J}\left(2^{m-1} + \frac{l-1}{2}\right) + 1$$

$$= 2 \cdot \left(2 \cdot \frac{l-1}{2} + 1\right) + 1$$

$$= 2l + 1$$

Q. Finding values of n where $\dot{J}(n) = n/3$

$$\dot{J}(n) = n/3$$

$$\Rightarrow 2l + 1 = \frac{2^m + l}{3}$$

$$\Rightarrow l = \frac{1}{5}(2^m - 3)$$

$$\boxed{\begin{aligned} 2n + 1 &= 2^m + l \\ \Rightarrow n &= \frac{2^m + l - 1}{2} \\ \therefore n &= 2^{m-1} + \frac{l-1}{2} \end{aligned}}$$

[II] Probability theory and random variable

Random variable: is a function that assigns each real number to a variable from the sample space.

$X = 0 \rightarrow$ occurance 1
$X = 1 \rightarrow$ " 3
$X = 2 \rightarrow$ " 3
$X = 3 \rightarrow$ " 1

Q. 2 dice rolled ; probability of getting sum = 7

$\rightarrow (4,3), (6,1), (5,2), (3,4), (1,6), (2,5) = 6$

$\therefore$ probability $= \frac{6}{36} = \frac{1}{6}$.

Q. numbers are different

$(1,1), (2,2), (3,3), (4,4), (5,5), (6,6) = 6 = 36-6 = 30$

$\therefore$ probability $= \frac{30}{36} = \frac{5}{6}$.

Q. Sum of 2 fair dice,

$P(X=2) = P(1,1) = \frac{1}{36}$

$P(X=3) = P\{(1,2),(2,1)\} = \frac{2}{36}$

$P(X=4) = P\{(1,3),(2,2),(3,1)\} = \frac{3}{36}$

$P(X=5) = P\{(1,4),(2,3),(3,2),(4,1)\} = \frac{4}{36}$

$P(X=6) = P\{(1,5),(2,4),(3,3),(4,2),(5,1)\} = \frac{5}{36}$

Q. two coins

$P(Y=0) = P\{(T,T)\} = \frac{1}{4}$

$P(Y=1) = P\{(T,H),(H,T)\} = \frac{2}{4}$

$P(Y=2) = P\{(H,H)\} = \frac{1}{4}$

$\therefore P\{Y=0\} + P\{Y=1\} + P\{Y=2\} = 1$.

Josephus Exercise:

Ex-1: given, $2l+1 = \frac{2^m + l}{2}$

$$\Rightarrow 2l = \frac{2^m + l}{2} - 1 = \frac{2^m + l - 2}{2}$$

$$\Rightarrow 4l + 2 = 2^m + l$$

$$\Rightarrow 4l - l + 2 = 2^m$$

$$\Rightarrow 3l + 2 = 2^m$$

$$\Rightarrow 3l = 2^m - 2$$

$$\therefore l = \frac{1}{3} \cdot (2^m - 2)$$

$m = 0$; $l = \frac{1}{3}(2^0 - 2) = 0$

$m = 1$; $l = \frac{1}{3}(2^1 - 2) = 0$; $n = 2^m + l = 2 + 0 = 2$; $J(n) = 2l+1$
$= \frac{n}{2} = \frac{2}{2} = 1$

$m = 2$; $l = \frac{1}{2}(4-2) = 2$; $n = 2^m + l = 6$; $J(n) = \frac{n}{2} = \frac{6}{2} = 3$

$m = 3$, $l = 10$; $n = 42$, $J(n) = 21$

$m = 4$, $l = 42$; $n = 170$; $J(n) = 85$

Q. smallest 3 values of n ; person at $n/3$ position

$\rightarrow$ $J(n) = \frac{n}{3}$

$\Rightarrow 2l+1 = \frac{2^m+l}{3}$

$\Rightarrow 6l+3 = 2^m + l$

$\Rightarrow 6l - l + 3 = 2^m$

$\Rightarrow 5l = 2^m - 3$

$\therefore l = \frac{1}{5} \cdot (2^m - 3)$

m value: l must be int ; $(2^m - 3)$ must div by 5.

$m = 1$; $l = \frac{1}{5}(2-3) = -1/5$

$m = 2$; $l = 1/5\ (4-3) = 1/5$

$m = 3$; $l = 1/5\ (8-3) = 5/5 = 1$ ✓

$m = 7$; $l = \cancel{25}\ \cancel{7}\ 25$ ✓

$m = 11$; $l = \cancel{4}\ 409$ ✓

$\therefore n = \cancel{2 \times 1 + 1}\ 2^3 + 1 = 9$

$\therefore n = 2^7 + 25 = 153$

$\therefore n = 2^{11} + 409 = 2457$

Verification:

$J(n) = 2l + 1$

$J(9) = 2 \times 1 + 1 = 3$

$J(153) = 2 \times 25 + 1 = 51$

$J(2457) = 2 \times 409 + 1 = 819$

Q, $J(n) = 25$

$\Rightarrow 2l + 1 = 25$

$\Rightarrow l = 12 \qquad \therefore n = 2^m + 12$

$m = 0;\ n = 2^0 + 12 = 13$

$m = 1;\ n = 2^1 + 12 = 14$

$m = 2;\ n = 2^2 + 12 = 16$

$m = 3;\ n = 2^3 + 12 = 20 \longrightarrow J(n) = 20 \Rightarrow 2^4 + 4 \Rightarrow 2\times 4 + 1 = 9 \neq 25$

$m = 4;\ n = 2^4 + 12 = 28 \longrightarrow J(n) = 25 \Rightarrow 28 = 2^4 + 12 \Rightarrow 2\times 12 + 1 = 25$

$m = 5;\ n = 2^5 + 12 = 44$

$m = 6;\ n = 2^6 + 12 = 76$

$m = 7;\ n = 2^7 + 12 = 140$

Q. $J(n) = n$

$\Rightarrow 2l + 1 = n \Rightarrow \cancel{l = n+1}$

$\Rightarrow 2l + 1 = 2^m + l \Rightarrow \boxed{l = 2^m - 1}$

$\therefore n = 2^m + 2^m - 1 = 2^{m+1} - 1$

m	$l = 2^m - 1$	$n = 2^{m+1} - 1$	$J(n) = n$
0	$2^0 - 1 = 1 - 1 = 0$	$2 - 1 = 1$	$2\times 0 + 1 = 1$
1	$2^1 - 1 = 2 - 1 = 1$	$4 - 1 = 3$	$2\times 1 + 1 = 3$
2	$2^2 - 1 = 4 - 1 = 3$	$8 - 1 = 7$	$2\times 3 + 1 = 7$
3	$2^3 - 1 = 8 - 1 = 7$	$16 - 1 = 15$	$2\times 7 + 1 = 15$
4	$2^4 - 1 = 16 - 1 = 15$	$32 - 1 = 31$	$2\times 15 + 1 = 31$
5	$2^5 - 1 = 32 - 1 = 31$	$64 - 1 = 63$	$2\times 31 + 1 = 63$