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Planning Demand-Responsive Transit to reduce inequality of accessibility

Duo Wang *, Andrea Araldo, Mounim El Yacoubi

SAMOVAR, Télécom SudParis, Institut Polytechnique de Paris, Paris, France

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ABSTRACT

Accessibility measures how well a location is connected to surrounding opportunities. We focus on accessibility provided by Public Transit (PT). There is an evident inequality in the distribution of accessibility between city centers or close to main transportation corridors and suburbs. In the latter, poor PT service leads to a chronic car-dependency. Demand-Responsive Transit (DRT) is better suited for low-density areas than conventional fixed-route PT. DRT can thus reduce accessibility inequality, but this potential has not yet been exploited. On the contrary, planning DRT without care to inequality (as in the methods proposed so far) may further exacerbate the accessibility gap in urban areas.

To the best of our knowledge, this paper is the first to propose a DRT planning strategy, aimed at the reduction of the inequality of the accessibility distribution. This objective is more complex than the usual generalized cost minimization. To achieve it, we combine a graph representation of conventional PT and a Continuous Approximation (CA) model of DRT. The two are combined in the same multi-layer graph, on which we compute accessibility. We then devise a scoring function to estimate the need of each area for an improvement, appropriately weighting population density and accessibility. Finally, we provide a bilevel optimization method, where the upper level is a heuristic to allocate DRT buses, guided by the scoring function, and the lower level performs transit assignment. Numerical results in a simplified model of 3 different cities show that inequality in the distribution of accessibility, measured with the Atkinson index, is reduced more than 20%.

1. Introduction

The *accessibility* of a location measures the ease for its residents to reach surrounding opportunities (e.g., schools, shops, or jobs). Here, we study Public Transit (PT) accessibility, i.e., how many opportunities per unit of time one can reach by using PT. Accessibility is unequally distributed in urban regions (Badeanlou et al., 2022): it is generally good in the city centers and areas close to main PT corridors, and poor in the suburbs, which are thus car-dependent. This makes cities environmentally, economically and socially unsustainable (Giuffrida et al., 2017; Saeidizand et al., 2022). Minimizing the inequality of the accessibility distribution in a territory is thus crucial and is the central focus of this study.

Poor accessibility in the suburbs is a chronic problem of conventional PT, which is based on fixed lines with predetermined schedules. Offering a good service would require an extremely large number of lines, stops, and frequencies. Considering the low population density, the cost per-capita would be prohibitive for a PT agency.

* Corresponding author.
E-mail address: duo.wang@ip-paris.fr (D. Wang).

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By dynamically building routes that adapt to the observed user requests, Demand-Responsive Transit (DRT) is better suited for low density areas (Quadrifoglio and Li, 2009). DRT is also a promising complement to conventional PT (Calabro' et al., 2023) for serving the first-mile and last-mile for suburbs. Autonomous vehicles are expected to reduce the cost, thus making DRT a feasible option for PT operators (Fielbaum, 2020). DRT is already deployed in real cities, often by private companies (e.g., Padam (2024), Texelhopper (2024), Ioki (2024)) making appropriate agreements with transport authorities (Iledefrance, 2024). However, to the best of our knowledge, such deployments are not planned having a precise quantification of reduction of accessibility inequality in mind. Even in the literature, most of the proposed planning strategies for DRT aim to minimize generalized cost, including both user and agency related cost (Luo and Nie, 2019; Calabro' et al., 2023). Limited attention has been paid to the quantitative assessment of how DRT allocation strategies influence the inequality of accessibility at the city-wide level. Moreover, there is a lack of research that formulates DRT allocation strategies grounded in clearly defined inequality indicators of accessibility. This study seeks to address this gap in the literature and addresses the deployment of DRT in the first and last mile.

Three aspects make this problem challenging. *First*, accessibility is the result of intricate and latent dependencies between different parts of the PT network. Therefore, a local change may affect accessibility of remote areas. Such dependencies are coupled in a non-trivial fashion with the topology of the PT network. *Second*, there is a complex demand–supply dependence, since the performance of DRT impacts the number of users that choose to use it, which in turn impacts the performance. *Third*, when considering accessibility inequality, one should not only examine the geographical distribution of accessibility, but also population density. This is important in order to prevent DRT planning decisions from being heavily driven by few people who have chosen to live in isolated areas. Population and accessibility distributions may be contrasting.

Our main contributions are the following:

- We combined a Continuous Approximation (CA) model of DRT and a graph model of conventional PT. CA allows to efficiently estimate the average performance indicators of DRT. On the other hand, the graph model of conventional PT represents the topology of the network that already exists in a city, thus capturing the geographical dependencies in the accessibility computation. This makes the findings obtained with our model relevant for the city under study. We obtain a single multilayer graph describing a multimodal PT, containing both DRT and conventional PT, in which accessibility can be computed.
- We propose an optimization method to support DRT planning, at the strategic level, in an urban conurbation. The method suggests in which areas to deploy DRT and with how many vehicles. The objective is to reduce the spatial inequality of the accessibility distribution. To solve the demand–supply circular dependency, our optimization procedure is bilevel: the upper level decides how many DRT vehicles to deploy in each area, while the lower level iteratively solves a transit assignment problem, to estimate how travelers distribute in the multimodal network.

Numerical results in scenarios representing three different cities show that our method effectively reduces accessibility inequality and shows superior performance when compared with state-of-the-art Bayesian optimization algorithms. We release our code to run our strategy and reproduce the analysis as open source (Wang, 2023). A simplified version of our code for more demonstration purposes is also available as open source (Araldo, 2023).

The paper is organized as follows: Section 2 reviews the related work. Section 3 introduces a multilayer graph which combines a Continuous Approximation model of DRT and a graph model of conventional PT. Section 4 formulates the problem of optimizing inequality of accessibility as an integer programming model. To solve it, we propose a bi-level optimization algorithm: the upper level addresses the allocation of Demand-Responsive Transit (DRT) resources, while the lower level updates the multi-layer network via a Traffic Assignment procedure. Section 5 compares its performance with the state-of-the-art Bayesian optimization method across three different cities. Finally, Sections 6 and 7 discuss future perspectives and conclude the paper, respectively.

2. Related work

In this work, we consider the integration of Public Transport (PT) and flexible mobility services. The latter are those provided by a fleet of vehicles with routes dynamically adapted to user requests. Examples of flexible mobility are Demand-Responsive Transit (DRT), provided with buses or minivans, and ride-sharing (e.g., Uber). In this work, in particular, We focus on DRT.

Public Transport Network Design Problems (PTNDPs) can be studied at two levels: planning level or operational level. At the *planning level*,[1] a PT planner decides the routes of the different lines as well as their frequencies. In case DRT is integrated into PT, a planner also decides in which areas to deploy DRT and how to allocate DRT fleet across those areas (Calabro' et al., 2023). At an *operational level*, a PT operator organizes the service in order to match the decisions taken at the strategic level, deciding precise timetables, as well as crew and vehicle scheduling and routing. We focus in this paper on PTNDPs at a planning level. A review of planning level PTNDPs is provided in Farahani et al. (2013).

Since the aim of our method is to reduce the inequality in the distribution of accessibility, we first present the different definitions of accessibility and equality (Section 2.1), we then review the work on measuring accessibility inequality (Section 2.2). We also introduce *what-to* methods, which aim to minimize generalized costs or accessibility inequality (Section 2.3). In this paper, we present a *what-to* method. *What-to* can be further classified into two categories based on the underlying modeling approach: fully analytical models and detailed simulations. Section 2.4 provides an overview of the different models and related studies used in the *what-to* method.

[1] With "planning level" we refer to strategic and tactical level decisions (Desaulniers and Hickman, 2007).

2.1. Accessibility and equality

A plethora of mathematical definitions exist for accessibility and equality (Miller, 2020; Biazzo et al., 2019; Badeanlou et al., 2022; Zuo et al., 2020). We focus on the equality of the geographical distribution of PT accessibility. It is a form of horizontal equity, i.e., it is agnostic to individual characteristics and income, as opposed to vertical equity, which instead considers them Tomasiello et al. (2020). Indeed, our focus is to offer good PT options to all individuals, to reduce their car-dependency, independent of whether they are disadvantaged or not and this is why we focus on equality instead of vertical equity. However, within our method, it would be possible to unplug the inequality computation and plug a vertical equity computation, if appropriate data are available, which is however outside of the scope of our work.

In general, PT planning methods can be classified into *what-if* and *what-to* methods. The former estimate change in performance with respect to few design alternatives. The latter, instead, aim to find the optimal design of PT according to some objective function.

2.2. What-if methods

Some works using *what-if* methods to measure the inequality of accessibility of PT are Badeanlou et al. (2022, 2023) and Zuo et al. (2020). Badeanlou et al. (2022) compute accessibility scores of four different cities via open data, using the method previously proposed by Biazzo et al. (2019). Badeanlou et al. (2023) compute the contribution of each PT line in the overall equality of accessibility. Zuo et al. (2020) use the Level of Traffic Stress (LTS) index (Crist et al., 2019) to measure whether a bicycle-lane can be arranged on a route and study the changes in accessibility of Hamilton County, Ohio, U.S. before and after bicycle-lane is arranged. Liezenga et al. (2024) combined datasets of microtransit ridership from two public transit agencies, transit surveys, land use data, and expert interviews in Minneapolis-St. Paul in Minnesota, USA, to conduct spatial analysis, accessibility analysis, and equity impact assessments for suburbs.

What-if methods have been presented to assess the impact of flexible mobility on cost-efficiency (Rahimi et al., 2018) and for cost–benefit analysis (Carreyre et al., 2023). We will focus here on *what-if* studies that estimate the impact of flexible mobility in the distribution of accessibility. Nahmias-Biran et al. (2021) and Zhou et al. (2021) both implement an activity-based accessibility measure in the SimMobility simulation for analyzing the performance of automated mobility on-demand (AMoD) services. Nahmias-Biran et al. (2021) conclude that introducing AMoD can improve accessibility. Zhou et al. (2021) conclude that AMoD could alleviate inequity in accessibility as it appears to benefit the disadvantaged socioeconomic groups. Diepolder et al. (2023) propose a geostatistical method to compute accessibility from DRT in a data-driven fashion. Applying this method on trips simulated in MATSim, they conclude that DRT has the potential to improve accessibility in areas where conventional PT is insufficient, thus suggesting potential for reduction in inequality. Ziemke and Bischoff (2023) calculate accessibility from AMoD. They compare in MATSim simulation different scenarios of deployment and confirm the potential to reduce inequity. Ahmed et al. (2020) propose a logsum-based measure of employment accessibility benefits for AMoD. The results indicate that workers living in lower density areas benefit more than workers living in high density areas from AMoD. We instead choose our accessibility as the sum of opportunities reachable per unit of time, which is easier to interpret. However, in our methodology it is possible to replace our formulation of accessibility, with a logsum formulation or any other well-known formulations. Eppenberger and Richter (2021) first uses a linear regression analysis to examine how different forms of accessibility correlate with socio-economic indicators (average income, unemployment rate and education level), which could inform policymakers in decisions related to deployment of shared autonomous vehicles (SAVs). Pereira et al. (2024) analyze the potential for ride-hailing services to improve employment accessibility when used as a standalone transportation mode and in conjunction with transit as a first-mile connection. They conclude that ride-hailing can improve urban accessibility, but without policies to address affordability, it risks reinforcing socio-economic disparities. Hawas et al. (2016) presents an approach for accessibility categorization in areas where there is no extensive data available.

2.3. What-to methods

What-to methods optimize PT according to some objective function. Most what-to approaches aim to optimize the generalized cost, such as the total of traveling time plus some cost for the PT agency. In our work, we focus on the inequality of accessibility, which is more complex than generalized cost. Indeed, reducing inequality of accessibility requires capturing the interrelation between the distribution of opportunities, of population and the PT graph topology.

Recent literature has studied integration of DRT and conventional PT at an operational level. For instance, Grahn et al. (2021) propose a framework to update DRT shuttle routes for the new incoming request by minimizing marginal costs, where marginal costs are the difference in total shuttle user costs before and after the new request is added to the shuttle queue.

We instead propose in this paper a planning-level what-to method. Related to this, most work aims to minimize generalized cost (Levin et al., 2019; Calabro' et al., 2023; Luo and Nie, 2019; Fielbaum, 2020; Fielbaum et al., 2022). Levin et al. (2019) formulate the problem of integration of DRT as a linear program of minimizing total passenger travel time using the link transmission model (Yperman et al., 2008), and find a suboptimal solution using a rolling horizon method. Calabro' et al. (2023) optimize jointly conventional PT line and DRT fleet deployment. Equity has also been taken into account for PT optimization, at an operational level: (Zhan et al., 2020) optimize timetables and ticketing to favor low-income passengers. Fielbaum and Alonso-Mora (2024) designs mixed fixed-flexible transit networks with the goal of minimizing the Total Generalized Cost.

We now discuss some work considering accessibility (Behbahani et al., 2019; Dai et al., 2023; Wang and Chen, 2021). Behbahani et al. (2019) add a constraint concerning equality in a classic cost minimization problem. Dai et al. (2023) optimize headway and stop spacing of a single PT line, considering equality. Wang and Chen (2021) use a geographically weighted regression model to estimate the local relationships between accessibility and previous transportation investments, and then find an optimal allocation of 10% growth in transportation investments by optimizing inequality of accessibility. Tong et al. (2015) employs a bi-level programming model to optimize the transportation network. In the upper level, the model seeks to maximize overall accessibility by strategically selecting network improvements. The lower level simulates user behavior, assuming that travelers choose routes that minimize their travel time within the constraints of the network. Fu et al. (2022) propose an Activity-based space–time accessibility measure, and use a heuristic algorithm to optimize headways and fares of conventional PT. None of these papers, however, consider DRT. We are the first to set the reduction of accessibility distribution inequality as the main optimization goal of the optimization of DRT planning.

2.4. Modeling frameworks

Traffic modeling methods can be divided into two categories: fully analytical models and detailed simulations.

2.4.1. Fully analytical models

We note that optimization of the design of Adaptive Transit (conventional PT + flexible modes) is often tackled with fully analytical models, e.g., continuous approximation (Calabro' et al., 2023; Luo and Nie, 2019; Levin et al., 2019) or similar models (Fielbaum, 2020; Rahimi et al., 2018; Kadem et al., 2024). Their main limit is that the PT network is represented with a regular geometrical pattern. For example, Calabro' et al. (2023) abstracts the city as a circular structure. Such idealized structures can loosely describe "any city", but at the end they describe "no city": indeed, a PT network is always much more complex than regular geometrical shapes. In our work we focus on accessibility, which results from an intimate interplay between the actual structure of PT deployed and how population and opportunities distribute around it. Idealizing PT with simple shapes would lead to findings with no real relevance. This rules out the aforementioned fully analytical approaches. This is why we model PT as a graph.

2.4.2. Detailed simulation

On the opposite side of the methodological spectrum, building a detailed simulation of the entire multimodal PT as in Pinto et al. (2020) is very complex to develop and heavy to run together with multiple optimization iterations. This would overshoot the needs of preliminary findings at a planning level. Dalmeijer and Van Hentenryck (2020) consider the network design problem for On-Demand Multimodal Transit Systems through transfer-expanded graphs, thus resulting in significant computational improvements. Gurumurthy et al. (2020) show that shared automated vehicles (SAVs) have the potential to help solving first-mile-last-mile transit problem when fare benefits are provided to transit users, via performing an agent-based simulation of Austin, Texas. However, since we are targeting new deployments of DRT services that are not currently operating, we must accept that even detailed simulation will not match the performance of such services in reality. For all these reasons, we decide to use Continuous Approximation (CA) to model DRT.

2.4.3. Hybrid model

To both capture the peculiarities of the PT topology of cities and achieve efficient computation for DRT planning, we propose a hybrid model that combines a graph to represent conventional PT and CA to model DRT. To the best of our knowledge, we are the first in doing so for PT. A similar work in this sense is Wong et al. (2003), which is however dealing with completely different dynamics (car flow dynamics, where a graph represents highways and CA models represent local streets). Moreover, we optimize DRT deployment, while Wong et al. (2003) does not involve any strategic decision.

3. Model

We study a multimodal urban transit system, as in Fig. 1:

1. Conventional PT, consisting of metro, train or bus lines, with regular routes and schedules.
2. Demand-Responsive Transit (DRT), provided by bus, acting as a feeder for conventional PT and serving the First Mile and Last Mile.

Fig. 2 presents the types of trips a user can perform in the considered PT system. A user can walk to the PT station or choose to go to the PT station via a DRT bus from the origin. Symmetrically, the "last mile" from a PT station to the user's destination can be performed by walking or DRT. A user can also walk directly from origin to destination. We assume a user always chooses the combination of modes with which they can go from origin to destination.

We now describe the model that we use as a base for RSG optimization. After explaining the tessellation of the study area (see Section 3.1), we first explain how we model conventional PT (see Section 3.2) and DRT (see Section 3.3) separately and how we merge such models into a single multilayer graph (see Section 3.4). We then explain how we generate the demand (see Section 3.5). Table of notation is in Appendix A.

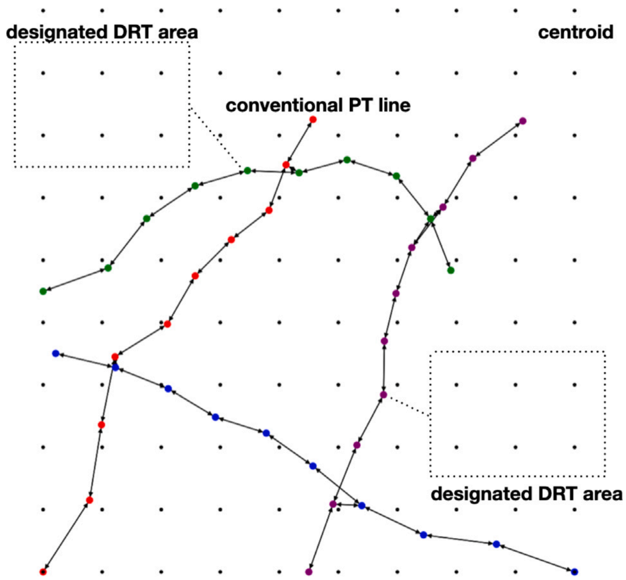


Fig. 1. Multimodal Public Transit.

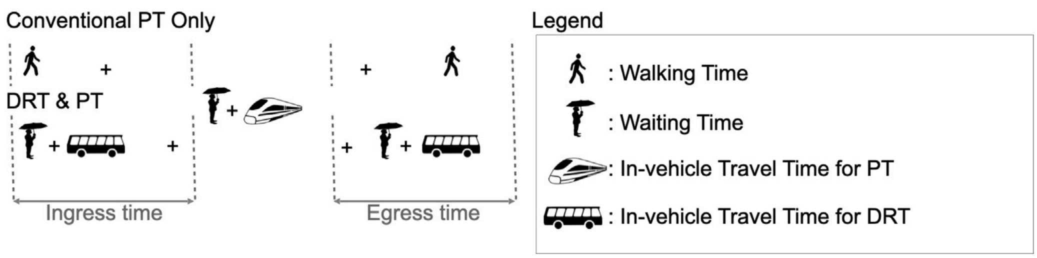


Fig. 2. Components of the access, egress and waiting time for the case with conventional PT only and the case with DRT.

3.1. Tessellation of the study area

We partition the study area with a regular tessellation. In the experiments, we adopt square tiles of 1 km^2, but any other regular shape could be employed. The center of each tile is called *centroid*. We denote by $\mathcal{C} := \{c_1, \ldots, c_m\}$ the set of all m *centroids* in the study area. By slight abuse of notation, we will sometimes refer to "tile c_i" to denote the "tile around centroid c_i". We will thus use "tile" and "centroid" interchangeably when this does not create ambiguity.

Opportunities are distributed in the tiles. They can be shops, restaurants, schools, etc. In the experiments. We assume tiles are small enough so that the distance between any point in a tile and the corresponding centroid in the center is negligible. We can thus approximately assume all trips generate from a centroid and end at another centroid. In some cases, approximating the destination to the closest centroid may overestimate the time of some trips and underestimate the time of other trips. These overestimated and underestimated times statistically compensate each other. The same argument applies to the origin. Note that, as in several studies on accessibility (Biazzo et al., 2019), we prefer regular tessellation to standard zoning. Regular tessellation is preferred in various studies due to its ability to simplify complex data and provide a clearer interpretation (Frigaard and Nouar, 2005). This will allow simple development of the CA model (Eqs. (2)–(8)).

3.2. Graph model of conventional PT

We model conventional PT as a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (see Fig. 1), $\mathcal{V}$ is the set of all nodes and $\mathcal{E}$ is the set of all edges. $\mathcal{G}$ is composed of all conventional PT lines. We will refer to them as "metro lines", for brevity, but the model does not change if a part of those lines is served via buses. Set $\mathcal{S} \subset \mathcal{V}$ is the set of all stations and set $\mathcal{E}_s \subset \mathcal{E}$ is the set of all edges of metro lines. A line l has a headway t_l, which is the time between two consecutive vehicles in the same direction. We approximate the average waiting time for line l as $w_l = t_l/2$ as in Pinto et al. (2020). Line l is a sequence of K_l stations, $s_1, \ldots, s_{K_l}$, linked by edges. Let us denote with $t(s_i, s_{i+1})$ the time spent by a vehicle going from station s_i to the next station s_{i+1}. To allow boarding and alighting at station s_i, the vehicle stops for a dwell time t_{s_i}. If a passenger is at a transfer station s_i and changes from PT line l to line l', we consider an average waiting time $w_{l'}$ at station s_i.

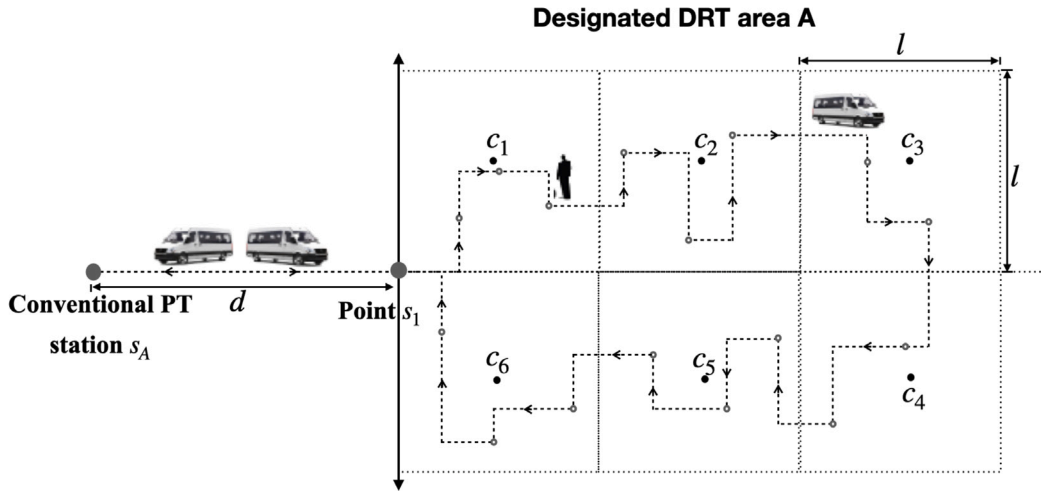


Fig. 3. Model of Demand Response Transit (DRT).

We also include in $\mathcal{V}$ the set of centroids $\mathcal{C}$. We also include in $\mathcal{E}$ the set $\mathcal{E}_c$ of all edges between to neighbor centroids, and the set $\mathcal{E}_d$ of edges between any centroid and all stations. Each edge $e \in \mathcal{E}_c \cup \mathcal{E}_d$ represents the possibility to walk with the time cost t_e. Time cost t_e is calculated by: $t_e = d_e/v_{\text{walking}}$, where d_e is the Euclidean distance between two nodes of edge e, and v_{walking} is the walking speed. Note that we are studying a geographic graph, and each node has its location coordinates, so the Euclidean distance between any nodes can be calculated. Therefore, in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we have $\mathcal{V} = \mathcal{C} \cup \mathcal{S}$ and $\mathcal{E} = \mathcal{E}_s \cup \mathcal{E}_c \cup \mathcal{E}_d$.

For a trip from centroid c_i and c_j, the traveler can choose between different modes of travel. For example, travelers could perform the entire trip by walking at speed v_{walking} or they could walk from the origin centroid c_i to a conventional PT station, go via conventional PT to another station, and from there walk to the destination centroid c_j. Travelers always select the shortest time path between the origin centroid and the destination centroid (for brevity, we will call it "shortest path"). This is more accurate than Luo and Nie (2019) and Calabro' et al. (2023) in which shortest distance path is selected.

Note that graph $\mathcal{G}$ represents a PT network in a certain time slot, e.g. 1 h, within which we can assume that line routes and headway values do not change. To obtain a planning over a day, the optimization we will present here should be repeated in the different time-slots.

3.3. Continuous approximation (CA) model of DRT

Due to financial constraints, there is a limitation on the total number of buses that can be deployed for DRT services. Using more buses means more expenses, such as repair costs and more employee salaries. Because of this, only selected *designated DRT areas* can be served via DRT. Vehicles will depart from a selected conventional PT station at regular times to pick up travelers in the designated DRT area. As a result, passengers in the service area will travel as shown in Fig. 2. A passenger may wait until a DRT bus arrives (waiting time) and spend some time in the bus (in-vehicle travel-time) to the conventional PT station. Symmetrically, to reach the final destination from an egress PT station, a passenger must walk or use another DRT service, if available.

The DRT model is shown in Fig. 3. For a generic designated DRT area A associated to conventional PT station s_A, a DRT bus departs from s_A and proceeds to point s_1, without performing any pick up/drop off between s_A and s_1. Starting from s_1, the bus serves each tile c_i in sequence and then returns to station s_A via point s_1.

Our DRT model is based on Sec. 4.4 of Calabro' et al. (2023) and relies on two assumptions:

- A DRT bus can only travel along the horizontal and the vertical direction following a no-backtracking policy.
- Demand is uniformly distributed within each tile, but it changes from tile to tile (see Section 3.5).[2]

Obviously, in reality, these aspects might be more complicated. However, simplifying assumptions are common (and necessary) in any strategic level decision making process, like the one we tackle in this paper. As shown in real world examples, going from high-level strategic decisions to detailed plans (Estrada et al., 2011), the findings we get thanks to those simplifying assumptions are an important guidance step toward realistic designs, where findings are adapted to the specific use case at hand, via appropriate methods (e.g. agent based models), which are outside the scope of this paper.

We extended the model of Calabro' et al. (2023) in the following terms:

[2] If tiles are sufficiently small this assumption is not problematic. In any case, the granularity of demand locations is limited by available data that, for privacy reason, usually aggregate over the space. As an example, the open dataset that we use in the numerical section (Table 1) are aggregated at a 1 km^2 level.

- We combine a CA for DRT in the first and last mile with a graph representation of conventional PT
- We do not just model population densities, but also opportunity densities.
- We allow the tiles contained in a DRT area to have different population and opportunity densities. We consequently allow the bus to have different number of pickups and dropoffs in each tile.
- We allow the main PT station associated to each DRT area to be outside of the area.

We now derive the equations of the Continuous Approximation of DRT. Let time $h(A)$ denote the DRT service headway, which means a new DRT bus leaves from the starting station s_A every time $h(A)$, e.g., if $h(A) = 5$ min, there a DRT bus will leave from the starting station s_A every 5 min.

Referring to Fig. 3, we denote with K the number of tiles in area A, with l the side of each tile. Let x_A be the number of DRT buses allocated to area A. Let v_{DRT} be the DRT bus speed, τ_s the time lost per stop (assuming a single passenger served per stop) and τ_T the dwell time of the DRT bus at the terminal s_A. Let d the distance between stop s_A and point s_1.

The following proposition allows, for any allocation of DRT buses, to compute $h(A)$. This result will be later used for transit assignment.

Proposition 1. *Let $n(A)$ denote the number of pickups and dropoffs requested within area A, generated during time $h(A)$. Thus, $n(A)$ is the number of requests to be served by one bus. Let $\phi_i^{DRT,in}$ denote the number of passengers per unit of time, having tile c_i as destination, and using DRT for their last mile. Let $C_{DRT}(A)$ be the cycle time, i.e., the expected time required to complete a cycle of a DRT bus, which corresponds the time between two departures of the same bus.*

The relation between those quantities is expressed by:

$$C_{DRT}(A) = h(A) \cdot x_A \tag{1}$$

$$n(A) = h(A) \cdot l^2 \Big(\sum_{c_i \in A} \phi_i^{DRT,\ out} + \sum_{c_i \in A} \phi_i^{DRT,\ in} \Big) \tag{2}$$

$$\begin{aligned} h(A) \cdot x_A = {} & \frac{Kl}{v_{DRT}} \cdot \frac{h(A) \cdot l^2 (\sum_{c_i \in A} \phi_i^{DRT,\ out} + \sum_{c_i \in A} \phi_i^{DRT,\ in})}{h(A) \cdot l^2 (\sum_{c_i \in A} \phi_i^{DRT,\ out} + \sum_{c_i \in A} \phi_i^{DRT,\ in}) + 1} \\ & + \Big(\frac{l}{3 v_{DRT}} + \tau_s\Big) \cdot h(A) \cdot l^2 \Big(\sum_{c_i \in A} \phi_i^{DRT,\ out} + \sum_{c_i \in A} \phi_i^{DRT,\ in} \Big) \\ & + \frac{4l}{3 v_{DRT}} + \frac{2d}{v_{DRT}} + \tau_T \end{aligned} \tag{3}$$

Proof. To prove the proposition, we will compute $C_{DRT}(A)$ in two ways: first, considering the point of view of multiple buses, second, considering the point of view of a single bus.

Point of view of multiple buses. The equation relating $h(A)$ and $C_{DRT}(A)$ is straightforward:

$$h(A) = \frac{C_{DRT}(A)}{x_A}$$

For example, if on average, each bus takes 1 h to complete its cycle, and we have 10 buses, then the headway $h(A)$ is 6 min.

Point of view of a single bus. We now calculate the number of requests $n(A)$. To ensure the stability of the system (i.e., no degenerating accumulation of unserved requests), on average, each bus needs to serve a number of requests equal to those generated during time $h(A)$. Since l is the side of a tile, each tile has an area of l^2, and the number of pickups and dropoffs in designated DRT area A is the following sum over all tiles c_i contained in designated DRT area A:

$$n(A) = h(A) \cdot l^2 \Big(\sum_{c_i \in A} \phi_i^{\text{DRT, out}} + \sum_{c_i \in A} \phi_i^{\text{DRT, in}} \Big) \tag{4}$$

The two terms in the above sum correspond to the number of pickups and dropoffs, respectively.

To calculate cycle time $C_{DRT}(A)$, we first need to estimate length $CL_{DRT}(A)$, which is the expected value of the distance traveled within the entire area A. Such length is represented by the circle in Fig. 3 and measures how many km the DRT bus travels from the moment it leaves point s_1 to the moment it comes back. The formula for $CL_{DRT}(A)$ is (refer to Appendix B for details):

$$CL_{DRT}(A) = Kl \frac{n(A)}{n(A)+1} + n(A) \cdot \frac{l}{3} + \frac{4l}{3} \tag{5}$$

Therefore, the expected time required to complete a cycle $C_{DRT}(A)$ is:

$$C_{DRT}(A) = \frac{CL_{DRT}(A) + 2d}{v_{DRT}} + \tau_s \cdot n(A) + \tau_T \tag{6}$$

where v_{DRT} is the DRT bus' speed, τ_s is the time lost per stop (assuming a single passenger served per stop) and τ_T is the terminal dwell time of the DRT bus at the terminal s_A. Term $2d$ accounts for the distance traveled from s_A to point s_1 and back.

Taking Eq. (5) into Eq. (6), we get:

$$C_{DRT}(A) = \frac{CL_{DRT}(A) + 2d}{v_{DRT}} + \tau_s \cdot n(A) + \tau_T$$

$$= \frac{(Kl\frac{n(A)}{n(A)+1} + n(A)\cdot\frac{l}{3} + \frac{4l}{3}) + 2d}{v_{DRT}} + \tau_s \cdot n(A) + \tau_T$$
$$= \frac{Kl}{v_{DRT}} \cdot \frac{n(A)}{n(A)+1} + \left(\frac{l}{3v_{DRT}} + \tau_s\right)\cdot n(A) + \frac{4l}{3v_{DRT}} + \frac{2d}{v_{DRT}} + \tau_T \tag{7}$$

Final step. Note that the only variable in Eq. (7) is $n(A)$, and the other parameters are fixed. Via Eq. (1), we can express the left size of the above equation in terms of $h(A)$ and x_A. We then replace $n(A)$ in the right side via Eq. (2). By doing so, we get the relationship equation between headway $h(A)$, number of DRT buses x_A, and demand flows $\phi_i^{\text{DRT, out}}$ and $\phi_i^{\text{DRT, in}}$:

$$h(A)\cdot x_A = \frac{Kl}{v_{DRT}} \cdot \frac{h(A)\cdot l^2(\sum_{c_i\in A}\phi_i^{\text{DRT, out}} + \sum_{c_i\in A}\phi_i^{\text{DRT, in}})}{h(A)\cdot l^2(\sum_{c_i\in A}\phi_i^{\text{DRT, out}} + \sum_{c_i\in A}\phi_i^{\text{DRT, in}}) + 1}$$
$$+ (\frac{l}{3v_{DRT}} + \tau_s)\cdot h(A)\cdot l^2(\sum_{c_i\in A}\phi_i^{\text{DRT, out}} + \sum_{c_i\in A}\phi_i^{\text{DRT, in}})$$
$$+ \frac{4l}{3v_{DRT}} + \frac{2d}{v_{DRT}} + \tau_T \quad \square$$

The previous Proposition gives the quantitative relation between headway $h(A)$, number of DRT buses x_A, and demand flows $\phi_i^{\text{DRT, out}}$ and $\phi_i^{\text{DRT, in}}$. For example, once we calculate demand flows, we can directly get the average headway $h(A)$ obtained by deploying x_A buses in area A, via Eq. (3). Then, we can compute the cycle time $C_{DRT}(A)$ via Eq. (1).

In the last part of Section 3.3, we will introduce how to calculate the excepted ingress time $T_{in}(c_i)$ at centroid $c_i \in A$, i.e., the time it would take for a user at tile c_i from the moment they requested a trip to the moment in which they arrive at the designated station s_A. Symmetrically, we define the egress time. Assume designated DRT area A has $2\times\frac{K}{2}$ tiles (K = 6 in Fig. 3). For centroid $c_i \in A$ let $n(c_i)$ denote the number of pickups and dropoffs requested within tile c_i between the departure of a DRT bus and the next. The expected value of ingress and egress access time for centroid c_i is:

$$T_{in}(c_i) = T_{out}(c_i) = \underbrace{\frac{h(A)}{2}}_{(1)} + \underbrace{\frac{\frac{n(c_i)}{2} + \sum_{j=i+1}^{K} n(c_j)}{n(A)} \cdot \frac{CL_{DRT}(A) + \tau_s\cdot n(A)}{v_{DRT}}}_{(2)} + \underbrace{\frac{d}{v_{DRT}}}_{(3)} \tag{8}$$

The three components (1), (2) and (3) of Eq. (8) calculate the average waiting time for the DRT bus, the average in-vehicle time from c_i to s_1, and the average in-vehicle time from s_1 to s_A, respectively (as shown in Figs. 2 and 3). For component (1), since the headway is $h(A)$, i.e., there will be a new DRT bus every $h(A)$, the average waiting time is $\frac{h(A)}{2}$. We now explain the calculation method of component (2). First, $\frac{CL_{DRT}(A)+\tau_s\cdot n(A)}{v_{DRT}}$ is the excepted time used by a DRT bus running whin the DRT area. Because the number of pickups and dropoffs $n(c_i)$ within different tiles c_i in the considered designated DRT area varies from a tile to another, the calculation formula differs from Quadrifoglio and Li (2009). The fraction of pickups and dropoffs demand in tile c_i is $\frac{n(c_i)}{n(A)}$. Therefore $\frac{n(c_i)}{n(A)}\cdot\frac{CL_{DRT}(A)+\tau_s\cdot n(A)}{v_{DRT}}$ is the average time the bus takes passing through tile c_i. A passenger who gets on the bus at c_i needs on average to go through half of tile c_i, so they will experience fraction $\frac{n(c_i)}{2n(A)}$ of stops within c_i. Then, such a passenger will go through all the subsequent tiles, which contain fraction $\frac{\sum_{j=i+1}^{K} n(c_j)}{n(A)}$ of pickups and dropoffs (Fig. 3). Component (3) is just the average in-vehicle time from s_1 to s_A.

3.4. Multilayer graph

We now include in $\mathcal{G}$ the set $\mathcal{E}_{DRT}$ of all *DRT edges*, therefore, we have $\mathcal{V} = \mathcal{C}\cup\mathcal{S}$ and $\mathcal{E} = \mathcal{E}_s\cup\mathcal{E}_c\cup\mathcal{E}_d\cup\mathcal{E}_{DRT}$. The edges in set $\mathcal{E}_{DRT}$ are between any centroid in any designated DRT area A and the corresponding conventional PT station s_A (Fig. 3). Their values represent the expected ingress and egress times (Eq. (8)). By optimizing the number x_A of buses deployed in designated DRT area A, we change the values of such DRT edges.

3.5. Demand model

The last part of Section 3, we will introduce how to calculate demand flow $\phi_i^{\text{DRT, out}}$, $\phi_i^{\text{DRT, in}}$ for each tile c_i based on our multilayer graph $\mathcal{G}$. We now explain how we obtain them for a certain time-slot. Let ρ_i be the population density in tile c_i and $Trip$ the trip rate, i.e., the percentage of residents performing any trip in that time-slot. The trip density ϕ_i originating in centroid c_i is

$$\phi_i = Trip\cdot\rho_i. \tag{9}$$

Let σ_j be the amount of opportunities in tile c_j. Note that σ_j represents the “attractiveness” of c_j. Adopting a singly constrained gravity model (formulas (1) and (2) in Abdel-Aal (2014)), the trip density ϕ_{ij} from centroid c_i to c_j is :

$$\phi_{ij} = \phi_i \cdot \frac{\sigma_j\cdot e^{-\beta T(c_i,c_j)}}{\sum_{k=1}^{M}(\sigma_k\cdot e^{-\beta T(c_i,c_k)})} \quad [trips/hour] \tag{10}$$

being β a constant called dispersion parameter and $T(c_i, c_j)$ the shortest time to go from c_i to c_j on the multilayer graph $\mathcal{G}$. Eq. (10) captures the fact that travelers tend to go to places with many reachable opportunities in a short time. Observe that, in order to go from c_i to c_j, the shortest path might use DRT in the first mile, or in the last mile, or in both, or in none of them. To account for this, let us assume that c_i is inside a designated DRT area A and that s_A is the associated conventional PT station (Fig. 1). We introduce the indicator function $\mathbb{I}^{s_A,\text{first}}_{(c_i,c_j)}$, which is 1 if the shortest path from c_i to c_j passes by s_A (a passenger would enter conventional PT via s_A). Symmetrically, indicator function $\mathbb{I}^{s_A,\text{last}}_{(c_j,c_i)}$ is 1 if the shortest path from c_j to c_i passes by s_A (a passenger would exit conventional PT from s_A to reach c_k via DRT). Otherwise, such indicators are 0. The numbers of DRT users originating and directed to centroid c_k are, respectively (see definitions)

$$\phi_i^{\text{DRT, out}} = \sum_{c_j \notin A} \phi_{i,j} \cdot \mathbb{I}^{s_A,\text{first}}_{(c_i,c_j)} \tag{11}$$

$$\phi_i^{\text{DRT, in}} = \sum_{c_j \notin A} \phi_{j,i} \cdot \mathbb{I}^{s_A,\text{last}}_{(c_j,c_i)} \tag{12}$$

4. DRT planning strategy

We first introduce accessibility, which is at the base of our approach, and the inequality indicator we will use to evaluate the quality of our method. We then describe our optimization procedure.

4.1. Accessibility and inequality

The accessibility of a centroid measures how well it is connected to the surrounding opportunities (schools, people, businesses). We select the classic *gravity-based* definition of accessibility (Miller, 2020), which can be intended as the number of opportunities reachable per unit of time. The accessibility of centroid $c_i \in \mathcal{C}$ is

$$acc(c_i) = \sum_{c_j} (f(T(c_i, c_j)) \cdot \sigma_j) \tag{13}$$

where $T(c_i, c_j)$ is the shortest time to go from c_i to c_j, $f(\cdot)$ is the impedance function and σ_j is the amount of opportunities of the tile having centroid c_j. There are many different impedance functions, we choose $f(T) = T^{-1}$ as our impedance function. The accessibility using T^{-1} as impedance function represents the number of opportunities that can be reached in a unit of time, which is easier to interpret. In broad terms, a large $acc(c_i)$ indicates that the centroid c_i is well connected to opportunities. Let $\mathcal{P}$ be the set of all individuals in the study area. For all individuals $p \in \mathcal{P}$ resident in tile c_i, we define their accessibility as

$$acc(p) = acc(c_i) \tag{14}$$

To quantify the benefits of our RSG strategy, we measure the inequality of the distribution of accessibility, before and after our optimization. We quantify inequality via the well-known Atkinson index on the set of individuals accessibility values $\{acc(p) | \text{individual } p \in \mathcal{P}\}$. From Costa and Pérez-Duarte (2019), setting $\epsilon = 2$, the Atkinson inequality index is:

$$Atk(\mathcal{G}) = 1 - \left(\frac{\overline{acc}(\mathcal{G})}{|\mathcal{V}|} \sum_{p \in \mathcal{P}} \frac{1}{acc(p)} \right)^{-1} \tag{15}$$

where $\overline{acc}(\mathcal{G}) = \frac{1}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} acc(p)$ is the average individual accessibility. The Atkinson index goes from 0 (perfect equality) to 1 (maximum inequality).

4.2. Accessibility inequality reduction problem

Let us consider a PT graph $\mathcal{G}$ and a set $\mathcal{A} := \{A_1, \ldots, A_{n_\mathcal{G}}\}$ of $n_\mathcal{G}$ candidate areas where DRT can be allocated. Due to limited resources, the total number of DRT buses that can be allocated is fixed to N. In broad terms, we consider the problem of the PT operator to deploy DRT, which is a vector $\mathbf{x} = (x_1, \ldots, x_{n_\mathcal{G}})$ of $n_\mathcal{G}$ natural numbers. Each number $x_k, k = 1, \ldots, n_\mathcal{G}$, is the number of buses allocated in area A_k. Let $\mathcal{G}(\mathbf{x})$ denotes the multilayer graph built as in Section 3.4, adding DRT deployment according to allocation $\mathbf{x}$. We aim to solve the following optimization

$$\max_{\mathbf{x}} Atk(\mathcal{G}(\mathbf{x})) \tag{16}$$

subject to the following constraints:

$$\sum_{k=1}^{n_\mathcal{G}} x_k = N, \tag{17}$$

$$x_k \in \mathbb{N}_0, k = 1, 2, \ldots, n_\mathcal{G}. \tag{18}$$

Performing an exhaustive search is infeasible, as the size of the entire solution space is $\binom{N+n_\mathcal{G}-1}{n_\mathcal{G}-1}$ by the calculation method from Heubach and Mansour (2004) and computing the accessibility of each solution is time-consuming in large networks. Our problem is NP-hard (as we show in Appendix C). For this reason, we propose a heuristic to solve our problem

4.3. Rank score greedy (RSG) allocation strategy

We propose a Rank Score Greedy (RSG) planning strategy (Algorithm 1) to allocate DRT resources efficiently, in order to reduce accessibility inequality.

Our basic idea is to identify the regions that need large accessibility improvements and deploy DRT there. If DRT is already deployed in that region, the number of DRT buses is increased. To find the regions in need of improvement, we need a score that quantifies such a need, taking into account both population and accessibility. Intuitively, the regions needing improvements are those that suffer from low accessibility and where many people live. We thus need a score function that matches this intuition in order to guide our planning strategy. To build such a score, we order tiles from the least populated to the most. Let us denote $Rank(\rho_i)$ the position of centroid c_i in the aforementioned list. Separately, we also order tiles from the least accessibility to the most, and denote $Rank(acc(c_i))$ the position of centroid c_i for accessibility rank list. For each centroid $c_i \in \mathcal{C}$, the score of centroid c_i:

$$Score(c_i) = \alpha \cdot Rank(\rho_i) + (1-\alpha) \cdot \left(|\mathcal{C}| - Rank(acc(c_i))\right), \tag{19}$$

where α is a constant that weights the importance given to accessibility in the score, with respect to population. If the population of c_i is large and the accessibility is low, the score will be large, denoting that c_i "needs" accessibility improvement, as it will benefit many people. Recall that DRT is not deployed within a single tile, but in designated DRT areas with K tiles each, as shown in Fig. 3. Therefore, we need to associate a score to each potential designated DRT area A:

$$SCORE(A) = \frac{1}{K} \sum_{c_i \in A} Score(c_i) \tag{20}$$

A high score for designated DRT area A means that large population suffers from low accessibility and that by deploying DRT there (or by increasing DRT buses) we can potentially reduce overall inequality.

Our approach is a bilevel optimization procedure. The upper level, Algorithm 1, performs planning of the DRT fleet over the study area, iteratively assigning DRT buses to the designated areas with the highest score. At each iteration, the lower level, Algorithm 2, is invoked to solve transit assignment. This is crucial, because increasing the number of DRT buses in a designated DRT area A_i reduces ingress and egress travel times $T_{in}(c_i), T_{out}(c_i)$ for passengers originated or directed to any tile c_i contained in area A_i (see Eq. (3)). However, this encourages more passengers to choose DRT, which in turn increases the aforementioned ingress and ingress travel times. Transit assignment implemented in Algorithm 2 is static and deterministic (travelers always deterministically choose the shortest time path), which makes our computation simple and computationally efficient. Algorithm 2 alternatively updates (i) traveler flows $\phi_i^{DRT,out}, \phi_i^{DRT,in}$ entering or exiting designated DRT area A and (ii) travel times until convergence is observed (flows do not change more than 5%).

Algorithm 1: Rank Score Greedy (RSG) planning strategy

```
Input Multilayer graph 𝒢 with stations 𝒮 and centroids 𝒞 = {c_1, …, c_M}.
      The limited number of DRT buses N.
      The set of all centroids' demand density: Φ = {ϕ_1, …, ϕ_M}.
Initialization
Partition the study region in a set 𝒜 of candidate designated DRT areas
Set the center point s_0 in the left side of each A ∈ 𝒜 (Fig. 3)
Associate to each A ∈ 𝒜 conventional PT stop s_A that is closest to s_0
Compute all travel times T(c_i, c_j) and shortest paths P(c_i, c_j) for all origin-destination pairs (c_i, c_j) assuming no DRT.
Compute all demand ϕ_{i,j} for all pairs (c_i, c_j) using Eq. (10).
For step i ← 1 to N:
    For each centroid c_i ∈ 𝒞, calculate Score(c_i) via Eq. (19).
    For each area A ∈ 𝒜, calculate SCORE(A) via Eq. (20).
    Find the area A_i with the largest SCORE(A).
    Update the number of DRT buses n_{A_i} = n_{A_i} + 1.
    Update multilayer graph 𝒢 via Algorithm 2.
End For
Return Updated multilayer graph 𝒢.
```

5. Evaluation

5.1. Considered scenario

We build the simplified model of three cities, which are Montreal, Budapest and Lisbon. We do not include in our analysis fixed-line buses, for simplicity. Fixed buses are identical to metro, from the modeling point of view. So adding them would only imply more time for pre-processing the data and running the optimization, but no additional methodological complexity.

Algorithm 2: Transit assignment within a designated DRT area A

Input Multilayer graph $\mathcal{G}$.
    Area A deployed DRT.
    Number of DRT buses x_A for area A.
While flows are not stable (either $\sum_{i=1}^{K} \phi_i^{\text{DRT, out}}$ or $\sum_{i=1}^{K} \phi_i^{\text{DRT, in}}$ change by more than 5%):
  For each tile c_i inside designated DRT area A:
    Update $\phi_i^{\text{DRT, out}}$ via Eq. (11).
    Update $\phi_i^{\text{DRT, in}}$ via Eq. (12).
  End For
  Taking $\phi_i^{\text{DRT, out}}$, $\phi_i^{\text{DRT, in}}$ ($\forall c_i \in A$) and x_A into Eq. (3) to find $h(A)$.
  If $h(A) > 0$:
   For each tile c_i inside designated DRT area A:
     Compute $T_{in}(c_i)$ and $T_{out}(c_i)$ from Eq. (8).
     Update DRT edge (c_i, s_A) $((s_A, c_i)) \in \mathcal{E}_{DRT}$ with value $T_{in}(c_i)$ $(T_{out}(c_i))$ in graph $\mathcal{G}$.
     Find all the shortest path value $T(c_i, c_j)$, $\forall c_i, c_j$.
     Update demand density $\phi_{i,j}$ and $\phi_{j,i}$ $\forall c_i \notin A$ via gravity model (Eq. (10)).
   End For
  ELSE:
    Exit the algorithm.
End For
Return Updated graph $\mathcal{G}$ with new DRT edges set $\mathcal{E}_{DRT}$.

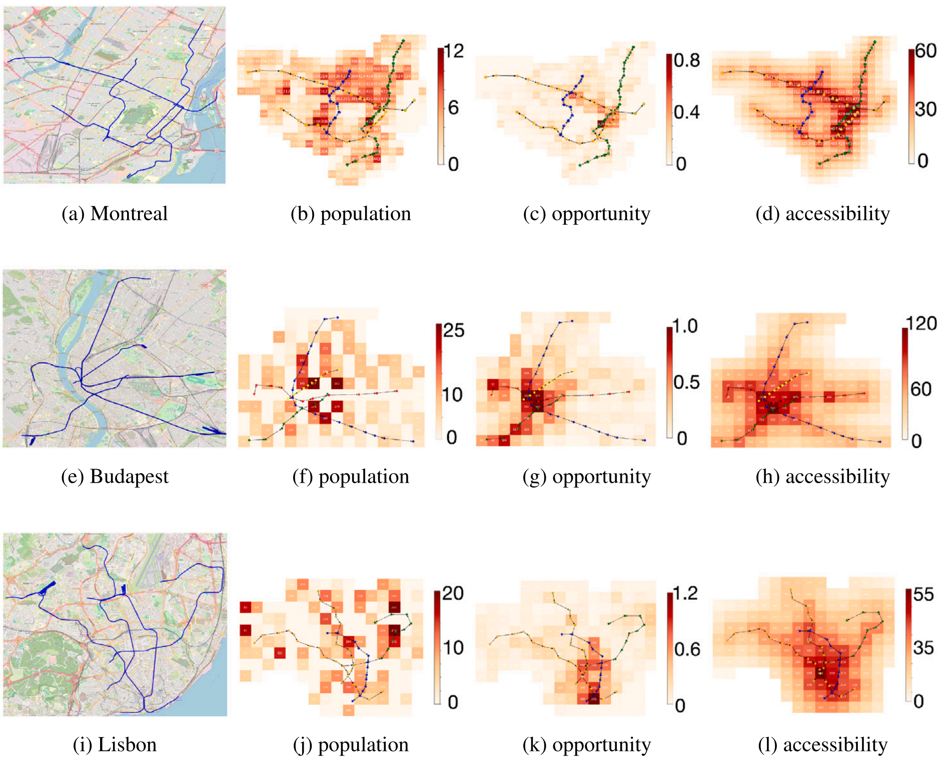


Fig. 4. OpenStreetMap, population ($K\ ppl./km^2$), opportunity ($K\ opps./km^2$), and initial accessibility ($K\ opps./h$) distribution of different cities.

For each city, we take the station locations, the sequence of stations of all lines, the dwell times at each station and the travel time between a station and the other from the General Transit Feed Specification (GTFS) data of Cities (STM, 2023; Socioeconomic Data and Applications Center, 2020). We then get the headway of each metro line during peak hours from Wikipedia (2024). We also take the gridded population of the city from Socioeconomic Data and Applications Center (2020). We extract opportunities, corresponding to Points of Interest (PoIs) from OpenStreetMap (2017), which have been shown to be a good proxy of area attractiveness (Ganter et al., 2022). We perform this extraction via the (Overpass API, 2017). We assign to the centroids the number of PoIs within their

Table 1
Scenario parameters.

Name	Parameter	Value	Reference
Headway for PT lines	t_l	4, 5, 7, 7.5 min	GTFS data (STM, 2023)
Average waiting time for PT lines	w_l	0.5, 1, 2 min	Eqn. 2 of Pinto et al. (2020)
Walking speed	$v_{walking}$	4.5 km/h	Google Maps
Average trip rate per peak hour	$Trip_{\text{peak}}$	0.16 Trips/person/h	Abdel-Aal (2014) and Calabro' et al. (2023)
PT share rate	$Trip_{\text{PT}}$	12.4%	Gilmore (2016)
Time period considered			
Peak hour	t_{peak}	6h30 to 8h30 and 17h00 to 20h30	Calabro' et al. (2023)
Off-peak hour	$t_{\text{off-peak}}$	8h30 to 17h00	Calabro' et al. (2023)
Dispersion parameter of Eq. (10)	β	0.12	Abdel-Aal (2014)
Side of a tile	l	1 km	Socioeconomic Data and Applications Center (2020)
Number of tiles in a DRT area	K	6	–
Time lost per DRT pickup or dropoff	τ_s	32 s	Calabro' et al. (2023)
Terminal dwell time	τ_T	1 min	Calabro' et al. (2023)
DRT bus speed	v_{DRT}	25 km/h	Calabro' et al. (2023)
Weight for the score of Eq. (19)	α	0, 0.25, 0.5, 0.7, 15	–

respective tile. For the PoIs, we select some of the main amenities in a city (schools, hospitals, police stations, libraries, cinemas, banks, restaurants, and bars). Fig. 4 shows the population, opportunities and initial accessibility of each tile (size: $1\,\text{km}^2$) in different cities.

In order to filter out non urbanized regions, we do not include in the study areas the centroids that are located more than 5 km far from a metro line and the tiles with 0 population. The scenario parameters are in Table 1.

5.2. Demand generation

Let $Trip_{peak}$ and $Trip_{\text{off-peak}}$ denote the trips/person/h in peak and off-peak time, respectively. From Abdel-Aal (2014), the average trip rate $Trip_{mean}$ is 1.32 trips/person/day. Assume that there are only trips during peak times and off-peak times in a day. In Calabro' et al. (2023), peak hours are: 6h30 to 8h30 and 17h00 to 20h30 (t_{peak} is 5.5 h in total), and Off-peak hours are: 8h30 to 17h00 ($t_{\text{off-peak}}$ is 8.5 h in total). And the ratio of average trip rate in one peak hour $Trip_{\text{peak}}$ and one off-peak hour $Trip_{\text{off-peak}}$ is 10:3.

$$Trip_{\text{peak}} \cdot t_{\text{peak}} + Trip_{\text{off-peak}} \cdot t_{\text{off-peak}} = Trip_{\text{mean}} \tag{21}$$

$$Trip_{\text{peak}} : Trip_{\text{off-peak}} = 10 : 3 \tag{22}$$

Therefore, $Trip_{\text{peak}} = 0.16$ trips/person/h is obtained by calculation. $Trip_{\text{peak}}$ also includes the trips generated by using private cars. Since we are considering the comparison of DRT and Montreal's conventional public transportation, we need to multiply $Trip_{\text{peak}}$ by the PT share rate $Trip_{\text{PT}} = 12.4\%$ (Gilmore, 2016) to get the trip rate using PT. We replace this value of trip rate into Eqs. (9)–(12).

5.3. Baselines

We did not find in the literature any approach that combines a graph that represents the real topology of a conventional PT network and a Continuous Approximation (CA) model of DRT. We did not either find any approach to optimize DRT planning to reduce inequality of accessibility. A straightforward comparison with the state-of-the-art is thus necessarily nonexistent. For this reason, to show the relevance of the performance of RSG, we compare it with three different baselines, which are based on advanced optimization methods. The fleet size values range from 0 to 200.

No matter how the upper-level algorithm chooses to allocate DRT buses in designed areas, the lower-level Algorithm 2 is needed to perform transit assignment and update the graph. However, transit assignment is time consuming, rendering learning methods that require extensive training data, as well as metaheuristics like genetic algorithms, impractical. Given allocation vector $\boldsymbol{x}$, transit assignment Algorithm 2 updates $\mathcal{G}(\boldsymbol{x})$ first. Then, we calculate the Atkinson index $Atk(\mathcal{G}(\boldsymbol{x}))$. The relation between $\boldsymbol{x}$ and $Atk(\mathcal{G}(\boldsymbol{x}))$ might be very complex and unknown. In this situations, $Atk(\mathcal{G}(\boldsymbol{x}))$ can be regarded as a *black box function*, whose evaluation in any point is very time consuming. In such cases, Bayesian optimization has recently been considered the most appropriate method, as it allows to efficiently trade off exploration and exploitation, thus limiting the number of needed function evaluations (Garnett, 2023). Bayesian Optimization cannot be used directly to optimize functions over high dimensional variables, which is the case of $\boldsymbol{x}$. It is thus necessary to reduce the dimension of $\boldsymbol{x}$. In other words, a subset $\mathcal{A}'$ of areas $\mathcal{A}$ needs to be selected in advance and Bayesian

Optimization will be employed to allocate the fleet therein. Algorithm 3 shows how Bayesian optimization finds an allocation $\boldsymbol{x}^*$, after inputting a pre-selected areas $\mathcal{A}'$.[3] We propose three different algorithms based on the different methods of preselecting $\mathcal{A}'$:

- Population Accessibility based Bayesian (PAB) optimization: We have

$$Atk(\mathcal{G}) = 1 - \left(\frac{\overline{acc}(\mathcal{G})}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} \frac{1}{acc(p)} \right)^{-1} = 1 - \left(\frac{\overline{acc}(\mathcal{G})}{|\mathcal{P}|} \sum_{c_i \in \mathcal{C}} \frac{\rho_i}{acc(c_i)} \right)^{-1}$$

  from Eq. (15), where the last equality holds because all ρ_i individuals living in tile c_i enjoy the same accessibility $acc(c_i)$ (see (14)). Therefore, the contribution of an area A to Atkinson can be measured by $\sum_{c_i \in A} \frac{\rho_i}{acc(c_i)}$. Guided by this finding, we select in $\mathcal{A}'$ the $\varpi \in \mathbb{N}$ areas with the largest value $\sum_{c_i \in A} \frac{\rho_i}{acc(c_i)}$.
- Rank Score based Bayesian (RSB) optimization: We calculate SCORE(A) of all areas $A \in \mathcal{A}$, based on the initial graph $\mathcal{G}$. We then select in $\mathcal{A}'$ the $\varpi \in \mathbb{N}$ areas with the largest value SCORE(A).
- Rank Score Greedy based Bayesian (RSGB) optimization: We first run our algorithm, RSG (Section 4.3), with the entire fleet size. RSG will select a list of areas. We take this list as our pre-selected $\mathcal{A}'$ and use Bayesian Optimization (Algorithm 3) to optimize the allocation of buses within $\mathcal{A}'$.

Value $\varpi \in \mathbb{N}$ is a hyperparameter. We set it to 10. Larger numbers led to high computation time, as typical for Bayesian Optimization (see Fig. 5).

Algorithm 3: Bayesian Optimization with pre-selected areas set $\mathcal{A}'$

Input Objective function $-Atk(\mathcal{G}(\boldsymbol{x}))$
    $\mathcal{A}'$ is the subset of $\mathcal{A}$
    Number of initial samples $n_{\text{init}} = 5$
    Hyperparameter κ (we use $\kappa = 2.576$, i.e. 99% confidence.)
Initialize dataset $\mathcal{H} = \emptyset$
Determine search space $\mathcal{D} = \{\boldsymbol{x} = (x_1, \dots, x_{n_{\mathcal{G}}}) \mid x_i = 0, \forall i \in \{i | A_i \notin \mathcal{A}'\}\}$
for $i = 1$ to n_{init} **do**
    Sample $\boldsymbol{x}_i$ randomly from search space $\mathcal{D}$
    Compute $\mathcal{G}(\boldsymbol{x}_i)$ via taking $\boldsymbol{x}_i$ and initial graph $\mathcal{G}$ into Algorithm 2
    Evaluate $y_i = -Atk(\mathcal{G}(\boldsymbol{x}_i))$
    Add $(\boldsymbol{x}_i, y_i)$ to $\mathcal{H}$
end for
Repeat
    Fit a surrogate model (Gaussian Process) on dataset $\mathcal{H}$
    Let $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ be its mean and standard deviation.
    Select next point $\boldsymbol{x}_i$ by maximizing acquisition function:

$$\boldsymbol{x}_i = \arg\max_{\boldsymbol{x}} [\mu(\boldsymbol{x}) + \kappa \cdot \sigma(\boldsymbol{x})]$$

    Compute $\mathcal{G}(\boldsymbol{x}_i)$ via taking $\boldsymbol{x}_i$ and initial graph $\mathcal{G}$ into Algorithm 2
    Evaluate $y_i = -Atk(\mathcal{G}(\boldsymbol{x}_i))$
    Add $(\boldsymbol{x}_i, y_i)$ to $\mathcal{H}$
Until Running time exceeds time T, e.g., T = 1 h
$\boldsymbol{x}^* \leftarrow \arg\max_{(\boldsymbol{x}_i, y_i) \in \mathcal{H}} y_i$
return $\boldsymbol{x}^*$ and $-Atk(\boldsymbol{x}^*)$

5.4. Results

We first study the performance of RSG with different $SCORE(A)$ (different α of Eq. (19)) in Montreal. When α increases, the weight of the population relative to accessibility increases. As a consequence, the PT structure resulting from our optimization will tend to have better $\overline{acc}(\mathcal{G})$ to the detriment of Atkinson, i.e., the optimization will favor efficiency over equality. Therefore, the choice of α requires a trade-off between efficiency and equality. When $\alpha = 0$, the optimization algorithm only considers areas with low accessibility and does not consider population at all. It usually selects areas with low accessibility but small population, which makes its efficiency very low compared with other α cases. When $\alpha = 1$, the optimization algorithm only considers areas with large population and always allocates DRT buses to the areas with the largest population, which will increase inequality. Therefore, these two extreme cases will not be considered in the following.

[3] We use the default values of the https://bayesian-optimization.github.io/BayesianOptimization/2.0.3/.

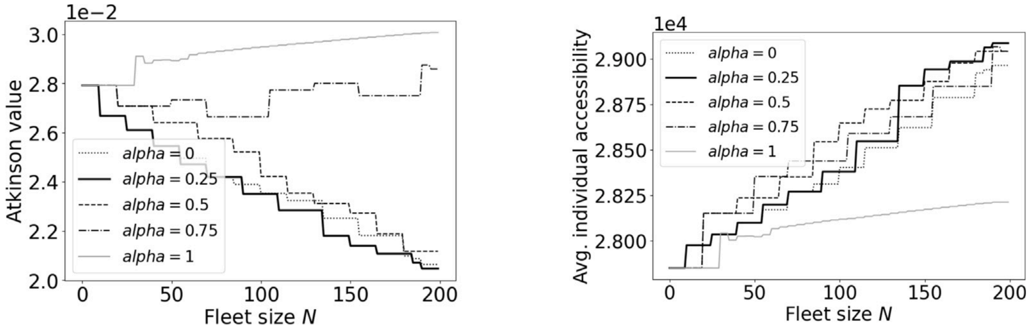
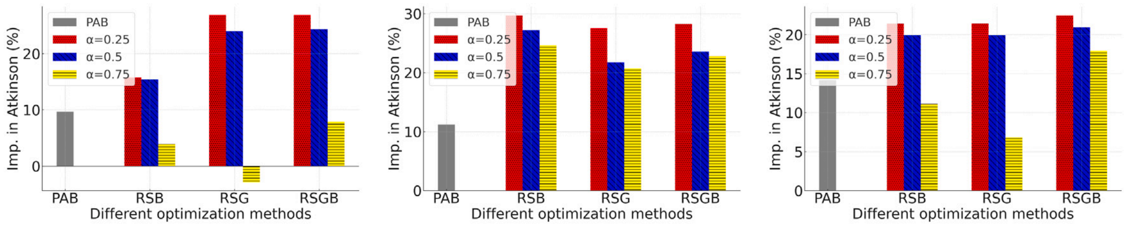


Fig. 5. The Atkinson index and $\overline{acc}(\mathcal{G})$ of RSG with different SCORE(A) (different α) in Montreal.



(a) Montreal (b) Budapest (c) Lisbon

Fig. 6. Improvement in Atkinson indices after optimization by different algorithms in different cities with fleet size = 200 buses (running time of PAB, RSB and RSGB are **1 h**, and running time of RSG is **5 min**).

Table 2
Improvement in terms of different indexs in different cities (RSG with $\alpha = 0.25$ and fleet size = 200 buses).

Name	Montreal	Budapest	Lisbon	Reference
Atkinson	26.98%	27.36%	21.43%	Costa and Pérez-Duarte (2019)
Theil	27.04%	26.49%	22.37%	Costa and Pérez-Duarte (2019)
Pietra	10.37%	8.29%	10.33%	Costa and Pérez-Duarte (2019)
Palma	8.63%	7.46%	8.55%	Sitthiyot and Holasut (2020)

Fig. 6 shows that hyperparameter α should be set ≤ 0.5. It is clear that the Population Accessibility based Bayesian optimization method is outperformed by the others. Our proposed method Rank Score Greedy (RSG) shows competitive performance with respect to the Bayesian Optimization-based baselines (namely RSB and RSGB). Observe, however, that the computation time of RSG is ≈ 5 minutes, versus ≈ 1 h for RSB and RSGB. Since RSGB is a method based on RSG, and further optimization is performed using the results of RSG as the initial starting point, the effect should theoretically be no worse than that of RSG. However, it can be seen from Fig. 6 that the improvement is negligible, which means that RSG has found a good solution. If we consider scenarios with more conventional PT lines, larger fleet sizes, in larger areas, the advantage of RSG in terms of computation cost may be very significant. On the other hand, if computation cost is not an issue, RSB and RSGB are also feasible choices.
To recap, our proposed RSG effectively reduced inequality and at the same time shows excellent computational efficiency.

The efficiency of RSG is also confirmed when using inequality indicators different from Atkinson's (Table 2). This indicates that RSG reduces the gap of accessibility among individuals.

Figs. 7 and 8 explain the origin of the inequality reduction brought by RSG ($\alpha = 0.25$) in different cities. In these figures, we compare the accessibility of each tile and each individual, before and after the introduction of DRT. RSG improves tiles and individuals that suffered from low accessibility with conventional PT only (those are the ones on the left of the x-axis of Figs. 7 and 8). These results confirm that the score functions we designed in Eqs. (19)–(20) are an effective guidance for the DRT bus allocation of Algorithm 1.

Finally, the changes observed in Montreal are taken as a case study. Fig. 9 shows all the areas deploying DRT after optimization by the RSG ($\alpha = 0.25$) model with 200 DRT buses. We observe that RSG can potentially allocate buses everywhere. However, as wished, it ends up deploying buses in only a few areas, which evidently are those that need most of the improvement. We also observe that the DRT gives priority to suburban areas. Detailed information on each area after the deployment of DRT can be found

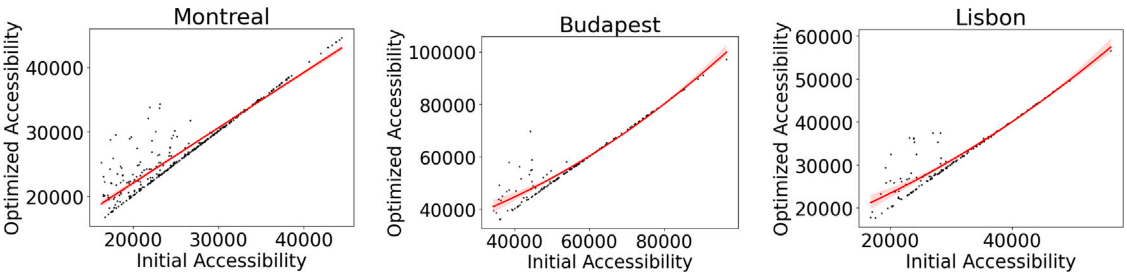


Fig. 7. Distribution of accessibility ($opp./h$) with the regression lines (RSG, with $N = 200$). Each point represents tile.

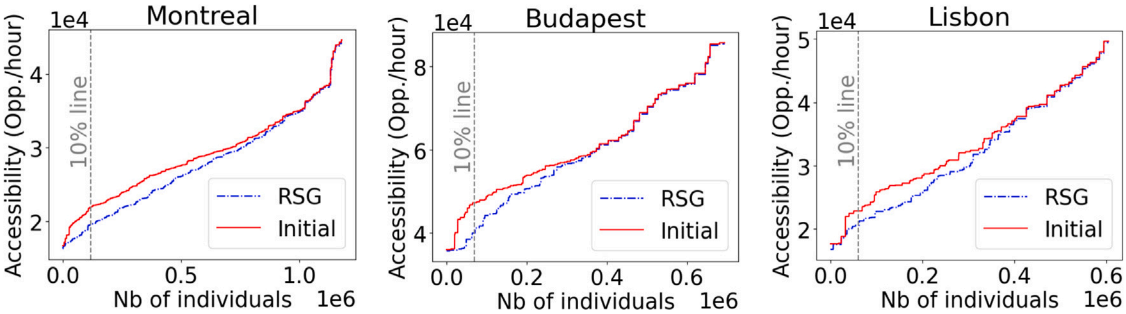


Fig. 8. Accessibility of each individual from small to large (RSG with $N = 200$).

Table 3
Information about DRT services in each area in Montreal.

Area	Number of DRT buses	Headway $h(A)$ (min)	Expected cycle time $C_{DRT}(A)$ (min)	Average number of people per bus
A_1	20	2	53	12
A_2	15	2	41	8
A_3	15	2	35	4
A_4	15	3	60	16
A_5	15	3	49	12
A_6	20	2	32	2
A_7	20	2	42	9
A_8	25	2	50	13
A_9	15	2	40	10
A_{10}	15	2	29	2
A_{11}	20	2	35	4

in Table 3. Furthermore, although we do not limit *a priori* bus capacity, the resulting bus allocation does not significantly increase bus occupation, with a maximum of not more than 16 people per bus in Table 3.

6. Perspectives

While the present work deals with optimization at the planning level, in our future work we will also test the performance of our design in operation, using (Horni et al., 2016; Lu et al., 2015). It will also be interesting to tackle, together with DRT deployment decisions, a deeper re-design of conventional PT lines, possibly removing lines that contribute less to accessibility equality (Badeanlou et al., 2023).

From the methodological point of view, our optimization is guided by a score that combines population and accessibility of the designated areas. While the structure of this score is fixed *a priori* (based on our trial and error experiments), we will investigate the possibility to “learn” optimal scores via Machine Learning (ML). By appropriately engineering features, ML would allow to consider complex local information (e.g. graph connectivity, topological or socio-economic data), which would be too difficult to capture with scoring functions constructed *a priori*, and which could automatically adapt to each city.

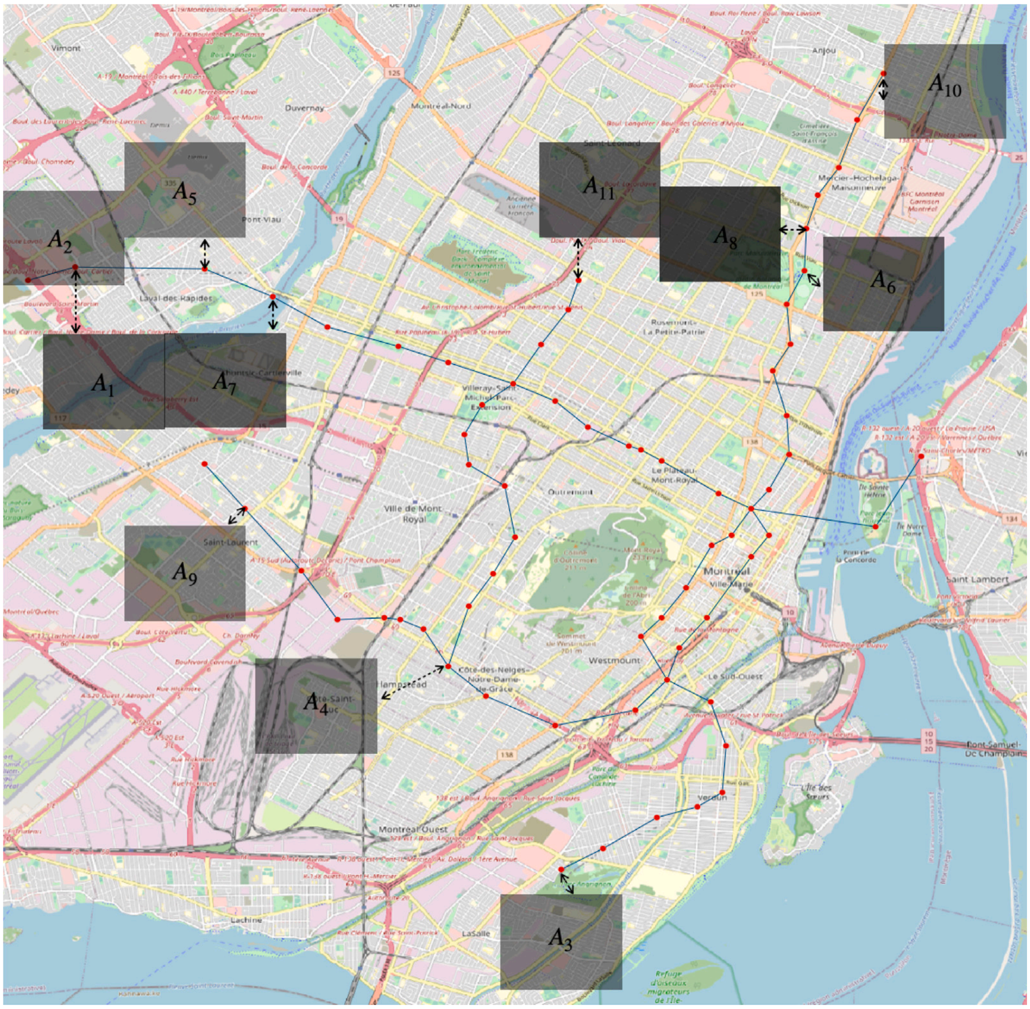


Fig. 9. Areas with DRT obtained via RSG ($\alpha = 0.25$, N = 200k) in Montreal. *The darker the color of the area, the more DRT buses are deployed in that area (exact allocated fleet per area are in Table 3).* (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

7. Conclusion

We presented a methodology to plan DRT in order to reduce accessibility inequality among individuals in an urban region. We combine a graph model of conventional PT, capable of capturing the characteristics of currently existing PT, with a Continuous Approximation (CA) model of DRT. We combine the two models in a single multilayer graph, on which we compute accessibility indicators. We propose a bilevel optimization method, where the upper level allocates DRT buses in designated areas scattered around the city and the lower level solves transit assignment. Numerical results shows strong improvement in the accessibility for those who were suffering from low accessibility with conventional PT, thereby reducing inequality.

We believe the method presented here is a first step toward optimizing accessibility inequality when planning DRT, thus exploiting the full potential of DRT in fighting against car dependency.

CRediT authorship contribution statement

Duo Wang: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation. **Andrea Araldo:** Writing – review & editing, Supervision, Project administration, Methodology, Funding acquisition, Conceptualization. **Mounim El Yacoubi:** Writing – review & editing, Supervision, Methodology.

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Appendix A. Table of notation

Symbol	Description
$\mathcal{G}$	A PT graph
$\mathcal{V}$	The set of all nodes
$\mathcal{E}$	The set of all edges
$\mathcal{S}$	The set of all stations
$\mathcal{C}$	The set of all centroids
$\mathcal{E}_s$	The set of all edges of metro lines
$\mathcal{E}_c$	The set of all edges between to neighbor centroids
$\mathcal{E}_d$	The set of edges between any centroid and all stations
$\mathcal{P}$	The set of all individuals in the study are
$\mathcal{A}$	The set of all candidate areas where DRT can be allocated
$\mathcal{A}'$	The set of pre-selected areas where DRT can be allocated
$n_{\mathcal{G}}$	The total number of areas in $\mathcal{A}$
m	Total number of centroids in study area
c_i	Centroid i $(1 \le i \le m)$
s_i	Station i
l	a Metro line
t_l	Headway of line l
w_l	Average waiting time for line l
$t_{(s_i, s_{i\times 1})}$	Time spent by a vehicle going from station s_i to the next station $s_{i\times 1}$
t_{s_i}	The dwell time a vehicle stops at station s_i
e	An edge
t_e	Time cost of an edge e
v_{walking}	Walking speed
s_A	The service station of DRT area A
$h(A)$	The difference between the departure time of each DRT vehicle and the next one from station s_A
x_A; $\boldsymbol{x}$	The number of DRT buses allocated in area A and the allocation vector
$C_{DRT}(A)$	The expected time required to complete a cycle of a DRT bus
$n(A)$	Number of pickups and dropoffs requested within area A generated in time $h(A)$
$\phi_i^{\text{DRT, out}}$	The number of passengers per unit of time that originate in tile c_i and that use DRT for their first mile
$\phi_i^{\text{DRT, in}}$	The number of passengers per unit of time having tile c_i as destination and using DRT for their last mile
l	The side of a tile
l^2	The area of a tile
K	The total number of tiles in an area
d	The distance shown in Fig. 3
τ_s	The time lost per stop
τ_T	The terminal dwell time of the DRT bus at the terminal s_A
v_{DRT}	DRT bus' speed
$T_{in}(c_i)$	The excepted ingress time of centroid c_i
$T_{out}(c_i)$	The excepted egress time of centroid c_i
$n(c_i)$	The number of pickups and dropoffs requested within tile c_i
ρ_i	The population density in tile c_i
ϕ_i	Trip density originating in centroid c_i
σ_j	The amount of opportunities in tile c_j
$T(c_i, c_j)$	The shortest time to go from c_i to c_j
$\mathcal{D}$	The search space of Bayesian Optimization

Appendix B. Calculation of circle length $CL_{DRT}(A)$

We now derive Eq. (5). Let $n(A)$ be the number of customers served per cycle by the DRC vehicle. Each customer $1 \leq p \leq n(A)$ has its position (u_p, v_p), uniformly distributed in area A with K tile, length of area is $\frac{Kl}{2}$, width of area is $2l$.

The maximum distance the bus can travel along the horizontal line is: $u_{max} = \max u_1, u_2, \dots, u_{n_A}$. The expectation of u_{max} is

$$\begin{aligned}
E(u_{max}) &= \int_0^{\frac{Kl}{2}} P(u_{max} \geq t)dt = \int_0^{\frac{Kl}{2}} (1 - P(u_{max} \leq t))dt \\
&= \int_0^{\frac{Kl}{2}} (1 - \prod_{p=1}^{n(A)} P(u_p \leq t))dt \\
\text{Assuming unif. distr. in the area} &= \int_0^{\frac{Kl}{2}} \left(1 - \left(\frac{2t}{Kl}\right)^{n(A)}\right) dt = \frac{Kl}{2} \cdot \frac{n(A)}{n(A)+1}.
\end{aligned}$$

So the horizontal expected journey length (to go and come back) is $2E(u_{max}) = Kl \cdot \frac{n(A)}{n(A)+1}$.

Let v be the random variable indicating the vertical distance between any pair of customers within the upper or lower half of the service area. We have

$$E(v) = \frac{1}{l^2} \int_0^l \int_0^l |v_p - v_{p'}| dv_p dv_{p'} = \frac{2}{l^2} \int_0^l \int_0^{v_p} (v_p - v_{p'}) dv_{p'} dv_p = \frac{1}{l^2} \int_0^l v_p^2 dv_p = \frac{l}{3}.$$

Let v' indicate the vertical distance between point s_0 and the first or last customer in the schedule, we have $E(v') = \int_0^l v_p dv_p = \frac{l}{2}$. Let v'' indicate the vertical distance between the last customer served in the upper half and the first customer served in the lower half. We have $E(v'') = l$.

Hence, we have

$$\begin{aligned}
CL_{DRT}(A) &= 2E(u_{max}) + (n(A) - 2) \cdot E(v) + 2E(v') + E(v'') \\
&= Kl \frac{n(A)}{n(A)+1} + (n(A) - 2) \cdot \frac{l}{3} + 2\frac{l}{2} + l = Kl \frac{n(A)}{n(A)+1} + n(A) \cdot \frac{l}{3} + \frac{4l}{3}.
\end{aligned}$$

The reason for $(n(A) - 2) \cdot E(v)$ is that there are $n(A)$ edges connecting all $n(A)$ customers in sequence, but we need to remove the edge between the last and first customer, as well as the edge between the last customer served in the upper half and the first customer served in the lower half, which are accounted for separately.

Appendix C. Proof of NP-hardness

Given a set of $n_{\mathcal{G}}$ areas, we can allocate a number $x_i = n_i$ of DRT buses to each area A_i with a bound N, i.e., $x_i \in \{0, 1, \dots, N\}$. We associate to area A_i a weight w_i. We aim to solve the following optimization

$$\max_{x_1, x_2, \dots, x_{n_{\mathcal{G}}}} Atk(\mathcal{G}((x_1, x_2, \dots, x_{n_{\mathcal{G}}}))) \tag{C.1}$$

subject to the following constraints:

$$\sum_{k=1}^{n_{\mathcal{G}}} x_i \cdot w_i \leq N \tag{C.2}$$

$$x_i \in \mathbb{N}_0, \qquad i = 1, 2, \dots, n_{\mathcal{G}} \tag{C.3}$$

$$w_i = 1, \qquad i = 1, 2, \dots, n_{\mathcal{G}} \tag{C.4}$$

The constraints of our problem are like the ones in an integer bounded quadratic knapsack problem, which is a generalization of a quadratic knapsack problem (QKP - Ch.12 of Kellerer et al. (2004)). However, our objective function is more complex than the quadratic form of the QKP. Since QKP is NP-complete, even if all weights are the same (as in our case (C.4)), our problem is strongly NP-hard (Pisinger, 2007).

Data availability

Within the paper, I have shared the link to the github repo, which contains code and data.

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