Conventional, Flexible, and Variable-Type Bus Services

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Abstract: To provide efficient public transportation services in areas with high demand variability over time, it may be desirable to switch vehicles between conventional services (with fixed routes and schedules) during peak periods and flexible route services during low-demand periods. This option is called *variable-type services*. Conventional, flexible, and variable-type service alternatives optimized for various conditions are compared to explore when variable-type bus services might be preferable to purely conventional or flexible service. The optimization models used for purely conventional or flexible service are adapted from previous studies. These models are integrated into a new model for optimizing variable-type bus service. The results of sensitivity analyses show how demand variability over time and other factors affect the relative effectiveness of conventional, flexible, and variable-type bus services. **DOI: 10.1061/(ASCE)TE.1943-5436.0000326.**

CE Database subject headings: Public transportation; Optimization; Urban areas; Transportation models.

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Introduction

Conventional transit service (defined as having fixed routes and fixed schedules) is most effective at high demand densities, which justify frequent service and high station density, which in turn reduce wait times and access times for passengers. Unconventional services (also called "demand-responsive" or "paratransit" services) are interesting to researchers and transit operators because they can provide high service quality (including "doorstop" pickups and dropoffs of passengers) even when demand densities are low. The recent literature (such as Diana et al. 2009; Horn 2002; Fu and Ishkhanov 2004; Quadrifoglio et al. 2006, 2008, 2009; Luo and Schonfeld 2011a, 2011b; Shen et al. 2011; Jung and Jayakrishnan 2011; Becker and Teal 2011; Kim and Haghani 2011; Baumgartner and Schofer 2011; Nourbakhsh and Ouyang 2011) confirms the continuing interest in various paratransit concepts.

Baumgartner and Schofer (2011) introduce the concept of Call-n-Ride, which is an operation without a central dispatcher in which drivers take requests for service directly on their cell phones and make all routing and scheduling decisions. They present the results of current Call-n-Ride services in the United States and provide a model for predicting the productivity of Call-n-Ride services. Becker and Teal (2011) study the service configuration aspects of next-generation demand-responsive transit (DRT) by focusing on the experiences of the Denver transit agency. Kim and Haghani (2011) focus mainly on developing algorithms to solve a static multidepot Dial-a-Ride problem with time-varying travel times and soft time windows. Jung and Jayakrishnan (2011) study an alternative transportation concept called High Coverage Point-to-Point Transit (HCPPT) that can reduce the number of transfers in urban transit systems, and note that HCPPT can be a good alternative to conventional fixed-route and conventional DRT services. Additionally, Shen and Quadrifoglio (2011) study a realistic coordinated decentralized paratransit system. Luo and Schonfeld (2011a, 2011b) develop performance models for demand-responsive many-to-many dial-a-ride services and rejected-reinsertion heuristics for dynamic multivehicle Dial-a-Ride Problems (DARPs). Aside from Chang and Schonfeld's (1991b) analysis of temporally integrated bus systems, it is difficult to find studies that consider variations in service type as demand changes. Thus, it seems worthwhile to examine not only the relative advantages and disadvantages of conventional and paratransit services, but also variable-type bus alternatives in which the service type changes in response to demand changes while using the same pool of resources (i.e., buses and drivers).

Flexible bus services typically provide Many-to-One and/or One-to-Many service with flexible route tours that operate on semifixed schedules. (The departure times from or arrival times at the One major trip generator are usually predetermined, and the tours may have cyclical schedules.) The relative advantages of conventional and flexible bus services have been compared using analytic optimization models in Chang and Schonfeld (1991a, cited henceforth as CS). The models for conventional and flexible services used in the present study are adapted from those developed in CS. To compare the costs of conventional bus and flexible bus services, CS assume that both conventional bus and flexible bus services either collect passengers from a local service area *or* distribute passengers to a local area. Introduced in this study is a controllable directional split factor, which enables us consideration of two-directional demands in various proportions.

A different approach to reducing bus transit cost is to use different fleets of buses as the demand varies, with larger buses used at higher demand densities. On the basis of this idea, Fu and Ishkhanov (2004) consider mixed bus fleet operation for paratransit services. Similarly, Lee et al. (1995) consider mixed bus fleets for urban conventional bus services. However, the potential advantages of variable-type bus service for integrated conventional and flexible bus operations have not been sufficiently explored. Those potential advantages are the subject of this paper, which seeks to quantify

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them. In this paper, cost functions are modified to (1) reflect two-directional demands in round-trip times; (2) develop an integrated model for variable-type bus services, and (3) compare conventional, flexible, and variable-type bus services under various assumed conditions. This model is intended for conceptual comparisons of services rather than detailed planning and operations.

Cost Functions

Here, the CS cost functions on which the present analysis relies are briefly presented and modified. Although some notation has been changed, the detailed formulations can be found in Chang (1990), CS, and Schonfeld et al (2010). This section explains how CS formulates the costs of conventional and flexible bus services. The notation used throughout the paper and the baseline input values used in our numerical analysis are provided under Notation.

Assumptions and Analytic Result

The assumptions for both conventional and flexible bus services are listed here. Definitions and baseline values of variables are provided under Notation.

For Both Conventional and Flexible Bus Services

1. A rectangular service area of length L and width W (as shown in Fig. 1) is J miles (1 mile = 1.61 km) away from a transportation terminal at its nearest corner.
2. The demand is fixed with respect to service quality and price.
3. The demand is uniformly distributed over space within the service area and over time within each specified period.
4. The vehicle size (S_c for conventional bus, S_f for flexible bus) is uniform throughout a system.
5. The estimated average waiting time of passengers is equal to half the headway (h_c for conventional bus, h_f for flexible bus).
6. Vehicle layover time is negligible.
7. Within the service area, the average speed (V_c for conventional bus, V_f for flexible bus) includes stopping times.
8. External costs are assumed to be negligible.

For Conventional Bus Service Only

1. The service area is divided into N parallel zones with width $r = W/N$ for conventional bus services, as shown in Fig. 1. Local routes branch from the line-haul route segment to run along the middle of each zone, at a route spacing $r = W/N$.

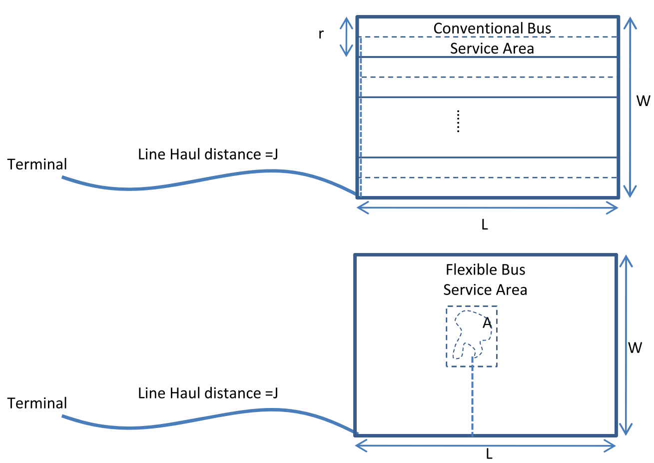

Fig. 1. Conventional and flexible bus services

2. A demand of Q trips/mi^2/h (1 mi^2 = 2.59 km^2), which is entirely channeled to (or through) the single terminal, is uniformly distributed over the service area.
3. In each round trip, as shown in Fig. 1, buses travel from the terminal a line-haul distance J at nonstop speed yV_c to a corner of the service area, then travel an average of $W/2$ miles (1 mile = 1.61 km) at local nonstop speed zV_c from the corner to the assigned zone, then run a local route of length L at local speed V_c along the central axis of the zone while stopping for passengers every d km, and then reverse this process in returning to the terminal.

For Flexible Bus Service Only

1. The service area is divided into N' equal zones, each having an optimizable zone area $A = LW/N'$. The zones should be "fairly compact and fairly convex" (Stein 1978).
2. Buses travel from the terminal a line-haul distance J at nonstop speed yV_f and an average distance $(L+W)/2$ miles (1 mile = 1.61 km) at local nonstop speed zV_f to the center of each zone. They collect (or distribute) passengers at their door steps through a tour of n stops and length D_c at local speed V_f The values of n and D_c are endogenously determined. D_c is approximated by Stein (1978), in which $D_c = \phi\sqrt{nA}$ and $\phi = 1.15$ for rectilinear space according to Daganzo (1984). To return to their starting point the buses retrace an average of $(L+W)/2$ miles at $zV > f$ mi/h and J miles at yV_f mi/h (1 mile = 1.61 km, 1 mi/h = 1.61 km/h).
3. Buses operate on preset schedules with flexible routing designed to minimize each tour distance D_c.
4. The tours are routed on the rectilinear street network.
5. Tour departure headways are equal for all zones in the service area and uniform within each period.

The formulation proposed by CS considered one-way service (i.e., only collecting passengers *or* only distributing passengers) in which total demand density is Q trips/mi^2 (1 mi^2 = 2.59 km^2). On the basis of these assumptions, the analytic optimization results CS obtained for conventional bus and flexible bus services are outlined in Table 1. For bus operating cost, a linear (i.e., $B = a + bS$) cost function was used (Jansson 1980; Oldfield and Bly 1988).

Total Cost Including Capital Cost

When the total system cost for bus services is computed, capital cost should be treated as another fixed cost. The capital cost, C_p, is the cost that satisfies peak-period vehicle requirements. Bus service cost is defined as the sum of bus operating cost C_o, user in-vehicle cost C_v, user waiting cost C_w, and user access cost C_x:

$$\text{total cost} = \text{captial cost} + \text{bus operating cost} + \text{user} \tag{1}$$

Table 1. Analytic Optimization Results

Conventional bus services		Flexible bus services	
Vehicle size S_c	$\sqrt[3]{\frac{8a^2D^2LQV_x}{v_w v_x V_c^2 l_c^3}}$	Vehicle Size S_f	$\sqrt[5]{\frac{a^3D_f^3Qu}{v_w\phi^2V_f l_f^3(b+\frac{v_v l_f}{2})^2}}$
Route spacing r	$\sqrt[3]{\frac{8aDv_wV_x^2}{v_x^2LQV_c}}$	Service Area A	$\sqrt[5]{\frac{av_w^3V_f^3D_f^3u^{8/3}l_f^4}{\phi^4Q^{7/3}Y^{10/3}(b+\frac{v_v l_f}{2})^2}}$
Service cost (conventional)	$3LWQ(\frac{v_w v_x D}{8LQV_cV_x})^{1/3} + \frac{bDLWQ}{l_cV_c} + \frac{v_vLWQM}{V_c} + \frac{v_xLWQd}{4V_x}$		
Service cost (flexible)	$LWQ\left[\frac{v_wa^2D_f^2\phi^2(b+\frac{v_v l_f}{2})^2}{ul_f^2V_f^4Q}\right]^{1/5} + 1.5LWQ\left[\frac{v_wa^2\phi^2}{l_f^2V_f^4}\right]^{1/5}\left(\frac{Y^2}{uQ}\right)^{1/3} + \frac{LWQD_f(b+\frac{v_v l_f}{2})}{V_f l_f}$		
Note	$Y = [a^2v_w\phi^2V_f l_f^3]^{1/5} + [uD_f^3Q(b+\frac{v_v l_f}{2})^3]^{1/5}$		

Table 2. Analytic Results with Capital Cost

Conventional bus service		Flexible bus service	
Vehicle size S_c	$\sqrt[3]{\frac{8\bar{a}^2 D^2 L\bar{Q}V_x}{v_w v_x V_c^2 l_c^3}}$	Vehicle Size S_f	$\sqrt[5]{\frac{\bar{a}^3 D_f^3 \bar{Q}u}{v_w \phi^2 V_f l_f^3 (\bar{b}+\frac{v_v l_f}{2})^2}}$
Routing space r	$\sqrt[3]{\frac{8\,\bar{a}\,Dv_w V_x^2}{v_x^2 L\bar{Q}V_c}}$	Service Area A	$\sqrt[5]{\frac{\bar{a}v_w^3 V_f^3 D_f^3 u^{8/3} l_f^4}{\phi^4 \bar{Q}^{7/3} Y^{10/3} (\bar{b}+\frac{v_v l_f}{2})^2}}$
Total cost (conventional)	$\frac{a_c DLWQ_p}{S_c V_c l_c} + \frac{b_c DLWQ_p}{V_c l_c} + \frac{D(a+bS_c)LW\sum_i^I Q_i t_i}{v_c s_c l_c} + \frac{v_v MLW\sum_i^I Q_i t_i}{v_c} + \frac{v_w l_c WS_c \sum_i^I Q_i t_i}{2r} + \frac{v_x(r+d)LW\sum_i^I Q_i t_i}{4v_x}$		
Total cost (flexible)	$\frac{LWQ_p D_f(a_c+b_c s_f)}{v_f s_f l_f} + \frac{\phi LWQ_p(a_c+b_c s_f)\sqrt{A/uS_f l_f}}{v_f} + \sum_i^l \left\{ \frac{LWQ_i t_i D_f(a+bS_f)}{V_f S_f l_f} + \frac{\phi LWQ_i t_i (a+bS_f)\sqrt{A/uS_f l_f}}{V_f} + \frac{V_v D_f LWQ_i t_i}{2V_f} + \frac{V_v LWQ_i t_i \phi\sqrt{AS_f l_f/u}}{2V_f} + \frac{V_w LWS_f l_f t_t}{2A} + - \right\}$		
Note	$\bar{Q}=\frac{\sum_i^I Q_i t_i}{\sum_i^I t_i},\ \bar{a}=\frac{a_c Q_p \sum_i^I a_i Q_i t_i}{\sum_i^I Q_i t_i},\ \bar{b}=\frac{b_c Q_p \sum_i^I b_i Q_i t_i}{\sum_i^I Q_i t_i},\ Y=[\bar{a}^2 v_w \phi^2 V_f l_f^3]^{1/5}+[uD_f^3\bar{Q}(\bar{b}+\frac{v_v l_f}{2})^3]^{1/5}$		

Relation (1) can be rewritten as

$$C_t = C_p + C_o + C_u = C_o + C_v + C_w + C_x \tag{2}$$

Analytic results with capital cost for conventional and flexible bus services are summarized in Table 2.

Limitations of Chang and Schonfeld (1991a) Study

Here, the goal is to overcome two main limitations in the CS bus service cost formulations. First, CS assume that trip demand for bus services is always one-directional (i.e., all demand either from terminal to local or from local to terminal). That assumption is modified here by introducing a directional demand split factor, f. Second, CS consider only maximum allowable headway (required to satisfy demand), rather than an optimized headway. It seems preferable to optimize the headway for each period, which should be the minimum of (1) the maximum feasible headway that satisfies the demand and (2) the headway that minimizes total costs.

Cost Function Modification and Headway Optimization

Here a directional demand split factor, f, is introduced conventional bus service only (because flexible service does not need a directional demand split factor unless passengers are collected and distributed in different tours), and optimized headway solutions are provided for both conventional bus and flexible bus services. If $f = 1.0$, all demand is one-directional. In other words, buses return without any passengers. Similarly, if $f = 0.5$, then demand is equal in the two directions. In flexible service, because passengers are collected and distributed within the same tours, no directional split factor is needed. Therefore, if demand density Q is assumed to be the sum of both collected passengers and distributed passengers, the CS flexible service cost functions can still be used.

Conventional Bus System Cost

In computation of total system cost for conventional bus service, the capital cost C_p should satisfy the peak-period fleet size requirement. Bus service cost is the sum of bus operating cost C_o, user in-vehicle cost C_v, user waiting cost C_w, and user access cost C_x:

$$\begin{aligned}\text{total cost} &= \text{captial cost} + \text{bus service cost}\\ &= \text{captial cost} + \text{bus operating system} + \text{user cost}\end{aligned} \tag{3}$$

Detailed cost component derivations for operator and user costs are provided in Appendix 1. Eq. (3) can be expressed as

$$C_t = C_p + C_o + C_u = C_o + C_v + C_w + C_x\,C \tag{4}$$

Although conventional bus cost is reformulated here, the overall procedure for computing total cost with capital cost is basically similar to that in CS. The capital cost for conventional bus systems should be computed on the basis of peak-period demand. Therefore, capital cost C_{pc} for conventional bus service is

$$C_{pc} = \frac{D}{V_c}\frac{W}{r}\frac{1}{h_p}B_c = \frac{D}{V_c}\frac{W}{r}\frac{rLfQ_p}{S_p l_c}B_c = \frac{D}{V_c}\frac{W}{r}\frac{rLfQ_p}{S_p l_c}(a_c + b_c S_c) \tag{5}$$

$$C_{\mathrm{tc}} = C_{\mathrm{pc}} + \sum_i^I \left\{ C_{oci} + C_{vci} + C_{wci} + C_{xci} \right\} \tag{6}$$

The total daily service cost for conventional bus service C_{tc} is formulated as Subscript i denotes time periods in the following equations and t_i represents the number of hours in Period i

Eq. (6) can be rewritten as

$$\begin{aligned}C_{\mathrm{tc}} = {} & \frac{a_c DLWfQ_p}{S_c V_c l_c} + \frac{b_c DLWQ_p}{V_c l_c} + \frac{D(a+bS_c)LWf\sum_i^I Q_i t_i}{S_c V_c l_c}\\ & + \frac{v_v MLW\sum_i^I Q_i t_i}{V_c} + \frac{v_w WS_c l_c \sum_i^I t_i}{2rf} + \frac{v_x(r+d)LW\sum_i^I Q_i t_i}{4V_x}\end{aligned} \tag{7}$$

Simultaneously solving the derivatives of C_{tc} in Eq. (7) with respect to route space r and vehicle size S_c reveals the optimal values of r^* and S_c^*:

$$r^* = \sqrt[3]{\frac{8\,\bar{a}\,\mathrm{D}v_w V_x^2}{v_x^2 L\bar{Q}V_c}} \tag{8}$$

$$\begin{aligned} S_c^* &= \frac{2f}{l_c}\sqrt[3]{\frac{\bar{a}^2 D^2 L\bar{Q}V_x}{v_w v_x V_c^2}}, \qquad \text{where } \bar{Q} = \frac{\sum_i^I Q_i t_i}{\sum_i^I t_i},\\ \bar{a} &= \frac{a_c Q_p + \sum_i^I a_i Q_i t_i}{\sum_i^I Q_i t_i}, \qquad \bar{b} = \frac{b_c Q_p + \sum_i^I b_i Q_i t_i}{\sum_i^I Q_i t_i}\end{aligned} \tag{9}$$

On the basis of optimized vehicle size S_c^* and route spacing r^*, bus service cost for Period i can be expressed as

$$\begin{aligned}C_{ci} = {} & \frac{DW(a+bS_c^*)}{r^* V_c h_{ci}} + \frac{v_v LWQ_i M}{V_c} + \frac{v_w LWQ_i h_{ci}}{2}\\ & + \frac{v_x LWQ_i (r^*+d)}{4V_x}\end{aligned} \tag{10}$$

The optimized headway h_{ci}^{opt} for Period i can be obtained by setting the first derivative of conventional bus service cost C_{ci} to zero:

$$h_{ci}^{\text{opt}} = \sqrt{\frac{2D(a + bS_c^*)}{v_w r^* V_c L Q_i}} \tag{11}$$

Therefore, the optimal headway h_{ci}^* for each Period i is the minimum of $h_{ci}^{\max}$ and h_{ci}^{opt}:

$$h_{ci}^* = \min\{h_{ci}^{\max}, h_{ci}^{opt}\} = \min\left\{\frac{S_c^* l_c}{r^* L f Q_i}, \sqrt{\frac{2D(a + bS_c^*)}{v_w r^* V_c L Q_i}}\right\} \tag{12}$$

The optimal fleet size F_{ci}^* for each period depends on the optimal headway of that period:

$$F_{ci}^* = \frac{DW}{r^* V_c h_{ci}^*} \tag{13}$$

The capital cost should be determined from the peak-period demand, which is defined here to be Period 1, as follows:

$$C_{\text{pc}}^* = \frac{DW}{r^* V_c h_{c1}^*} B_c = \frac{DW(a_c + b_c S_c^*)}{r^* V_c h_{c1}^*} \tag{14}$$

The bus service cost C_{ci} for each Period i can be formulated using the optimal headway of that period. Therefore, the conventional bus service cost C_c for all periods can be expressed as

$$C_c^* = \sum_i \left\{\frac{DW(a + bS_c^*)}{r^* V_c h_{ci}^*} + \frac{v_v L W Q_i M}{V_c} + \frac{v_w L W Q_i h_{ci}^*}{2} + \frac{v_x L W Q_i (r^* + d)}{4V_x}\right\} t_i \tag{15}$$

The total cost including capital cost can be found by substituting optimal route spacing r^* and optimal vehicle size S_c^* into Eq. (7):

$$C_{\text{tc}}^* = \frac{DW(a_c + b_c S_c^*)}{r^* V_c h_{c1}^*} + \sum_i \left\{\frac{DW(a + bS_c^*)}{r^* V_c h_{ci}^*} + \frac{v_v L W Q_i M}{V_c} + \frac{v_w L W Q_i h_{ci}^*}{2} + \frac{v_x L W Q_i (r^* + d)}{4V_x}\right\} t_i \tag{16}$$

Flexible Bus System Cost

When considering capital cost for flexible bus service, the optimized vehicle size S_f^* and vehicle service area A^* from CS are provided in Table 1. In this section, headways for flexible bus service are optimized, unlike in CS, which used only the maximum allowable headway. The optimal headway should be the minimum of (1) the maximum allowable headway and (2) the minimum cost headway.

The maximum allowable headway $h_{fi}^{\max}$ for demand Period i is a function of optimized vehicle size S_f^*, load factor l_f, service area A^*, and demand density Q_i:

$$h_{fi}^{\max} = \frac{S_f^* l_f}{A^* Q} \tag{17}$$

From Table 1, the flexible bus service cost for Period i, C_{fi}, can be rewritten as

$$C_{fi} = \frac{LW(a + bS_f^*)\left(D_f + \phi A^* \sqrt{\frac{Q_i h_{fi}}{u}}\right)}{A^* V_f h_{fi}} + \frac{v_v L W Q_i \left(D_f + \phi A^* \sqrt{\frac{Q_i h_{fi}}{u}}\right)}{2V_f} + \frac{v_w L W Q_i h_{fi}}{2} \tag{18}$$

The optimized service headway h_f^{opt} can be obtained by setting the first derivative equal to zero:

$$\frac{\partial C_{fi}}{\partial h_{fi}} = -\frac{LW(a + b_f^* S_f^*) D_f}{A^* V_f} \frac{1}{h_{fi}^2} - \frac{LW(a + bS_f^*)\phi A^* \sqrt{Q_i/u}}{2A^* V_f} \frac{1}{\sqrt{h_{fi}^3}} + \frac{v_v L W Q_i \phi A^* \sqrt{Q_i/u}}{4V_f} \frac{1}{\sqrt{h_{fi}}} + \frac{v_w L W Q_i}{2} = 0 \tag{19}$$

Eq. (19) is a quartic equation with respect to headway. Optimized headway is denoted as h_{fi}^{opt}. The solution to Eq. (19) is presented in Appendix 2. Only one of the four solutions to this equation is feasible.

There are now two headway solutions. Therefore, the optimal headway h_{fi}^* is the minimum of the (1) maximum allowable headway and (2) optimized headway obtained by solving Eq. (19):

$$h_{fi}^* = \min\left(\frac{S_f l_f}{A^* Q_i}, h_{fi}^{\text{opt}}\right) \tag{20}$$

On the basis of the optimal headway for Period i, the required fleet size F_{fi}^* is

$$F_{fi}^* = \frac{LW\left(D_f + \phi A^* \sqrt{\frac{Q_i h_i^*}{u}}\right)}{A^* V_f h_{fi}^*} \tag{21}$$

Finally, the total cost C_{tf}^*, which is the sum of capital cost and bus service cost for all periods, can be expressed as

$$C_{tf}^* = \frac{LW\left(a_c + b_c S_f^*\right)\left(D_f + \phi A^* \sqrt{\frac{Q_i h_{fi}^*}{u}}\right)}{A^* V_f h_{f1}^*} + \sum_i \left\{\frac{LW\left(a + bS_f^*\right)\left(D_f + \phi A^* \sqrt{\frac{Q_i h_{fi}^*}{u}}\right)}{A^* V_f h_{fi}^*} + \frac{v_v L W Q_i \left(D_f + \phi A^* \sqrt{\frac{Q_i h_{fi}^*}{u}}\right)}{2V_f} + \frac{v_w L W Q_i h_{fi}^*}{2}\right\} t_i \tag{22}$$

Variable-Type Bus Operation Using Conventional and Flexible Bus Services

Conceptually, conventional services using relatively large buses are expected to have lower cost per passenger trip than flexible services at higher demand densities, and vice versa. In this section, the demand boundary between conventional and flexible bus services is investigated. Below this boundary, service type is switched conventional to flexible. Pure conventional and pure flexible service costs are also compared with variable-type services.

Integer Solution for Variable-Type Bus Service

Only one service type (either conventional or flexible) is used in each period in the objective function in

$$C_t = \min\left\{C_{pc} + \sum_i (C_{ci} + C_{fi}) t_i\right\} \tag{23}$$

subject to

$$S_c^* = S_f^* = \text{integer} \tag{24}$$

$$\frac{W}{r}, \frac{LW}{A} = \text{integer} \tag{25}$$

$$F_{ci}^*, F_{fi}^* = \text{integer} \tag{26}$$

$$\frac{F_{ci}^*}{N}, \frac{F_{fi}^*}{N'} = \text{integer} \tag{27}$$

The constraints in Eqs. (24)–(27) are required to obtain integer values for the number of routes and fleet sizes per route. To obtain total cost with integer solutions, we first optimize the decision variables, and then compare their neighboring integer solutions to satisfy such constraints.

Numerical Analysis

In this section, bus operation costs (pure conventional, pure flexible, and variable-type bus services) are computed and compared. In this numerical analysis the cumulative demand distribution over time has four values, as shown in Fig. 2.

For pure conventional service cost, Eqs. (3)–(16) are used in this numerical example. For flexible service cost, optimized decision variables (optimized vehicle size and service area) can be found from Tables 1 and 2. The CS study used the maximum allowable headway, without considering the minimum cost headway. Here, the flexible bus headway was analytically optimized in Eqs. (17)–(20). More details can be found in Schonfeld et al. (2010). Appendix 2 shows how flexible service headways for this formulation can be optimized.

Variable-Type Bus Service Boundary

For variable-type bus operation, the optimized bus size is usually determined by the conventional service requirements. As mentioned for constraint (25), integer values for both W/r and LW/A are required to obtain integer fleets. The resulting possible values of decision variables r and A are listed in Table 3.

The values in Table 3 are used to search for the minimum cost route spacing r^* for conventional bus and minimum cost service area A^* for flexible bus services.

Procedure for Finding Minimum Variable-Type Service Cost
In general, if the demand distribution has k periods, then there are $k+1$ possible boundaries between periods (i.e., boundary $1 \ldots k+1$) when switching from one service type to another. Thus, the numerical example has five possible boundaries because it has four cumulative demand periods. Variable-type service is provided when $1 < k < 5$, where $k = 1$ means that service is always purely flexible service and $k = 5$ implies purely conventional service in every period.

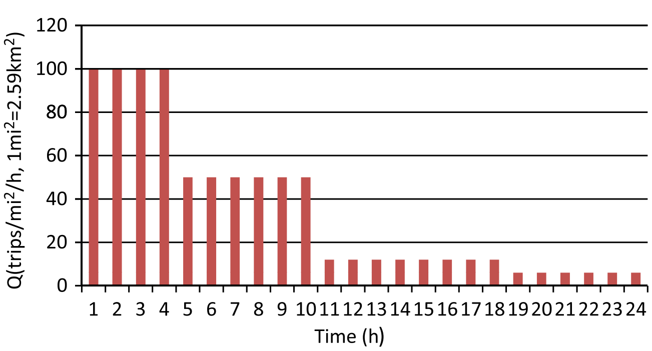

Fig. 2. Demand density variation over time

The computation procedures for variable-type service are as follows:

1. Set up boundary $k = 1$.
2. On the basis of boundary, optimize decision variables, namely, vehicle sizes and route spacing for conventional operations or service area for flexible service.
3. Optimize headway for variable-type service.
4. Compute total cost using results in steps 2 and 3.
5. Change boundary k to $k+1$ (i.e., one more period has conventional service and the remaining periods have flexible service).
6. Continue steps 2–4 until the total cost starts increasing. The optimal boundary that minimizes total cost for variable-type service is thus reached.

Numerical Analysis Result

The results obtained with baseline inputs are provided under Notation. The optimized pure conventional bus service costs \$107,166/day, including capital costs and user time costs. To operate conventional bus service with given demand density, 60 buses are required. Vehicle size is optimized to satisfy all demand periods, at 40 seats/bus. The local area route spacing is jointly optimized (subject to a constraint requiring an integer number of zones) at one mile (1 mile = 1.61 km).

Pure flexible bus results show that the optimized total cost for serving this demand is about \$118,377/day, which is much costlier than pure conventional bus service. The reason is that the optimized flexible services use many more zones (9 versus 4) and vehicles, but much smaller vehicles, than optimized conventional services. Moreover, pure flexible service requires more buses to cover peak demand, because its optimized vehicles are smaller than those used for conventional bus service. As shown in Table 4, flexible service requires 108 buses in the peak period, which increases capital cost.

For variable-type service, the pure conventional bus size of 40 seats is used in all periods for both conventional and flexible operations as well as for capital cost computation. In this numerical analysis, variable-type service found to be preferable to pure bus services. Therefore, select conventional service in Periods 1 and 2 and flexible service in Periods 3 and 4, using the same bus size.

With variable-type service (using flexible service in Periods 3 and 4), cost is reduced compared with both pure conventional and pure flexible services. Compared with pure conventional service, variable-type service saves \$1,382/day. Similarly, variable-type service costs about \$12,600/day less than pure flexible service.

Sensitivity Analysis

Sensitivity analyses are conducted to explore the relative merits of conventional, flexible, and variable-type bus services in different circumstances. Seven cases are presented below.

Table 3. Possible Values of Decision Variables r and A

N, N	$r = \mathrm{W}/\mathrm{N}$ for conventional bus periods	$A = \mathrm{LW}/\mathrm{N}'$ for flexible bus periods
1	4	20
2	2	10
3	1.333	6.667
4	1	5
5	0.8	4
6	0.667	3.333
—	—	—

Table 4. Numerical Results with Baseline Inputs

	Pure conventional service	Pure flexible service	Variable-type service	
S_c, S_f (seats/bus)	40	23	40	
r, A	1	2.222	0.8	6.667
N	4	9	5	3
h1 (h)	0.078	0.089	0.097	
h2 (h)	0.146	0.177	0.194	
h3 (h)	0.389	0.510		0.269
h4 (h)	0.583	0.476		0.340
F1(vehicles)	60	108	60	
F2(vehicles)	32	54	30	
F3(vehicles)	12	18		12
F4(vehicles)	8	18		9
C1 ($/h)	10,676.7	11,175.1	10,430.0	
C2 ($/h)	5,822.7	8,111.1	5,798.3	
C3 ($/h)	1,911.6	5,085.4		1,927.3
C4 ($/h)	1,171.8	1,824.9		1,109.3
t1 (h)	4	4	4	
t2 (h)	6	6	6	
t3 (h)	8	8		8
t4 (h)	6	6		6
C_p ($/day)	7,200.0	12,042.0	7,200.0	
TC ($/day)	107,166.3	118,376.8	105,784.3	
% Change	1.290%	10.64%		

Table 5. Sensitivity Analysis Results of Directional Demand Split Factor

	Pure conventional service	Pure flexible service	Variable-type service	
S_c, S_f (seats/bus)	31	23	31	
r, A	1	2.857	0.8	6.667
N	4	7	5	3
h1 (h)	0.078	0.066	0.097	
h2 (h)	0.146	0.154	0.194	
h3 (h)	0.389	0.334		0.269
h4 (h)	0.583	0.487		0.340
F1(veh)	60	112	60	
F2(veh)	32	49	30	
F3(veh)	12	21		12
F4(veh)	8	14		9
C1 ($/h)	10,568.7	11,481.7	10,322.0	
C2 ($/h)	5,765.1	6,087.9	5,744.3	
C3 ($/h)	1,890.0	1,990.4		1,897.1
C4 ($/h)	1,157.4	1,220.0		1,087.7
t1 (h)	4	4	4	
t2 (h)	6	6	6	
t3 (h)	8	8		8
t4 (h)	6	6		6
C_p ($/day)	6,930.0	12,488.0	6,930.0	
TC($/day)	105,859.5	118,185.3	104,387.2	
% Change	1.39%	11.67%	—	

Case I—Directional Demand Split Factor (f) for Conventional Service

The directional demand split was changed to 75 and 25% (versus 100 and 0% in the baseline). In Table 5, the total costs of conventional and variable-type services in Case I decrease compared with the baseline results in Table 4. In this case, variable-type service reduces total cost by 1.39% from pure conventional service and 11.67% from pure flexible service. In case I, with $f = 0.75$, a directional demand split factor can slightly reduce costs below the baseline case.

Case II—Load Factors

In case II, maximum load factors for both conventional and flexible services are increased from 1 to 1.25 (implying that some standees are allowed). Table 5 outlines the resulting costs. Note that the costs of pure conventional service in Table 6 are below the baseline case (Table 4). Similarly, pure flexible and variable-type services benefit from higher load factors. However, similarly to Case I, variable-type service saves about 1.41 and 9.71% compared with pure conventional and pure flexible services, respectively.

Case III—Service Period Demand Variation [Q1=10, Q2=5, Q3=1.2, Q4=0.6 trips=mi^2 (1 mi^2=2.59 km^2)]

This case explores the effect of very low demand density (i.e., 10% of baseline value). Here the costs of pure conventional and flexible operation are very close. Variable-type service, as shown in Table 7, saves 3.19 and 4.06% compared with pure conventional and flexible services, respectively. It is interesting here that conventional service is used only during the highest demand period, leaving the other three periods to flexible service.

Table 6. Sensitivity Analysis Results of Load Factors

	Pure conventional service	Pure flexible service	Variable-type service	
S_c, S_f (seats/bus)	32	23	32	
r, A	1	3.333	0.667	6.667
N	4	6	6	3
h1 (h)	0.078	0.068	0.117	
h2 (h)	0.146	0.118	0.194	
h3 (h)	0.389	0.340		0.209
h4 (h)	0.583	0.494		0.340
F1(veh)	60	96	60	
F2(veh)	32	54	36	
F3(veh)	12	18		15
F4(veh)	8	12		9
C1 ($/h)	10,580.7	11,391.4	10,247.3	
C2 ($/h)	5,771.5	6,149.2	5,808.7	
C3 ($/h)	1,892.4	1,940.2		1,896.5
C4 ($/h)	1,159.0	1,176.9		1,090.1
t1 (h)	4	4	4	
t2 (h)	6	6	6	
t3 (h)	8	8		8
t4 (h)	6	6		6
C_p ($/day)	6,960.0	10,704.0	6,960.0	
TC ($/day)	106,004.7	115,747.8	104514.3	
% Change	1.41%	9.71%	—	

Case IV—Service Period Time Variation (t1=2, t2=4, t3=4, t4=14)

In the baseline case (Table 4) there are 4, 6, 8, and 6 h, respectively, in Periods 1, 2, 3 and 4. In Case IV the effect of higher demand variability is explored changing those four periods to 2, 4, 4 and

Table 7. Sensitivity Analysis Results of Demand Variation

	Pure conventional service	Pure flexible service	Variable-type service	
S_c, S_f (seats/bus)	17	16	17	
r, A	2.0	10.0	2.0	10.0
N	2	2	2	2
h1 (h)	0.167	0.111	0.167	
h2 (h)	0.292	0.198		0.253
h3 (h)	1.167	0.472		0.472
h4 (h)	1.167	0.944		0.944
F1(veh)	14	18	14	
F2(veh)	8	10		8
F3(veh)	2	4		4
F4(veh)	2	2		2
C1 ($/h)	1,653.9	1,692.3	1,653.9	
C2 ($/h)	935.4	930.2		903.0
C3 ($/h)	353.2	317.7		318.7
C4 ($/h)	210.0	192.8		193.4
t1 (h)	4	4	4	
t2 (h)	6	6		6
t3 (h)	8	8		8
t4 (h)	6	6		6
C_p ($/day)	1,519.0	1,944.0	1,519.00	
TC ($/day)	17,832.1	17,992.9	17,262.8	
% Change	3.19%	4.06%		

Table 8. Sensitivity Analysis Results of Service Time Variation

	Pure conventional service	Pure flexible service	Variable-type service	
S_c, S_f (seats/bus)	40	21	40	
r, A	1.333	3.333	0.667	10.0
N	3	6	6	2
h1 (h)	0.058	0.052	0.117	
h2 (h)	0.117	0.105	0.194	
h3 (h)	0.389	0.340		0.179
h4 (h)	0.583	0.494		0.259
F1(veh)	60	120	60	
F2(veh)	30	60	36	
F3(veh)	9	18		12
F4(veh)	6	12		8
C1 ($/h)	11,243.3	11,775.8	10,343.3	
C2 ($/h)	5,971.7	6,202.7	5,866.3	
C3 ($/h)	1,893.6	1,931.7		1,958.9
C4 ($/h)	1,143.8	1,171.4		1,096.0
t1 (h)	2	2	2	
t2 (h)	4	4	4	
t3 (h)	4	4		4
t4 (h)	14	14		14
C_p($/day)	7,200.0	13,260.0	7,200.0	
TC($/day)	77,160.9	85,748.9	74531.6	
% Savings	3.41%	13.08%		

14 h. The results in Table 8 indicate that variable-type bus service now achieves much greater savings compared with the baseline case (Table 4)—about 3.41 and 13.08%—whereas in the baseline case (Table 4), variable-type bus service is 1.29% less than pure conventional service.

On the basis of the sensitivity of results in these cases, it is found that there are significant advantages to variable-type bus service when there are long periods of low demand that is far below peak levels.

Case V—Operating Cost Parameters (a=45, b=0.3)

In Case V, the sensitivity of total cost and other results to bus operating cost, which is a linear function of number of seats (i.e., $B = a + bS$), is explored. Here, parameter a and b values are increased by 50%. The results in Table 9 indicate that the cheapest total cost is provided by variable-type service. Variable-type service results in 1.379 and 14.86% savings compared with pure conventional and flexible services, respectively.

Case VI—Length of Service Region [L=6 miles (9.6 km)]

In Case VI, the service region length is increased 20% [from 5 to 6 miles (8 to 9.6 km)]. For variable-type service, note that conventional bus service operates in Periods 1–3; flexible bus service operates only in Period 4. This result shows that as the local service region lengthens, the potential savings of variable-type service decrease because, as region lengthens, demand also increases, thus favoring conventional service. In Table 10, Period 3 in variable-type service is served by conventional service, unlike in the baseline case (Table 4).

Case VII—Line-haul Distance [J=20 miles (32.2 km), J/L=4]

In Case VII, sensitivity to line-haul distance [from 10 to 20 miles (16.1 to 32.2 km)] is analyzed. Here the ratio line-haul distance/length of local area (i.e., J/L) is increased from 2 to 4. The results

Table 9. Sensitivity Analysis Results of Operating Cost Input Parameters

	Pure conventional service	Pure flexible service	Variable-type service	
S_c, S_f (seats/bus)	50	27	50	
r, A	1	2.857	1	6.667
N	4	7	4	3
h1 (h)	0.097	0.077	0.097	
h2 (h)	0.194	0.154	0.194	
h3 (h)	0.583	0.527		0.269
h4 (h)	1.167	0.487		0.340
F1(veh)	48	98	48	
F2(veh)	24	49	24	
F3(veh)	12	14	12	
F4(veh)	4	14		3
C1 ($/h)	11,510.0	13,381.0	11,510.0	
C2 ($/h)	6,338.3	7,153.1	6,338.3	
C3 ($/h)	2,215.6	2,357.3	2,175.6	
C4 ($/h)	1,527.8	1,510.0		1,312.3
t1 (h)	4	4	4	
t2 (h)	6	6	6	
t3 (h)	8	8	8	
t4 (h)	6	6		6
C_p ($/day)	6,000.0	11,123.0	6,000.0	
TC ($/day)	116,961.6	135,483.6	115348.9	
% Savings	1.379%	14.86%		

in Table 11 indicate that variable-type service reduces total cost by 0.704 and 11.12% compared with pure services. By increasing line-haul distance (without changing demand), round trip time increases for both conventional and flexible sevices, favoring larger vehicles

Table 10. Sensitivity Analysis Results of Service Region Length

	Pure conventional service	Pure flexible service	Variable-type service	
S_c, S_f (seats/bus)	45	25	45	
r, A	1	3.429	1	8
N	4	7	4	3
h1 (h)	0.075	0.057	0.075	
h2 (h)	0.141	0.122	0.141	
h3 (h)	0.422	0.352	0.422	
h4 (h)	0.633	0.511		0.358
F1(veh)	68	133	68	
F2(veh)	36	63	36	
F3(veh)	12	21	12	
F4(veh)	8	14		9
C1 ($/h)	12,980.9	14,299.2	12,980.9	
C2 ($/h)	7,045.3	7,524.1	7,045.3	
C3 ($/h)	2,308.3	2,370.2	2,308.3	
C4 ($/h)	1,414.6	1,435.0		1,351.4
t1 (h)	4	4	4	
t2 (h)	6	6	6	
t3 (h)	8	8	8	
t4 (h)	6	6		6
C_p ($/day)	8,330.0	14,962.5	8,330.00	
TC ($/day)	129,479.7	144,875.7	129,100.6	
% Savings	0.29%	10.89%		

Table 11. Sensitivity Analysis Results of Line-haul Distance

	Pure conventional service	Pure flexible service	Variable-type service	
S_c, S_f (seats/bus)	50	31	50	
r, A	1	4	0.8	10
N	4	5	5	2
h1 (h)	0.096	0.066	0.123	
h2 (h)	0.191	0.139	0.246	
h3 (h)	0.431	0.405		0.239
h4 (h)	0.574	0.523		0.324
F1(veh)	72	125	70	
F2(veh)	36	60	35	
F3(veh)	16	20		14
F4(veh)	12	15		10
C1 ($/h)	14,269.3	15,954.5	14,037.3	
C2 ($/h)	7,708.7	8,347.3	7,756.7	
C3 ($/h)	2,488.9	2,546.6		2,539.7
C4 ($/h)	1,507.8	1,523.0		1,422.5
t1 (h)	4	4	4	
t2 (h)	6	6	6	
t3 (h)	8	8		8
t4 (h)	6	6		6
C_p ($/day)	9,000.0	14,437.5	8,750.0	
TC ($/day)	141,287.5	157,850.3	140,292.1	
% Savings	0.704%	11.12%		

because bus operators want to carry more passengers in round-trip time. Thus, in Table 11, vehicle size is 50 seats/bus for variable-type service, but only 40 seats/bus in the baseline case (Table 4). With variable-type service, service cost is less in Periods 1 and 4 compared with pure conventional service. These service cost savings and capital cost savings favor variable-type service over pure services.

Conclusions

In this paper optimization models are developed for analyzing and integrating conventional services (having fixed routes and schedules) and flexible bus services. The optimization models are improved from those of Chang and Schonfeld (1991a), mainly by (1) optimizing the flexible service headways rather than just using maximum allowable headways, and (2) introducing directional demand split factors. These models are used to compare pure conventional services, pure flexible services, and variable-type services, which can switch between conventional and flexible services as demand changes over time.

The numerical analysis indicates that variable-type bus operation can reduce total cost compared with pure conventional or pure flexible bus service. In the baseline case, variable-type service can reduce costs by about 1.29% compared with pure conventional service and about 10.64% compared with pure flexible service. Moreover, various sensitivity analyses are put forward explore how major parameter changes affect the optimized results. In Case IV (when service periods are adjusted to increase the variability of demand over time), it is found that variable-type service can reduce costs by more than 3.41 and 13.08%, respectively, compared with pure conventional and flexible services. These results confirm that such variable-type services are especially promising for systems whose demand (1) varies greatly over time and (2) straddles the threshold between conventional and flexible services.

To summarize, this paper confirms that conventional service with large buses is preferable when demand is high. Similarly, flexible service is less costly at relatively low demand. A public bus system alternating among these two service concepts based on demand variation and other conditions can be used to improve service efficiency. As an extension of this study, how service type can be best matched to demand in regions where the demand varies over space as well as time should be explored.

Appendix 1: Conventional Service Cost Formulation

As is shown in Fig. 1, buses travel from the terminal a line-haul distance J at nonstop speed yV_c to a corner of the service area, then travel an average of $W/2$ miles (1 mile = 1.61 km) at local nonstop speed zV_c from the corner to the assigned zone, run a distribution route of length L at local speed V_c along the central axis of the zone while stopping for passengers every d km, and then reverse the process in returning. Therefore, the buses' average round-trip time is

$$R_c = \frac{2J}{yV_c} + \frac{W}{zV_c} + \frac{2L}{V_c} \tag{28}$$

This round-trip time can be rewritten as

$$R_c = \left\{\frac{2J}{y} + \frac{W}{z} + 2L\right\} / V_c \tag{29}$$

In Eq. (29), the expression in parentheses represents an equivalent vehicle round trip distance, D.

The total cost of conventional bus service includes operator cost, C_{oc}, and the user cost, C_{uc}. To calculate operator cost, fleet size, N, which is the total vehicle round-trip time divided by the headway, is determined. With the equivalent vehicle round-trip distance D, a controllable directional split factor f, and conventional bus speed V_c, required fleet size, F_c, is obtained as

$$F_c = \frac{DW}{rh_c V_c}, \quad \text{where } D = 2J/y + W/z + 2L \tag{30}$$

The hourly conventional bus operator cost, C_{oc}, is equal to required fleet size multiplied by bus operating cost:

$$C_{\text{oc}} = F_c B \tag{31}$$

Bus operating cost, B, is formulated as

$$B = a + bS_c \tag{32}$$

and required service headway, h_c, is

$$h_c = \frac{S_c l_c}{rLfQ} \tag{33}$$

The operating cost, C_{oc}, can be reformulated by substituting Eqs. (30)–(33) into Eq. (31):

$$C_{\text{oc}} = \frac{D(a + bS_c)LWfQ}{l_c V_c S_c} \tag{34}$$

The hourly user cost for the conventional bus system, C_{uc}, is the sum of in-vehicle cost c, C_{vc}; waiting cost, C_{wc}; and access cost, C_{xc}:

$$C_{\text{uc}} = C_{\text{vc}} + C_{\text{wc}} + C_{\text{xc}} \tag{35}$$

The hourly in-vehicle cost for the conventional system is then

$$C_{\text{vc}} = v_v LWQt \tag{36}$$

The average travel time, t, per passenger trip is formulated as

$$t = \frac{J}{yV_c} + \frac{W}{2zV_c} + \frac{L}{2V_c} = \frac{M}{V_c}, \quad \text{where } M = J/y + W/2z + L/2 \tag{37}$$

Then Eq. (36) can be written as

$$C_{\text{vc}} = v_v LWQ \frac{M}{V_c} \tag{38}$$

The average waiting time is assumed to be half the headway. Therefore, the hourly user waiting cost for a conventional system, C_{wc}, is

$$C_{\text{wc}} = v_w LWQ \frac{h_c}{2} = v_w LWQ \frac{S_c l_c}{2rLfQ} = \frac{v_w WS_c l_c}{2rf} \tag{39}$$

Because the spacing between adjacent branches of local bus service is r, and because service trip origins (or destinations) are uniformly distributed over the area, the average access distance to the nearest route is one-fourth of route spacing, $r/4$. Similarly, the access distance alongside the route to the nearest transit stop is one-fourth of the bus stop spacing, $d/4$. Therefore, the hourly access cost for the conventional bus system, C_{xc}, is

$$C_{\text{xc}} = \frac{v_x LWQ(r + d)}{4V_x} \tag{40}$$

The total cost for the conventional system, C_c, is the sum of operating cost and user cost:

$$C_c = \frac{D(a + bS_c)LWfQ}{l_c V_c S_c} + \frac{v_v LWQM}{V_c} + \frac{v_w LWQS_c l_c}{2rf} + \frac{v_x LWQ(r + d)}{4V_x} \tag{41}$$

In Eq. (41), the optimizable variables are routing space r and vehicle size S_c, which are optimized by taking partial derivatives of C_c in Eq. (41). Setting the partial derivatives equal to zero and solving simultaneously results in

$$S_c^* = \frac{2f}{l_c} \sqrt[3]{\frac{a^2 D^2 LQV_x}{v_w v_x V_c^2}} \tag{42}$$

$$r^* = \sqrt[3]{\frac{8aDv_w V_x^2}{v_x^2 LQV_c}} \tag{43}$$

The second derivatives of Eq. (41) with respect to vehicle size S_c and routing space r are positive for any reasonable inputs. Therefore, Eqs. (42) and (43) provide the globally minimal total cost. From Eqs. (42) and (43) it can be observed that the product of optimized vehicle size and optimized route spacing is constant [i.e., $S_c^* \times r^* = (4faDv_x)/(l_c v_x V_c) = \text{constant}$].

After optimization of vehicle size S_c^* and route spacing r^*, the headway h_c^*, which minimizes total cost C_c. Optimal headway h_c^* should be the minimum of the maximum allowable headway and minimum cost headway. The maximum allowable headway, $h_c^{\max}$, can be found by substituting Eqs. (42) and (43) in Eq. (33):

$$h_c^{\max} = \frac{S_c^* l_c}{r^* LfQ} \tag{44}$$

The optimized headway, h^{opt}, can be obtained from the total cost function, which is provided in Eq. (45), by setting its first derivative equal to zero. The second derivative is positive. Therefore, the optimized headway will yield the globally minimal total cost:

$$C_c = \frac{DW(a + bS_c)}{rV_c h_c} + \frac{v_v LWQM}{V_c} + \frac{v_w LWQh_c}{2} + \frac{v_x LWQ(r + d)}{4V_x} \tag{45}$$

The resulting minimum cost headway is

$$h_c^{\text{opt}} = \sqrt{\frac{2D(a + bS_c^*)}{v_w r^* V_c LQ}} \tag{46}$$

Overall, the optimal headway h_c^* is then

$$h_c^* = \min\left\{\frac{S_c l_c}{r^* LfQ}, \sqrt{\frac{2D(a + bS_c^*)}{v_w r^* V_c LQ}}\right\} \tag{47}$$

Substitution of Eqs. (43) and (44) in Eq. (30) yields optimal fleet size F_c^* for the conventional bus system:

$$F_c^* = \frac{DW}{r^* h_c^* V_c} \tag{48}$$

Therefore, the bus service cost based on the jointly optimized vehicle size S_c^*, route spacing r^*, and optimal headway h_c^* is

$$C_c = \frac{DW(a+bS_c^*)}{r^*V_c h_c^*} + \frac{v_v LWQM}{V_c} + \frac{v_w LWQh_c^*}{2} + \frac{v_x LWQ(r^*+d)}{4V_x} \tag{49}$$

Appendix 2: Optimized Headway for Flexible Service

When t is substituted for $1/\sqrt{h_{fi}}$, Eq. (19) becomes

$$-\frac{LW(a+b_f^*S_f^*)D_f}{A^*V_f}t^4 - \frac{LW(a+bS_f^*)\phi A^*\sqrt{Q_i/u}}{2A^*V_f}t^3 + \frac{v_v LWQ_i\phi A^*\sqrt{Q_i/u}}{4V_f}t + \frac{v_w LWQ_i}{2} = 0 \tag{50}$$

Eq. (50) can be rewritten as

$$At^4 + Bt^3 + Ct^2 + Dt + E = 0 \tag{51}$$

where $A = -\frac{LW(a+bS_f^*)D_f}{A^*V_f}$, $B = -\frac{LW(a+bS_f^*)\phi A^*\sqrt{\frac{Q_i}{u}}}{2A^*V_f}$, $D = \frac{v_v LWQ_iA^*\sqrt{\frac{Q_i}{u}}}{4V_f}$, $E = \frac{v_w LWQ_i}{2}$.

To solve Eq. (51), P, Q, R, S, T, and V values must be computed using A, B, C, D, and E.

$P = \frac{B}{4A}$, $Q = \frac{2C}{3A}$, $R = C^2 - 3BD + 12AE$, $S = 2C^2 - 9BCD + 27AD^2 + 27EB^2 - 72ACE$

$$T = -\frac{B^3}{A^3} + \frac{4BC}{A^2} - \frac{8D}{A}, V = \frac{\sqrt[3]{2}R}{3A\sqrt[3]{S+\sqrt{-4R^3+S^2}}} + \frac{\sqrt[3]{S+\sqrt{-4R^3+S^2}}}{3\sqrt[3]{2}A} \tag{52}$$

After the values of P, Q, R, S, T and V are computed,

$$\begin{aligned}
X1 &= -P - \frac{1}{2}\sqrt{4P^2-Q+V} - \frac{1}{2}\sqrt{8P^2-2Q-V-\frac{T}{4\sqrt{4P^2-Q+V}}}\\
X2 &= -P - \frac{1}{2}\sqrt{4P^2-Q+V} + \frac{1}{2}\sqrt{8P^2-2Q-V-\frac{T}{4\sqrt{4P^2-Q+V}}}\\
X3 &= -P + \frac{1}{2}\sqrt{4P^2-Q+V} - \frac{1}{2}\sqrt{8P^2-2Q-V+\frac{T}{4\sqrt{4P^2-Q+V}}}\\
X4 &= -P + \frac{1}{2}\sqrt{4P^2-Q+V} + \frac{1}{2}\sqrt{8P^2-2Q-V+\frac{T}{4\sqrt{4P^2-Q+V}}}
\end{aligned} \tag{53}$$

$X1$–$X4$ correspond to $t\ (=1/\sqrt{h_{fi}})$. Therefore, among the four solutions, the only feasible solution satisfying both $t > 0$ and $h_{fi} > 0$ is the optimized headway h_{fi}^{opt}.

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Notation

The following symbols are used in this paper:

a = hourly fixed cost coefficient for operating bus service (\$/bus h) 30.0;
a_c = fixed cost coefficient for bus ownership (capital cost) (\$/bus day) 100.0;
B = bus operating cost (\$/bus h), for conventional and flexible service ($= a + bS_c$, $a + bS_f$);
B_c = bus operator cost for owning bus (capital cost) (\$/bus day);
$\bar{b}$ = weighted fixed cost coefficient defined in Table 2;
C_c, C_f = service costs for conventional and flexible services (\$/h);
C_{ci}, C_{fi} = service costs in period i for conventional and flexible services (\$/h);
C_o, C_{oc}, C_{of} = operating cost; for conventional and flexible service (\$/h);
C_p, C_{pc}, C_{pf} = capital cost; for conventional and flexible services (\$/day);
C_t, C_{tc}, C_{tf} = total cost; for conventional and flexible service (\$/day);
C_u, C_{uc}, C_{uf} = user cost; for conventional and flexible service (\$/h);
C_v, C_{vc}, C_{vf} = in-vehicle cost; for conventional and flexible service (\$/h);
C_w, C_{wc}, C_{wf} = waiting cost; for conventional and flexible service (\$/h);
C_x, C_{xc} = access cost; for conventional service (\$/h);
D = equivalent average bus round-trip distance for conventional bus service ($=2J/y + W/z + 2L$) (miles);
D_c = distance of one tour of flexible bus service at local area (miles);
D_f = equivalent line-haul distance for flexible bus service [$= (L+W)/z + 2J/y$], (miles);
d = bus stop spacing (miles) 0.2;
F_c, F_f = fleet size for conventional and flexible services (buses);
F_{ci}, F_{fi} = fleet size at period i for conventional and flexible bus services (buses);
f = directional demand split factor 1.0;
h_c, h_f = headway for conventional and flexible service (h/bus);
$h_{ci}^{\max}$, $h_{fi}^{\max}$ = maximum allowable headway in period i for conventional and flexible services (h/bus);
h_{ci}^{opt}, h_{fi}^{opt} = optimized headway in period i for conventional and flexible services (h/bus);
h_{ci}, h_{fi} = headway in period i for conventional and flexible services (h/bus);
i, k = period index;
J = line-haul distance (miles) 10.0;
L, W = length and width of service area [miles (1 mile = 1.61 km)] 5.0, 4.0;
l_c, l_f = load factors for conventional services and flexible services (passengers/seat) 1.0;

M = equivalent average trip distance (miles) $(J/y_c + W/2z_c + L/2)$;
N, N' = number of zones in service area for conventional and flexible services;
n = number of passengers in one collection tour;
Q = round-trip demand density ($\text{trips}/\text{mi}^2/\text{h}$);
Q_i = round-trip demand density in Period i ($\text{trips}/\text{mi}^2/\text{h}$);
Q_p = demand density at peak time ($\text{trips}/\text{mi}^2/\text{h}$);
$\bar{Q}$ = average round-trip demand density, as defined in Table 2;
R_c = round-trip time for conventional service (h);
r = route spacing (miles);
S_c, S_f = bus size for conventional and flexible services (seats/bus);
t_i = duration of Period i;
u = average number of passengers per stop for flexible service 1.2;
V_c, V_f = local service speed for conventional and flexible bus [mi/h (1 mi/h = 1.61 km/h)] 20, 18;
V_x = average access speed [mi/h (1 mi/h = 1.61 km/h)] 2.5;
v_v, v_w, v_x = value of in-vehicle time, wait time, and access time (\$/passenger h) 5, 12, 12;
Y = term used in Tables 2 and 3;
y = express speed/local speed ratio for conventional service conventional bus = 1.8, flexible bus = 2.0;
z = nonstop ratio = local nonstop speed/local speed; same values as y;
$\varnothing$ = constant in the collection distance equation (Daganzo 1984) for flexible bus service 1.15; and
$*$ = superscript indicating optimal value.

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