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Optimal zone sizes and headways for flexible-route bus services

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ABSTRACT

Flexible-route bus systems serving passengers at their doorsteps may be preferable to fixed-route bus systems in areas with low demand densities or whose roads cannot accommodate relatively large fixed-route buses. Flexible-route systems may also be preferable for elderly or handicapped riders for whom accessing the pre-determined stops on fixed routes is difficult. Since bus systems with flexible demand-responsive routes retain the economic and environmental advantages of public transportation, it is important to analyze them and optimize their characteristics to match their operating environments. This study shows how the total cost can be minimized for a flexible-route bus system with a many-to-one demand pattern by jointly optimizing its headway and service zone size. Numerical results demonstrate the model's applicability and indicate how such flexible-route systems should be adapted to demand characteristics and planning constraints.

Introduction

Since public transit systems can transport many passengers efficiently, they have an important role in urban areas. Hence, many researchers have sought to improve the costs and service quality of transit systems. Bus transit operations may be classified as fixed-route or flexible-route services. Fixed-route bus services have predetermined routes, stops and schedules, and they are very widely used in densely populated urban areas since their average cost for serving high demand densities is relatively low (Kim and Schonfeld, 2012, 2013). When the demand densities or roadway geometric characteristics are unsuitable for fixed-route bus operations, flexible-route services, which can operate on-demand without predetermined stop locations, may offer a practical alternative. When the demand density is relatively low, flexible-route service may have lower average system cost than conventional fixed-route bus services. In an early study, Wilson and Hendrickson (1980) reviewed models for predicting the performance of flexible transportation services such as taxi and dial-a-ride. Their review covered methodological approaches such as simulation, empirical, deterministic queuing and stochastic processes. With rapid developments of information technology and computation capabilities in recent years, various studies (such as Luo and Schonfeld, 2007; Markovic et al, 2015) explored micro-level transit service modeling by solving dial-a-ride problems or vehicle routing problems for public transit systems.

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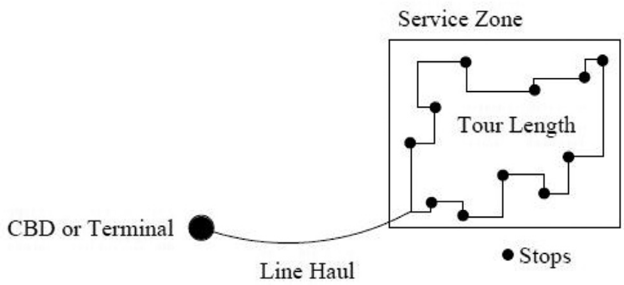

Fig. 1. Configuration of flexible-route bus service module.

However, we observe that the relations among optimal zone sizes, headways and relevant exogenous factors for flexible-route services have not been sufficiently explored, especially in considering how zone sizes may be optimized based on the local demand densities. This paper presents a planning model for optimizing a flexible-route bus operation serving many-to-one and one-to-many demand patterns, as sketched in Fig. 1. The rest of the paper includes a literature review for flexible transportation operations, problem formulations, numerical studies and a concluding summary.

Literature review

Flexible-route bus services, including demand-responsive dial-a-ride services, are widely considered to improve the service quality for disabled passengers by providing door-to-door services or to reduce the cost of public transit systems for regions with low demand densities. In this section, we present the studies that are closely related to flexible-route bus operations. General reviews of fixed route bus transit operations and network design may be found in Ceder and Wilson (1986) and in Ibarra-Rojas et al (2015). An extensive overview of dial-a-ride problems, including recent developments, is provided in Molenbruch et al (2017).

Nourbakhsh and Ouyang (2012) analyze the costs of flexible bus routes and their competitiveness versus fixed bus routes and taxis. The optimal network layout, service area, and bus headway are used to minimize total system cost. It is noted that in low to moderate demand areas, flexible bus route systems have the lowest system cost among flexible-route bus, fixed-route bus and taxi services. Daganzo and Ouyang (2019) present analytic models for door-to-door transit services, including non-shared taxi and demand-responsive transportation as special variants of the model. Several objectives are considered in the paper as follows: Firstly, the case of non-shared taxi is analyzed to emphasize the level of service quality. Secondly, dial-a-ride services are analyzed for minimizing resources. Thirdly, the case of shared taxi is analyzed for maximizing the number of served passenger trips. The model formulations assume a uniform and steady many-to-many demand pattern. The deterministic analysis framework provides approximated closed form solutions for the travel time and fleet size thresholds among dial-a-ride, ridesharing, and non-shared taxi services.

Amirgholy and Gonzales (2016) employ an analytic model to approximate the operation cost of demand-responsive transit (DRT) operations with time-dependent demand patterns. The cost of demand-responsive transit services is estimated using the fleet size, the vehicle miles traveled, and the vehicle hours traveled. The study also formulates the total cost that users experience as a result of the operating decisions. This study analyzes the dynamic equilibrium that is associated with oversaturated conditions in which the demand rate exceeds the operating capacity of the DRT system. A dynamic pricing policy is considered that improves the efficiency of the DRT system by shifting the demand to a different time of the day and avoiding the underutilization of capacity during off-peak times. Bakas et al (2016) formulate a dial-a-ride problem with the assumption that customer pickups and drop-offs are allowed only at pre-defined bus stops. With this assumption, the benefit of demand responsive transportation services in low demand areas is compared to fixed-route bus operations.

Adebisi (1980) develops a model for estimating the travel time for fully- and partially flexible bus routes. His model incorporates randomness in the number of passengers and their locations on grid networks. Pei et al (2019) explore flexible bus operation by allowing turn-backs for dispatched vehicles as well as skipping some bus stops. By adjusting the length of the service route with demand-responsive turn-backs, the travel time for all passengers, including the onboard and waiting time, is minimized. This study finds that a flexible bus operation performs well when demand density is low to medium. Qui et al (2015) explore a flag-stop bus operation policy in which bus services do not provide complete curb-to-curb services, but still offer some flexibility to transit riders. By comparing this operation policy to fixed- and flexible-route bus operations, the flag-stop policy is found to be advantageous in low-demand regions, such as suburban and rural areas. Zheng et al (2018) use two analytical models for comparing services with limited route deviation versus more flexible point deviation. Stiglic et al (2015) analyze a flexible dial-a-ride system in which passengers are picked up at their origin and dropped off at their destinations, or passengers come to the designated points to be picked up or dropped off. Although a possible inconvenience for such systems is that passengers have to arrive at the meeting point before their vehicle arrives, the proposed model substantially improves the system performance. Gomes et al (2015) develop a simulation-based optimization model for demand-responsive transportation scheduling. Yu et al (2015) incorporate a dynamic vehicle routing algorithm into an agent-based traffic simulation and compare demand responsive transportation services with conventional bus operations. They note that demand-responsive service increases mobility by decreasing the travel time perceived by passengers. Several case studies are found on implementing flexible transportation services (Horn, 2002; Fernandez et al, 2008).

Pan et al., (2015) propose a mathematical model for designing the service area and routing plan for a flexible bus feeder system that is connected to a rail transit line. By assuming the fleet size and demand as inputs, this model approximates the service area and routing schedule without assuming a grid street geometry and uniform demand distributions. It is assumed that the passengers within the zone report to the designated pickup or drop points, also called gates Thus, the tour within each zone is not modeled; instead, a gravity-based heuristic approach for optimizing vehicle routes among gates is proposed. As an extension from Pan et al., (2015), Lu et al (2016) present a model for deviated fixed-route transit operations. Their main purpose is to assign random requests to the nearest possible routes by allowing route deviations based on the travelling salesman problem. Bus travel time is minimized with a genetic algorithm. Saeed and Kurauch (2015) formulate mixed-integer problems to analyze a dial-a-ride (DAR) system for rural areas. A branch-and-cut algorithm-based solution is proposed to minimize the operating costs as well as user's travel times for the DAR system. They note that flexible-route DAR operation is suitable for low demand density and wide spatial coverage in complex road networks. Their DAR results show decreased waiting times for rural area users.

The relative advantages of fixed-route, flexible-route and variable-type bus services are analyzed by Chang and Schonfeld (1991) and by Kim and Schonfeld (2012). find that at low demand densities, flexible-route services have lower average cost per passenger than conventional fixed-route bus operations. Kim and Schonfeld (2013) integrate fixed and flexible bus services using a genetic algorithm and analytic optimization while determining the type of service based on demand density. Chen and Nie (2017) analyze hybrid systems with fixed and flexible services using simulation. They use flexible services to increase accessibility to fixed-route bus systems. Häll et al. (2008) propose a simulation-based analysis for integrated bus operations combining a fixed-route service and demand-responsive service. They find that for the integrated bus operations, the number of transfers as well as the pricing policy for demand-responsive service strongly affect the performance of such integrated transit services.

Chang and Schonfeld (1993) develop a model for jointly optimizing the dimensions for served zones and headways, but only for fixed-route services whose characteristics and resulting model formulations differ considerably from those for flexible-route services. They assume that zone shapes are rectangular and find that zone lengths, zone widths and headways should increase with distance from a major terminal. Some related studies such as Chang and Schonfeld (1991), Kim and Schonfeld (2013,2014,2015), assume that each served zone is rectangular, while larger service regions are divided into multiple zones, based on the demand levels. It should be noted that the zone size is a very important factor (and not only for rectangular zones) in planning efficient flexible route services. Broome et al. (2012) completed a study showing the public's positive perception of flexible bus route systems. Designed for elderly persons who cannot travel to and from fixed bus stops, these flexible bus systems saw ridership double over the course of the study. Satisfaction was also significantly increased and the flexible bus system began to attract younger users.

From the literature review we note the importance of flexible-route bus operation for serving low demand regions or handicapped passengers. However, the main interests in those studies are new solution algorithms or heuristics for dial-a-ride or demand responsive transportation services. It is clear that the relations among optimal zone sizes, headways and relevant exogenous factors have not been sufficiently explored for flexible-route services. This paper contributes to the literature mathematical relations for optimally matching headways and service zone sizes to exogenous local characteristics such as demand density, distance from major terminals, unit costs and travel speeds, in order to minimize average costs per passenger, including the supplier and user costs. The model presented in this paper can help guide the design of relatively simple flexible-route bus services, especially where the demand density is relatively low or where street geometry cannot accommodate the relatively large buses used for fixed-route bus services.

Flexible-route service formulation

Each flexible-route module considered in this study includes one bus route that connects a local service zone to a major terminal, through an express segment, as shown in Fig. 1. The route's headway and the area of the local service zone are jointly optimized as functions of demand density, distance from the major terminal, applicable unit costs, bus speeds, and other relevant exogenous parameters. The major terminal may be a Central Business District (CBD), another major trip attractor, or a transfer terminal along a rail transit route. The route within each module primarily serves a Many-to-One (M-to-1) or One–to-Many (1-to-M) demand pattern. However, with multiple modules optimized for urban regions around the major terminal(s), their converging routes would enable Many-to-Many (M-to-M) demand patterns to be served, as briefly discussed below. This paper focuses on the optimization of single modules while a future study is expected to optimize the development of multi-module flexible route systems that cover substantial urban and suburban areas.

For each module, trip origins and destinations are assumed to be randomly and uniformly distributed over the local zone and over time. An example of this type of bus system could be a suburban neighborhood outside a large city. Passengers residing in that neighborhood or "service zone" take the bus to or from a train station, a downtown terminal or an airport, as shown in Fig. 1. Neighborhoods may be divided into smaller subsections, depending on the results of the joint optimization of the decision variables that are analyzed here. Fig. 1 shows a zone with a rectangular shape, but the analysis is also applicable to other zone shapes as long as the zones are fairly compact and fairly convex.

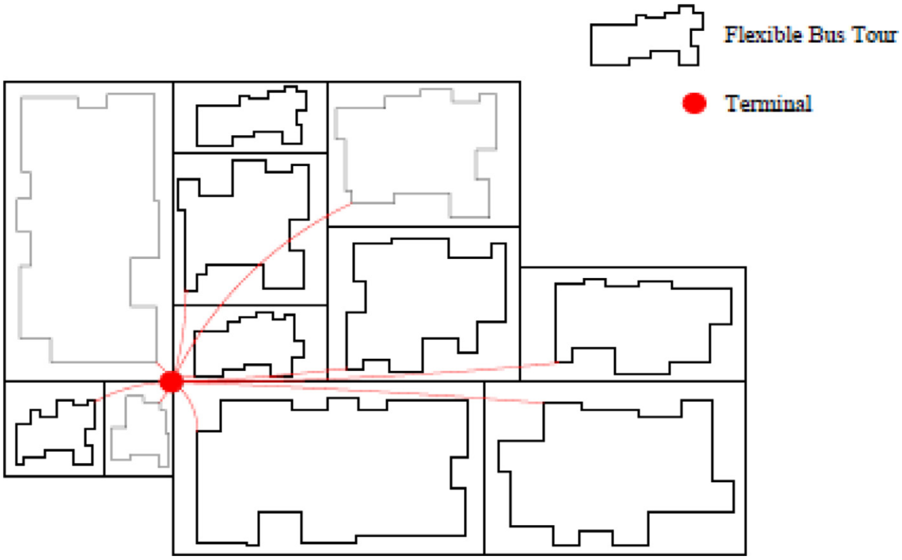

Fig. 2. Potential extension to multi-zone flexible-route system.

Table 1
Notation and baseline values for inputs.

Symbol	Variable	Units	Base value	Range for sensitivity analysis
A	Zone size (Area)	Square miles	–	–
a	Parameter for bus operating cost	\$/bus hour	30	–
b	Parameter for bus operating cost	\$/seat hour	0.3	–
C_A	Average total cost	\$/passenger	–	–
C_T	Total cost	\$/hour	–	–
C_S	Supplier cost	\$/hour	–	–
C_V	In-vehicle cost	\$/hour	–	–
C_W	Waiting cost	\$ per hour	–	–
c	Unit bus operating cost ($=a + +b \times S$)	\$/bus hour	–	–
D_C	Tour length within zone	Miles	–	–
h	Headway	Hours	–	–
J	Line haul distance	Miles	10	6–15
l	Load FACTOR	Dimensionless	1.0	–
N	Fleet size	Buses	–	–
$\emptyset$	Stein's constant	Dimensionless	1.15	–
Q	Demand density	Trips/square mile/hour	10	5–50
q	Hourly demand	Trips/hour	–	–
R	Round trip time	Hours	–	–
S	Bus capacity	Seats/bus	45	10–50
u	Number of passengers per stop	Number of passengers	1	–
V_X	Line haul speed	Miles/hour	30	15–60
V_L	Average local speed	Miles/hour	–	10–40
v_v	Value of in-vehicle time	\$/passenger hour	12	6–20
v_w	Value of waiting time	\$/passenger hour	15	6–20
w	waiting time	Hours	–	–
y	Ratio of local speed to express speed	Dimensionless	0.9	0.5–1.0

We noted that the flexible-route service model presented in this paper and illustrated in Fig. 1 can constitute a modular building block in designing larger and more comprehensive systems, which serve M-to-M demand patterns. As an example, the region in Fig. 2 can be subdivided into multiple zones, roughly in accordance to the guidelines developed in this paper on jointly optimizing zone areas and headways as functions of demand densities and distances from the central terminal (or central business district). Then the M-to-M demand patterns can be served with transfers at the central terminal, possibly with coordinated headways to minimize transfer delays, while minimizing system-wide costs.

The formulated total cost for flexible bus services is the sum of supplier (operator) cost, user in-vehicle cost, and user waiting cost. Since flexible-route services are assumed to provide door-to-door service, they have no access time or cost. The relevant notation is defined in Table 1. In order to provide a relatively simple and widely applicable model for planning and preliminary system design, the following simplifying assumptions are made:

1. Stein (1978)'s formula is assumed to provide an acceptable approximation of the shortest tour distance within a zone, with a constant $k = 1.15$ according to Daganzo (1984) for rectilinear movements within the zone.
2. The service zone is fairly compact and fairly convex.
3. Destinations and origins are fairly uniformly distributed over time and space within the service zone.
4. The number of stopping points in a zone for each vehicle tour exceeds five.
5. Dwell times and stopping times within the service zone are included in the average speed.
6. The average wait time is approximately half of the service headway.
7. Passenger pickups and drop-offs are intermingled within each tour.

The operator's service cost C_S is formulated as the product of the required fleet size N and the unit operating cost c:

$$C_S = N \times c \tag{1}$$

The required fleet size is the vehicle round trip time divided by headway:

$$N = \frac{R}{h} \tag{2}$$

The round trip time R is the sum of 2-directional trip time for the line haul (i.e., express) segment and trip time in the local zone. Stein's approximation for tour length within the zone D_c is expressed as:

$$D_c = \emptyset \sqrt{nA} \tag{3}$$

The approximation in Eq. (3) is very useful in this study for optimizing the headway and zone size for flexible bus routes.

The number of stops during the tour n is expressed as a function of demand density Q, zone size, A, headway h, and the number of passengers (boarding or alighting) per stop u:

$$n = \frac{QAh}{u} \tag{4}$$

As the actual hourly demand per zone q (in passenger trips/hour) is product of the demand density Q and the zone size A, the demand q is expressed as:

$$q = QA \tag{5}$$

Then, the tour length D_c is:

$$D_c = \emptyset \sqrt{\frac{qAh}{u}} \tag{6}$$

The line haul distance J, the express speed V_X, and local speed V_L are used to compute the vehicle round trip time R:

$$R = \frac{2J}{V_X} + \frac{D_c}{V_L} \tag{7}$$

We assume the local speed is a fraction of the express speed V_X. We denote the ratio of local speed V_L, to express speed V_X, as y. Then:

$$V_L = y\, V_X \tag{8}$$

The service cost, C_S is then formulated as:

$$C_S = \frac{2Jc}{hV_X} + \frac{\emptyset c}{y\, V_X}\sqrt{\frac{qA}{uh}} \tag{9}$$

The in-vehicle cost C_V is the product of the value of passenger's in-vehicle time v_v, the demand q, and passenger's trip time, which is assumed to be half of the vehicle round trip time R:

$$C_V = v_v q\, \frac{R}{2} \tag{10}$$

Eq. (10) can be re-written as:

$$C_V = \frac{v_v qJ}{V_x} + \frac{\emptyset v_v}{2y\, V_x}\sqrt{\frac{q^3 hA}{u}} \tag{11}$$

Since we are formulating a planning-level model, we approximate the average wait time as half of the headway h. This assumption is widely applied in urban transit services, in which the headways are relatively short. It should also be noted that for flexible-route bus services much of the waiting may be inside the users' homes or workplaces rather than at remote bus stops. The resulting waiting cost for passenger C_W is product of the value of waiting time v_w, the demand q, and the average waiting time $h/2$:

$$C_W = v_w q \frac{h}{2} \tag{12}$$

The total cost for the flexible service is the sum of operating cost, in-vehicle cost and waiting cost:

$$C_T = C_S + C_V + C_W \tag{13}$$

Eq. (13) is detailed as:

$$C_T = \frac{2Jc}{hV_X} + \frac{\emptyset c}{y\ V_X}\sqrt{\frac{qA}{uh}} + \frac{v_v qJ}{V_x} + \frac{\emptyset v_v}{2y\ V_X}\sqrt{\frac{q^3 hA}{u}} + v_w q\frac{h}{2} \tag{14}$$

The average cost per passenger C_A can be found by dividing the total cost function in Eq. (14) by the passenger flow q:

$$C_A = \frac{C_T}{QA} = \frac{C_T}{q} \tag{15}$$

Thus, the average cost for the service is:

$$C_A = \frac{2Jc}{hV_X}\frac{1}{QA} + \frac{\emptyset c}{y\ V_X QA}\sqrt{\frac{QA\cdot A}{uh}} + \frac{1}{QA}\frac{v_v QA\cdot J}{V_x} + \frac{\emptyset v_v}{2yV_X}\frac{1}{QA}\sqrt{\frac{(QA)^3 hA}{u}} + \frac{v_w QAh}{2QA} \tag{16}$$

Eq. (16) is rewritten as:

$$C_A = \frac{2Jc}{V_X QAh} + \frac{\emptyset c}{y\ V_X}\sqrt{\frac{1}{Qhu}} + \frac{v_v J}{V_X} + \frac{\emptyset v_v A}{2yV_X}\sqrt{\frac{Qh}{u}} + \frac{v_w h}{2} \tag{17}$$

In Eq. (17), the average cost per passenger is a function of two decision variables, namely the zone size A and the headway h.

The first order derivative of the average cost with respect to the zone size A, shown in Eq. (18), is set equal to zero:

$$\frac{\partial C_A}{\partial A} = -\frac{2Jc}{V_X QhA^2} + \frac{\emptyset v_v\sqrt{Qh}}{2yV_X\sqrt{u}} = 0 \tag{18}$$

The necessary condition for the global optimality of the zone size A is that Eq. (19) should have a positive value. Since Eq. (19) is always positive, we confirm that the optimal value of the zone size A yields the global minimum in Eq. (17).

$$\frac{\partial^2 C_A}{\partial {C_A}^2} = \frac{4Jc}{V_X QhA^3} > 0 \tag{19}$$

The first order derivative of the average cost with respect to the headway h is:

$$\frac{\partial C_A}{\partial h} = -\frac{2Jc}{V_X QAh^2} - \frac{\emptyset c}{2yV_X\sqrt{Quh^3}} + \frac{\emptyset v_v A\sqrt{Q}}{4yV_X\sqrt{uh}} + \frac{v_w}{2} = 0 \tag{20}$$

Similarly, the second-order derivative for the headway, which should be positive, is shown in Eq. (21).

$$\frac{\partial^2 C_A}{\partial h^2} = \frac{4Jc}{V_X QAh^3} + \frac{3\emptyset c}{4yV_X\sqrt{Quh^5}} - \frac{\emptyset v_v A\sqrt{Q}}{8yV_X\sqrt{uh^3}} \tag{21}$$

We seek the solution by solving Eq. (18) and Eq. (20) simultaneously. From Eq. (18), we obtain the following relation:

$$\frac{1}{A^2\sqrt{h^3}} = \frac{\emptyset v_v\sqrt{Q^3}}{4yJc\sqrt{u}} \tag{22}$$

By denoting t to substitute for the right hand side of Eq. (22), we find a simplified relation between the headway and zone size, as shown in Eq. (23):

$$A = \frac{1}{\sqrt{t}\sqrt[4]{h^3}} \tag{23}$$

Eq. (23) is inserted in Eq. (20), and Eq. (20) is re-written as:

$$\frac{\partial C_A}{\partial h} = -\frac{2Jc\sqrt{t}}{V_X Q\sqrt[4]{h^5}} - \frac{\emptyset c}{2yV_X\sqrt{Qu}\sqrt[4]{h^6}} + \frac{\emptyset v_v A\sqrt{Q}}{4yV_X\sqrt{ut}\sqrt[4]{h^5}} + \frac{v_w}{2} = 0 \tag{24}$$

By using X to substitute for $\sqrt[4]{h}$, Eq. (25) is obtained:

$$\frac{v_w}{2}X^6 + \left\{\frac{\emptyset v_v\sqrt{Q}}{4yV_X\sqrt{ut}} - \frac{2Jc\sqrt{t}}{V_X Q}\right\}X - \frac{\emptyset c}{2yV_X\sqrt{Qu}} = 0 \tag{25}$$

Eq. (25) is re-arranged, and denoted as Y:

$$Y = \frac{v_w}{2}X^6 - \left\{ \frac{\emptyset v_v \sqrt{Q}}{4yV_X\sqrt{ut}} \right\} X - \frac{\emptyset c}{2yV_X\sqrt{Qu}} \tag{26}$$

To investigate the convexity of Eq. (26), we find the partial derivative of Y with respect to X as follows:

$$\frac{\partial Y}{\partial X} = 3v_w X^5 - \left\{ \frac{\emptyset v_v \sqrt{Q}}{4yV_X\sqrt{ut}} \right\} \tag{27}$$

By setting Eq. (27) to zero, we find only one root, as shown in Eq. (28):

$$X = \left\{ \frac{\emptyset v_v \sqrt{Q}}{12 v_w y V_X \sqrt{ut}} \right\}^{1/5} \tag{28}$$

Since we find only one root, shown in Eq. (28), for which Eq. (27) equals zero, and since the coefficient of the X^6 term in Eq. (26), which is $v_w/2$, is positive, we confirm that the function in Eq. (26) is convex. We apply Newton's Method to find the optimal headway as $h^* = X^4$. The pseudo-algorithm is shown as follows (Press et. al., 2007):

Step 1: Pick the initial value ($x1$) of X, which must be greater than zero.

Step 2: Evaluate Eq. (26), $f(x1)$ at this point ($x1$)

Step 3: Assuming $f(x1)$ was not the root of the equation, compute the following equation to determine the next evaluation point, $x2$.

$$x2 = x1 - \frac{f(x1)}{f'(x1)}$$

Step 4: Evaluate Eq. (26) at $x2$, and obtain $f(x2)$.

Step 5: Repeat this iterative process until the root is determined or the tolerance criterion is satisfied.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In Eq. (19), the optimal zone size A^*, yields the global minimum of the objective function. We must also ensure that the optimal headway h* yields the globally optimal solution by investigating the previously derived Eq. (21).

To ensure the optimal headway yields the global minimum, Eq. (21) should be positive. To achieve the closed-form equation of headway boundary that yields the global solution, we note that either of the following two conditions should be positive:

$$\frac{4Jc}{V_X Q A h^3} - \frac{\emptyset v_v A \sqrt{Q}}{8yV_X\sqrt{uh^3}} > 0 \tag{29}$$

or

$$\frac{3\emptyset c}{4yV_X\sqrt{Quh^5}} - \frac{\emptyset v_v A \sqrt{Q}}{8yV_X\sqrt{uh^3}} > 0 \tag{30}$$

Eq. (29) is rearranged in Eq. (31), and Eq. (30) is rearranged in Eq. (32):

$$h < \sqrt[3]{\frac{1}{A^4}\left\{ \frac{32yJc\sqrt{u}}{\emptyset v_v \sqrt{Q^3}} \right\}^2} \tag{31}$$

or

$$h < \frac{6c}{v_v Q A} \tag{32}$$

Since both Eqs. (31) and (32) provide closed-form conditions, they can provide insights for decision makers regarding headway planning. The visual representation of the global solution boundary will be discussed in the numerical analysis section.

When the maximum allowable headway policy is considered ($h = \frac{Sl}{QA}$), the complexity of the problem is reduced to analytically optimizing a single variable, which is the zone size A. The optimal zone size A^* is found in closed-form, as shown in Eq. (33):

$$A^* = \sqrt[3]{\left(\frac{v_w Sl}{2Q} \Big/ \left\{ \frac{\emptyset c}{2yV_X\sqrt{uSl}} + \frac{\emptyset v_v \sqrt{Sl}}{4yV_X\sqrt{u}} \right\} \right)^2} \tag{33}$$

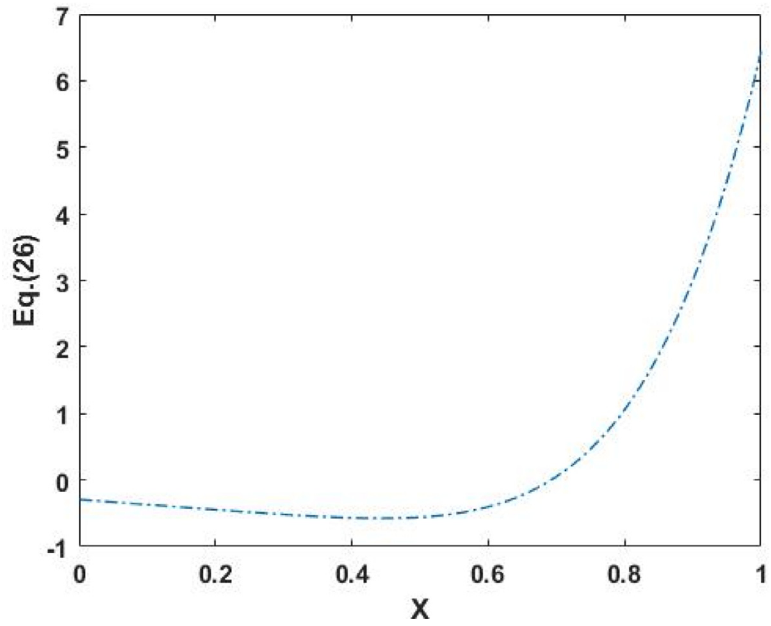

Fig. 3. Convexity of Eq. (26).

The global optimality condition of the zone size A is presented in Eq. (34):

$$A < \sqrt[3]{\{v_w Sl\}^2 / Q^2 \left\{ \frac{\o c}{4yV_X\sqrt{uSl}} + \frac{\o v_v\sqrt{Sl}}{8yV_X\sqrt{u}} \right\}^2} \tag{34}$$

The details of the derivations of A, and the necessary condition for its global optimality are summarized in Appendix A.

Numerical analysis

Numerical cases are used to explore the relations between the decision variables, which are the headway h and the zone size A, for the problem illustrated in Fig. 1. This section explores how various input parameters affect the design of flexible-route bus services. We present a baseline case and an elasticity analysis for the sensitivity of solutions to service design parameters. The baseline values of the parameter for the formulations and numerical cases are provided in Table 1.

Visual representation of convexity in the objective function

We seek to verify the global optimality of the solutions. By solving Eq. (27) using the baseline values from Table 1, the minimum of X is obtained as 0.4426. As shown in Fig. 3, the value of Eq. (26) decreases until X is 0.4426, and then Eq. (26) increases beyond X values of 0.4426. Therefore, we graphically confirm that Eq. (26) is a convex function with only one root (i.e., 0.692) as discussed for Eq. (28). In Fig. 3, we also note that the root of X, at which Eq. (26) equals zero, is 0.692. Thus, the optimal headway h^* (=0.229 h) is found from X^4 (=0.692^4). From the optimal headway* (i.e., 0.229 h), we find the optimal zone size A^* using Eq. (23), as 5.72 sq. miles.

Either Eq. (31) or Eq. (32) must be satisfied to guarantee the global optimality of the solution. As shown in Fig. 4, we note that the headway condition specified in Eq. (32) always satisfies the condition specified in Eq. (31). Thus, we take Eq. (31) as the criterion which guarantees the globally optimal solution for the headway. With baseline values, the optimal headway h^* is 0.229 h, and we obtain the upper limit (i.e., condition) of the headway from Eq. (31) as 0.917 h. Therefore, we find the global solution.

Joint optimization vs. maximum allowable headway solutions

For the baseline case, the optimal zone size A^* is found as 5.72 sq. miles and the optimal headway h^* is found as 0.23 h. As shown in Table 2, the average cost per person trip includes an operating cost of 3.44 \$/person trip, an in-vehicle time of 6.21 \$/person trip, and a waiting time cost of 1.72 \$/person trip, resulting a total cost for flexible service of 11.37 \$/person trip. The operating cost, in-vehicle cost and waiting cost components of the average system cost are about 30 %, 55% and 15%, respectively. Fig. 3 shows that the cost function is convex, and its optimal zone size and headway are the global solution.

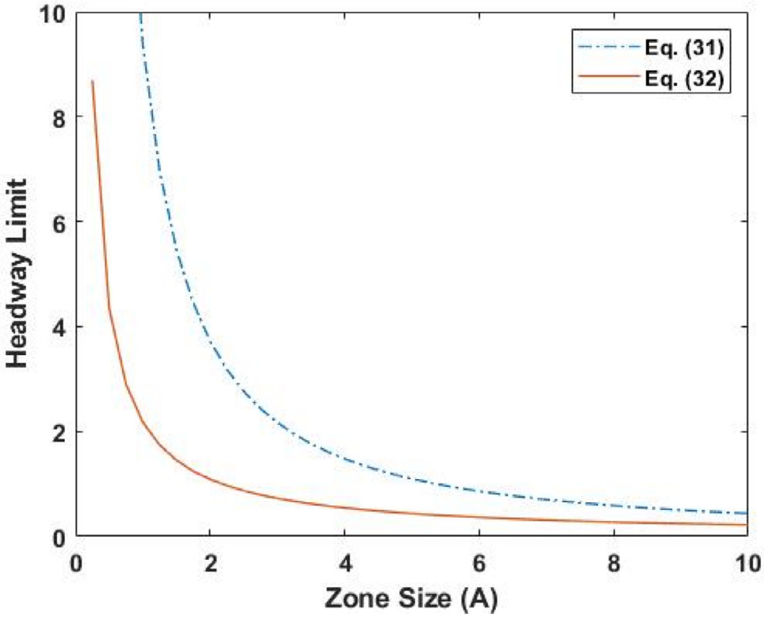

Fig. 4. Headway limits constrained by Eqs. (31) and (32).

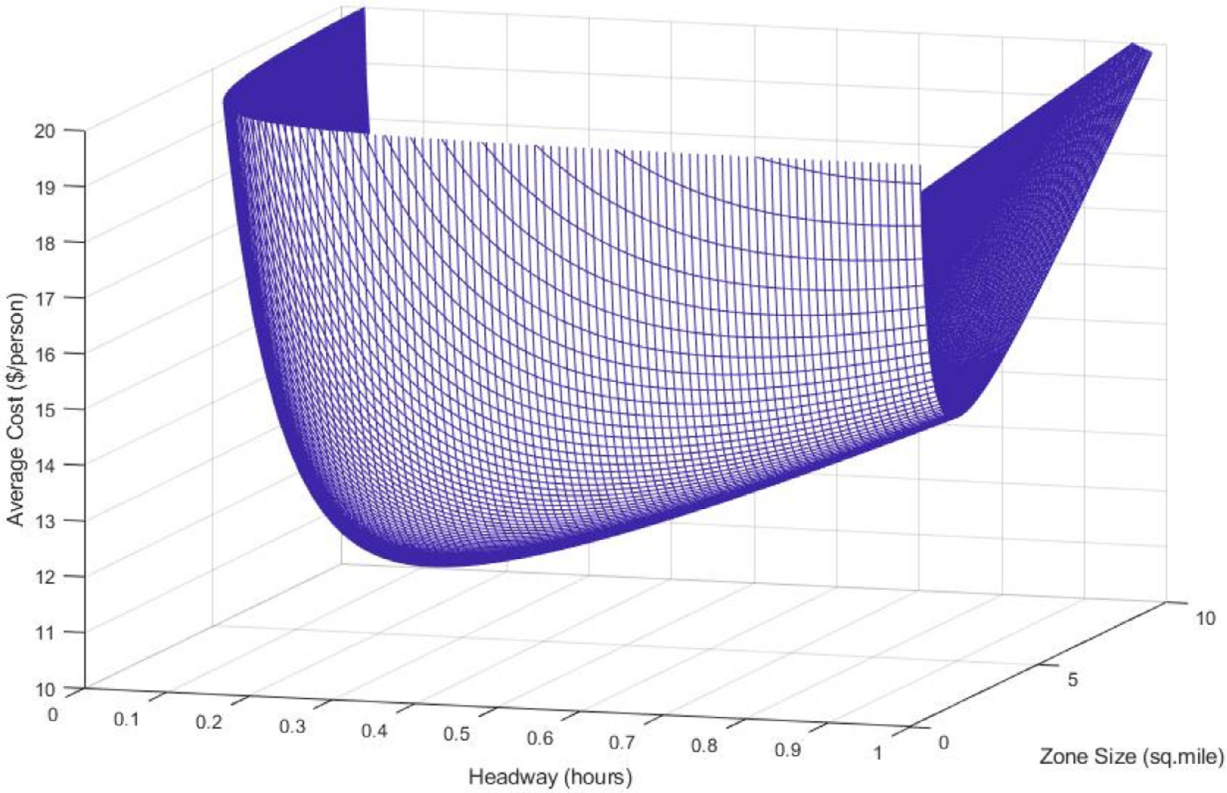

Fig. 5. Fig. 5Average cost plot for baseline case.

Table 2
Cost comparison between the maximum allowable headway and joint optimal solution for zone size and headway.

	Zone size, A (sq. miles)	Headway, h (h)	Operator cost ($/person)	In-vehicle cost ($/person)	Waiting cost ($/person)	Average cost ($/person)
Maximum allowable headway solution (1)	10.48	0.43	1.54	9.55	3.22	14.31
Joint optimal solution (2)	5.72	0.23	3.44	6.21	1.72	11.37
Cost difference, {(1) – (2)}/(2)	83.2%	87.8%	−55.2%	53.7%	87.5%	25.9%

Table 3
Joint optimal solution for demand variations.

Demand density (persons/sq. mile)		5	10	15	20	25	30	35	40	45	50
Zone size (sq. miles)	Joint optimal solution	8.42	5.72	4.56	3.88	3.42	3.08	2.83	2.62	2.45	2.31
	% Change	47.3	0.0	−20.3	−32.2	−40.3	−46.1	−50.6	−54.2	−57.1	−59.6
Headway (hours)	Joint optimal solution	0.27	0.23	0.21	0.19	0.18	0.17	0.17	0.16	0.16	0.15
	% Change	19.4	0.0%	−9.7	−16.0	−20.5	−24.0	−26.8	−29.2	−31.2	−33.0
Average operating cost ($/h)	Joint optimal solution	4.10	3.44	3.10	2.89	2.73	2.61	2.51	2.43	2.36	2.30
	% Change	19.4	0.0	−9.7	−16.0	−20.5	−24.0	−26.8	−29.2	−31.2	−33.0
Average in-vehicle cost ($/h)	Joint optimal solution	6.52	6.21	6.05	5.94	5.86	5.80	5.75	5.71	5.67	5.64
	% Change	4.9	0.0	−2.6	−4.3	−5.6	−6.6	−7.5	−8.2	−8.8	−9.3
Average waiting cost ($/h)	Joint optimal solution	2.05	1.72	1.55	1.44	1.37	1.31	1.26	1.22	1.18	1.15
	% Change	19.4	0.0	−9.7	−16.0	−20.5	−24.0	−26.8	−29.2	−31.2	−33.0
Total cost (Average cost in $ per person)	Joint optimal solution	12.67	11.37	10.71	10.27	9.96	9.72	9.52	9.36	9.21	9.09
	% Change	11.5	0.0	−5.8	−9.6	−12.4	−14.5	−16.3	−17.7	−18.9	−20.0

When the maximum headway policy is applied, a closed-form solution is achievable, which offers insights regarding the relations among decision variables and input parameters. With the closed-form solution shown in Eq. (33), the optimal solution for the zone size A^* is analytically found as 10.48 sq. miles. Using Eq. (34), the necessary condition (i.e., the maximum zone size A) is obtained as 26.4 sq. miles, and the optimal zone size (10.48 sq. miles) satisfies the condition. Thus, we confirm that the solution with the maximum allowable headway policy finds the global minimum solution. The maximum allowable headway h^* is obtained as 0.43 h, using the relation $h = Sl/QA$.

Table 2 provides comparisons between the two solution approaches, namely the joint optimal solution and maximum allowable headway solution. The optimal zone size A^* based on the maximum allowable headway is 83% larger than that for the joint optimal solution. The maximum allowable headway h^* is almost double that of the joint optimal solution. For this base case, the maximum allowable headway can reduce the operator cost compared to the joint optimal solution while the in-vehicle and waiting costs exceed those in the joint optimal solution. The average cost per person trip is 26% higher for the maximum allowable headway than for the joint optimal solution.

Sensitivity analysis

Demand density

Table 3 shows the sensitivity of costs to the demand density Q. In it the in-vehicle cost is the highest among the cost components and is approximately double the operating cost, while the waiting cost is the lowest cost component. Table 3 shows that as Q increases, the optimal zone size and headway decrease. The resulting A^* decreases from 8.42 to 2.31 sq. miles, and it shows greater variations (between 47.3% increase and −59.6% decrease) than the headway variations (between 19% increase and −33% decrease) over the range of demand changes. For instance, when the hourly demand density increases from 5 to 10 person trips/sq. miles, A^* decreases from 8.42 to 5.72 sq. miles, while the demand density increase from 45 to 50 person trips/sq. miles reduces the zone size from 2.45 to 2.31 sq. miles. When the demand density increases from 10 to 50 persons/sq. mile, the average cost per person decreases by 20% from 11.37 to 9.09 $/person.

We also explore the effects of demand densities on headways while the zone size is not optimizable, as presented in Table 4. We hold the zone size at $A = =5.72$ sq. miles, as a fixed input parameter, and explore the relations between the demand density and headway. When Q increases from 10 (in the baseline conditions) to 50, we find that h^* decreases from 0.23 h to 0.07 h, thus resulting in more frequent bus service, while the total cost per trip decreases by 14.7%. When Q increases from 10 to 50 while fixing the zone size, the in-vehicle cost increase by 8.8% and the operating cost decreases by 30%. As h^* decreases from 0.23 to 0.07 h, the waiting cost decreases significantly by 68.9%.

Table 4
Effects of demand density with zone size on hold.

Demand density (persons/sq. mile)		5	10	15	20	25	30	35	40	45	50
Zone size (sq. miles)	Zone size on hold (Input)	5.72	5.72	5.72	5.72	5.72	5.72	5.72	5.72	5.72	5.72
Headway (hours)	Zone size on hold (Input)	0.35	0.23	0.18	0.14	0.12	0.11	0.09	0.09	0.08	0.07
	% Change	52.9	0.0	−23.5	−37.4	−46.7	−53.5	−58.7	−62.8	−66.1	−68.9
Average operating cost	Zone size on hold (Input)	4.29	3.44	3.07	2.86	2.72	2.62	2.55	2.49	2.44	2.40
($/h)	% Change	25.0	0.0	−10.6	−16.8	−20.8	−23.7	−25.8	−27.5	−28.9	−30.0
Average in−vehicle cost	Zone size on hold (Input)	5.93	6.21	6.37	6.48	6.55	6.61	6.66	6.70	6.73	6.76
($/h)	% Change	−4.5	0.0	2.5	4.2	5.5	6.4	7.2	7.8	8.4	8.8
Average waiting cost	Zone size on hold (Input)	2.63	1.72	1.31	1.08	0.92	0.80	0.71	0.64	0.58	0.53
($/h)	% Change	52.9	0.0	−23.5	−37.4	−46.7	−53.5	−58.7	−62.8	−66.1	−68.9
Total cost (Average cost	Zone Size on Hold (Input)	12.86	11.37	10.76	10.41	10.19	10.03	9.92	9.83	9.76	9.70
in $ per person)	% Change	13.1	0.0	−5.4	−8.4	−10.4	−11.7	−12.7	−13.5	−14.2	−14.7

Table 5
Table 5Effects of vehicle capacity on costs.

Vehicle capacity (# of seats)	Optimal zone size (sq. miles)	Optimal headway (h)	Average operating cost ($/h)	Average in-vehicle cost ($/h)	Average waiting cost ($/h)	Total cost (Average cost in $ per person)
10	5.23	0.19	3.22	5.85	1.43	10.50
15	5.56	0.20	3.06	6.03	1.53	10.62
20	5.59	0.21	3.12	6.06	1.56	10.75
25	5.62	0.21	3.19	6.09	1.59	10.88
30	5.65	0.22	3.25	6.12	1.63	11.00
35	5.67	0.22	3.31	6.15	1.66	11.13
40	5.70	0.23	3.38	6.18	1.69	11.25
45	5.72	0.23	3.44	6.21	1.72	11.37
50	5.74	0.23	3.50	6.24	1.75	11.49
55	5.77	0.24	3.55	6.27	1.78	11.60

Table 6
Effects of vehicle capacity on optimal zone size and headway.

Vehicle capacity (# of seats)	Optimal zone size (sq. mile)	% Increase of optimal zone size per unit of vehicle capacity (+1 seat/veh)	% Increase of optimal zone size from baseline result	Optimal headway (h)	% Increase of optimal zone size per unit of vehicle capacity (+1 seat/veh)	% Change of optimal headway from baseline result
10	5.23	–	−8.50	0.191	–	−19.92
15	5.56	6.52	−2.79	0.204	0.26	−12.38
20	5.59	0.59	−2.28	0.208	0.09	−10.01
25	5.62	0.56	−1.79	0.213	0.09	−7.78
30	5.65	0.54	−1.31	0.217	0.08	−5.67
35	5.67	0.52	−0.86	0.221	0.08	−3.68
40	5.70	0.50	−0.42	0.225	0.08	−1.79
45	5.72	0.48	0.00	0.229	0.08	0.00
50	5.74	0.47	0.41	0.233	0.08	1.70
55	5.77	0.45	0.80	0.237	0.08	3.33

Vehicle capacity

The vehicle size S is a critical design parameter for planning and scheduling public transportation operations. When S increases from 10 to 15 seats/bus, the optimal zone size A^* increases by 6.3% while the optimal headway h^* increases by 5%. As expected, when the vehicle capacity increases, the optimal zone size A^* and optimal headway h^* both increase to cover a larger area for each bus tour while decreasing service frequency.

Figs. 6 and 7 each show a sharp turning point for optimal zone size and headway. These changes are based on the vehicle capacity constraints (i.e., $h \leq \frac{Sl}{QA}$). When the vehicle capacity is small (e.g., between 10 and 15 seats/bus), the vehicle capacity constraint is bounded so that the optimal zone size increase sharply. When the capacity constraint is not binding (e.g., vehicle capacity larger than 15 seats/bus), the optimal zone size A* increases less rapidly.

Fig. 7 shows how the optimal headway h^* varies with given vehicle capacity S. We note that the headway increases rapidly with vehicle capacities between 5 and 15 seats/bus. When vehicles have sufficient capacity (i.e., above 15 seats/bus) to satisfy the demand, the increase in optimal headway ranges between 0.08% and 0.09% as vehicle capacity increases by one seat. Table 6 summarizes the resulting effects of vehicle capacity.

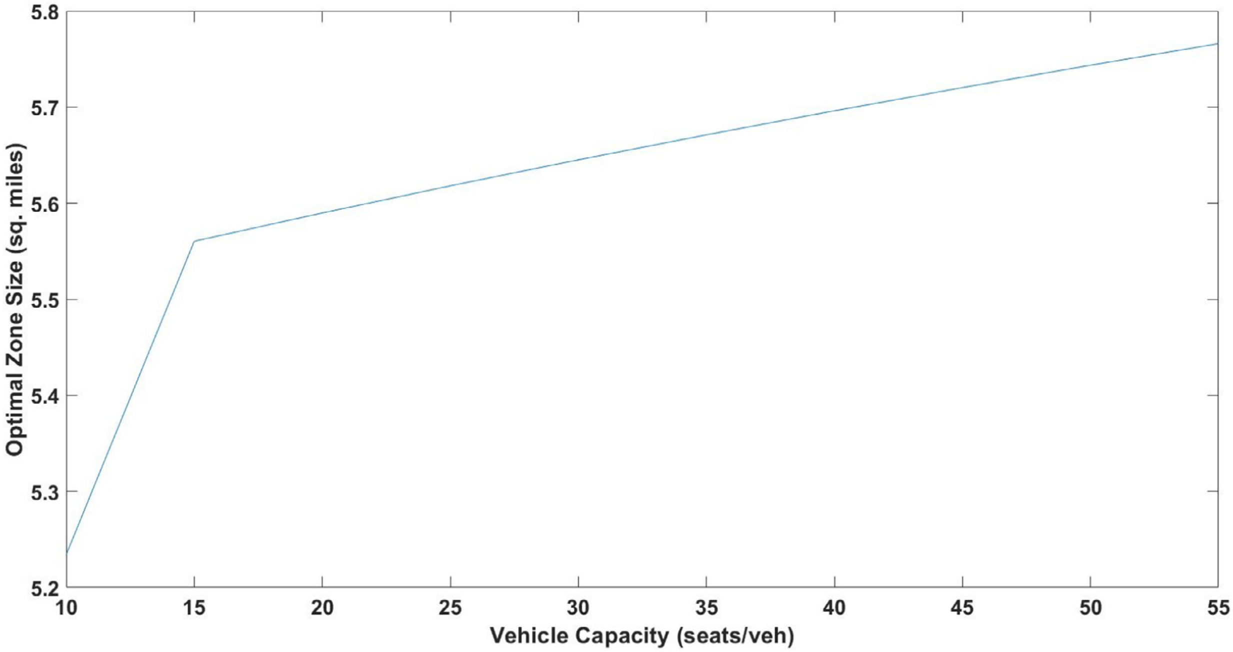

Fig. 6. Optimal zone size versus vehicle capacity.

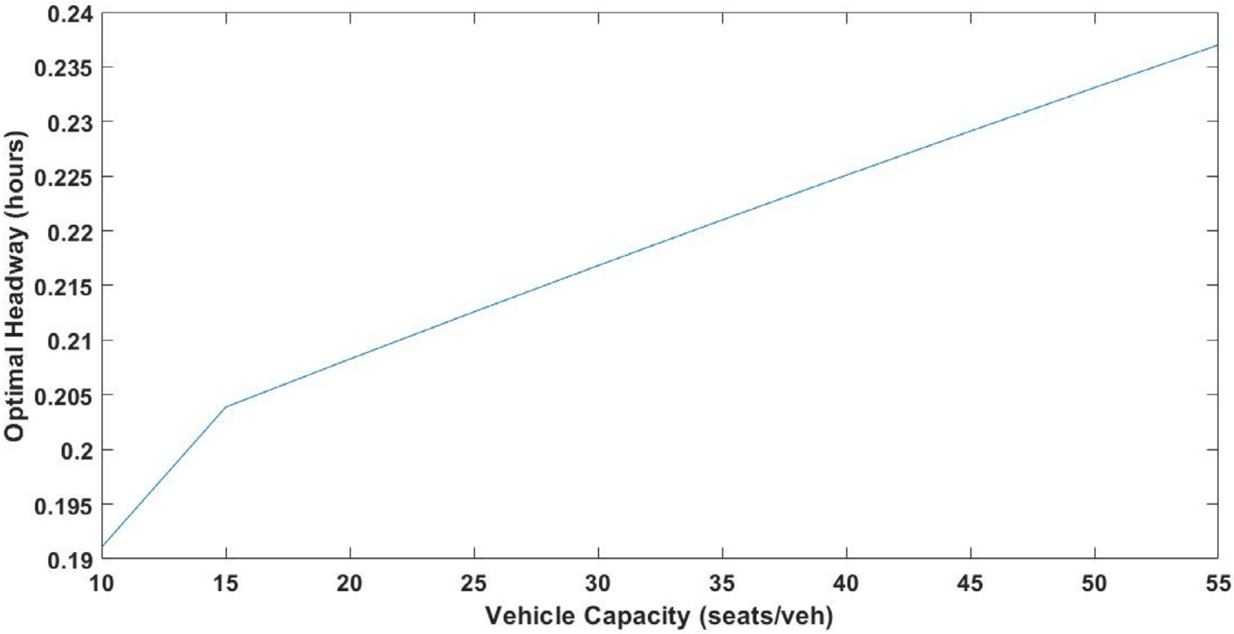

Fig. 7. Optimal headway versus vehicle capacity.

Elasticity analysis of input parameters

Table 7 shows the elasticity of the optimized zone size A^* and headway h^* with respect to 10% increases in various design parameters. For example, when demand density Q increases by 10%, A^* decreases by 5.20%, implying a -0.52 elasticity of A^* to Q. Similarly, a 10% increase in Q, reduces h^* by 2.31%, implying the elasticity of h^* to Q is -0.23. In this case, the average cost per person-trip decreases from 11.368 to 11.260 \$.

The sensitivities of A^* and h^* to other input parameters are also shown in Table 7. Increases in the parameters a and b for the unit operating cost both increase A^* and h^*. The elasticities of A^* and h^* to vehicle capacity are 0.04 and 0.16,

Table 7
Elasticities of optimal zone size and headway with respect to various design parameters.

Baseline input values		Q 10.0	a 30.0	B 0.30	S 45.0	J 10.0	v_v 12.0	v_w 15.0	Vx 30.0	y 0.90
Baseline case	A^*					5.72				
	h^*					0.23				
	TC*					11.37				
Elasticity results	+10% change of x	11.00	33.00	0.33	49.50	11.00	13.20	16.50	33.00	0.99
	A^*	5.42	5.77	5.74	5.74	5.90	5.36	6.03	6.03	6.22
	elasticity of A^* w.r.t. +10% change of x	−0.52	0.08	0.04	0.04	0.31	−0.63	0.55	0.55	0.87
	h^*	0.22	0.24	0.23	0.23	0.23	0.23	0.21	0.21	0.22
	Elasticity of h^* w.r.t. +10% change of x	−0.23	0.35	0.16	0.16	0.24	0.24	−0.68	−0.68	−0.47
	TC*	11.26	11.60	11.47	11.47	11.98	11.98	11.53	10.49	11.05

respectively. An increase in the line haul distance increases the round trip time. Thus, the supplier cost and in-vehicle cost are increased, which increase A^* and h^*.

Table 7 shows that, with respect to the value of the in-vehicle time, the elasticity of A^* is negative while the elasticity of h^* is positive. As expected, the optimal vehicle round trip time decreases when the passengers' value of in-vehicle time increases. With respect to the value of waiting time v_w and the vehicle operating speed V_X, the elasticities of A^* are positive and the elasticities of h^* are negative.

Conclusions

In this paper a model for optimizing the headway and zone size is formulated for a flexible-bus service. It minimizes the average cost per passenger trip, which is the sum of the operator cost, in-vehicle cost, and waiting cost, while considering a vehicle capacity constraint. An analytic relation is obtained between optimal headway and optimal zone size, and the minimum cost is found by solving a sixth order polynomial function using Newton's method. We also analyze the case in which the zone size is unchangeable as a policy constraint. For it, we treat the headway as the only decision variable, and compare the results with the joint optimal solutions. We also consider the flexible-route bus service based on the maximum allowable headway policy. Since its headway is determined based on the zone size A, the average cost formulation, as shown in Eq. (A4), is reduced to a problem with one decision variable, namely A. The maximum allowable headway approach provides a closed-form solution, and thus useful insights into the relations among decision variables and input parameters.

The optimized relations presented here for zone sizes and headways may be used to design multi-zone systems that serve many-to-many demand patterns (as sketched in Fig. 2), as well as simpler single-zone systems (as sketched in Fig. 1). For the baseline case analysis, the cost percentages for vehicle operation, in-vehicle time and waiting time are 30%, 55% and 16%, respectively. When demand density increases from 10 to 50 persons/sq. mile, we find that the share of in-vehicle cost increases to 62%. The numerical analyses confirm that, in general, the cost of in-vehicle time exceeds operating and waiting cost components. Sensitivity analyses also explore how changes in design parameters affect the operating decisions as well as costs. Thus, the obtained relations and results from numerical analysis can be used as planning guidelines in designing flexible-bus route systems. A useful extension of this work would explore how flexible-route modules which are separately optimized in this paper can be integrated to serve larger regions with many-to-many demand patterns. Such analysis might consider additional transfer stations away from the central one (e.g. at some zone boundaries) and possible coordination of headways for different modules to reduce passenger wait times at transfer stations.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.trb.2019.10.006.

Appendix A. Optimal solution with maximum allowable headway

The average cost in Eq. (17) is presented in Eq. (A1):

$$C_A = \frac{2Jc}{V_X QAh} + \frac{\varnothing c}{y\,V_X}\sqrt{\frac{1}{Qhu}} + \frac{v_v J}{V_X} + \frac{\varnothing v_v A}{2yV_X}\sqrt{\frac{Qh}{u}} + \frac{v_w h}{2} \tag{A1}$$

The maximum allowable headway $h_{\max}$ depends on the vehicle capacity S, the load factor l, the demand density Q, and the zone size A:

$$h_{max} = \frac{Sl}{QA} \tag{A2}$$

By substituting h in Eq. (A1) the maximum allowable headway $h_{\max}$ from Eq. (A2), we obtain the average cost C_A as

$$C_A = \frac{2JcQA}{V_X QASl} + \frac{\emptyset c}{y\ V_X}\sqrt{\frac{QA}{QuSl}} + \frac{v_v J}{V_X} + \frac{\emptyset v_v A}{2yV_X}\sqrt{\frac{QSl}{uQA}} + \frac{v_w Sl}{2QA} \tag{A3}$$

Eq. (A3) is simplified as

$$C_A = \frac{2Jc}{V_X Sl} + \frac{\emptyset c}{y\ V_X}\sqrt{\frac{A}{uSl}} + \frac{v_v J}{V_X} + \frac{\emptyset v_v}{2yV_X}\sqrt{\frac{ASl}{u}} + \frac{v_w Sl}{2QA} \tag{A4}$$

The optimal zone size A^* is found by taking the derivative of the average cost C_A, with respect to A, as follows:

$$\frac{\partial C_A}{\partial A} = \frac{\emptyset c}{2yV_X}\sqrt{\frac{1}{uSlA}} + \frac{\emptyset v_v}{4yV_X}\sqrt{\frac{Sl}{uA}} - \frac{v_w Sl}{2QA^2} = 0 \tag{A5}$$

By substituting t for $\sqrt{A}$:

$$\frac{\emptyset c}{2yV_X}\sqrt{\frac{1}{uSl}}\frac{1}{t} + \frac{\emptyset v_v}{4yV_X}\sqrt{\frac{Sl}{u}}\frac{1}{t} - \frac{v_w Sl}{2Qt^4} = 0 \tag{A6}$$

We multiply Eq. (A6) by t^4:

$$\frac{\emptyset c}{2yV_X}\sqrt{\frac{1}{uSl}}t^3 + \frac{\emptyset v_v}{4yV_X}\sqrt{\frac{Sl}{u}}t^3 - \frac{v_w Sl}{2Q} = 0 \tag{A7}$$

Eq. (A7) is rewritten as:

$$t^3\left\{\frac{\emptyset c}{2yV_X\sqrt{uSl}} + \frac{\emptyset v_v\sqrt{Sl}}{4yV_X\sqrt{u}}\right\} = \frac{v_w Sl}{2Q} \tag{A8}$$

From Eq. (A8), the value of t is found as

$$t = \sqrt[3]{\frac{v_w Sl}{2Q}\Big/\left\{\frac{\emptyset c}{2yV_X\sqrt{uSl}} + \frac{\emptyset v_v\sqrt{Sl}}{4yV_X\sqrt{u}}\right\}} \tag{A9}$$

The optimal value of the zone size A^* is then found as

$$A^* = \sqrt[3]{\left(\frac{v_w Sl}{2Q}\Big/\left\{\frac{\emptyset c}{2yV_X\sqrt{uSl}} + \frac{\emptyset v_v\sqrt{Sl}}{4yV_X\sqrt{u}}\right\}\right)^2} \tag{A10}$$

After the optimal value of the zone size A^* is obtained from Eq. (A10), the maximum allowable headway is obtained using Eq. (A2).

The global optimality of the solution for the zone area, A should be verified. The second derivative of the average cost C_A with respect to the zone size A is expressed as follows, and Eq. (A11) should be positive to guarantee the globally minimal solution.

$$\frac{\partial^2 C_A}{\partial A^2} = -\frac{\emptyset c}{4yV_X}\frac{1}{\sqrt{uSl}}\frac{1}{\sqrt{A^3}} - \frac{\emptyset v_v}{8yV_X}\frac{\sqrt{Sl}}{\sqrt{u}}\frac{1}{\sqrt{A^3}} + \frac{v_w Sl}{QA^3} > 0 \tag{A11}$$

As we substitute $\sqrt{A^3}$ with P, Eq. (A11) becomes:

$$-\frac{\emptyset c}{4yV_X}\frac{1}{\sqrt{uSl}}\frac{1}{P} - \frac{\emptyset v_v}{8yV_X}\frac{\sqrt{Sl}}{\sqrt{u}}\frac{1}{P} + \frac{v_w Sl}{QP^2} > 0 \tag{A12}$$

By multiplying P^2 in Eq. (A12),

$$P\left\{\frac{\emptyset c}{4yV_X\sqrt{uSl}} + \frac{\emptyset v_v\sqrt{Sl}}{8yV_X\sqrt{u}}\right\} < \frac{v_w Sl}{Q} \tag{A13}$$

As we substitute $\sqrt{A^3}$ for P in Eq. (A13):

$$\sqrt{A^3} < \left\{\frac{v_w Sl}{Q}\right\}\Big/\left\{\frac{\emptyset c}{4yV_X\sqrt{uSl}} + \frac{\emptyset v_v\sqrt{Sl}}{8yV_X\sqrt{u}}\right\} \tag{A14}$$

From Eq. (A14), we find the condition of the zone size A for global optimality:

$$A < \sqrt[3]{\{v_w Sl\}^2\Big/Q^2\left\{\frac{\emptyset c}{4yV_X\sqrt{uSl}} + \frac{\emptyset v_v\sqrt{Sl}}{8yV_X\sqrt{u}}\right\}^2} \tag{A15}$$

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