SIR Model

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The SIR Model

Consider an epidemic spreading in a population of P individuals. In the so-called SIR model, a discrete-time stochastic process $(S_t, I_t, R_t)_{t\in\mathbb{N}}$ keeps track of the numbers:

- S_t of susceptible individuals at the beginning of day t, i.e., the individuals who have not got the infection before day t;
- I_t of infected individuals at the beginning of day t;
- R_t of removed individuals at the beginning of day t, i.e., the individuals who have either recovered-and, consequently, have become immune to the disease—or died before day t, and therefore cannot become infected again.

We assume that the population is isolated, in the sense that at each day t, $P = S_t + I_t + R_t$.

The epidemic evolves as follows. The outbreak starts with an initial number I_0 of cases, yielding $R_0 = 0$ and $S_0 = P - I_0$. Then, during day t, an additional number Δ_t^I of individuals become infected, which means that the number S_t of susceptible ones decreases by the same amount. At the same time, an additional number Δ_t^R of infected individuals either recover or die, which means that the number R_t of removed individuals increases by the same amount. Consequently, the group of infected individuals at time t is subjected to an inflow of Δ_t^I individuals and an outflow of Δ_t^R individuals. This dynamics is summarised by the state equations:

$$\begin{cases} S_{t+1} = S_t - \Delta_t^I \\ I_{t+1} = I_t + \Delta_t^I - \Delta_t^R \\ R_{t+1} = R_t + \Delta_t^R \end{cases} \tag{1}$$

Distribution of Δ_t^R

Regarding Δ_t^R, we assume that each of the I_t infected individuals at time t either recovers or dies the until next day with a constant probability $p^{i\to r} \in (0,1)$ (since we assume that a recovered individual cannot become susceptible again, we do not need to make any distinction between the cases of recovery and death). We may hence assign Δ_t^R a binomial distribution with parameters I_t and $p^{i\to r}$, i.e.

$$\Delta_t^R \sim Bin(I_t, p^{i\to r}). \tag{2}$$

Distribution of Δ_t^I

Since the epidemic is transmitted between individuals, we may assume that Δ_t^I, the number of individuals that become infected during day t, could, conditionally on the states (S_t, I_t, R_t), be assumed to follow another binomial distribution with parameters S_t and $p^{s\to i} \in (0,1)$, where

$$p_t^{s\to i} = 1 - e^{-\lambda(t)\frac{I_t}{P}} \tag{3}$$

is the probability that a susceptible individual becomes infected during day t. Here $\lambda(t) > 0$ is a time-dependent parameter that reflects the average number of interactions per individual on day t. Still, there is a problem of using a binomial distribution in this modelling step, since for large populations, such as the inhabitants of an entire country, the model becomes close to deterministic, while real data usually exhibit significant noise. Thus, a better choice is to use a (generalised) negative binomial distribution:

$$\Delta_t^I \sim NegBin(\kappa_t, \varphi) \tag{4}$$

with parameters $\varphi \in (0,1)$ and

$$\kappa_t = \left(\frac{1}{\varphi} - 1\right) S_t p_t^{s \to i}. \tag{5}$$

Under (4), the mean of Δ_t^I is, for every φ, the same, $S_t p_t^{s\to i}$, as in the binomial case, while the variance $S_t p_t^{s\to i}/(1-\varphi)$ is always larger than in the binomial case. Furthermore, the closer φ is to one, the larger the variance, which gives us a way to obtain a sufficiently dispersed distribution of Δ_t^I.

Finally, the parameter $\lambda(t)$ might be considered constant as long as the individuals of the population do not change their social habits; however, in the event of a serious disease with extensive spread, the government will typically enforce restrictions of various kinds, which will most certainly effect the interaction patterns in the population. Thus, we assume that there are $d1$ breakpoints that divide the time frame into d intervals during which $\lambda(t)$ is constant. More precisely, let T denote the last day of the modelling period; then, given integer breakpoints $\boldsymbol{t} = (t_i)_{i=1}^{d-1}$ such that $0 = t_0 < t_1 < \cdots < t_{d1} < t_d = T$ and positive parameters $\boldsymbol{\lambda} = (_i)_{i=1}^{d}$ we let

$$\lambda(t) = \sum_{i}^{d-1} \lambda_i \mathbb{I}_{[t_{i-1}, t_i)}(t) + \lambda_d \mathbb{I}_{[t_{d-1}, t_d]}(t) \tag{6}$$

In the model described in this section, the parameters $\theta = (\boldsymbol{\lambda}, \boldsymbol{t}, p^{i\to r})$ are unknown and need to be estimated on the basis of a given record of observed epidemic states up to T.

Prior Distributions

For this project, we will assume the parameters are *a priori* independent (i.e., $\pi(\theta) = \pi(\boldsymbol{\lambda}) \cdot \pi(\boldsymbol{t}) \cdot \pi(p^{i\to r})$) and characterised by the following prior distributions:

$$\pi(\boldsymbol{t}) \propto \mathbb{I}_{\{0<t_1<\cdots<t_{d-1}<T\}}(\approx) \tag{7}$$

$$\pi(\boldsymbol{\lambda}) = \prod_{i=1}^{d} \pi(\lambda_i) = \prod_{i=1}^{d} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} e^{-\beta_i \lambda_i} \tag{8}$$

$$\pi(p^{i\to r}) = \frac{1}{B(a,b)} \left(p^{i\to r}\right)^{a-1} \left(1 - p^{i\to r}\right)^{b-1} \tag{9}$$

Full-Conditionals

The transition probabilities of the Markov chain $(S_t, I_t)_{t\geq 0}$ are computed as:

$$\begin{aligned}
q_\theta\left(s_t, i_t; s_{t+1}, i_{t+1}\right) &\doteq \mathbb{P}_\theta\left(S_{t+1} = s_{t+1}, I_{t+1} = i_{t+1} \mid S_t = s_t, I_t = i_t\right) = \\
&= \mathbb{P}_\theta\left(S_t - \Delta_t^I = s_{t+1}, I_t + \Delta_t^I - \Delta_t^R = i_{t+1} \mid S_t = s_t, I_t = i_t\right) = \\
&= \mathbb{P}_\theta\left(s_t - \Delta_t^I = s_{t+1}, i_t + \Delta_t^I - \Delta_t^R = i_{t+1} \mid S_t = s_t, I_t = i_t\right) = \\
&= \mathbb{P}_\theta\left(\Delta_t^I = s_t - s_{t+1}, \Delta_t^R = i_t - i_{t+1} + s_t - s_{t+1} \mid S_t = s_t, I_t = i_t\right) = \\
&= \frac{\Gamma(s_t - s_{t+1} + \kappa_t)}{(s_t - s_{t+1})!\,\Gamma(\kappa_t)} (1-\varphi)^{\kappa_t} \varphi^{s_t - s_{t+1}} \cdot \\
&\quad \binom{i_t}{i_t - i_{t+1} + s_t - s_{t+1}} \left(p^{i\to r}\right)^{i_t - i_{t+1} + s_t - s_{t+1}} \left(1 - p^{i\to r}\right)^{i_{t+1} - s_t + s_{t+1}}
\end{aligned}$$

Therefore, assuming that $\mathbb{P}(S_0 = s_0, I_0 = i_0 \mid \theta) = 1$, the likelihood is given by:

$$
\begin{aligned}
f(\boldsymbol{y} \mid \theta) \doteq \mathbb{P}(S_0 = s_0, I_0 = i_0, S_1 = s_1, I_1 = i_1, \ldots, S_T = s_T, I_T = i_T \mid \theta) = \\
= \mathbb{P}(S_0 = s_0, I_0 = i_0 \mid \theta) \cdot \mathbb{P}(S_1 = s_1, I_1 = i_1 \mid \theta, S_0 = s_0, I_0 = I_0) \cdot \\
\mathbb{P}(S_2 = s_2, I_2 = i_2 \mid \theta, S_1 = s_1, I_1 = i_1) \ldots \mathbb{P}(S_T = s_T, I_T = i_t \mid \theta, S_{T-1} = s_{T-1}, I_{T-1} = i_{T-1}) = \\
= \prod_{t=0}^{T-1} q_\theta(s_t, i_t; s_{t+1}, i_{t+1}) = \\
= \prod_{t=0}^{T-1} \left(\frac{\Gamma(s_t - s_{t+1} + \kappa_t)}{(s_t - s_{t+1})!\,\Gamma(\kappa_t)} (1-\varphi)^{\kappa_t} \binom{i_t}{i_t - i_{t+1} + s_t - s_{t+1}} \right) \cdot \varphi^{s_0 - s_T} \cdot \left(p^{i\to r}\right)^{i_0 - i_T + s_0 - s_T} \cdot \\
\left(1 - p^{i\to r}\right)^{\sum_{t=1}^T i_t + s_T - s_0}
\end{aligned}
$$

As a consequence, the joint law of the data and the parameters is given by:

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{\lambda}, \boldsymbol{t}, p^{i\to r}, \boldsymbol{y}) = f(\boldsymbol{y} \mid \theta) \cdot \pi(\boldsymbol{\lambda}) \cdot \pi(\boldsymbol{t}) \cdot \pi(p^{i\to r}) \propto \\
\propto \prod_{t=0}^{T-1} \left(\frac{\Gamma(s_t - s_{t+1} + \kappa_t)}{\Gamma(\kappa_t)} (1-\varphi)^{\kappa_t} \right) \cdot \\
\left(p^{i\to r}\right)^{a + i_0 - i_T + s_0 - s_T - 1} \cdot \left(1 - p^{i\to r}\right)^{b + \sum_{t=1}^T i_t + s_T - s_0 - 1} \cdot \mathbb{I}_{\{0 < t_1 < \cdots < t_{d-1} < T\}}(\boldsymbol{t}) \cdot \prod_{i=1}^d \lambda_i^{\alpha_i - 1} e^{-\beta_i \lambda_i}
\end{aligned}
$$

All in all, the full-conditionals of the parameters are given by:

$$
\pi(p^{i\to r} \mid \lambda, \boldsymbol{t}, \boldsymbol{y}) \propto \left(p^{i\to r}\right)^{a + i_0 - i_T + s_0 - s_T - 1} \cdot \left(1 - p^{i\to r}\right)^{b + \sum_{t=1}^T i_t + s_T - s_0 - 1} \implies
$$

$$
p^{i\to r} \mid \boldsymbol{\lambda}, \boldsymbol{t}, \boldsymbol{y} \sim Beta\left(a + i_0 - i_T + s_0 - s_T, b + \sum_{t=1}^T i_t + s_T - s_0 \right)
$$

$$
\pi(\boldsymbol{\lambda} \mid p^{i\to r}, \boldsymbol{t}, \boldsymbol{y}) \propto \prod_{t=0}^{T-1} \left(\frac{\Gamma(s_t - s_{t+1} + \kappa_t)}{\Gamma(\kappa_t)} (1-\varphi)^{\kappa_t} \right) \cdot \prod_{i=1}^d \lambda_i^{\alpha_i - 1} e^{-\beta_i \lambda_i}
$$

$$
\pi(\boldsymbol{t} \mid p^{i\to r}, \boldsymbol{\lambda}, \boldsymbol{y}) \propto \prod_{t=0}^{T-1} \left(\frac{\Gamma(s_t - s_{t+1} + \kappa_t)}{\Gamma(\kappa_t)} (1-\varphi)^{\kappa_t} \right) \cdot \mathbb{I}_{\{0 < t_1 < \cdots < t_{d-1} < T\}}
$$

Therefore, it is possible to sample from the posterior of θ using a standard Gibbs step for $p^{i\to r}$ while updating each of the components of $\boldsymbol{\lambda}$ and $\boldsymbol{t}$ using local Metropolis-Hastings moves.