CSCC37 TUT3 WEEK5
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https://snoopysnipe.github.io/ta/c37f20/

1. Conditioning of functions (derive the expression with Taylor Series, but there are many other methods)
2. Matrix and vector norms

1) $f(x) = f(\hat{x} + x - \hat{x})$

$= f(\hat{x}) + (x-\hat{x}) f'(\hat{x}) + \frac{(x-\hat{x})^2}{2!} f''(\xi) \qquad \xi \in [\hat{x}, x]$

$\Rightarrow f(x) \approx f(\hat{x}) + (x - \hat{x}) f'(\hat{x})$

Taylor Series of f centered around $\hat{x}$
where $\hat{x} = fl(x) = x(1-\delta_x)$

The approximation is good when $\hat{x}$ is near x

$\Rightarrow |f(x) - f(\hat{x})| \approx |f'(\hat{x})| |x - \hat{x}|$

$\Rightarrow \frac{|f(x) - f(\hat{x})|}{|f(x)|} \approx \frac{|x f'(x)|}{|f(x)|} \frac{|x - \hat{x}|}{|x|}$

$\therefore$ The relative error in $f(\hat{x})$ is proportional to the relative error in $\hat{x}$ with proportionality factor

$cond(f) = \frac{|x f'(\hat{x})|}{|f(x)|}$

In context of measuring condition of functions, $\hat{x}$ is usually sufficiently close to x, allowing us to use the approximation

$$\text{Cond}(f) \approx \frac{|x f'(x)|}{|f(x)|}$$

notice it is $f'(x)$ NOT $f'(\hat{x})$

ex: $f(x) = \sqrt{x}$

$$\text{cond}(f) = \frac{\left| x \frac{1}{2\sqrt{x}} \right|}{|\sqrt{x}|}$$

$$= \frac{|x|}{|2\sqrt{\hat{x}}\sqrt{x}|}$$

$$= \frac{1}{2}$$ well-conditioned

ex: $f(x) = \frac{10}{1-x^2}$

$$\text{cond}(f) = \frac{\left| x \frac{20\hat{x}}{(1-\hat{x}^2)^2} \right|}{\left| \frac{10}{1-x^2} \right|}$$

$$= \frac{|20x^2|}{|(1-x^2)^3|}$$

poorly conditioned when $|x| \approx 1$

2) Vector Norms $\bar{x} \in \mathbb{R}^n$

$$\|\bar{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|\bar{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$$

$$\|\bar{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

in general, for $p > 0$

$$\|\bar{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

Properties

1) $\|\bar{x}\| > 0$ if $\bar{x} \neq \bar{0}$
2) $\|\gamma \bar{x}\| = |\gamma| \, \|\bar{x}\|$ for any scalar γ
3) $\|\bar{x} + \bar{y}\| \le \|\bar{x}\| + \|\bar{y}\|$ triangle inequality

also,

$$\|\bar{x}\|_1 \geq \|\bar{x}\|_2 \geq \|\bar{x}\|_\infty$$

$$\|\bar{x}\|_1 \le \sqrt{n}\,\|\bar{x}\|_2$$

$$\|\bar{x}\|_2 \le \sqrt{n}\,\|\bar{x}\|_\infty$$

$$\|\bar{x}\|_1 \le n\,\|\bar{x}\|_\infty$$

ex: $\bar{x} = [3, 5, -7, 8]$

$\|\bar{x}\|_1 = 23$

$\|\bar{x}\|_2 = \sqrt{147}$

$\|\bar{x}\|_\infty = 8$

Matrix Norms $\quad A \in \mathbb{R}^{n \times m}$

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \quad = \text{max absolute column sum}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{m} |a_{ij}| \quad = \text{max absolute row sum}$$

Properties

1) $\|A\| > 0$ if $A \neq 0$
2) $\|\gamma A\| = |\gamma| \|A\|$ for any scalar γ
3) $\|A + B\| \leq \|A\| + \|B\|$
4) $\|AB\| \leq \|A\| \|B\|$
5) $\|Ax\| \leq \|A\| \|x\|$ for any vector x

ex: $A = \begin{bmatrix} 1 & -7 \\ -2 & -3 \end{bmatrix}$

$\|A\|_1 = 10$

$\|A\|_\infty = 8$

Additional Notes:

In general, $\|A\| = \max_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$

Also, $\|A\|_2 = \sigma_{max}(A)$; the largest singular value of matrix A

$\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2}$; the frobenius norm