Chapter 1

Preliminaries: Set theory and categories

Problem 1.1. Locate a discussion of Russel's paradox, and understand it.

Solution. Recall that, in naive set theory, any collection of objects that satisfy some property can be called a set. Russel's paradox can be illustrated as follows. Let R be the set of all sets that do not contain themselves. Then, if $R \notin R$, then by definition it must be the case that $R \in R$; similarly, if $R \in R$ then it must be the case that $R \notin R$. $\square$

Problem 1.2. $\triangleright$ Prove that if $\sim$ is an equivalence relation on a set S, then the corresponding family $\mathscr{P}_\sim$ defined in §1.5 is indeed a partition of S; that is, its elements are nonempty, disjoint, and their union is S. [§1.5]

Solution. Let S be a set with an equivalence relation $\sim$. Consider the family of equivalence classes w.r.t. $\sim$ over S:

$$\mathscr{P}_\sim = \{ [a]_\sim \mid a \in S \}$$

Let $[a]_\sim \in \mathscr{P}_\sim$. Since $\sim$ is an equivalence relation, by reflexivity we have $a \sim a$, so $[a]_\sim$ is nonempty. Now, suppose a and b are arbitrary elements in S such that $a \not\sim b$. For contradiction, suppose that there is an $x \in [a]_\sim \cap [b]_\sim$. This means that $x \sim a$ and $x \sim b$. By transitivity, we get that $a \sim b$; this is a contradiction. Hence the $[a]_\sim$ are disjoint. Finally, let $x \in S$. Then $x \in [x]_\sim \in \mathscr{P}_\sim$. This means that

$$\bigcup_{[a]_\sim \in \mathscr{P}_\sim} [a]_\sim = S,$$

that is, the union of the elements of $\mathscr{P}_\sim$ is S. $\square$

Problem 1.3. $\triangleright$ Given a partition $\mathscr{P}$ on a set S, show how to define a relation $\sim$ such that $\mathscr{P} = \mathscr{P}_\sim$. [§1.5]

Solution. Define, for $a, b \in S$, $a \sim b$ if and only if there exists an $X \in \mathscr{P}$ such that $a \in X$ and $b \in X$. We will show that $\mathscr{P} = \mathscr{P}_\sim$.

1. ($\mathscr{P} \subseteq \mathscr{P}_\sim$). Let $X \in \mathscr{P}$; we want to show that $X \in \mathscr{P}_\sim$. We know that X is nonempty, so choose $a \in X$ and consider $[a]_\sim \in \mathscr{P}_\sim$. We need to show that $X = [a]_\sim$. Suppose $a' \in X$ (it may be that $a' = a$.) Since $a, a' \in X$, $a \sim a'$, so $a' \in [a]_\sim$. Now, suppose $a' \in [a]_\sim$. We have $a' \sim a$, so $a' \in X$. Hence $X = [a]_\sim \in \mathscr{P}_\sim$, so $\mathscr{P} \subseteq \mathscr{P}_\sim$.
2. ($\mathscr{P}_\sim \subseteq \mathscr{P}$). Let $[a]_\sim \in \mathscr{P}_\sim$. From exercise I.1.1 we know that $[a]_\sim$ is nonempty. Suppose $a' \in [a]_\sim$. By definition, since $a' \sim a$, there exists a set X such that $a, a' \in X$. Hence $[a]_\sim \subseteq X$. Also, if $a, a' \in X$ (not necessarily distinct) then $a \sim a'$. Therefore, $\mathscr{P}_\sim \subseteq \mathscr{P}$, and with 1. we get that the sets $\mathscr{P}$ and $\mathscr{P}_\sim$ are equal.

$\square$

Problem 1.4. How many different equivalence relations can be defined on the set $\{1, 2, 3\}$?

Solution. From the definition of an equivalence relation and the solution to problem **I.1.3**, we can see that an equivalence relation on S is equivalent to a partition of S. Thus the number of equivalence relations on S is equal to the number of partitions of S. Since $\{1, 2, 3\}$ is small we can determine this by hand:

$$\mathscr{P}_0 = \{\,\{1, 2, 3\}\,\}$$

$$\mathscr{P}_1 = \{\,\{1\}, \{2\}, \{3\}\}\,\}$$
$$\mathscr{P}_2 = \{\,\{1, 2\}, \{3\}\,\}$$
$$\mathscr{P}_3 = \{\,\{1\}, \{2, 3\}\,\}$$
$$\mathscr{P}_4 = \{\,\{1, 3\}, \{2\}\,\}$$

Thus there can be only 5 equivalence relations defined on $\{1, 2, 3\}$. $\square$

Problem 1.5. Give an example of a relation that is reflexive and symmetric but not transitive. What happens if you attempt to use this relation to define a partition on the set? (Hint: Thinking about the second question will help you answer the first one.)

Solution. For $a, b \in \mathbf{Z}$, define $a \diamond b$ to be true if and only if $|a - b| \leq 1$. It is reflexive, since $a \diamond a = |a - a| = 0 \leq 1$ for any $a \in \mathbf{Z}$, and it is symmetric since $a \diamond b = |a - b| = |b - a| = b \diamond a$ for any $a, b \in \mathbf{Z}$. However, it is not transitive. Take for example $a = 0, b = 1, c = 2$. Then we have $|a - b| = 1 \leq 1$, and $|b - c| = 1 \leq 1$, but $|a - c| = 2 > 1$; so $a \diamond b$ and $b \diamond c$, but not $a \diamond c$.

When we try to build a partition of $\mathbf{Z}$ using $\diamond$, we get "equivalence classes" that are not disjoint. For example, $[2]_\diamond = \{1, 2, 3\}$, but $[3]_\diamond = \{2, 3, 4\}$. Hence $\mathscr{P}_\diamond$ is not a partition of $\mathbf{Z}$. $\square$

Problem 1.6. Define a relation $\sim$ on the set $\mathbf{R}$ of real numbers by setting $a \sim b \iff b - a \in \mathbf{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbf{R}/\sim$. Do the same for the relation $\approx$ on the plane $\mathbf{R} \times \mathbf{R}$ by declaring $(a_1, b_1) \approx (a_2, b_2) \iff b_1 - a_1 \in \mathbf{Z}$ and $b_2 - a_2 \in \mathbf{Z}$. [§II.8.1, II.8.10]

Solution. Suppose $a, b, c \in \mathbf{R}$. We have that $a - a = 0 \in \mathbf{Z}$, so $\sim$ is reflexive. If $a \sim b$, then $b - a = k$ for some $k \in \mathbf{Z}$, so $a - b = -k \in \mathbf{Z}$, hence $b \sim a$. So $\sim$ is symmetric. Now, suppose that $a \sim b$ and $b \sim c$, in particular that $b - a = k \in \mathbf{Z}$ and $c - b = l \in \mathbf{Z}$. Then $c - a = (c - b) + (b - a) = l + k \in \mathbf{Z}$, so $a \sim c$. So $\sim$ is transitive.

An equivalence class $[a]_\sim \in \mathbf{R}/\sim$ is the set of integers $\mathbf{Z}$ transposed by some real number $\epsilon \in [0,1)$. That is, for every set $X \in \mathbf{R}/\sim$, there is a real number $\epsilon \in [0,1)$ such that every $x \in X$ is of the form $k+\epsilon$ for some integer k.

Now we will show that $\approx$ is an equivalence relation over $\mathbf{R}\times\mathbf{R}$. Supposing $a_1, a_2 \in \mathbf{R}\times\mathbf{R}$, we have $a_1 - a_1 = a_2 - a_2 = 0 \in \mathbf{Z}$, so $(a_1,a_2) \approx (a_1,a_2)$. If we also suppose that $b_1, b_2, c_1, c_2 \in \mathbf{R}\times\mathbf{R}$, then symmetry and transitivity can be shown as well: $(a_1,a_2) \approx (b_1,b_2) \implies b_1 - a_1 = k$ for some integer k and $b_2 - a_2 = l$ for some integer l, hence $a_1 - b_1 = -k \in \mathbf{Z}$ and $a_2 - b_2 = -l \in \mathbf{Z}$, so $(b_1,b_2) \approx (a_1,a_2)$; also if $(a_1,a_2) \approx (b_1,b_2)$ and $(b_1,b_2) \approx (c_1,c_2)$, then $(b_1,b_2) - (a_1,a_2) = (k_1,k_2) \in \mathbf{Z}\times\mathbf{Z}$ as well as $(c_1,c_2) - (b_1,b_2) = (l_1,l_2) \in \mathbf{Z}\times\mathbf{Z}$, so $(c_1,c_2) - (a_1,a_2) = (c_1,c_2) - (b_1,b_2) + (b_1,b_2) - (a_1,a_2) = (k_1+l_1, k_2+l_2 \in \mathbf{Z}\times\mathbf{Z}$. Thus $\approx$ is an equivalence relation.

The interpretation of $\approx$ is similar to $\sim$. An equivalence class $X \in \mathbf{R}\times\mathbf{R}/\approx$ is just the 2-dimensional integer lattice $\mathbf{Z}\times\mathbf{Z}$ transposed by some pair of values $(\epsilon_1,\epsilon_2) \in [0,1)\times[0,1)$. □

2 Functions between sets

Problem 2.1. How many different bijections are there between a set with n elements and itself?

Solution. A function $f : S \to S$ is a graph $\Gamma_f \subseteq S \times S$. Since f is bijective, then for all $y \in S$ there exists a unique $x \in S$ such that $(x,y) \in \Gamma_f$. We can see that $|\Gamma_f| = n$. Since each x must be unique, all the elements $x \in S$ must be present in the first component of exactly one pair in Γ_f. Furthermore, if we order the elements (x,y) in Γ_f by the first component, we can see that Γ_f is just a permutation on the n elements in S. For example, for $S = \{1,2,3\}$ one such Γ_f is:

$$\{\,(1,3),(2,2),(3,1)\,\}$$

Since $|S| = n$, the number of permutations of S is $n!$. Hence there can be $n!$ different bijections between S and itself. □

Problem 2.2. ▷ Prove statement (2) in Proposition 2.1. You may assume that given a family of disjoint subsets of a set, there is a way to choose one element in each member of the family. [§2.5, V3.3]

> **Proposition 2.1.** Assume $A \neq \emptyset$, and let $f : A \to B$ be a function. Then (1) f has a left-inverse if and only if f is injective; and
> (2) f has a right-inverse if and only if f is surjective.

Solution. Let $A \neq \emptyset$ and suppose $f : A \to B$ is a function.

($\implies$) Suppose there exists a function g that is a right-inverse of f. Then $f \circ g = \mathrm{id}_B$. Let $b \in B$. We have that $f(g(b)) = b$, so there exists an $a = g(b)$ such that $f(a) = b$. Hence f is surjective.

($\Longleftarrow$) Suppose that f is surjective. We want to construct a function $g : B \to A$ such that $f(g(a)) = a$ for all $a \in A$. Since f is surjective, for all $b \in B$ there is an $a \in A$ such that $f(a) = b$. For each $b \in B$ construct a set Λ_b of such pairs:

$$\Lambda_b = \{ (a, b) \mid a \in A, f(a) = b \}$$

Note that Λ_b is non-empty for all $b \in B$. So that we can choose one pair (a, b) (a not necessarily unique) from each set in $\Lambda = \{ \Lambda_b \mid b \in B \}$ to define $g : B \to A$:

$$g(b) = a, \text{ where } a \text{ is in some } (a, b) \in \Lambda_b$$

Now, g is a right-inverse of f. To show this, let $b \in B$. Since f in surjective, g has been defined such that when $a = g(b)$, $f(a) = b$, so we get that $f(g(b)) = (f \circ g)(b) = b$, thus g is a right-inverse of f. $\square$

Problem 2.3. Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection.

1. Suppose $f : A \to B$ is a bijection, and that $f^{-1} : B \to A$ is its inverse. We have that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$. Hence f is the left- and right-inverse of f^{-1}, so f^{-1} must be a bijection.

2. Let $f : B \to C$ and $g : A \to B$ be bijections, and consider $f \circ g$. To show that f is injective, let $a, a' \in A$ such that $(f \circ g)(a) = (f \circ g)(a')$. Since f is a bijection, $f(g(a)) = f(g(a')) \implies g(a) = g(a')$. Also, since g is a bijection, $g(a) = g(a') \implies a = a'$. Hence $f \circ g$ is injective. Now, let $c \in C$. Since f is surjective, there is a $b \in B$ such that $f(b) = c$. Also, since g is surjective, there is an $a \in A$ such that $g(a) = b$; this means that there is an $a \in A$ such that $(f \circ g)(a) = c$. So $f \circ g$ is bijective.

Problem 2.4. $\triangleright$ Prove that 'isomorphism' is an equivalence relation (on any set of sets.) [§4.1]

Solution. Let S be a set. Then id_S is a bijection from S to itself, so $S \cong S$. Let T be another set with $S \cong T$, i.e. that there exists a bijection $f : S \to T$. Since f is a bijection, it has an inverse $f^{-1} : T \to S$, so $T \cong S$. Finally, let U also be a set, and assume that there exists bijections $f : S \to T$ and $g : T \to U$, i.e. that $S \cong T$ and $T \cong U$. From exercise **I.2.3** we know that the composition of bijections is itself a bijection. This means that $g \circ f : S \to U$ is a bijection, so $S \cong U$. Hence $\cong$ is an equivalence relation. $\square$

Problem 2.5. $\triangleright$ Formulate a notion of *epimorphism*, in the style of the notion of *monomorphism* seen in §2.6, and prove a result analogous to Proposition 2.3, for epimorphisms and surjections.

Solution. A function $f : A \to B$ is an *epimorphism* iff for all sets Z and all functions $\beta', \beta'' : Z \to B$, this holds:

$$\beta' \circ f = \beta'' \circ f \implies \beta' = \beta''$$

Now we will show that f is a surjection iff it is an epimorphism.

($\implies$) $f : A \to B$ is a surjection, by Proposition 2.1.2 f has a right-inverse. Let $g : B \to A$ be a right-invese for f. Then suppose $\beta', \beta'' : B \to Z$ for all sets Z such that $\beta' \circ f = \beta'' \circ f$. Then:

$$\beta' \circ f = \beta'' \circ f \iff \beta' \circ f \circ g = \beta'' \circ f \circ g \iff \beta' = \beta''$$

So f is an epimophism.

($\impliedby$) Suppose f is not a surjection. Then $(\exists b^* \in B)$ $(\forall a \in A)$ $b^* \neq f(a)$. Since f is an epimorphism, for all sets Z and for all $\beta', \beta'' : B \to Z$:

$$\beta' \circ f = \beta'' \circ f \implies \beta' = \beta''$$

But this can't be true, because $b^* \notin imf$, so even if $\beta' \circ f = \beta'' \circ f$ it doesn't imply that $\beta'(b^*) = \beta''(b^*)$. □

Problem 2.6. With notation as in Example 2.4, explain how any function $f : A \to B$ determines a section of π_A.

Solution. Let $f : A \to B$ and let $\pi_A : A \times B \to A$ be such that $\pi_A(a, b) = a$ for all $(a, b) \in A \times B$. Construct $g : A \to A \times B$ defined as $g(a) = (a, f(a))$ for all $a \in A$. The function g can be thought of as 'determined by' f. Now, since $(\pi_A \circ g)(a) = \pi_A(g(a)) = \pi_A(a, f(a)) = a$ for all $a \in A$, g is a right inverse of π_A, i.e. g is a section of π_A as required. □

Problem 2.7. Let $f : A \to B$ be any function. Prove that the graph Γ_f of f is isomorphic to A.

Solution. Recall that sets Γ_A and A are *isomorphic*, written $\Gamma_A \cong A$, if and only if there exists a bijection $g : \Gamma_A \to A$. Let's construct such a function g, defined to be $g(a, b) = a$. Keep in mind that here $(a, b) \in \Gamma_f \subseteq A \times B$.

Let $(a', b'), (a'', b'') \in \Gamma_f$ such that $f(a', b') = f(a'', b'')$. For contradiction, suppose that $(a', b') \neq (a'', b'')$. Since $f(a', b') = a' = a'' = f(a'', b'')$, it must be that $b' \neq b''$. However, this would mean that both (a', b') and (a', b'') are in Γ_f; this would mean that $f(a') = b' \neq b'' = f(a')$, which is impossible since f is a function. Hence g is injective.

Let $a' \in A$. Since f is a well-defined function with A as its domain, there must exists a pair $(a', b') \in \Gamma_f$ for some $b' \in B$, in particular that $g(a', b') = a'$; thus g is surjective, so it is a bijection. □

Problem 2.8. Describe as explicitly as you can all terms in the canonical decomposition (cf. §2.8) of the function $\mathbf{R} \to \mathbf{C}$ defined by $r \mapsto e^{2\pi i r}$. (This exercise matches one previously. Which one?)

Solution. Let $f : \mathbf{R} \to \mathbf{C}$ be as above. The first piece in the canonical decomposition is the equivalence relation $\sim$ defined as $x \sim x' \iff f(x) = f(x')$, i.e. $[x]_\sim$ is the set of all elements in $\mathbf{R}$ that get mapped to the same element in $\mathbf{C}$ by f as x.

The second piece is the set $\mathscr{P}_\sim$. This set is the set of all equivalence classes of $\mathbf{R}$ over equality up to f. Note that, since $f(x) = e^{2\pi i x} = \cos(2\pi x) + i\sin(2\pi x)$, f is periodic with period 1. That is, $f(x) = e^{2\pi i x} = e^{2\pi i x + 2\pi} = e^{2\pi i(x+1)} = f(x+1)$. In other words, we can write $\mathscr{P}_\sim$ as,

$$\mathscr{P}_\sim = \{\, \{\, r + k \mid k \in \mathbf{Z} \,\} \mid r \in [0,1) \subseteq \mathbf{R} \,\},$$

and it is here when we notice uncanny similarities to exercise **1.6** where $x \sim y$, for $x, y \in \mathbf{R}$, if and only if $x - y \in \mathbf{Z}$, in which we could have written $\mathscr{P}_\sim$ in the same way.

Now we will explain the mysterious $\tilde{f} : \mathscr{P}_\sim \to \mathrm{im} f$. This function is taking each *equivalence class* $[x]_\sim$ over the reals w.r.t. $\sim$ and mapping it to the element in $\mathbf{C}$ that f maps each element $x' \in [x]_\sim$ to; indeed, since $x \sim x'$ is true for $x, x' \in \mathbf{R}$ if and only if $f(x) = f(x')$, we can see that for any $x \in \mathbf{R}$, for all $x' \in [x]_\sim$, there exists a $c \in \mathbf{C}$ such that $f(x') = c$. To illustrate with the equivalence class over $\mathbf{R}$ w.r.t. $\sim$ corresponding to the element $0 \in \mathbf{R}$, we have $[0]_\sim = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. We can see that $e^{-4\pi i} = e^{-2\pi i} = e^{0\pi i} = 1 = e^{2\pi i} = e^{4\pi i}$, etc; so the function would map $[0]_\sim \mapsto 1 \in \mathbf{C}$, and so on. Furthermore, we can see that $\tilde{f}$ is surjective, since for y to be in $\mathrm{im} f$ is to say that there is an $x \in \mathbf{R}$ such that $f(x) = y$; so there must be an equivalence class $[x]_\sim$ which is mapped to y by $\tilde{f}$.

Finally, the simple map from $\mathrm{im} f \to \mathbf{C}$ that simply takes $c \mapsto c$. This can be thought of as a potential "expansion" of the domain of $\tilde{f}$. It is obviously injective, since (trivially) $c \neq c' \implies c \neq c'$. However, it may not be surjective: for example, $2 \in \mathbf{C}$ is not in $\mathrm{im} f$ as it is defined above. $\square$

Problem 2.9. $\triangleright$ Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the operation $A \amalg B$ is well-defined up to *isomorphism* (cf. §2.9) [§2.9, 5.7]

Solution. Let A', A'', B', B'' be sets as described above. Since $A' \cong A''$ and $B' \cong B''$, we know there exists respective bijections $f : A' \to A''$ and $g : B' \to B''$. Now, we wish to show that $A' \cup B' \cong A'' \cup B''$. Define a function $h : A' \cup B' \to A'' \cup B''$ such that $h(x) = f(x)$ if $x \in A'$ and $g(x)$ if $x \in B'$.

We will now show that h is a bijection. Let $y \in A'' \cup B''$. Then, since $A'' \cap B'' = \emptyset$, either $y \in A''$ or $y \in B''$. Without loss of generality suppose that $y \in A''$. Then, since $f : A' \to A''$ is a bijection, it is *surjective*, so there exists an $x \in A' \subseteq A' \cup B'$ such that $h(x) = f(x) = y$. So h is surjective. Now, suppose that $x \neq x'$, for $x, x' \in A' \cup B'$. If $x, x' \in A'$, then since f is injective and $h(x) = f(x)$ for all $x \in A'$, then $h(x) \neq h(x')$. Similarly for if $x, x' \in B'$. Now, without loss of generality if $x \in A'$ and $x' \in B'$, then $h(x) = f(x) \neq g(x') = h(x')$ since $A'' \cap B'' = \emptyset$. Hence h is a bijection, so $A' \cup B' \cong A'' \cup B''$.

Since these constructions of A', A'', B', B'' correspond to creating "copies" of sets A and B for use in the disjoint union operation, we have that disjoint union is a well-defined function *up to isomorphism*. In particular, since $\cong$ is an equivalence relation, we can consider $\amalg$ to be well-defined from $\mathscr{P}_{\cong}$ to $A' \cup B'$. $\square$

Problem 2.10. $\triangleright$ Show that if A and B are finite sets, then $\left|B^A\right| = |B|^{|A|}$. [§2.1, 2.11, I.4.1]

Solution. Let A and B be sets with $|A| = n$ and $|B| = m$, with n, m being non-negative integers. Recall that B^A denotes the set of functions $f : A \to B$. Now, if $A = B = \emptyset$ or $A = \emptyset$ and $|B| = 1$, we get one function, the empty function $\Gamma_f = \emptyset$, and $0^0 = 1^0 = 1$. If $|A| = |B| = 1$, then we get the singleton function $\Gamma_f = \{(a, b)\}$, and $1^1 = 1$. If $A \neq \emptyset$ and $B = \emptyset$, then no well-defined function can exist from A to B since there will be no value for the elements in A to take; this explains $\left|B^A\right| = |B|^{|A|} = 0^{|A|} = 0$.

Suppose that $B \neq \emptyset$ and B is finite. We will show inductively that $\left|B^A\right| = |B|^{|A|}$. First, suppose that $|A| = 1$. Then there are exactly $|B|$ functions from A to B: if $B = \{ b_1, b_2, \dots, b_m \}$, then the functions are $\{(a, b_1)\}, \{(a, b_2)\}$, etc. Hence $\left|B^A\right| = |B|^{|A|} = |B|$. Now, fix $k \geq 2$, and assume that $\left|B^A\right| = |B|^{|A|}$ for all sets A such that $|A| = k - 1$. Suppose that $|A| = k$. Let $a \in A$. (We can do this since $|A| = k \geq 2$.) Then, by the inductive hypothesis, since $|A \backslash \{a\}| = k - 1$, $\left|B^{(A \backslash \{a\})}\right| = |B|^{|A|-1}$. Let F be the set of functions from $A \backslash \{a\}$ to B. Then, for each of those functions $f \in F$, there is $|B|$ "choices" of where to assign a: one choice for each element in B. Hence, $\left|B^A\right| = |B|\,|B|^{|A|-1} = |B|^{|A|}$ as required. $\square$

Problem 2.11. $\triangleright$ In view of Exercise 2.10, it is not unreasonable to use 2^A to denote the set of functions from an arbitrary set A to a set with 2 elements (say $\{0, 1\}$). Prove that there is a bijection between 2^A and the *power set* of A (cf. §1.2). [§1.2, III.2.3]

Solution. Let $S = \{0, 1\}$, and consider $f : \mathcal{P}(A) \to 2^A$, defined as

$$f(X) = \{ (a, 1) \text{ if } a \in X, \text{ and } (a, 0) \text{ otherwise} \}$$

We will show that f is bijective. Let $g \in 2^A$. Then f is a function from A to S. Let $A_1 = \{ a \in A \mid g(a) = 1 \}$. Then A_1 is a set such that $A_1 \in \mathcal{P}(A)$, and $f(A_1) = g$. Hence f is surjective.

Now, suppose that $X, Y \subseteq A$ and $f(X) = f(Y)$. Then, for all $a \in A$, $a \in X \iff f(X)(a) = 1 \iff f(Y)(a) = 1 \iff a \in Y$. Hence f is injective, so $2^A \cong \mathcal{P}(A)$. $\square$

3 Category theory

Problem 3.1. ▷ Let C be a category. Consider a structure C^{op} with

1. $\mathrm{Obj}(\mathsf{C}^{op}) = \mathrm{Obj}(\mathsf{C})$
2. For A, B objects of C^{op} (hence objects of C), $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A)$.

Show how to make this into a category (that is, define composition of morphisms in C^{op} and verify the properties listed in §3.1).

Intuitively, the 'opposite' category C^{op} is simply obtained by 'reversing all the arrows' in C. [5.1, §III.1.1, §IX.1.2, IX.1.10]

Solution. For objects $A, B, C \in \mathrm{Obj}(\mathsf{C}^{op})$, the set of morphisms between A and B in C^{op}, $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$, is defined as $\mathrm{Hom}_{\mathsf{C}}(B, A)$. Similarly for the morphisms between B and C. So for morphisms $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$, to define composition we recall the set-function $\circ_{\mathsf{C}} : \mathrm{Hom}_{\mathsf{C}}(C, B) \times \mathrm{Hom}_{\mathsf{C}}(B, A) \to \mathrm{Hom}_{\mathsf{C}}(C, A)$ that is defined for the objects $A, B, C \in \mathrm{Obj}(\mathsf{C}) = \mathrm{Obj}(\mathsf{C}^{op})$; we shall define the composition of morphisms $f : A \to B$ and $g : B \to C$ in C^{op} with this function. Precisely, we define

$$\circ_{\mathsf{C}^{op}} : \mathrm{Hom}_{\mathsf{C}^{op}}(A, B) \times \mathrm{Hom}_{\mathsf{C}^{op}}(B, C) \to \mathrm{Hom}_{\mathsf{C}^{op}}(A, C)$$

to be

$$\circ_{\mathsf{C}^{op}}(f, g) = \circ_{\mathsf{C}}(g, f)$$

for all $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$ and $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$. The domain and codomain of $\circ_{\mathsf{C}}$ and $\circ_{\mathsf{C}^{op}}$ match (up to transposing the coordinates in the domain) due to the equality of $\mathrm{Hom}_{\mathsf{C}}(A, B)$ with $\mathrm{Hom}_{\mathsf{C}^{op}}(B, A)$.

To show that this composition makes C^{op} a category, first we note that the fact that C is a category implies the existence of a morphism 1_A taking A to itself where $A \in \mathrm{Obj}(\mathsf{C})$; this morphism is thus also present in $\mathrm{Hom}_{\mathsf{C}^{op}}(A, A) = \mathrm{Hom}_{\mathsf{C}}(A, A)$. Secondly, for objects $A, B, C, D \in \mathrm{Obj}(\mathsf{C})$, any morphisms $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$, $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$, and $h \in \mathrm{Hom}_{\mathsf{C}^{op}}(C, D)$ are associative, since

$$(h \circ_{\mathsf{C}^{op}} g) \circ_{\mathsf{C}^{op}} f = f \circ_{\mathsf{C}} (g \circ_{\mathsf{C}} h) = (f \circ_{\mathsf{C}} g) \circ_{\mathsf{C}} h = h \circ_{\mathsf{C}^{op}} (g \circ_{\mathsf{C}^{op}} f).$$

Finally, for any morphism $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$ we have,

$$f \circ_{\mathsf{C}^{op}} 1_A = 1_A \circ_{\mathsf{C}} f = f \text{ and } 1_B \circ_{\mathsf{C}^{op}} f = f \circ_{\mathsf{C}} 1_B = f;$$

hence the identities are "identities with respect to composition". Last, for objects $A, B, C, D \in \mathrm{Obj}(\mathsf{C})$ where $A \neq C$ and $B \neq D$, clearly $\mathrm{Hom}_{\mathsf{C}}(B, A) \cap \mathrm{Hom}_{\mathsf{C}}(D, C) = \emptyset$ is true iff $\mathrm{Hom}_{\mathsf{C^{op}}}(A, B) \cap \mathrm{Hom}_{\mathsf{C^{op}}}(C, D) = \emptyset$. Hence C^{op} is a category. □

Problem 3.2. If A is a finite set, how large is $\mathrm{End}_{\mathsf{Set}}(A)$?

Solution. The set $\text{End}_{\mathsf{Set}}(A)$ is the set of functions $f : A \to A$. Since A is finite, write $|A| = n$ for some $n \in \mathbf{Z}$. By exercise 2.10, we know that $\left|A^A\right| = |A|^{|A|} = n^n$. So the the set $\text{End}_{\mathsf{Set}}(A)$ has size n^n. $\square$

Problem 3.3. $\triangleright$ Formulate precisely what it means to say that 1_a is an identity with respect to composition in Example 3.3, and prove this assertion. [§3.2]

Solution. Let S be a set and $\sim$ be a binary relation on the set S. Then, for elements $a, b \in S$, $\text{Hom}(a, b)$ is the pair $(a, b) \in S \times S$ if $a \sim b$, or $\emptyset$ otherwise. Composition of morphisms (a, b) and (b, c) is simply the pair (a, c), which captures the transitivity of $\sim$. We will say that $1_a = (a, a)$, for $a \in S$, is *an identity with respect to composition* if, for any $b \in S$, $(a, b)(a, a) = (a, b)$. Now, if $a \sim a$ and $a \sim b$, then trivially it is the case that $a \sim b$; hence $(a, b)(a, a) = (a, b)$, and 1_a is an identity w.r.t. composition as required. $\square$

Problem 3.4. Can we define a category in the style of Example 3.3 using the relation $<$ on the set $\mathbf{Z}$?

Solution. No, we can't. This is because $<$ isn't reflexive: $x \not< x$ for any $x \in \mathbf{Z}$. $\square$

Problem 3.5. $\triangleright$ Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3. [§3.2]

Solution. Let S be a set. Example 3.4 considers the category $\hat{S}$ with objects $\text{Obj}(\hat{S}) = \mathscr{P}(S)$ and morphisms $\text{Hom}_{\hat{S}}(A, B) = \{\, (A, B) \,\}$ if $A \subseteq B$ and $\emptyset$ otherwise, for all sets $A, B \in \mathscr{P}$. The category $\hat{S}$ is an instance of the categories explained in Example 3.3 because $\subseteq$ is a reflexive and transitive operation on the power set of any set S. Indeed, for $X, Y, Z \subseteq S$, we have that $X \subseteq X$ and, if $X \subseteq Y$ and $Y \subseteq Z$, then if $x \in X$, then $x \in Y$ and $x \in Z$ so $X \subseteq Z$. $\square$

Problem 3.6. $\triangleright$ (Assuming some familiarity with linear algebra.) Define a category V by taking $\text{Obj}(\mathsf{V}) = \mathbf{N}$ and letting $\text{Hom}_{\mathsf{V}}(m, n) =$ the set of $m \times n$ matrices with real entries, for all $m, n \in \mathbf{N}$. (We will leave the reader the task of making sense of a matrix with 0 rows or columns.) Use product of matrices to define composition. Does this category 'feel' familiar? [§VI.2.1, §VIII.1.3]

Solution. Yes! It is yet another instance of Example 3.3. The binary relation $\sim$ on $\mathbf{N} \times \mathbf{N}$ holds for all values $(n, m) \in \mathbf{N} \times \mathbf{N}$, and means that a matrix of size $m \times n$ "can be built". It is reflexive trivially. It is transitive trivially as well—a matrix of any size can be built. However, it would also hold, for example, if we had to in some sense "deduce" that a 3×3 matrix could be built using the fact that 3×1 and 1×3 matrices can be built. $\square$

Problem 3.7. ▷ Define carefully the objects and morphisms in Example 3.7, and draw the diagram corresponding to compositon. [§3.2]

Solution. Let C be a category, and $A \in \mathsf{C}$. We want to define C^A. Let $\mathrm{Obj}(\mathsf{C}^A)$ include all morphisms $f \in \mathrm{Hom}_{\mathsf{C}}(A, Z)$ for all $Z \in \mathrm{Obj}(\mathsf{C})$. For any two objects $f, g \in \mathrm{Obj}(\mathsf{C}^A)$, $f : A \to Z_1$ and $g : A \to Z_2$, we define the morphisms $\mathrm{Hom}^A_{\mathsf{C}}(f, g)$ to be the morphisms $\sigma \in \mathrm{Hom}_{\mathsf{C}}(Z_1, Z_2)$ such that $g = \sigma f$. Now we must check that these morphisms satisfy the axioms.

1. Let $f \in \mathrm{Obj}(\mathsf{C}^A) \in \mathrm{Hom}_{\mathsf{C}}(A, Z)$ for some object $Z \in \mathrm{Obj}(\mathsf{C})$. Then there exists an identity morphism $1_Z \in \mathrm{Hom}_{\mathsf{C}}(Z, Z)$ since C is a category. This is a morphism such that $f = 1_z f$, so $\mathrm{Hom}^A_{\mathsf{C}}(f, f)$ is also nonempty.
2. Let $f, g, h \in \mathrm{Obj}(\mathsf{C}^A)$ such that there are morphisms $\sigma \in \mathrm{Hom}^A_{\mathsf{C}}(f, g)$ and $\tau \in \mathrm{Hom}^A_{\mathsf{C}}(g, h)$. Then there is a morphism $\upsilon \in \mathrm{Hom}^A_{\mathsf{C}}(f, h)$, namely $\tau\sigma$, which exists because of morphism composition in C. For clarity, we write that $f : A \to Z_1$, $g : A \to Z_2$, $h : A \to Z_3$, with $\sigma : Z_1 \to Z_2$ and $\tau : Z_2 \to Z_3$. We have $g = \sigma f$ and $h = \tau g$. Hence, $\upsilon f = \tau\sigma f = \tau g = h$ as required.
3. Lastly, let $f, g, h, i \in \mathrm{Obj}(\mathsf{C}^A)$ with Z_1, Z_2, Z_3, Z_4 codomains respectively, and with $\sigma \in \mathrm{Hom}^A_{\mathsf{C}}(f, g)$, $\tau \in \mathrm{Hom}^A_{\mathsf{C}}(g, h)$, and $\upsilon \in \mathrm{Hom}^A_{\mathsf{C}}(h, i)$. Since σ, τ, and υ are morphisms in C taking $Z_1 \to Z_2$, etc., morphism composition is associative; hence morphism composition is associative in C^A as well.

□

Problem 3.8. ▷ A *subcategory* C' of a category C consists of a collection of objects of C, with morphisms $\mathrm{Hom}'_{\mathsf{C}}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all objects $A, B \in \mathrm{Obj}(\mathsf{C}')$, such that identities and compositions in C make C' into a category. A subcategory C' is *full* if $\mathrm{Hom}'_{\mathsf{C}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all $A, B \in \mathrm{Obj}(\mathsf{C}')$. Construct a category of *infinite sets* and explain how it may be viewed as a full subcategory of Set. [4.4,§VI.1.1, §VIII.1.3]

Solution. Let InfSet be a subcategory of Set with $\mathrm{Obj}(\mathsf{InfSet})$ being all infinite sets and $\mathrm{Hom}_{\mathsf{Inf}}\mathsf{Set}(A, B)$ for infinite sets A, B being the functions from A to B. Since $\mathrm{Hom}_{\mathsf{Set}}(A, B)$ is just the set of all functions from A to B and not, say, the set of all functions from subsets of A that are in $\mathrm{Obj}(\mathsf{Set})$ to B, InfSet is full since $\mathrm{Hom}_{\mathsf{Inf}}\mathsf{Set}(A, B) = \mathrm{Hom}_{\mathsf{Set}}(A, B)$ for all infinite sets $A, B \in \mathrm{Obj}(\mathsf{InfSet})$. □

Problem 3.9. ▷ An alternative to the notion of *multiset* introduced in §2.2 is obtained by considering sets endowed with equivalence relations; equivalent elements are taken to be multiple instance of elements 'of the same kind'. Define a notion of morphism between such enhanced sets, obtaining a category MSet containing (a 'copy' of) Set as a full subcategory. (There may be more than one reasonable way to do this! This is intentionally an open-ended exercise.) Which objects in MSet determine ordinary multisets as defined in §2.2 and how? Spell out what a morphism of multisets would be from this point of view. (There are several natural motions of morphisms of multisets. Try to define morphisms in MSet so that the notion you obtain for ordinary multisets captures your intuitive understanding of these objects.) [§2.2, §3.2, 4.5]

Solution. Define $\mathrm{Obj}(\mathsf{MSet})$ as all tuples $(S, \sim)$ where S is a set and $\sim$ is an equivalence relation on S. For two multisets $\hat{S} = (S, \sim), \hat{T} = (T, \approx) \in \mathrm{Obj}(\mathsf{MSet})$, we define a morphism $f \in \mathrm{Hom}_{\mathsf{MSet}}(\hat{S}, \hat{T})$ to be a set-function $f : S \to T$ such that, for $x, y \in S$, $x \sim y \implies f(x) \approx f(y)$, and morphism composition the same way as set-functions. Now we verify the axioms:

1. For a multiset $(S, \sim)$, we borrow the set-function $1_S : S \to S$ and note that it necessarily preserves equivalence, i.e. $x \sim y \implies 1_S(x) \sim 1_S(y)$.
2. Let there be objects $\hat{S} = (S, \sim), \hat{T} = (T, \approx), \hat{U} = (U, \cong)$ with morphisms $f \in \mathrm{Hom}_{\mathsf{MSet}}(\hat{S}, \hat{T})$ and $g \in \mathrm{Hom}_{\mathsf{MSet}}(\hat{T}, \hat{U})$. Note that $gf : S \to U$ is a set-function since Set is a category. Now, since f is a morphism in MSet, for $x, y \in S$, if $x \sim y$, then $f(x) \approx f(y)$, and since $f(x), f(y) \in T$ and g is a morphism in MSet, $g(f(x)) \cong g(f(y))$.
3. Associativity can be proven similarly.

Hence MSet as defined above is a category. Now, recall that multisets are defined in §2.2 as a set S and a *multiplicity function* $m : S \to \mathbf{N}$. So, for any set S and function $m : S \to \mathbf{N}$, if we define the equivalence relation corresponding to m as $\sim_m$ then the tuple $(S, \sim_m) \in \mathrm{Obj}(\mathsf{MSet})$. The objects in MSet which *don't* correspond to any multiset as defined in §2.2 are sets S with equivalence relations $\sim$ such that both S and $\mathscr{P}_\sim$ are uncountable; this way, one cannot construct a function $m : S \to \mathbf{N}$ corresponding to each set in the partition $\mathscr{P}_\sim$, since $\mathbf{N}$ is countable. □

Problem 3.10. Since the objects of a category C are not (necessarily) sets, it is not clear how to make sense of a notion of 'subobject' in general. In some situations it *does* make sense to talk about subobjects, and the subobjects of any given object A in C are in one-to-one correspondence with the morphisms $A \to \Omega$ for a fixed, special object Ω of C, called a *subobject classifier.* Show that Set has a subobject classifier.

Solution. Let $A \in \mathrm{Obj}(\mathsf{Set})$. Any set $X \subseteq A$ corresponds to a mapping $A \to \{0, 1\}$; the elements $x \in A$ that are also in X are mapped to 1, and the elements $x \in A$ that aren't in X are mapped to 0. Hence the "subobject classifier" for Set is $\Omega = \{0, 1\}$. □

Problem 3.11. ▷ Draw the relevant diagrams and define composition and identities for the category $\mathsf{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathsf{C}^{\alpha,\beta}$ mentioned in Example 3.10. [§5.5, 5.12]

Solution. Let C be a category, with $A, B \in \mathrm{Obj}(\mathsf{C})$. The objects of $\mathsf{C}^{A,B}$ are then diagrams:

$$\begin{tikzcd}
A \arrow[dr, "f"] & \\
& Z \\
B \arrow[ur, "g"'] &
\end{tikzcd}$$

Namely, tuples (Z, f, g) where $Z \in \mathrm{Obj}(\mathsf{C}), g \in \mathrm{Hom}_{\mathsf{C}}(A, Z)$, and $f \in \mathrm{Hom}_{\mathsf{C}}(B, Z)$. For objects $O_1 = (Z_1, f_1, g_1)$ and $O_2 = (Z_2, f_2, g_2)$ in $\mathrm{Obj}(C^{A,B})$, the morphisms between them are morphisms $\sigma \in \mathrm{Hom}_{\mathsf{C}}(Z_1, Z_2)$ such that $\sigma f_1 = f_2$ and $\sigma g_1 = g_2$. This forms the following commutative diagram:

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
& Z_1 \arrow[r, "\sigma"] & Z_2 \\
B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}$$

Given a third object $O_3 = (Z_3, f_3, g_3)$, with another morphism $\tau : O_2 \to O_3$ (which is a morphism from $Z_2 \to Z_3$), composition in $\mathsf{C}^{A,B}$ is defined the same way as composition in C: $\tau\sigma : Z_1 \to Z_3$. Since σ and τ both commute (i.e. $\sigma f_1 = f_2$, $\sigma g_1 = g_2$, $\tau f_2 = f_3$, and $\tau g_2 = g_3$), then $\tau\sigma$ also commutes: $\tau\sigma f_1 = \tau f_2 = f_3$ and $\tau\sigma g_1 = \tau g_2 = g_3$. This is how we can define composition the same in $\mathsf{C}^{A,B}$ as in C. Diagrammatically, this is like "taking away" the (Z_2, f_2, g_2) object in the joint commutative diagram for σ and τ:

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] \arrow[drrr, bend left, "f_3"] & & & \\
& Z_1 \arrow[r, "\sigma"] & Z_2 \arrow[r, "\tau"] & Z_3 \\
B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] \arrow[urrr, bend right, "g_3"'] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_3"] & & \\
& Z_1 \arrow[r, "\tau\sigma"] & Z_3 \\
B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_3"'] & &
\end{tikzcd}$$

$\square$

Solution. Let C be a category. Fix two morphisms $\alpha \in \mathrm{Hom}_{\mathsf{C}}(C, A)$ and $\beta \in \mathrm{Hom}_{\mathsf{C}}(C, B)$ with the same source C, and where $A, B, C \in \mathrm{Obj}(\mathsf{C})$. We wish to formalize the *fibered* version of $\mathsf{C}^{A,B}$: $\mathsf{C}^{\alpha,\beta}$, where instead of specifying specific objects in C we use morphisms α and β directly.

The objects in $\mathsf{C}^{\alpha,\beta}$ are triples (Z, f, g) where $Z \in \mathrm{Obj}(\mathsf{C})$, $f \in \mathrm{Hom}_{\mathsf{C}}(A, Z)$, and $g \in \mathrm{Hom}_{\mathsf{C}}(B, Z)$ such that $f\alpha = g\beta$; intuitively, starting with object C we can use α and β to map to objects A and B, respectively, and the objects in $\mathsf{C}^{\alpha,\beta}$ specify a fourth object Z and morphisms $f : Z \leftarrow A$ and $g : Z \leftarrow B$ that both map to Z.

Morphisms in $\mathsf{C}^{\alpha,\beta}$ between objects (Z_1, f_1, g_1) and (Z_2, f_2, g_2) are morphisms $\sigma \in \mathrm{Hom}_{\mathsf{C}}(Z_1, Z_2)$ such that everything commutes: $\sigma f_1\alpha = f_2\alpha$ and $\sigma g_1\beta = g_2\beta$. In short, we diverge to A and B from C, then simultaneously converge to Z_1 and Z_2 in such a way that we can continue to Z_2 from Z_1 mapping with σ.

□

4 Morphisms

Problem 4.1. ▷ Composition is defined for *two* morphisms. If more than two morphisms are given, e.g.,

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E$$

then one may compose them in several ways, for example,

$$(ih)(gf), \quad (i(hg))f, \quad i((hg)f), \quad \text{etc.}$$

so that at every step one is only composing two morphisms. Prove that the result of any such nested composition is independent of the placement of the parentheses.

Solution. For three morphisms f, g, h in a category C:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we have that $(hg)f = h(gf)$ due to C being a category. Now, fix $n \geq 4$ and suppose that all parenthesizations of $n-1$ morphisms are equivalent. Imagine that $f_1, \ldots, f_n$ are morphisms in a category C:

$$Z_1 \xrightarrow{f_1} Z_2 \xrightarrow{f_2} \cdots Z_n \xrightarrow{f_n} Z_{n+1}$$

Suppose that some parenthesization of $f_n, f_{n-1}, \ldots, f_1$ is f and furthermore that $f = hg$, where h is some parenthesization of $f_n, \ldots, f_{i+1}$, and g is some parenthesization of $f_i, \ldots, f_1$, where $1 \leq i \leq n$. Since h and g are parenthesizations of $n-i$ and i morphisms, respectively, they can be written in the following forms:

$$h = ((\cdots((f_n f_{n-1})f_{n-2})\cdots)f_{i+1})$$

$$g = (f_i(f_{i-1}(\cdots(f_2 f1)\cdots))) = f_i g'$$

in hence $f = hg = h(f_i g') = (hf_i)g'$. Inductively, we can "pop" morphisms off the left hand side of g' and add them to the right hand side of h, resulting in the canonical form:

$$f = ((\cdots((f_n f_{n-1})f_{n-2})\cdots)f_1)$$

□

Problem 4.2. $\triangleright$ In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (c.f. Example 4.6)? [§4.1]

Solution. Recall that, in order to construct a category from a set S endowed with a reflexive and transitive relation $\sim$, we take $\text{Obj}(\mathsf{C}) = S$ and set $\text{Hom}_{\mathsf{C}}(x, y) = (x, y)$ if $x \sim y$, and $\emptyset$ otherwise. This way every object x has a unique identity morphism $\text{Hom}_{\mathsf{C}}(x, x) = (x, x)$ since $\sim$ is reflexive. Composition of morphisms can be defined since $\sim$ is transitive: for morphisms (x, y) and (y, z), their composition is $(y, z)(x, y) = (z, x)$.

Now, a groupoid is a category in which every morphism is an isomorphism. We will show that we can take a pair $(S, \sim)$ where S is a set and $\sim$ is an *equivalence relation* in order to create a groupoid C. Since $\sim$ is reflexive and transitive, we have seen above how to construct a category C. In order to show that every morphism is an isomorphism, let $(x, y) \in \text{Hom}_{\mathsf{C}}(x, y)$. Hence $x \sim y \implies y \sim x$, since $\sim$ is an equivalence relation. Now, $(x, y)(y, x) = (y, y)$ and $(y, x), (x, y) = (x, x)$ due to the morphism composition rule above; hence (x, y) is an isomorphism, so C is a groupoid. $\square$

Problem 4.3. Let A, B be objects of a category C, and let $f \in \text{Hom}_{\mathsf{C}}(A, B)$ be a morphism.

- Prove that if f has a right-inverse, then f is an epimorphism.
- Show that the converse does not hold, by giving an explicit example of a category and an epimorphism without a right-inverse.

Solution. Let A, B, C, and f be as above.

- Suppose that f has a right-inverse $g : B \to A$ so that $f \circ g : B \to B = \text{id}_B$. Let $Z \in \text{Obj}(\mathsf{C})$ and $\beta', \beta'' : A \to Z$, and suppose that $\beta' \circ f = \beta'' \circ f$. Then we apply g to both sides to get $\beta' \circ (f \circ g) = \beta'' \circ (f \circ g) \implies \beta' \circ \text{id}_B = \beta'' \circ \text{id}_B$ since $fg = \text{id}_B$, which in turn implies that $\beta' = \beta''$ since id_B is the identity.
- Let C be such that $\text{Obj}(\mathsf{C}) = \mathbf{Z}$, $\text{Hom}_{\mathsf{C}}(a, b) = \{(a, b)\}$ if $a \leq b$ and $\emptyset$ otherwise, and for any objects a, b, c and morphisms $f : a \to b$ and $g : b \to c$, define $g \circ f = \{(c, a)\}$. Then every morphism $f \in \text{Hom}_{\mathsf{C}}(a, b)$ is an epimorphism; this is given in the text. However, if $f : a \to b = (a, b)$ for $a \neq b$ (hence $a \leq b$,) we have that $\text{Hom}_{\mathsf{C}}(b, a) = \emptyset$; so f in general. This implies that epimorphisms do not in general have right inverses.

$\square$

Problem 4.4. Prove that the composition of two monomorphisms is a monomorphism. Deduce that one can define a subcategory C_{mono} of a category C by taking the same objects as in C and defining $\text{Hom}_{\mathsf{C}_{\text{mono}}}(A, B)$ to be the subset of $\text{Hom}_{\mathsf{C}}(A, B)$

consisting of monomorphisms, for all objects A, B. (Cf. Exercise 3.8; of course, in general $\mathsf{C}_{\mathsf{mono}}$ is not full in C.) Do the same for epimorphisms. Can you define a subcategory $\mathsf{C}_{\mathsf{nonmono}}$ of C by restricting to morphisms that are not monomorphisms?

Solution. Let C be a category with $A, B, C \in \mathrm{Obj}(\mathsf{C})$, and let $f : A \to B$ and $g : B \to C$ be monomorphisms. Let $Z \in \mathrm{Obj}(\mathsf{C})$ and $\alpha', \alpha'' : Z \to A$. Suppose $gf\alpha' = gf\alpha''$. Since g is a mono, $f\alpha' = f\alpha''$. Since f is a mono, $\alpha' = \alpha''$. Therefore $(gf)\alpha' = (gf)\alpha'' \implies \alpha' = \alpha''$, so gf is a mono.

This means that we can take the category $\mathsf{C}_{\mathsf{mono}}$ as detailed in the question. Since identities are isomorphisms, they are also monomorphisms, so we still have identities. We just proved that the composition of monomorphisms is a monomorphism, so the composition of any two appropriate monomorphisms in $\mathsf{C}_{\mathsf{mono}}$ between, say A and B, and B and C, respectively, will also be a monomorphism hence in $\mathrm{Hom}_{\mathsf{C}_{\mathsf{mono}}}(A, C)$, so composition "works" in $\mathsf{C}_{\mathsf{mono}}$.

The $\mathsf{C}_{\mathsf{nonmono}}$ as described above is not a category since it doesn't have any identities (since all identities are monomorphisms.)

Now, fix $f : A \to B$ and $g : B \to C$ to be epimorphisms. Let $Z \in \mathrm{Obj}(\mathsf{C})$ and $\beta', \beta'' : C \to Z$. Suppose $\beta' gf = \beta'' gf$. Since f is an epi, $\beta' g = \beta'' g$. Since g is an epi, $\beta' = \beta''$. Hence gf is an epi as above.

By the same reasoning as above we deduce that $\mathsf{C}_{\mathsf{epi}}$ is a category and $\mathsf{C}_{\mathsf{nonepi}}$ is not a category. □

Problem 4.5. Give a concrete description of monomorphisms and epimorphisms in the category MSet you constructed in Exercise 3.9. (Your answer will depend on the notion of morphism you defined in that exercise!)

Solution. Recall that, for two multisets $\hat{S} = (S, \sim), \hat{T} = (T, \approx)$ (where S, T are sets and $\sim, \approx$ are equivalence relations on S and T, respectively,) we defined a morphism $f : \hat{S} \to \hat{T}$ in MSet to be a normal set-function except with the extra condition that for any $s, s' \in S$, we require that f preserves equivalence, so if $s \sim s'$ then $f(s) \approx f(s')$.

The notions of monomorphism and epimorphism transfer over as follows.

1. A multiset-function $f : \hat{S} \to \hat{T}$ is a monomorphism iff for all $s_1, s_2 \in S$, if $f(s_1) \approx f(s_2)$ then $s_1 \sim s_2$.
2. A multiset-function $f : \hat{S} \to \hat{T}$ is an epimorphism iff for all $t \in T$, there is an $s \in S$ such that $f(s) \approx t$.

(Since $\sim$ and $\approx$ are equivalence relations and since these definitions are analogous to monos and epis in Set, the proof that these really are monos and epis is analogous.)

□

5 Universal Properties

Problem 5.1. Prove that a final object in a category C is initial in the opposite category C^{op} (cf. Exercise 3.1).

Solution. Let C be a category and suppose that $A \in \mathrm{Obj}(\mathsf{C})$ is final in C. Then $\mathrm{Hom}_{\mathsf{C}}(Z, A) = \mathrm{Hom}_{\mathsf{C}^{op}}(A, Z)$ is a singleton for all $Z \in \mathrm{Obj}(\mathsf{C})$, so A is initial in C^{op}. □

Problem 5.2. ▷ Prove that $\emptyset$ is the unique initial object in Set. [§5.1].

Solution. Suppose there is another set I which is initial in Set. Then $\emptyset \simeq I$, so $|\emptyset| = 0 = |I|$. But then vacuously we get that $\emptyset = I$ (since all the elements in $\emptyset$ are in I and vice versa,) so $\emptyset$ is the unique initial object in Set. □

Problem 5.3. ▷ Prove that final objects are unique up to isomorphism. [§5.1]

Solution. Let C be a category and F_1, F_2 be two final objects in C. Then there are unique morphisms $f : F_1 \to F_2$ and $g : F_2 \to F_1$. Since there are only one of each identities 1_{F_1} and 1_{F_2}, then necessarily $gf = 1_{F_2}$ and $fg = 1_{F_1}$, hence f is an isomorphism. □

Problem 5.4. What are initial and final objects in the category of 'pointed sets' (Example 3.8)? Are they unique?

Solution. Recall that Set^* is the set of pairs (S, s) where S is a set and $s \in S$. We claim that objects $(\{s\}, s)$, i.e. pointed singleton sets, are the initial and final objects in Set^*. Note that there can be no "empty function" between pointed sets, since each set has to have a point. Suppose $(T, t) \in \mathrm{Obj}(\mathsf{Set}^*)$. Then there is only one function $f : S \to T$ such that $f(s) = t$: the function $f = \{(s, t)\}$. There is also only one function $f : T \to S$, namely the function that maps each element t in T to s. Hence singleton pointed sets are initial and final.

Furthermore, clearly morphisms between pointed sets (S, s) and (T, t) such that $|S|, |T| \geq 2$, there are more than one function $f : S \to T$ and $g : T \to S$: we could take $f(s) = f(s') = t$, or $f(s) = t, f(s') = t'$.

They are not unique; any singleton pointed set is initial and final. □

Problem 5.5. What are the final objects in the category considered in §5.3? [§5.3]

Solution. Consider coslice category A/Set. There was shown in §5.3 that $A/\sim$ is initial object in this category. We know that in Set final objects are singletons. Easy to see, that a pair $(const, \{*\})$ is final object in this coslice category. Indeed, for any object in Set we can consider it with some constant map, which codomain is $\{*\}$. This map is obviously unique, so such pair is indeed the final object. Obviously, such final object is not unique, but as has been proven all terminal objects are isomorphic. □

Problem 5.6. Consider the category corresponding to endowing (as in Example 3.3) the set $\mathbf{Z}^+$ of positive integers with the divisibility relation. Thus there is exactly one morphism $d \to m$ in this category if and only if d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their 'conventional' names? [§VII.5.1]

Solution. Let Div be the above category. Let $m, n \in \mathrm{Obj}(\mathsf{Div})$. We claim that $\gcd(m,n)$ corresponds to a final object (namely $(\gcd(m,n), m, n)$) in $\mathsf{Div}_{m,n}$. Note that for any $z \in \mathrm{Obj}(\mathsf{Div})$ such that $z \mid m$ and $z \mid n$, $z \mid \gcd(m,n)$ (by definition of gcd;) hence $\mathrm{Hom}_{\mathsf{Div}_{m,n}}((z,m,n),(\gcd(m,n),m,n))$ is non-empty. Furthermore, since there can only be at most 1 morphism between any two objects in Div, $(\gcd(m,n),m,n)$ is final. The conventional name for this is the 'greatest common divisor.'

The coproducts in Div are the 'least common multiple'. For any $z \in \mathbf{Z}^+$, if $m \mid z$ and $n \mid z$, then $\mathrm{lcm}(n,m) \mid z$. Hence $((\mathrm{lcm}(m,n),m,n),(z,m,n))$ is the unique morphism from $(\mathrm{lcm}(m,n),m,n)$ in $\mathsf{Div}^{m,n}$, so $(\mathrm{lcm}(m,n),m,n)$ is initial. □

Problem 5.7. Redo Exercise 2.9, this time using Proposition 5.4.

Solution. Suppose A, B, A', B' are sets with $A \cap B = \emptyset$, $A' \cap B' = \emptyset$, $A \cong A'$, and $B \cong B'$. We will show that there are two isomorphic disjoint unions corresponding to $A \cup B$ and $A' \cup B'$.

First, take $i_A : A \to A \cup B, i_A(a) = a$ for all $a \in A$ and analogous for B. Then if Z is a set with morphisms $f_A : A \to Z$ and $f_B : B \to Z$, we can take $\sigma : A \amalg B = A \cup B \to Z, \sigma(x)$ to be $f_A(x)$ if $x \in A$ and $f_B(x)$ otherwise. This is analogous to the proof for disjoint union being a coproduct, hence $A \amalg B = A \cup B$ is a disjoint union.

Second, since $A \cong A'$ and $B \cong B'$, let $f : A \to A'$ and $g : B \to B'$ be isomorphisms. We can take $i_{A'} : A \to A' \cup B', i_{A'}(a) = f(a)$ for all $a \in A$ and similar for $i_{B'}$. Then if Z is a set with morphisms $f_A : A \to Z$ and $f_B : B \to Z$, we can take $\sigma : A' \amalg B' = A' \cup B' \to Z, \sigma(x)$ to be $f_A \circ f^{-1}$ if $x \in A'$ and $f_B \circ g^{-1}$ otherwise (which works since $A' \cap B' = \emptyset$.) Hence $A' \amalg B' = A' \cup B'$ is a disjoint union.

By Proposition 5.4, since both $A \amalg B$ and $A' \amalg B'$ are initial objects in some auxiliary category of Set, they are isomorphic, as required. □

Problem 5.8. Show that in every category C the products $A \times B$ and $B \times A$ are isomorphic, if they exist. (Hint: Observe that they both satisfy the universal property for the product of A and B; then use Proposition 5.4.)

Solution. Let C be a category with products $A \times B$ and $B \times A$. First, consider how $f : A \times B \to B \times A, f(a,b) = (b,a)$ is an isomorphism between $A \times B$ and $B \times A$ (with inverse $f^{-1}(b,a) = (a,b)$.) Since $B \times A$ is a product in C, for each object $Z \in \mathrm{Obj}(\mathsf{C})$ with morphisms $f_B : B \to Z$ and $f_A : A \to Z$, there is a unique morphism $\tau : Z \to B \times A$ such that everything commutes. However, using f we can construct a unique morphism $\sigma : Z \to A \times B$ in terms of τ by taking $\sigma = f^{-1} \circ \tau$. Hence $B \times A$ is a product for $A \times B$ as well, i.e. $B \times A$ is a final object in some auxiliary category.

Hence, by Proposition 5.4, $A \times B$ and $B \times A$ are isomorphic. $\square$

Problem 5.9. Let C be a category with products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of three objects of C ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

Solution. Let C be a category with products, and let $A, B, C \in \mathrm{Obj}(\mathsf{C})$. The three-product is an object $A \times B \times C \in \mathrm{Obj}(\mathsf{C})$ with morphisms $\pi_A : A \times B \times C \to A$, $\pi_B : A \times B \times C \to B$, and $\pi_C : A \times B \times C \to C$ such that for all $Z \in \mathrm{Obj}(\mathsf{C})$ with morphisms $f_A : Z \to A, f_B : Z \to B, f_C : Z \to C$, there is a unique morphism $\sigma : Z \to A \times B \times C$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & A & \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f_A"] \arrow[drr, bend right, "f_B"'] \arrow[ddrrr, bend right, "f_C"'] & A \times B \times C \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"] \arrow[ddrr, bend left, "\pi_C"] & & \\
 & & B & \\
 & & & C
\end{tikzcd}$$

First, we will show that $(A \times B) \times C$ is a three-product. Since $A \times B$ and $Z \times C$ are products, there are a unique morphisms $\tau : A \times B \to Z$ and $\upsilon : Z \times C \to Z$ for every object Z. We can use these two morphisms to build $\sigma : A \times B \times C \to Z$ for any object Z as follows: $\sigma : (A \times B) \times C \to Z, \sigma(a, b, c) = \upsilon(\tau(a, b), c)$. Since υ and τ are well-defined and unique, σ is well-defined and unique. Hence $(A \times B) \times C$ is a three-product.

Now, consider $A \times (B \times C)$. Similarly, this corresponds to unique morphisms $\tau : A \times Z \to Z$ and $\upsilon : B \times C \to Z$ from which we can construct $\sigma : A \times (B \times C) \to Z, \sigma(a, b, c) = \tau(a, \upsilon(b, c))$. By the same logic as above, $A \times (B \times C)$ is a three product.

Thus by Proposition 5.4, $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic. $\square$

Problem 5.10. Push the envelope a little further still, and define products and coproducts for families (i.e., indexed sets) of objects of a category.

Do these exist in Set?

It is common to denote the product $A \times \cdots \times A$ (n times) by A^n.

Solution. Let C be a category and I be a set. Consider $\{A_i\}_{i\in I}$ with each $A_i \in \operatorname{Obj}(\mathsf{C})$. An *infinitary product* $\prod_{i\in I} A_i \in \operatorname{Obj}(\mathsf{C})$ with morphisms $\{\pi_{A_i}\}_{i\in I}$ must satisfy the universal property that, for all $Z \in \operatorname{Obj}(\mathsf{C})$ and morphisms $\{f_{A_i}\}_{i\in I}$, there must be a unique $\sigma : Z \to \prod_{i\in I} A_i$ such that $\sigma\pi_{A_i} = f_{A_i}$ for all $i \in I$.

These should exist in Set as long as we have the axiom of choice. $\square$

Problem 5.11. Let A, resp. B, be a set, endowed with an equivalence relation $\sim_A$, resp. $\sim_B$. Define a relation $\sim$ on $A \times B$ by setting

$$(a_1, b_1) \sim (a_2, b_2) \iff a_1 \sim_A a_2 \text{ and } b_1 \sim_B b_2.$$

(This is immediately seen to be an equivalence relation.)

- Use the universal property for quotients (§5.3) to establish that there are functions
$$(A \times B)/{\sim} \to A/{\sim_A}, (A \times B)/{\sim} \to B/{\sim_B} .$$
- Prove that $(A\times B)/{\sim}$, with these two functions, satisfies the universal property for the product of $A/{\sim_A}$ and $B/{\sim_B}$.
- Conclude (without further work) that $(A \times B)/{\sim} \cong (A/{\sim_A}) \times (B/{\sim_B})$.

Solution. Let $A, B, \sim, \sim_A, \sim_B$ be as above. Let $\pi_A : A\times B \to A$ and $\pi_B : A\times B \to B$ be the product canonical projections for A and B. Let $\pi^Z_\sim : Z \to Z/\sim$ be the canonical quotient mapping for all objects Z and equivalence relations $\sim$. Then we can apply the universal property for quotients twice to get the required two functions:

$$\begin{tikzcd}
(A \times B)/{\sim} \arrow[rr, "\overline{\pi^A_{\sim_A} \circ \pi_A}"] & & A/{\sim_A} \\
& A \times B \arrow[ul, "\pi^{A\times B}_\sim"] \arrow[ur, "\pi^A_{\sim_A} \circ \pi_A"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B)/{\sim} \arrow[rr, "\overline{\pi^B_{\sim_B} \circ \pi_B}"] & & A/{\sim_A} \\
& A \times B \arrow[ul, "\pi^{A\times B}_\sim"] \arrow[ur, "\pi^B_{\sim_B} \circ \pi_B"'] &
\end{tikzcd}$$

Now, we wish to show that $(A\times B)/{\sim}$ satisfies the universal property for the product of $A/{\sim_A}$ and $B/{\sim_B}$. Rename the two functions proved above to be $*^A$ and $*^B$. Let Z be a set with morphisms $f_A : Z \to A/{\sim_A}$ and $f_B : Z \to B/{\sim_B}$. We wish to construct a function σ so that the following diagram commutes:

$$\begin{tikzcd}
 & & A/{\sim_A} \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f_A"] \arrow[drr, bend right, "f_B"'] & (A \times B)/{\sim} \arrow[ur, "*^A"] \arrow[dr, "*^B"'] & \\
 & & B/{\sim_B}
\end{tikzcd}$$

First, define $f : Z \to A/\sim_A \times B/\sim_B$ to be $f(z) = (f_A(z), f_B(z))$. Next, we observe that we can use the quotient universal property with $A/\sim_A$ to get a map $\overline{1_A} : A/\sim_A \to A$ and likewise for $\overline{1_B} : B/\sim_B \to B$. Define $\overline{1_{A\times B}} : A/\sim_A \times B/\sim_B \to A \times B$ to be $\overline{1_{A\times B}}([a]_{\sim_A}, [b]_{\sim_B}) = (\overline{1_A}([a]_{\sim_A}), \overline{1_B}([b]_{\sim_B}))$. Finally, we can take $\sigma = \pi_{\sim}^{A\times B} \circ \overline{1_{A\times B}} \circ f : Z \to (A \times B)/\sim$ to satisfy the universal property for product of $A/\sim_A$ and $B/\sim_B$ (it is uniquely determined by its respective pieces.)

Therefore, by Proposition 5.4, $(A \times B)/\sim \cong A/\sim_A \times B/\sim_B$. $\square$

Problem 5.12. Define the notions of fibered products and fibered coproducts, as terminal objects of the categories $\mathsf{C}^{\alpha,\beta}, \mathsf{C}_{\alpha,\beta}$ considered in Example 3.10 (cf. also Exercise 3.11), by stating carefully the corresponding universal properties.

As it happens, Set has both fibered products and coproducts. Define these objects 'concretely', in terms of naive yet theory. [II.2.9, III.6.10, III.6.11]

Solution. I'm not really sure how to phrase this yet. I think I'll come back to it later when we've dealt with fibered categories a little more. $\square$