Multiparameter persistence computation: review

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In topological data analysis, one may use persistent homology to extract the topological features of the data. Its construction isn't too hard: one needs to build a simplicial complex K over the data, obtain a filtration of the complex (see fig. 1), that is a sequence of its subspaces $K_0 \subseteq K_1 \subseteq ... \subseteq K_n = K$, and then apply the i-th homology functor with coefficients in some field k to it, obtaining the next sequence of modules:

$$H_i(K_0) \to H_i(K_1) \to ... \to H_i(K).$$

In such setting, the algorithm for computing the persistent homology relies on Gaussian elimination and has its roots in the structure theorem for finitely generated modules over a PID. This is because the persistent module, as above, can be seen as a finitely generated module over $k[t]$. This gives the decomposition of persistent homology into the direct sum of indecomposable summands, which are usually called *interval modules*.

But in practice we often encounter richer structures that are described by multiple parameters. Such structures may be modeled with *multifiltrations*, like on the fig. 2. Such setting is much more complicated [2]. But a complete solution to the problem of computing multiparameter persistence[1] exists and is provided by the Gröbner bases.

Let's give the precise definitions. We say that a topological space X is *multifiltered* if we're given a family of subspaces $\{X_v\}_{v \in \mathbb{N}^n}$ with inclusions $X_u \subseteq X_w$ whenever $u \leq w$ so that the diagrams

$$\begin{tikzcd} X_v \arrow[r] \arrow[d] & X_{v1} \arrow[d] \\ X_{v2} \arrow[r] & X_w \end{tikzcd}$$

commute. We will consider such multifiltered complexes, where each has has a unique minimal critical grade at which it enters the complex. Such multifiltrations are called *one-critical* and mostly arise in practice.

Given a multifiltration $\{X_u\}_u$, $i : X_u \to X_v$ induces a map $i_* : H_i(X_u) \to H_i(X_v)$ at the homology level. The *ith persistent homology* H_i^{pers} is the image of i_* for all pairs $i \leq v$.

As was mentioned above, in the setting with a single filtration, persistent homology corresponds to a graded $k[t]$-module. In the same way, persistent homology in the multifiltered setting corresponds to a finitely generated n-graded module over $k[t_1, ..., t_n]$. Moreover, the next theorem holds:

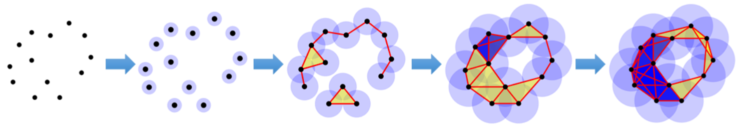

Figure 1 Example of a filtration of a simplicial complex built over the data. Image credit: [3]

[1] The name "multiparameter persistence" is due to H. Schenck [5], in the work of Carlsson et al. this is called "multidimensional persistence"

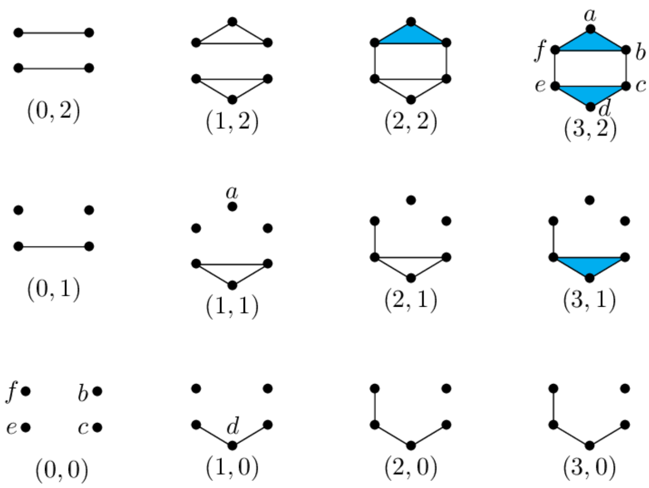

Figure 2 An example of a bifiltration of a complex at coordinate $(3, 2)$. Image credit: [4]

Theorem 1 (Realization [2]). *Let $k = \mathbb{F}_p$, $i \in \mathbb{N}$, M be an n-graded module over $k[t_1, ..., t_n]$. Then there's a multifiltered finite simplicial complex X so that $H_i^{pers}(X, k) \cong M$.*

One can build the *ith chain module* over $k[t_1, ..., t_n]$ of a multifiltered complex $\{K_u\}_u$ as

$$C_i = \bigoplus_u C_i(K_u),$$

where the k-module structure is provided via the universal property of direct sum and $x^{v-u} : C_i(K_u) \to C_i(K_v)$ is the inclusion $K_u \to K_v$. For one-critical filtrations, these modules are free; the *standard basis* for the ith chain module C_i is given by the set of i-simplices in critical grades.

Then, given standard bases, we may write the boundary operator $\delta_i : C_i \to C_{i-1}$ explicitly as a matrix with polynomial entries. This gives us a new n-graded chain complex that encodes the multifiltration. The homology of this chain complex is precisely the persistent homology of the multifiltration. By definition, homology can be computed in three steps:

1. Compute im δ_{i+1}: this problem is the *submodule membership problem*, which may be solved by computing the *reduced Gröbner bases* using the `Buchberger`, reduction and division algorithms.

2. Compute ker δ_i: the *(first) syzygy module* can be computer using `Schreyer's` algorithm

3. Compute H_i: one the above two tasks are complete, this is simple: we need to test whether the generators of the syzygy submodule are in the boundary submodule.

The submodule membership problem is a generalization of the *Polynomial Ideal Membership Problem*, which is `Exspace`-complete. But the multifiltrations provide the additonal structure that is used to simplify the algorithms; they key property is *homogeneity*: a matrix M with monomial entries is *homogeneous* if:

1. every column f of M is associated with a coordinate in the multifiltration u_f, and thus a corresponding monomial x^{u_f},
2. every non-zero element M_{jk} may be expressed as the quotient of the monomials associated with column k and row j. resp.

Any vector f endowed with a coordinate u_f that may be written as above is *homogeneous.*

With this in mind, one can simplify the algorithms [1]:

Lemma 1. *For a one-critical multifiltration, the matrix of $\delta_i : C_i \to C_{i-1}$ written in terms of the standard bases is homogeneous.*

Corollary 1. *For a one-critical multifiltration, the boundary matrix δ_i in terms of the standard bases has monomial entries.*

Lemma 2. *The S-polynomial $S(f, g)$ of homogeneous vectors f and g is homogeneous.*

Lemma 3. *The reminder of the division of homogeneous vector f by the tuple of the homogeneous vectors $(f_1, ..., f_t)$ is homogeneous.*

Theorem 2 (homogeneous Gröbner)**.** *The `Buchberger` algorithm computes a homogeneous Gröbner basis for a homogeneous matrix.*

Theorem 3 (homogeneous syzygy)**.** *For a homogeneous matrix, all matrices encountered in the computation of the syzygy module are homogeneous.*

Using optimization techniques (e.g. proper data structures), one can achieve the following result:

Theorem 4 ([1])**.** *Multiparameter persistence may be computed in polynomial time.*

References

[1] Carlsson, G., Singh, G., Zomorodian, A. (2009). Computing Multidimensional Persistence. In: Dong, Y., Du, DZ., Ibarra, O. (eds) Algorithms and Computation. ISAAC 2009. Lecture Notes in Computer Science, vol 5878. Springer, Berlin, Heidelberg.

[2] Carlsson, G., Zomorodian, A. The Theory of Multidimensional Persistence. Discrete Comput Geom 42, 71–93 (2009).

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