Problem list 1

April 18, 2022

1 Problem list 1

1.1 Lecture 2: Polynomials and Affine Space

1. Consider the polynomial $g(x, y) = x^2y + y^2x \in \mathbb{F}_2[x, y]$. Show that $g(x, y) = 0$ for every $(x, y) \in \mathbb{F}_2^2$, and explain why this does not contradict Proposition 5.

∘ Let's just use Sage in order to check if this polynomial is zero-function:

[1]:
```
R.<x,y> = PolynomialRing(GF(2))
poly = x**2 * y + y**2 * x
for el in AffineSpace(R):
    print(poly(*el))
```

```
0
0
0
0
```

So, $g(x, y) = 0$. This doesn't contradict Prop. 5 though since $\mathbb{F}_2$ is not an infinite field. •

2. Find a nonzero polynomial in $\mathbb{F}_2[x, y, z]$ which vanishes at every point of $\mathbb{F}_2^3$. Try to find one involving all three variables.

∘ Seems, $g(x, y, z) = x(y + z) + y(x + z) + z(x + y)$ does the job:

[2]:
```
R.<x,y,z> = PolynomialRing(GF(2))
poly = x * (y + z) + y * (x + z) + z * (x + y)
for el in AffineSpace(R):
    print(poly(*el))
```

```
0
0
0
0
0
0
0
0
```

3. Consider $f(x, y, z) = x^5y^2z - x^4y^3 + y^5 + x^2z - y^3z + xy + 2x - 5z + 3$.

(a) Write f as a polynomial in x with coefficients in $\mathbb{K}[y,z]$.

(b) Write f as a polynomial in y with coefficients in $\mathbb{K}[x,z]$.

(c) Write f as a polynomial in z with coefficients in $\mathbb{K}[x,y]$.

- (a): $f(x) = (y^2z)x^5 - (y^3)x^4 + (z)x^2 + (y+2)x + (y^5 - y^3z - 5z + 3)$,

(b): $f(y) = y^5 - (x^4+z)y^3 + (x^5z)y^2 + (x)y + (x^2z + 2x - 5z + 3)$,

(c): $f(z) = (x^5y^2 + x^2 - y^3 - 5)z - (x^4y^3 + y^5 + xy + 2x + 3)$. •

1.2 Lecture 3: Affine Varieties

1. In the plane $\mathbb{R}^2$, draw a picture to illustrate

$$V(x^2+y^2-4) \cap V(xy-1) = V(x^2+y^2-4, xy-1),$$

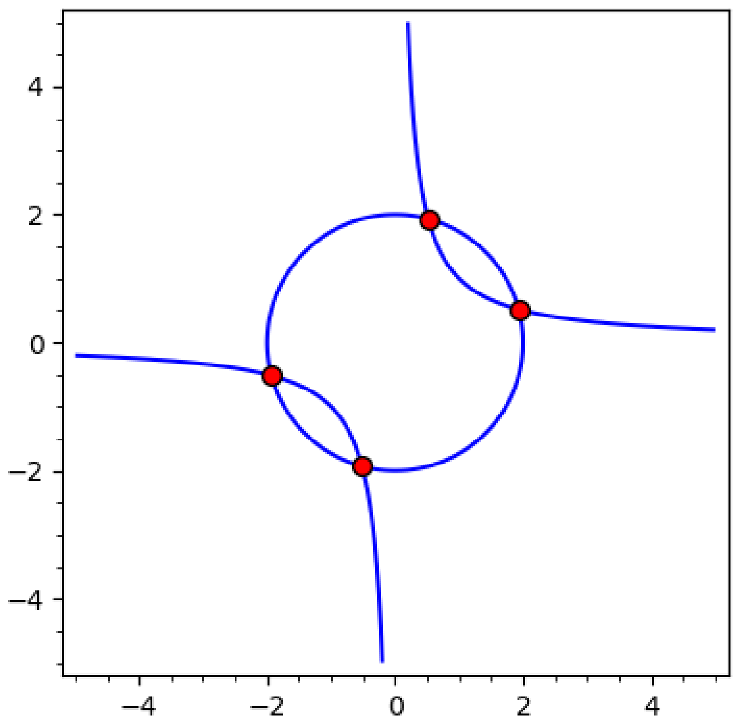
and determine the points of intersection.

- Let's draw it:

[3]:
```
x,y = var('x y')
poly1 = x**2 + y**2 - 4
p = [implicit_plot(poly1, xrange=(-5,5), yrange=(-5,5))]

poly2 = x * y - 1
p.append(implicit_plot(poly2, xrange=(-5,5), yrange=(-5,5)))

R.<x,y> = RR[]
J = R.ideal(x**2 + y**2 - 4,x * y - 1) #intersection = solve([poly1, poly2],␣
 ↪[x,y])

int_x = [j[x] for j in J.variety()] #[point[0].rhs() for point in intersection]
int_y = [j[y] for j in J.variety()] #[point[1].rhs() for point in intersection]
p.append(scatter_plot(list(zip(int_x, int_y)), facecolor='red'))
sum(p)
```

```
verbose 0 (2285: multi_polynomial_ideal.py, variety) Warning: falling back to
very slow toy implementation.
verbose 0 (2285: multi_polynomial_ideal.py, variety) Warning: falling back to
very slow toy implementation.
```

[3]:



So, blue varieties – are exactly the varieties in left hand-side. Four red points are the variety on the right hand-side, and, of course, is the intersection of blue ones •

2. Sketch $V((x-2)(x^2-y), y(x^2-y), (z+1)(x^2-y))$ in $\mathbb{R}^3$.

∘ From the previous problem,

$$V((x-2)(x^2-y), y(x^2-y), (z+1)(x^2-y)) = V((x-2)(x^2-y) \cap V(y(x^2-y)) \cap V((z+1)(x^2-y))$$

So we can look at all these affine varieties, more concretely, at their intersections

[4]:
```
x,y,z = var('x,y,z')

V = [(x-2)*(x^2-y), y*(x^2-y), (z+1)*(x^2-y)]
c=['red','blue', 'green']

p=add([implicit_plot3d(V[i],[x,-10,10],[y,-10,10],[z,-10,10], color=c[i]) for i␣
 ↪in [0..2]])

show(p)
```

```
Graphics3d Object
```

Let's solve the appropriate system and plot the variety itself using Sage:

[5]:
```
soln = solve([(x-2)*(x^2-y), y*(x^2-y), (z+1)*(x^2-y)], [x,y,z])
print(soln)

plots = []

for sol in soln:
    plots.append(parametric_plot3d(list(map(lambda x: x.rhs() ,sol)), (-5,5),
 ↪(-5, 5), color='orange'))

sum(plots)
```

```
[
[x == r1, y == r1^2, z == r2],
[x == 0, y == 0, z == -1],
[x == 2, y == 0, z == -1]
]
```

[5]: `Graphics3d Object`

3. Let us show that all finite subsets of $\Bbbk^n$ are affine varieties

(a) Prove that a single point $(a_1, ..., a_n) \in \Bbbk^n$ is an affine variety.

(b) Prove that every finite subset of $\Bbbk^n$ is an affine variety

∘ Let's start with (a):

(a): Recall that V is an affine variety if $\exists f_1, ..., f_s \in \Bbbk[x_1, ..., x_n]$ such that $\forall v \in V : f_i(v) = 0$. Thus a single point $(a_1, ..., a_n)$ is surely an affine variety(or, to be correctly, *the set* that consists of that one point): we need just to show appropriate system of polynomial equations such that it has only one solutions – this point. The system

$$\begin{cases} x_1 - a_1 = 0, \\ x_2 - a_2 = 0, \\ \ldots \\ x_n - a_n = 0 \end{cases}$$

is such one.

(b): We know, that if V and W are affine varieties, then so is $V \cup W$.

Consider a finite subset V of $\Bbbk^n$. Since V is finite, $V = \{v_1, ..., v_n\}$. Each $\{v_i\}$ is an affine variety by (a). Then so is $V = \bigcup \{v_i\}$. •

4. Let $R = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ be the upper half plane. Prove that R is not an affine variety.

∘ Suppose R is an affine variety. Let V be an affine variety $V(y-x^2)$, which is just a line that passes through $(0, 0)$, so it's just a variety defined by $f(x, y) = y - x^2$. Then by previous considerations, $C = R \cap V$ is an affine variety as well. Now, consider $g(t) = f(t, t^2)$. Then g is zero-function on C, so it's a zero polynomial. Thus, $g(0) = f(0, 0) = 0$, and therefore R is not an affine variety. •

5. Let $V \subset \Bbbk^n$ and $W \subset \Bbbk^m$ be two affine varieties, and let

$$V \times W = \{(x_1, ..., x_n, y_1, ..., y_m) \in \Bbbk^{n+m} | (x_1, ..., x_n) \in V, (y_1, ..., y_m) \in W\}$$

be their Cartesian product. Prove that $V \times W$ is an affine variety in $\Bbbk^{n+m}$.

∘ Since V, W are varieties, then let $V = V(f_1, ..., f_s)$ and $W = V(g_1, ..., g_r)$, where $f_i \in \Bbbk[x_1, ..., x_n]$ and $g_i \in \Bbbk[y_1, ..., y_m]$. Then we can regard f_i and g_i as polynomials in $\Bbbk[x_1, ..., x_n, y_1, ..., y_m]$.

Then consider $V(f_1, ..., f_s, g_1, ..., g_r)$. I claim that $V \times W = V(f_1, ..., f_s, g_1, ..., g_r)$.

The $\subseteq$ part: let $(a_1, ..., a_n, b_1, ..., b_m) \in V \times W$. Clearly, both f_i and g_i vanish at this point.

The $\supseteq$ part: let $(a_1, ..., a_n, b_1, ..., b_m) \in V(f_1, ..., f_s, g_1, ..., g_r)$. Then it's also clear that $(a_1, ..., a_n, b_1, ..., b_m) \in V \times W$ since f_i vanishes on $(a_1, ..., a_n)$ (then $(a_1, ..., a_n) \in V$)and g_i vanishes on $(b_1, ..., b_m)$ (then $(b_1, ..., b_m) \in W$). •

1.3 Lecture 4: Parametrizations of Affine Varieties

1. Given $f \in \Bbbk[x]$, find a parametrization of $V(y - f(x))$.

∘ Parametrization of $V(y - f(x))$ is a system

$$\begin{cases} x = r_1(t), \\ y = r_2(t) \end{cases}$$

Since $V(y - f(x)) = \{(x, y) \in \Bbbk^2 : y - f(x) = 0\}$, the next parametrization should work:

$$\begin{cases} x = t, \\ y = f(t) \end{cases}$$

2. Consider the curve defined by $y^2 = cx^2 - x^3$, where c is some constant.

(a) Show that a line will meet this curve at either 0, 1, 2, or 3 points. Illustrate your answer with a picture.

(b) Show that a nonvertical line through the origin meets the curve at exactly one other point when $m^2 \neq c$. Draw a picture to illustrate this, and see if you can come up with an intuitive explanation as to why this happens.

(c) Now draw the vertical line $x = 1$. Given a point $(1, t)$ on this line, draw the line connecting $(1, t)$ to the origin. This will intersect the curve in a point (x, y). Draw a picture to illustrate this, and argue geometrically that this gives a parametrization of the entire curve.

(d) Show that the geometric description from part (c) leads to the parametriza- tion

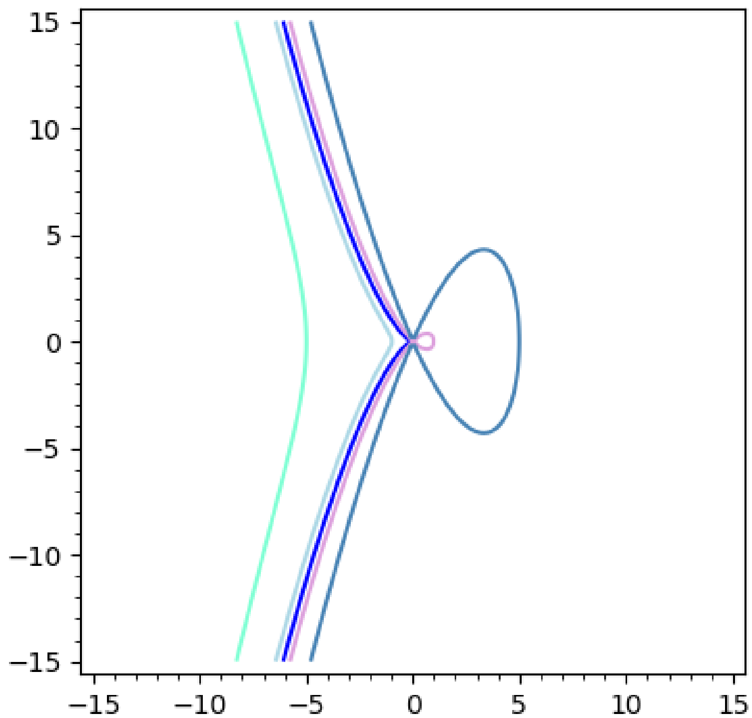
$$\begin{cases} x = c - t^2, \\ y = t(c - t^2). \end{cases}$$

∘ Let's start with (a):

Let's at first look at this curve with different values of c:

[6]:
```
x,y = var('x,y')

cs = [-5, -1, 0, 1, 5]
colors = ['aquamarine', 'automatic', 'blue', 'plum', 'steelblue']
plots = []

for i in range(len(cs)):
    plots.append(
        implicit_plot(y^2 == cs[i]*x^2 - x^3, (x,-15,15), (y,-15,15), color =
 ↪colors[i])
    )

sum(plots)
```

[6]:



Easy to see, that there exist lines, that meet the curve at $0, 1, 2$ or 3 points.

Formally, let $x = a$ for some a. Then $y^2 = a^2(c - a)$. Then there are 3 cases: * $a > 0$: RHS is negative, and so line doesn't meet the curve * $a = 0$: The only meeting poing then is $(0, 0)$. * $a < 0$: RHS is positive, so line meets the curve at 2 points

If $y = mx + b$, then $(mx + b)^2 = cx^2 - x^3$, which simplifies to

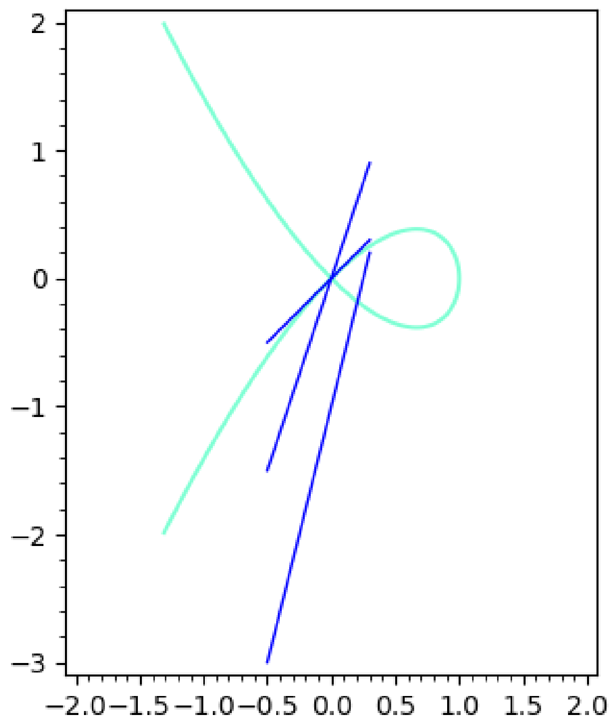
$$x^3 + (m^2 - c)x^2 + 2mbx + b^2 = 0.$$

And by fundamental theorem of algebra, it has at most 3 roots.

(b): Let's draw a picture first. For this, fix $c = 1$ and draw several lines.

[29]:
```
x,y = var('x,y')

curve = y^2 == x^2 - x^3
lines = [x, 3*x, 4*x-1]

plots = [implicit_plot(curve, (x,-2,2), (y,-2,2), color = 'aquamarine')]

for line in lines:
    plots.append(plot(line, -0.5, 0.3))

sum(plots)
```

[29]:



Let's work it out formally: we have

$$x^2(x + (m^2 - c)) = 0$$

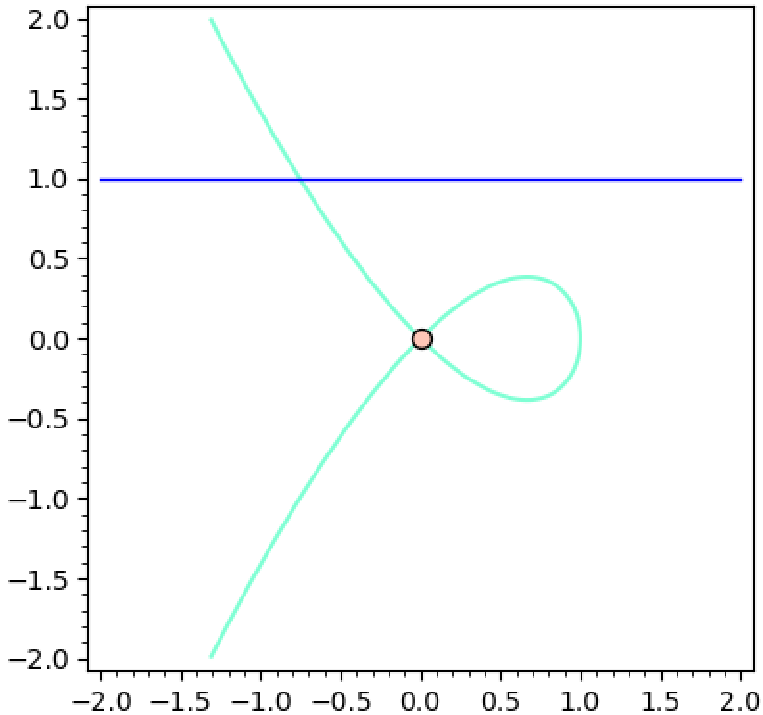
So the roots are only 0 and $m^2 - c$

(c): Let's plot again:

[35]:
```
x,y = var('x,y')

curve = y^2 == x^2 - x^3
```

```
plots = [implicit_plot(curve, (x,-2,2), (y,-2,2), color = 'aquamarine'),
         plot(1, -2,2),
         scatter_plot([(0,0)])]


sum(plots)
```

[35]:



Let's consider $(1, t)$ which lies on the line $x = 1$ and connect it with $(0, 0)$. Then the line that passes through these 2 points has the equation

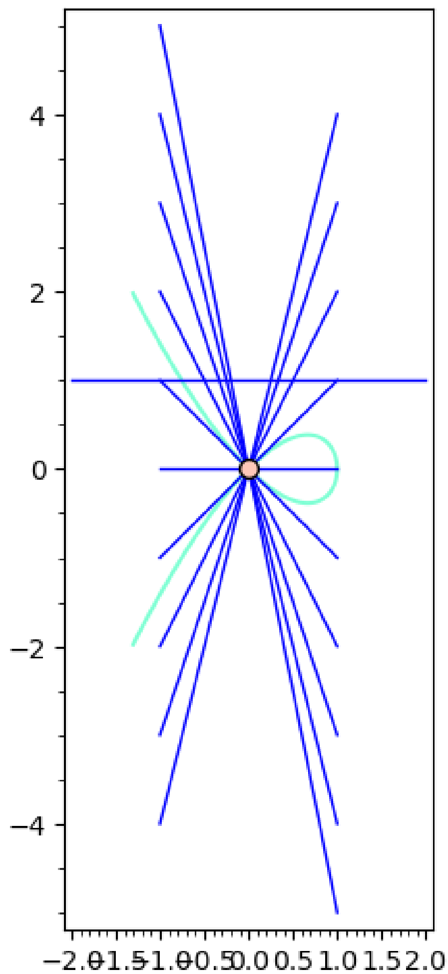
$$y = tx$$

Let's draw it:

[61]:
```
x,y, t = var('x,y, t')

curve = y^2 == x^2 - x^3
connect_line = t*x

plots = [implicit_plot(curve, (x,-2,2), (y,-2,2), color = 'aquamarine',␣
 ↪figsize=8),
         plot(1, -2,2),
         scatter_plot([(0,0)])]

for p in range (-5,5):
```

```
    plots.append(plot(connect_line.subs(t==p), -1,1))


sum(plots)
```

[61]:



as t ranges from $-\infty$ to ∞ the line meets the curve at all points.

(d): Let's parametrize the curve:

Since $y = tx$, we can put it in the equation:

$$t^2x^2 = cx^2 - x^3,$$

cancelling out x^2, we get that $x = c - t^2$, thus $y = t(c - t^2)$.

1.4 Lecture 5: Ideals

1. Show that $V(x + xy, y + xy, x^2, y^2) = V(x, y)$.

∘ Observe that $x+xy = x(1+y)$, and therefore $x+xy \in <x>$. Similarly, $y+xy = y(1+x) \in <y>$; $x^2 \in <x>$ and $y^2 \in <y>$, which is obvious. Thus, $<x,y> = <x+xy, y+xy, x^2, y^2>$, therefore varieties are equal. •

2. Show that $I(V(x^n, y^m)) = \langle x, y \rangle$ for any positive integers n and m

∘ From Hilbert's Nullstellensatz, $I(V(\alpha)) = \sqrt{(}\alpha)$, where $\sqrt{(}\alpha)$ is the radical of the ideal. And $\sqrt{(}<x^n, y^m>) = <x,y>$.

Another way: Since $V(x^n, y^m) = V(x^n) \cap V(y^m)$, we can write down each variety: * $V(x^n) = \{(0,y) : y \in \Bbbk\} = V(x)$ * $V(y^m) = \{(x,0) : x \in \Bbbk\} = V(y)$

Thus, $V(x^n, y^m) = V(x^n) \cap V(y^m) = V(x) \cap V(y) = V(x,y)$, therefore $<x,y> \subsetneq I(V(x,y)) = I(V(x^n, y^m))$

For converse inclusion, notice that $(0,0) \in V(x^n, y^m)$, so any polynomial $f \in I(V(x^n, y^m))$ can't have nonzero constant term, hence $f \in <x,y>$. Thus, $I(V(x^n, y^m)) = <x,y>$•

3. Let $V \subset \mathbb{R}^3$ be the curve parametrized by (t^2, t^3, t^4).

(a) Prove that V is an affine variety.

(b) Determine $I(V)$.

∘ Let's start with (a): In order to prove that V is an affine variety, I need to find the defining polynomials $f_1, f_2, ..., f_s$ in $\Bbbk[x,y,z]$ such that $V(f_1, ..., f_s) = V$.

So, since $x = t^2$, $y = xt$ and $z = x^2$, so $f_1 = z - x^2$. To find another polynomial, consider what we have now:

$$\begin{cases} x = t^2, \\ y = xt. \end{cases}$$

So $t = \frac{y}{x}$, and $x = \frac{y^2}{z}$, from which we can derive that $f_2 = y^2 - xz$. Thus,

$$V \subseteq V(y^2 - xz, z - x^2)$$

To prove the converse, let $P = (a,b,c) \in V(y^2 - xz, z - x^2)$, so $ac = b^2$ and $a^2 = c$. If $a = 0$, then $P = (0,0,0)$ and in this case, $P \in V(\text{corr. to } t = 0)$. If $a \neq 0$, then because of the relations, let $r = \frac{b}{a}$, and then

$$P = (a,b,c) = (r^2, r^3, r^4).$$

Hence, $V = V(y^2 - xz, z - x^2)$.

[73]:
```
""" something strange on the plot; red parametrized line doesn't match with the␣
 ↪intersection of 2 surfaces """

x,y,z,t = var("x,y,z,t")
plots = [implicit_plot3d(y^2-x*z,xrange=(-15,15), yrange=(-15,15),␣
 ↪zrange=(-15,15)),
         implicit_plot3d(z-x^2,xrange=(-15,15), yrange=(-15,15),␣
 ↪zrange=(-15,15)),
         parametric_plot3d((t^2,t^3,t^4), (-2,2), color='red', thickness=5)]

sum(plots)
```

[73]: Graphics3d Object

(b) Now, let's determine $I(V)$. Recall that

$$I(V) = \{f \in k[x_1, x_2, x_3] : \forall x \in V : f(x) = 0\}.$$

The considerations above doesn't depend on the ring, thus by Hilbert's Nullstellensatz, $I(V(J)) = \sqrt{J}$, so

$$I(V(y^2 - xz, z - x^2)) = \sqrt{\langle y^2 - xz, z - x^2\rangle}.$$

But

$$\Bbbk[x,y,z]/\left\langle y^2 - xz, z - x^2\right\rangle = \Bbbk[x,y]/\left\langle y^2 - x^3\right\rangle.$$

But $y^2 - x^3$ is irreducible: using Eisenstein's criterion, consider $y^2 - x^3 \in \Bbbk[x][y]$ and prime ideal $p = (x)$. Then criterion implies that $y^2 - x^3$ is irreducible, thus $\langle y^2 - x^3\rangle$ is prime, and therefore

$$I(V(y^2 - xz, z - x^2)) = \sqrt{\langle y^2 - xz, z - x^2\rangle} = \left\langle y^2 - xz, z - x^2\right\rangle. \bullet$$

4. The system of equations $f_1 = ... = f_s = 0$ gives the ideal $I = \langle f_1, ..., f_s\rangle$ of polynomial consequences. Now suppose that the system has a consequence of the form $f = g$ and we take the m-th power of each side to obtain $f^m = g^m$. In terms of the ideal I, this means that $f - g \in I$ should imply $f^m - g^m \in I$. Prove this by factoring \$f^m − g^m \$.

∘ Let's just divide $f^m - g^m$ by $f - g$, we obtain:

$$f^m - g^m = (f - g)(f^{m-1} + f^{m-2}g + ... + fg^{m-2} + g^{m-1}),$$

So, indeed, $f - g \in I$ implies $f^m - g^m \in I$. •

5. (a) Prove that $xy \notin \langle x^2, y^2\rangle$

(b) Prove that $1, x, y, xy$ are the only monomials not contained in $\langle x^2, y^2\rangle$.

∘ (a): Suppose, $xy \in \langle x^2, y^2\rangle$. Then $xy = ax^2 + by^2$ for some $a, b \in \Bbbk[x, y]$. But degree in RHS in either x or y can't be less than 2, whileas the degree in LHS in either x or y is only 1. So $xy \notin \langle x^2, y^2\rangle$, but in rad $\langle x^2, y^2\rangle = \langle x, y\rangle$.

(b): Let a be the other monomial that is not contained in $\langle x^2, y^2\rangle$. Since it's not $1, x, y, xy$, it has degree in either x or y at least 2. But then it factors by the generators, and thus lies inside the ideal. •