Задача нечеткой кластеризации: Метод декомпозиционного дерева

Preliminaries

In [102]:

```
import numpy as np
from typing import Callable
import networkx as nx
from networkx.drawing.nx_agraph import graphviz_layout
import matplotlib.pyplot as plt
import seaborn as sns

colors = ['#1f77b4', '#ff7f0e', '#2ca02c', '#d62728',
          '#9467bd', '#8c564b', '#e377c2', '#7f7f7f',
          '#bcbd22', '#17becf']
sns.set(style='whitegrid', context='notebook', palette=tuple(colors),
        rc={'axes.labelsize': 16})

%matplotlib inline
%config InlineBackend.figure_formats = {'png', 'retina'}
```

Определим Расстояние Хемминга:

In [103]:

```
def Hamming(i:np.int, j:np.int, table: np.ndarray) -> np.ndarray:
    """
    Computes relative Hamming distance between A_i and A_j
    Where A_i \in [0,1]^n -- n-dim. vector that contains estimation that
    i-th element has P_k feature
    """
    A_i = table[i,:] # table[:,i]
    A_j = table[j,:] # table[:,j]
    return sum(abs(A_i - A_j)) / A_i.shape[0]
```

Определим (max-min)-композицию:

In [104]:

```
def mu(ix: np.int, iy: np.int, arr1: np.ndarray, arr2=np.nan) -> np.float:
    """
    Returns (max-min)-composition for single pair
    """
    if arr2 is np.nan:
        arr2 = arr
    xz = arr1[ix,:]
    zy = arr2[:,iy]
    choose = np.array(list(map(lambda x: min(x), list(zip(xz,zy)))))
    val = max(choose)
    return val

def maxmin(arr1:np.ndarray, arr2=np.nan) -> np.ndarray:
    """
    Returns (max-min)-composition for given matrices
    """
    if arr2 is np.nan:
        arr2 = arr1
    x = arr1.shape[0]
    y = arr2.shape[1]
    pot = np.empty((x,y))
    for ix in range(0,x):
        for iy in range(0, y):
            pot[ix,iy] = mu(ix,iy,arr1,arr2)
    return pot
```

Определим (max-T)-композицию:

In [105]:

```
def T(x:np.float, y:np.float, a=-5, b=0.2) -> np.float:
    """
    Computes given T-norm
    """
    numer = x*y - (1-a)*(1-b)*(1-x)*(1-y)
    denom = 1 + a*b*(1-x)*(1-y)
    return max(0, numer/denom)

def muT(ix: np.int, iy: np.int, arr1: np.ndarray, arr2=np.nan) -> np.float:
    """
    Returns (max-T)-composition for single pair
    """
    if arr2 is np.nan:
        arr2 = arr
    xz = arr1[ix,:]
    zy = arr2[:,iy]
    choose = np.array(list(map(lambda x: T(*x), list(zip(xz,zy)))))
    val = max(choose)
    return val

def maxT(arr1:np.ndarray, arr2=np.nan, T=T) -> np.ndarray:
    """
    Computes (max-T)-composition for given matrices and given T-norm
    """
    if arr2 is np.nan:
        arr2 = arr1
    x = arr1.shape[0]
    y = arr2.shape[1]
    pot = np.empty((x,y))
    for ix in range(0,x):
        for iy in range(0, y):
            pot[ix,iy] = np.round(muT(ix,iy,arr1,arr2),2)
    return pot
```

Определим транзитивное замыкание:

In [106]:

```
def union(arr1:np.ndarray, arr2:np.ndarray) -> np.ndarray:
    """
    Computes union of 2 fuzzy relations
    """
    shape = arr1.shape
    pot = np.empty(shape)
    for ix in range(0,shape[0]):
        for iy in range(0,shape[1]):
            pot[ix,iy] = max(arr1[ix,iy],arr2[ix,iy])
    return pot

def trans_closure(arr:np.ndarray, comp = maxmin, show = False) -> np.ndarray:
    """
    Computes transitive closure of a fuzzy relation
    """
    pot = arr
    new_arr = comp(arr)
    count = 1
    if show:
        print(count)
        print(new_arr)
        print('----------')
    if (new_arr==arr).all():
        return pot
    else:
        pot = union(new_arr,arr)
        while True:
            prev_arr = new_arr
            new_arr = comp(new_arr,arr)
            count += 1
            if show:
                print(count)
                print(new_arr)
                print('----------')
            if (prev_arr==new_arr).all():
                return pot
            else:
                pot = union(new_arr,arr)

def transitive(arr: np.ndarray, comp = maxmin) -> bool:
    """
```

```
    Checks whether given matrix is transitive or not
    """
    arr_squared = comp(arr)
    return (arr_squared <= arr).all()
```

Определим процедуру декомпозиции отношения эквивалентности:

In [107]:

```
def alphas(arr:np.ndarray) -> np.ndarray:
    """
    Returns valid alpha for future alpha-decomposition for decomposing tree
    """
    mask = (arr > 0) & (arr < 1)
    alphs = np.unique(np.append(arr[mask], [1])) # add 1 separately in case if r>1 ∀r ∈ R
    return alphs

def decompose(arr:np.ndarray) -> np.ndarray:
    """
    Returns pairs (alpha, R_aplha) that are given during decomposition for given relation
    Such that for given R: R = max_alpha{alpha * R_aplha}, where alpha \in (0,1]
    """
    alphs = alphas(arr)
    cuts = []
    for a in alphs:
        mask = (arr >= a)
        cuts.append(mask.astype(np.int))
    return list(zip(alphs,cuts))
```

Определим процедуру группировки строк в матрице, чтобы определить классы эквивалентности:

In [108]:

```
def equiv_class(arr: np.ndarray, U: list) -> np.ndarray:
    """
    Determine equivalence classes by grouping rows in a matrix
    Such that if arr[i:] = arr[j:], then U[i] ~ U[j]
    """

    # get uniq rows with counts
    unq, count = np.unique(arr, axis=0, return_counts=True)
    repeated_groups = unq[count > 1]  # choose rows with counts > 1

    classes = []
    unlisted = np.linspace(0,arr.shape[0]-1,arr.shape[0]).astype('int64')
    for repeated_group in repeated_groups:
        # get idx for repeated rows
        repeated_idx = np.argwhere(np.all(arr == repeated_group, axis=1))
        unlisted = [e for e in unlisted if e not in repeated_idx.ravel()]
        # add them to list of equiv classes
        classes.append([U[i] for i in repeated_idx.ravel()])
    # find elements of U which are not in classes
    # so it is the elements which are single in its equiv class
    for i in unlisted:
        classes.append([U[i]])  # add them to classes
    return classes
```

Определим функцию построения декомпозиционного дерева:

In [109]:

```
def find_prev(name:str, prev:list) -> list:
    """
    Find several ancestors for given node
    """
    name_prop = name[:name.index('.')-2]
    for prev_name in prev:
        prev_name_prop = prev_name[:prev_name.index('.')-2]
        if set(name_prop).issubset(set(prev_name_prop)):
            return [prev_name]

    res = []
    for prev_name in prev:
        prev_name_prop = prev_name[:prev_name.index('.')-2]
        if set(prev_name_prop).issubset(set(name_prop)):
            res.append(prev_name)
    return res

def tree(arr: list) -> nx.DiGraph:
    """
    Returns networkx.DiGraph instance -- decomposing tree -- based on given set of alpha-cuts
    Even alpha-cut is an equivalence relation, thus every cut can be factorized
    """
    #get factor sets
    factor_arr = [ equiv_class(arr[i][1], Obj) for i in range(len(arr))]

    G = nx.DiGraph()
    prev = []  # labels of nodes from previous level
    for f in range(len(factor_arr)):  # for every factor set
        curr = []  # labels of nodes from current level
        for eq_class in factor_arr[f]:  # for every equivalence class
            name = " ".join(eq_class)  # create label for a node
            name += " ({:.3f})".format(arr[f][0])  # with alpha-value
            curr.append(name)
            G.add_node(name)
            #prev_node = False  # prev node is not found atm
            #----
            prevs = find_prev(name, prev)
            for prev_name in prevs:
                G.add_edge(prev_name,name)

        prev = curr
```

```
    return G
```

Определим функцию, которая рисует декомпозиционное дерево:

In [110]:

```
def treeplot(G:nx.DiGraph, title: str, shape = (15,15)):
    """
    Plot given networkx.Graph
    """
    plt.figure(figsize=shape)
    plt.title(title)
    pos=graphviz_layout(G, prog='dot')
    nx.draw(G, pos, with_labels=True, arrows=True, node_size=1500, node_shape = 's')
    plt.savefig(title + ".jpg")
    plt.show()
```

Main Part

Имеется следующая таблица:

In [111]:

```
U = ('a', 'b', 'c', 'd', 'e')
Obj = ['A1', 'A2', 'A3', 'A4', 'A5', 'A6']
given = "0.8 0.7 0.7 0.3 0.0 | 0.5 1 0.5 0.0 0.0 | 0.5 0.6 0.8 0.4 0.2 | \
         0.9 0.5 0.3 0.2 0.2 | 0.6 0.8 0.9 0.3 0.1 | 0.2 0.4 0.6 0.8 0.9"
lines = given.split("|")
elements = [line.split() for line in lines]
table = np.array([list(map(lambda x: float(x), line)) for line in elements])
table
```

Out[111]:

```
array([[0.8, 0.7, 0.7, 0.3, 0. ],
       [0.5, 1. , 0.5, 0. , 0. ],
       [0.5, 0.6, 0.8, 0.4, 0.2],
       [0.9, 0.5, 0.3, 0.2, 0.2],
       [0.6, 0.8, 0.9, 0.3, 0.1],
       [0.2, 0.4, 0.6, 0.8, 0.9]])
```

С помощью расстояния Хемминга построим матрицу отношения несходства R, а после перейдем к матрице сходства:

In [112]:

```
card = table.shape[0]  # 1 is for {a,b,c,d,e}
R = np.empty((card,card))
for i in range(0,card):
    for j in range(0,card):
        R[i,j] = Hamming(i,j,table)
#R = normalize(R)
Rhat = 1 - R
print("R matrix:\n{}\n\nRhat matrix:\n{}".format(R,Rhat))
```

```
R matrix:
[[0.   0.22 0.16 0.2  0.12 0.48]
 [0.22 0.   0.26 0.3  0.22 0.54]
 [0.16 0.26 0.   0.24 0.12 0.36]
 [0.2  0.3  0.24 0.   0.28 0.48]
 [0.12 0.22 0.12 0.28 0.   0.48]
 [0.48 0.54 0.36 0.48 0.48 0.  ]]

Rhat matrix:
[[1.   0.78 0.84 0.8  0.88 0.52]
 [0.78 1.   0.74 0.7  0.78 0.46]
 [0.84 0.74 1.   0.76 0.88 0.64]
 [0.8  0.7  0.76 1.   0.72 0.52]
 [0.88 0.78 0.88 0.72 1.   0.52]
 [0.52 0.46 0.64 0.52 0.52 1.  ]]
```

Найдем транзитивное замыкание $\hat{R}$ с двумя разными композициями:

In [113]:

```
ClR = trans_closure(Rhat)
TClR = trans_closure(Rhat, comp=maxT)
print('Transitive closure with (max-min)-composition:\n{}\n \
Transitive closure with (max-T)-composition:\n{}'.format(ClR, TClR))
```

```
Transitive closure with (max-min)-composition:
[[1.   0.78 0.88 0.8  0.88 0.64]
 [0.78 1.   0.78 0.78 0.78 0.64]
 [0.88 0.78 1.   0.8  0.88 0.64]
 [0.8  0.78 0.8  1.   0.8  0.64]
 [0.88 0.78 0.88 0.8  1.   0.64]
 [0.64 0.64 0.64 0.64 0.64 1.  ]]
 Transitive closure with (max-T)-composition:
[[1.   0.78 0.84 0.8  0.88 0.52]
 [0.78 1.   0.74 0.7  0.78 0.46]
 [0.84 0.74 1.   0.76 0.88 0.64]
 [0.8  0.7  0.76 1.   0.72 0.52]
 [0.88 0.78 0.88 0.72 1.   0.52]
 [0.52 0.46 0.64 0.52 0.52 1.  ]]
```

Убедимся в транзитивности:

In [114]:

```
print('Transitive closure with (max-min)-composition is transitive') if transitive(ClR) \
else print('Transitive closure with (max-min)-composition is not transitive')
print('Transitive closure with (max-T)-composition is transitive') if transitive(TClR,
                                                                                  comp=maxT) \
else print('Transitive closure with (max-T)-composition is not transitive')
```

```
Transitive closure with (max-min)-composition is transitive
Transitive closure with (max-T)-composition is transitive
```

Построим декомпозиционное дерево для (max-min)-композиции

Разложим полученное с помощью (max-min)-композиции отношение по теореме о декомпозиции через систему α-срезов:

In [115]:

```
decClR = decompose(ClR)
for alpha, cut in decClR:
    print("For \\alpha = {:.3f} there is a cut:\n{}\n".format(alpha,cut))
```

```
For \alpha = 0.640 there is a cut:
[[1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]]

For \alpha = 0.780 there is a cut:
[[1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [0 0 0 0 0 1]]

For \alpha = 0.800 there is a cut:
[[1 0 1 1 1 0]
 [0 1 0 0 0 0]
 [1 0 1 1 1 0]
 [1 0 1 1 1 0]
 [1 0 1 1 1 0]
 [0 0 0 0 0 1]]

For \alpha = 0.880 there is a cut:
[[1 0 1 0 1 0]
 [0 1 0 0 0 0]
 [1 0 1 0 1 0]
 [0 0 0 1 0 0]
 [1 0 1 0 1 0]
 [0 0 0 0 0 1]]

For \alpha = 1.000 there is a cut:
[[1 0 0 0 0 0]
 [0 1 0 0 0 0]
 [0 0 1 0 0 0]
 [0 0 0 1 0 0]
 [0 0 0 0 1 0]
 [0 0 0 0 0 1]]
```

Каждый полученный α-срез -- отношение эквивалентности на универсальном множестве U, тогда построим фактор-множества:

In [116]:

```
factor_decClR = [ equiv_class(decClR[i][1], Obj) for i in range(len(decClR))]
'''
for i in range(len(decClR)):
    print("For \\alpha = {:.3f} there is a factor set:\n{}\n".\
          format(decClR[i][0],factor_decClR[i]))
'''
factor_decClR
```

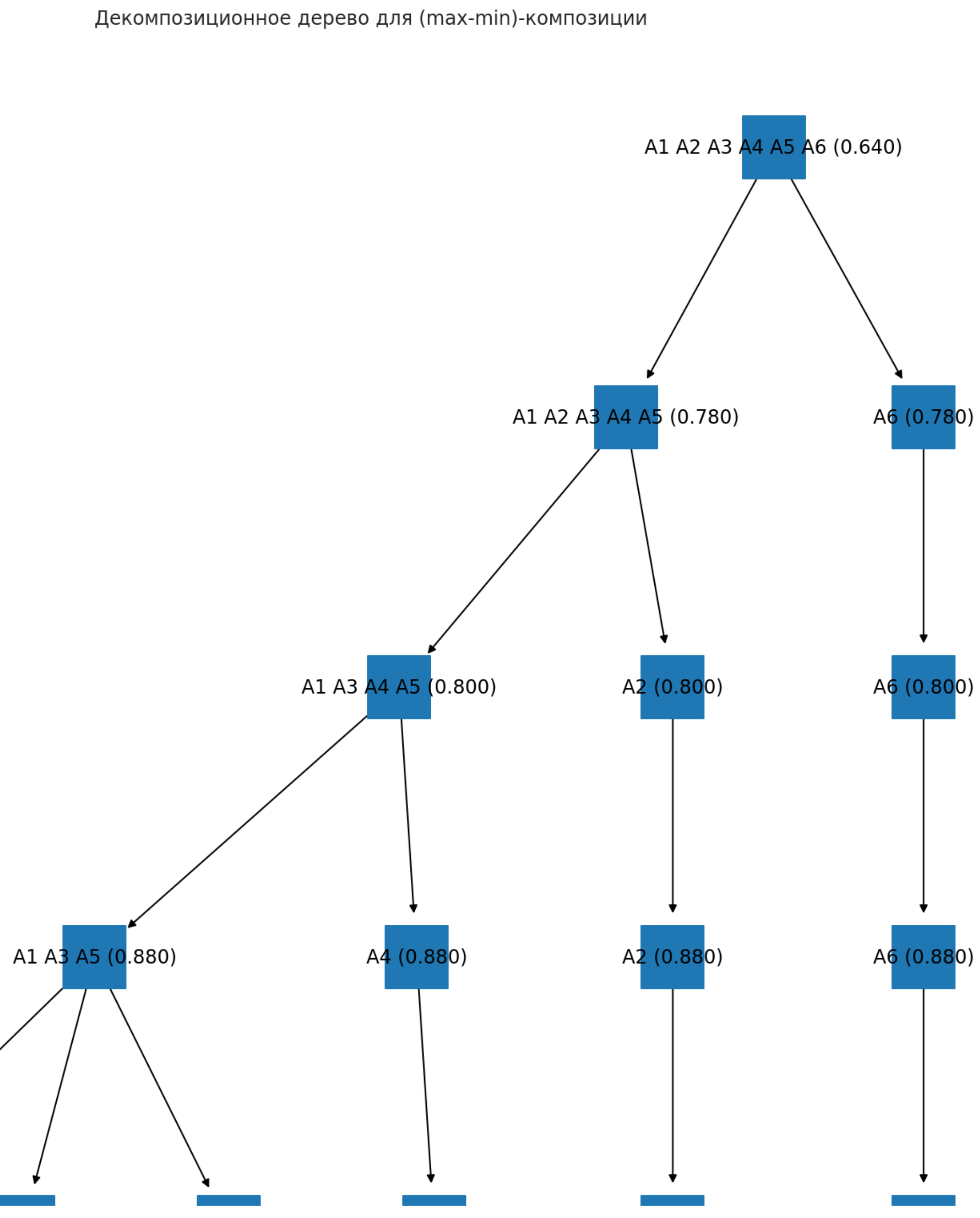
Out[116]:

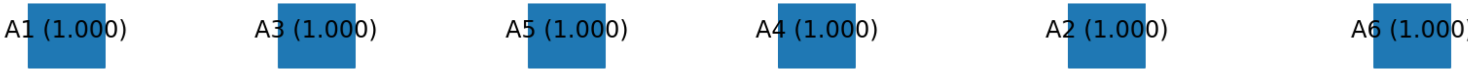
```
[[['A1', 'A2', 'A3', 'A4', 'A5', 'A6']],
 [['A1', 'A2', 'A3', 'A4', 'A5'], ['A6']],
 [['A1', 'A3', 'A4', 'A5'], ['A2'], ['A6']],
 [['A1', 'A3', 'A5'], ['A2'], ['A4'], ['A6']],
 [['A1'], ['A2'], ['A3'], ['A4'], ['A5'], ['A6']]]
```

Построим декомпозиционное дерево по полученным фактор-множествам(фактор-множества строятся непосредственно в функции $ {\it tree}$):

In [117]:

```
G_ClR = tree(decClR)
treeplot(G_ClR, title = 'Декомпозиционное дерево для (max-min)-композиции')
```


Декомпозиционное дерево для (max-min)-композиции
A1 A2 A3 A4 A5 A6 (0.640)
A1 A2 A3 A4 A5 (0.780)
A6 (0.780)
A1 A3 A4 A5 (0.800)
A2 (0.800)
A6 (0.800)
A1 A3 A5 (0.880)
A4 (0.880)
A2 (0.880)
A6 (0.880)



Построим декомпозиционное дерево для (max-T)-композиции

Разложим полученное с помощью (max-T)-композиции отношение по теореме о декомпозиции через систему α-срезов:

In [118]:

```
decTClR = decompose(TClR)
for alpha, cut in decTClR:
    print("For \\alpha = {:.3f} there is a cut:\n{}\n".format(alpha,cut))
```

```
For \alpha = 0.460 there is a cut:
[[1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]]

For \alpha = 0.520 there is a cut:
[[1 1 1 1 1 1]
 [1 1 1 1 1 0]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 1 1 1 1 1]
 [1 0 1 1 1 1]]

For \alpha = 0.640 there is a cut:
[[1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [1 1 1 1 1 1]
 [1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [0 0 1 0 0 1]]

For \alpha = 0.700 there is a cut:
[[1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [1 1 1 1 1 0]
 [0 0 0 0 0 1]]

For \alpha = 0.720 there is a cut:
[[1 1 1 1 1 0]
 [1 1 1 0 1 0]
 [1 1 1 1 1 0]
 [1 0 1 1 1 0]
 [1 1 1 1 1 0]
 [0 0 0 0 0 1]]

For \alpha = 0.740 there is a cut:
[[1 1 1 1 1 0]
 [1 1 1 0 1 0]
```

```
 [1 1 1 1 1 0]
 [1 0 1 1 0 0]
 [1 1 1 0 1 0]
 [0 0 0 0 0 1]]

For \alpha = 0.760 there is a cut:
[[1 1 1 1 1 0]
 [1 1 0 0 1 0]
 [1 0 1 1 1 0]
 [1 0 1 1 0 0]
 [1 1 1 0 1 0]
 [0 0 0 0 0 1]]

For \alpha = 0.780 there is a cut:
[[1 1 1 1 1 0]
 [1 1 0 0 1 0]
 [1 0 1 0 1 0]
 [1 0 0 1 0 0]
 [1 1 1 0 1 0]
 [0 0 0 0 0 1]]

For \alpha = 0.800 there is a cut:
[[1 0 1 1 1 0]
 [0 1 0 0 0 0]
 [1 0 1 0 1 0]
 [1 0 0 1 0 0]
 [1 0 1 0 1 0]
 [0 0 0 0 0 1]]

For \alpha = 0.840 there is a cut:
[[1 0 1 0 1 0]
 [0 1 0 0 0 0]
 [1 0 1 0 1 0]
 [0 0 0 1 0 0]
 [1 0 1 0 1 0]
 [0 0 0 0 0 1]]

For \alpha = 0.880 there is a cut:
[[1 0 0 0 1 0]
 [0 1 0 0 0 0]
 [0 0 1 0 1 0]
 [0 0 0 1 0 0]
 [1 0 1 0 1 0]
```

```
 [0 0 0 0 0 1]]

For \alpha = 1.000 there is a cut:
[[1 0 0 0 0 0]
 [0 1 0 0 0 0]
 [0 0 1 0 0 0]
 [0 0 0 1 0 0]
 [0 0 0 0 1 0]
 [0 0 0 0 0 1]]
```

Каждый полученный α-срез -- отношение эквивалентности на универсальном множестве U, тогда построим фактор-множества:

In [124]:

```
factor_decTClR = [ equiv_class(decTClR[i][1], Obj) for i in range(len(decTClR))]
for i in range(len(decTClR)):
    print("For \\alpha = {:.3f} there is a factor set:\n{}\n".\
          format(decTClR[i][0],factor_decTClR[i]))
```

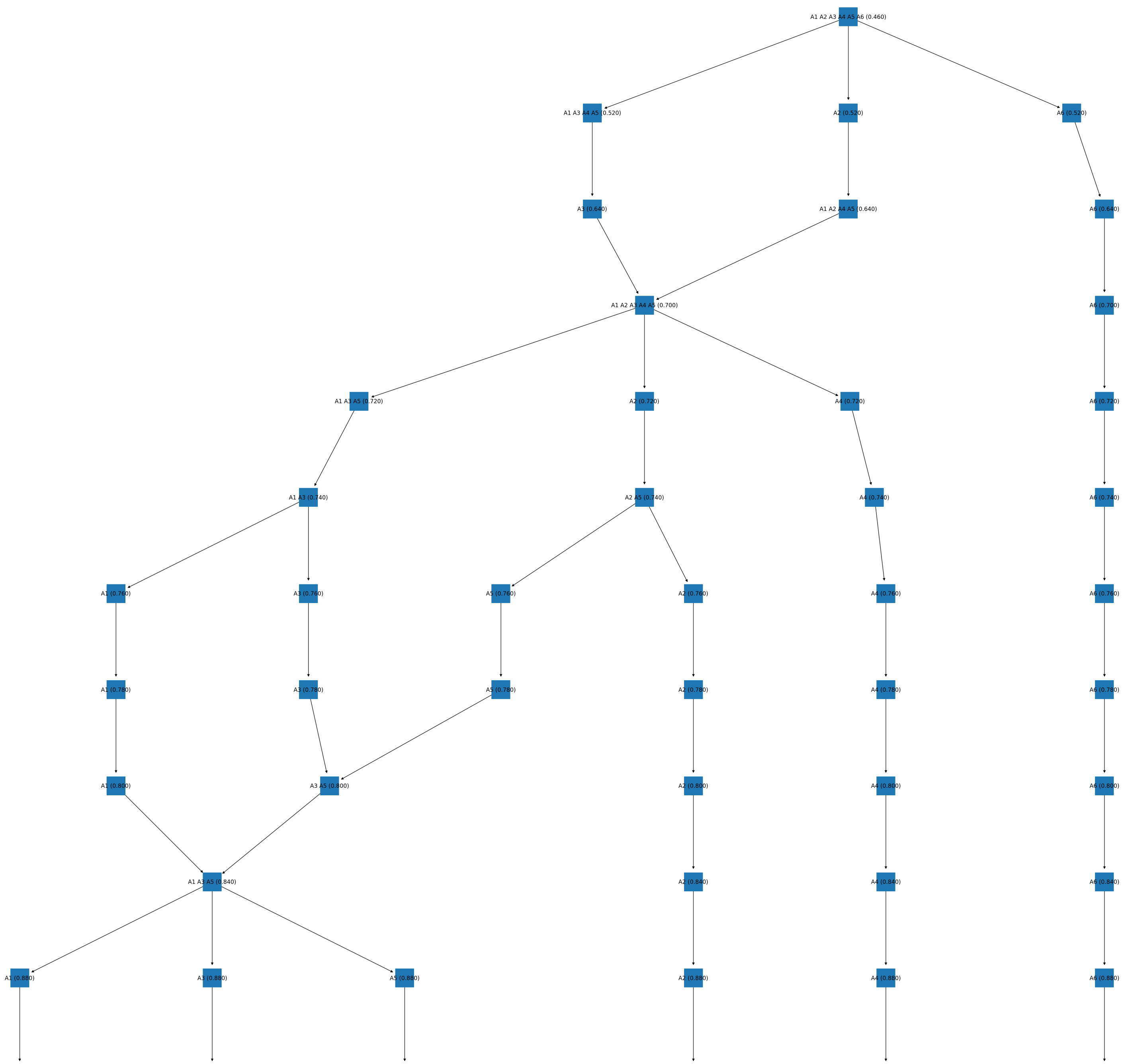
```
For \alpha = 0.460 there is a factor set:
[['A1', 'A2', 'A3', 'A4', 'A5', 'A6']]

For \alpha = 0.520 there is a factor set:
[['A1', 'A3', 'A4', 'A5'], ['A2'], ['A6']]

For \alpha = 0.640 there is a factor set:
[['A1', 'A2', 'A4', 'A5'], ['A3'], ['A6']]

For \alpha = 0.700 there is a factor set:
[['A1', 'A2', 'A3', 'A4', 'A5'], ['A6']]

For \alpha = 0.720 there is a factor set:
[['A1', 'A3', 'A5'], ['A2'], ['A4'], ['A6']]

For \alpha = 0.740 there is a factor set:
[['A2', 'A5'], ['A1', 'A3'], ['A4'], ['A6']]

For \alpha = 0.760 there is a factor set:
[['A1'], ['A2'], ['A3'], ['A4'], ['A5'], ['A6']]

For \alpha = 0.780 there is a factor set:
[['A1'], ['A2'], ['A3'], ['A4'], ['A5'], ['A6']]

For \alpha = 0.800 there is a factor set:
[['A3', 'A5'], ['A1'], ['A2'], ['A4'], ['A6']]

For \alpha = 0.840 there is a factor set:
[['A1', 'A3', 'A5'], ['A2'], ['A4'], ['A6']]

For \alpha = 0.880 there is a factor set:
[['A1'], ['A2'], ['A3'], ['A4'], ['A5'], ['A6']]

For \alpha = 1.000 there is a factor set:
[['A1'], ['A2'], ['A3'], ['A4'], ['A5'], ['A6']]
```

Построим декомпозиционное дерево по полученным фактор-множествам(фактор-множества строятся непосредственно в функции $ {\it tree}$):

In [120]:

```
G_TClR = tree(decTClR)
treeplot(G_TClR, title = 'Декомпозиционное дерево для (max-T)-композиции', shape = (50,50))
```

Декомпозиционное дерево для (max-T)-композиции



Заполним таблицу со значениями:

	Анализ	(max-min)	(max-T)
1	Укажите значение α, когда впервые появился тривиальный класс	0.78	0.42
2	Укажите значение α, когда появилось тривиальное разбиение	0.88	0.740
3	Сколько имеется зон неустойчивой классификации	0	7
4	Укажите зоны неустойчивой классификации в форме промежутков значений α	-	[0.64;0.84]
	Укажите объекты, для которых образуется зона неустойчивой классификации	-	$\{A_1,A_2,A_3,A_4,A_5\}$
5	Укажите зону роста в форме промежутка для α	[0.64; 1]	[0.46;0.88]
6	Для каждого объекта A_i укажите значение α, когда данный объект впервые образует тривиальный класс	A_1:1 A_2:0.8 A_3:1 A_4:0.88 A_5:1 A_6:0.78	A_1:0.76 A_2:0.52 A_3:0.64 A_4:0.72 A_5:0.76 A_6:0.52
7	Для каждого объекта A_i укажите значение α, при котором данный объект входит в класс максимальной мощности	A_1:0.78 A_2:0.78 A_3:0.78 A_4:0.78 A_5:0.78 A_6:0.78	A_1:0.7 A_2: A_3:0.7 A_4:0.7 A_5:0.7 A_6:0.52
8	Укажите мощность максимального класса(не принимаем во внимание первоначальный класс)	5	5