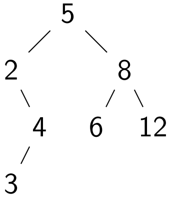
Balancing binary trees

Binary search trees – refresher

- Each node stores a value.
- Values in left subtree are smaller; right, bigger.



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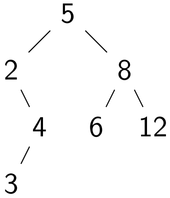

- Insertion, deletion, membership queries all take time $O(\mathrm{height})$.
- In a typical binary search tree, $\mathrm{height} \approx \log n$.
- In the worse case, $\mathrm{height} = n - 1$.

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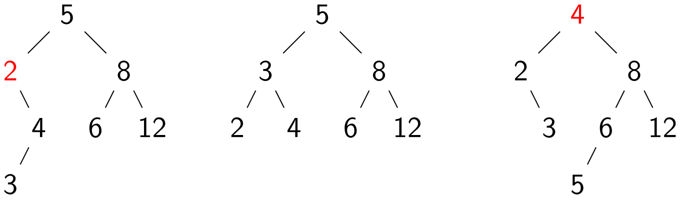

unbalanced   balanced   unbalanced

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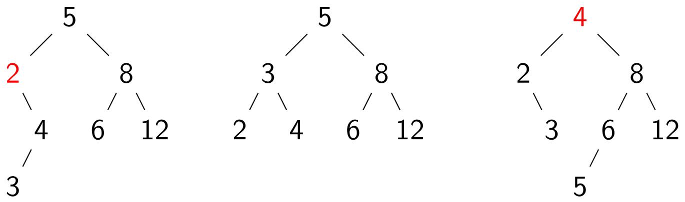
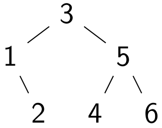


unbalanced | balanced | unbalanced

A balanced binary tree with n nodes has height exactly $\lfloor \log_2 n \rfloor$.

The cost of keeping BSTs balanced

Inserting into a balanced BST may unbalance it.



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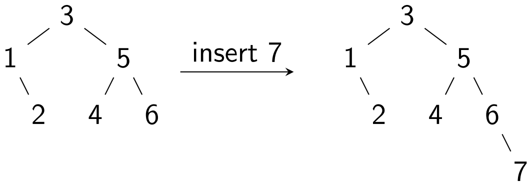

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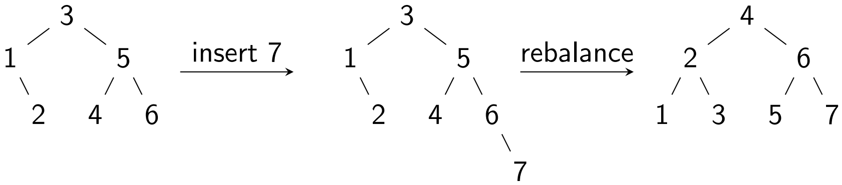

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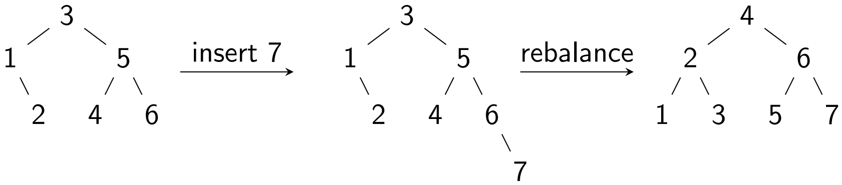

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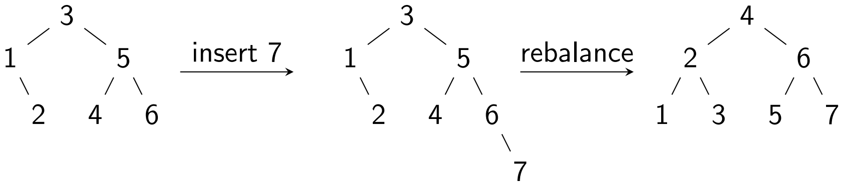

- This is the only balanced BST storing 1–7.
- Every vertex has moved and every vertex except 7 now has a different parent!
- This shows rebalancing takes time $\Theta(n)$ – too expensive.

AVL-balance

A binary search tree is **AVL-balanced** if:

- the heights of every node's left and right subtrees differ by at most 1.

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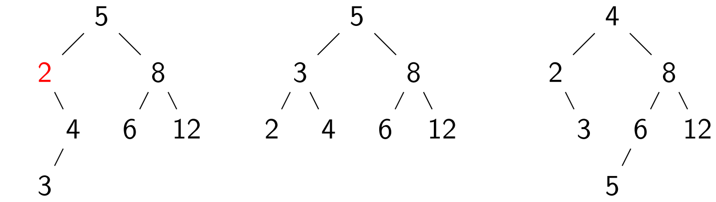

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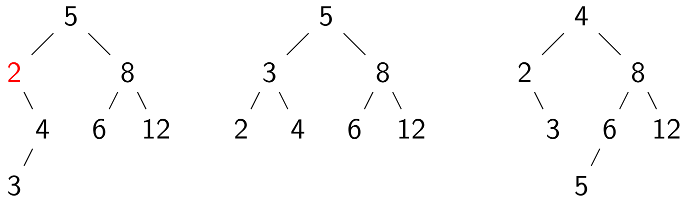

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AVL = Adelson-Velski and Landis, the two inventors.

Balance factors

To track whether a BST is AVL-balanced, define each node's **balance factor** as

$$\mathrm{BF}(x) = \mathrm{height}(x.\mathrm{right}) - \mathrm{height}(x.\mathrm{left})\,.$$

- For AVL-balanced trees, $\mathrm{BF}(x) \in \{-1, 0, +1\}$ for every node x.

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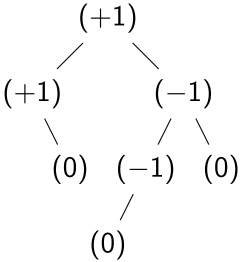
- For AVL-balanced trees, $\mathrm{BF}(x) \in \{-1, 0, +1\}$ for every node x.
- If $\mathrm{BF}(x) < 0$, we say x is **left-heavy** – its left subtree is taller.
- If $\mathrm{BF}(x) > 0$, we say x is **right-heavy** – its right subtree is taller.

Updating balance factors

Balance factors can be stored in tree nodes and updated after inserting.

In this diagram, the numbers are balance factors, not node values.



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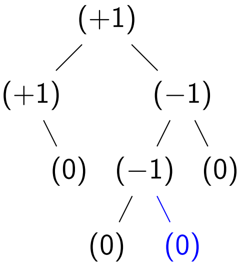

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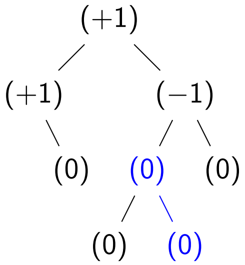

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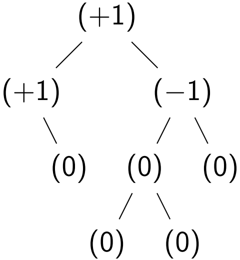

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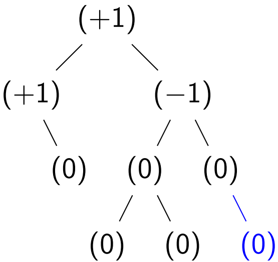

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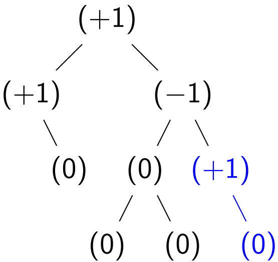

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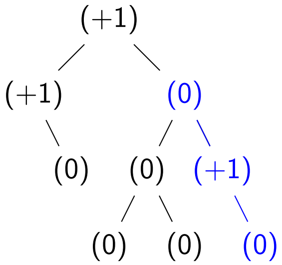

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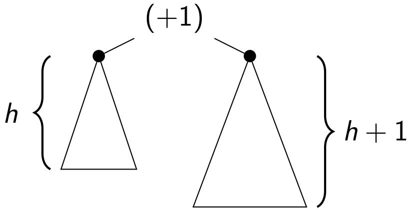

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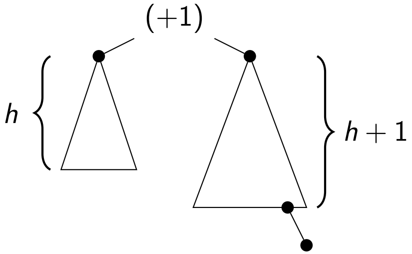

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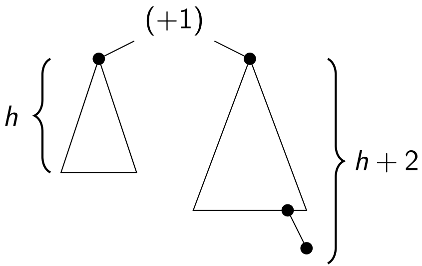

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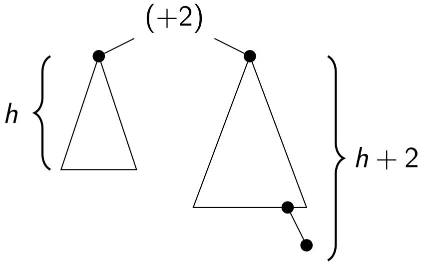

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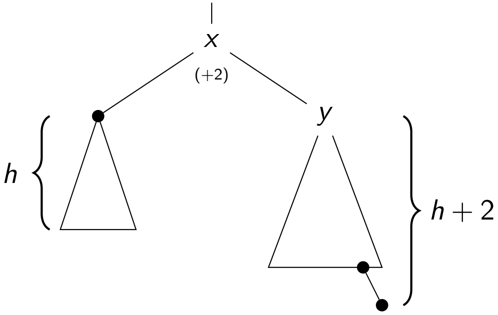
Suppose we're updating balance factors and x is the first node where we set $\mathrm{BF} = +2$.



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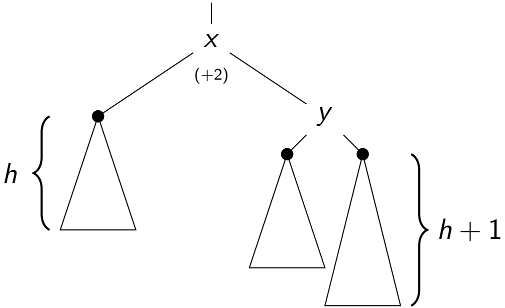

Look more closely at y's subtrees.
We inserted into the right subtree: it must now have height $h + 1$.

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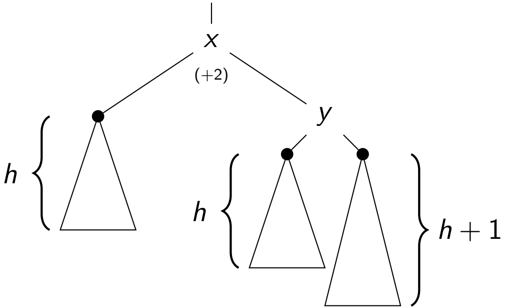

y's left subtree must have height h. If not, we would have set $\mathrm{BF}(y) = 0$ and stopped updating.

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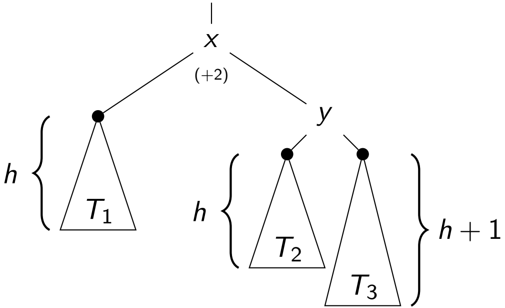

Nodes in T_1 have value v with $v < x$
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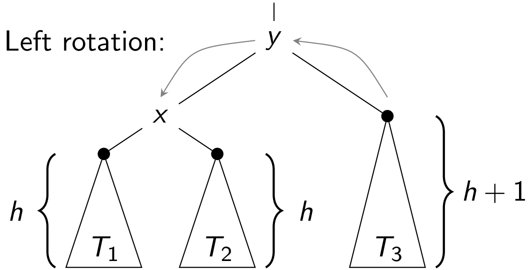

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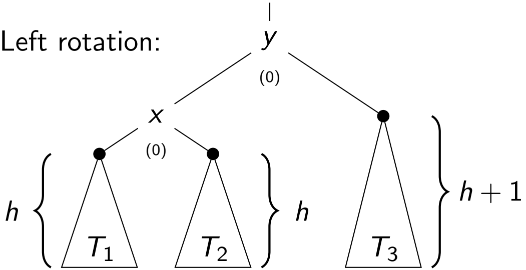
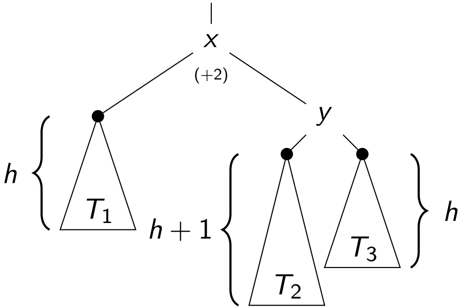


We have a valid BST and $\mathrm{BF}(y) = 0$ so the tree is AVL-balanced again.

Rebalancing unbalanced trees (2)

But what if it was y's left subtree that had height $h+1$?



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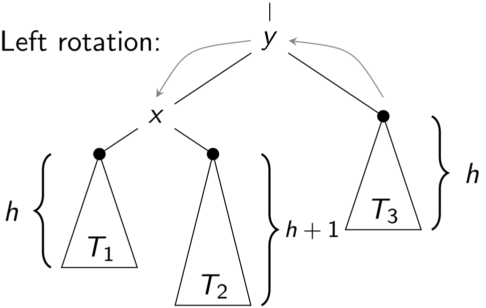

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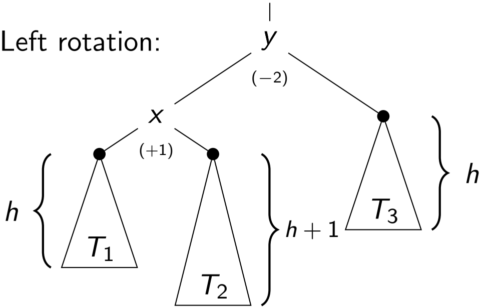

Left rotation has failed: now y's balance factor is bad instead of x's.

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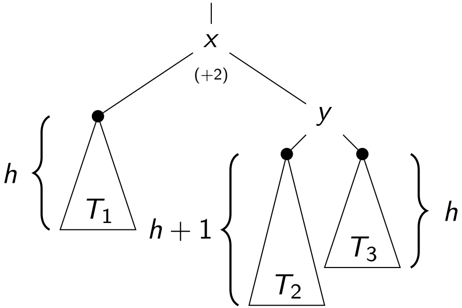

We must look at y's left subtree T_2 more closely.

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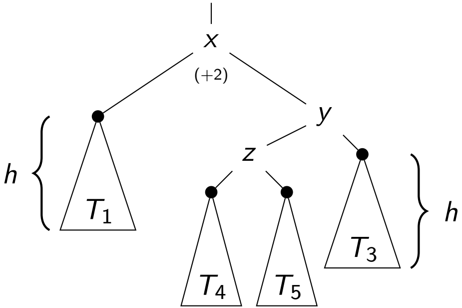

One of T_4 and T_5 has height h; the other has height $h-1$.
(They both used to have height $h-1$ but we just inserted into one of them to cause the imbalance.)

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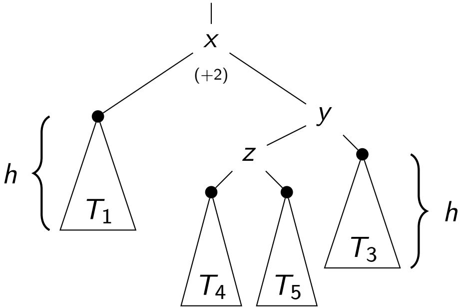

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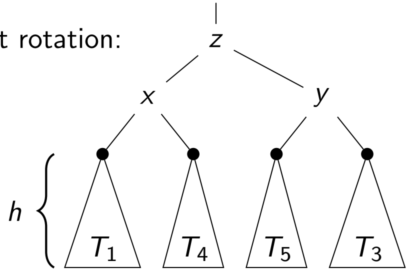

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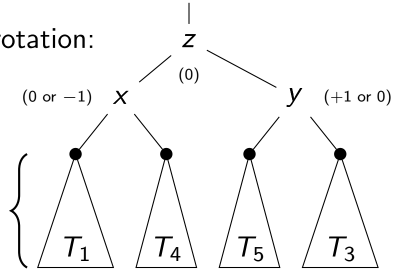

One of T_4 and T_5 has height $h-1$.
Either $\mathrm{BF}(x)=0$ and $\mathrm{BF}(y)=+1$ or $\mathrm{BF}(x)=-1$ and $\mathrm{BF}(y)=0$.
In both cases, $\mathrm{BF}(z)=0$ so the tree is AVL-balanced again.

AVL rebalancing

The $+2/-2$ node is	Its taller subtree is	You rotate
left-heavy (-2)	left	right
	right	left-right
right-heavy $(+2)$	left	right-left
	right	left

Right-rotation and left-right-rotation are the mirror-images of the previous slides.

AVL tree insertion

- Insert as in an ordinary BST.
- Recalculate balance factors up path from the insertion point towards the root.
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- Rotation requires adjusting at most three nodes' references: takes time $O(1)$.

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- It can be shown that an AVL-balanced tree has height $\Theta(\log n)$.
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- The height $\Theta(n)$ worst-case of ordinary BSTs is impossible in an AVL tree.
- Therefore, insert runs in time $O(\log n)$.
- Search is also $O(\text{height}) = O(\log n)$.
- Deletion can also be done in time $O(\log n)$, using rotations appropriately.

AVL trees – summary

- Balance factor = (height of right) – (height of left).
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- AVL trees are BSTs where $\mathrm{BF}(x) \in \{-1, 0, +1\}$ for all nodes x.
- Balance factor is stored in nodes and updated on inserts.
- Unbalanced nodes require rotations.
- Insert, search, delete in time $O(\log n)$.
- Avoids the $\Theta(n)$ worst-case of BSTs.