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Quicksort sorts a list by:

- picking a value called the **pivot**, p;
- rearranging the array so all values $\leq p$ come first;
- recursively sorting the values $\leq p$ and the values $> p$.

Quicksort: choosing a pivot

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- e.g., mean of first and last values; median of first, last, middle; a random element.
- We still get $O(n \log n)$ performance if the pivot is fairly close to the median.

Quicksort example

To partition, we look for an element that's $> p$ to the left of an element that's $< p$ and swap them.

Use $(\text{first} + \text{last})/2 =$ **7.5 as pivot**. (Median is 9.)

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After recursion on the two regions, the whole array is sorted.

Quicksort in Java

Source: adapted from Wikipedia.

```
static void qSort (int[] ints, int left, int right) {
    if (right - left < 1) return;

    int i = left-1;
    int j = right+1;
    int pivot = (ints[left] + ints[right])/2;

    while (i<j) {
        do { i++; } while (ints[i] < pivot);
        do { j--; } while (ints[j] > pivot);
        if (i < j) swap (ints, i, j);
    }
    qSort (ints, left, j);
    qSort (ints, j+1, right);
}
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Quicksort running time (1)

- Partitioning the array takes time $\Theta(n)$.
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- Typical running time is $O(n \log n)$.
- In the worst case, pivot is always largest or smallest element and we partition into arrays of length 1 and $n - 1$.
- Then, recursion depth is n and running time $O(n^2)$.

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- Suppose the split is never worse than 25%–75%.
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- Recursion depth is the number of times n can be multiplied by $3/4$ before the answer is 1.
- This is the number of times it can be divided by $4/3$.
- So the depth is $\log_{4/3} n = \Theta(\log n)$.

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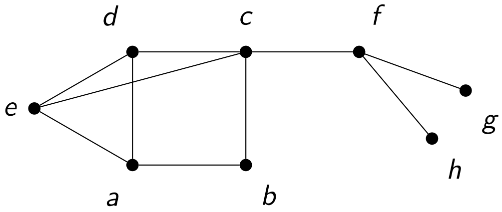
- Sort is in-place.
- Not well-suited to lists.
- Health warning: do not try to implement quicksort if you can avoid it.

Graphs

Graphs

A **graph** is a collection of **vertices** (singular: vertex) and **edges**. Each edge links two vertices.



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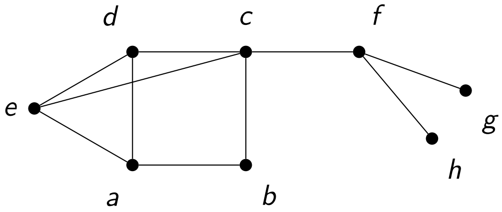

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Alternative terminology: network=graph, node=vertex, arc=edge.

Examples of graphs

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- computer networks (computers and connections),
- molecular structures (atoms and bonds),
- general binary relations (e.g. for a dating site, vertices are people and an edge indicates compatibility)

Paths

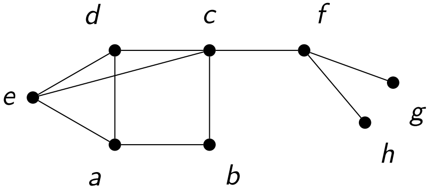
A **path** between vertices x and y is a sequence of edges

$$(x, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, y),$$

such that all the vertices $x, v_1, \ldots, v_k, y$ are distinct (i.e., no repetitions).

We often just list the vertices in order, e.g., $xv_1v_2 \ldots v_{k-1}v_ky$.

The **length** of a path is the number of edges (not vertices).



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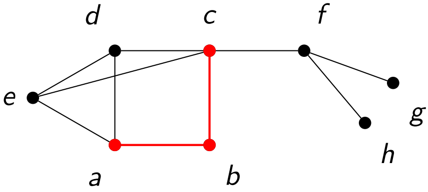

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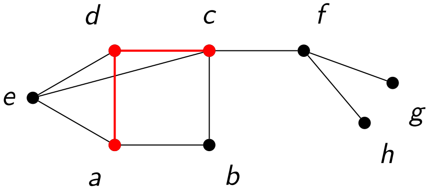

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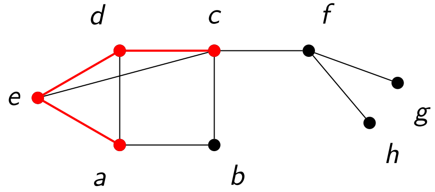

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A **cycle** is a sequence of edges

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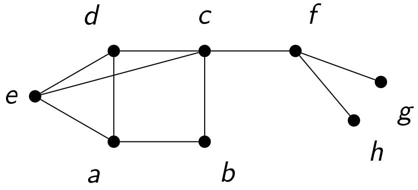

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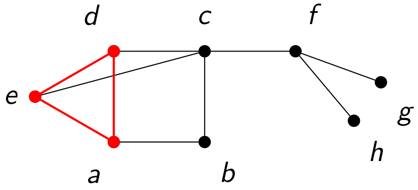

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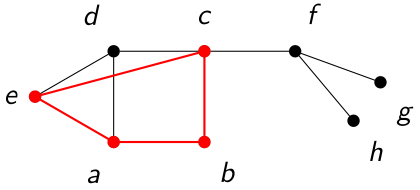

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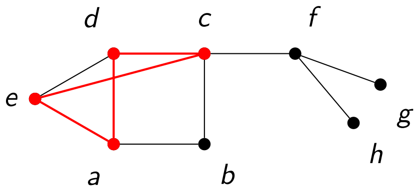

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Connected graphs

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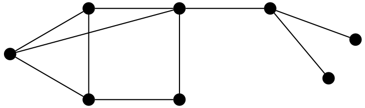

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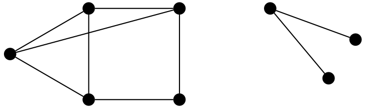

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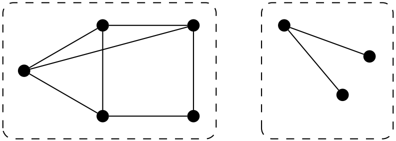

Components

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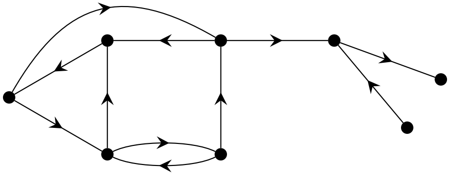

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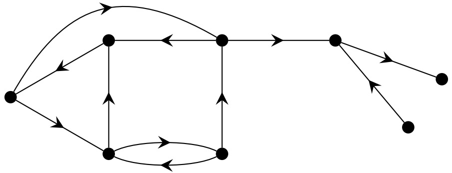

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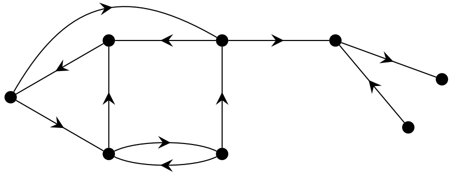

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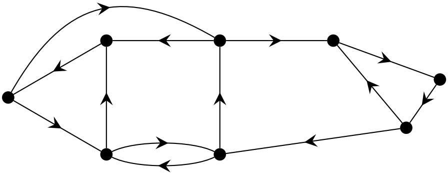

Weakly and strongly connected

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Let x be a vertex of a graph.

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In a weighted directed graph, (x, y) and (y, x) can have different weights.

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There are two main representations of graphs. In both cases, we assume the vertices are labelled with the integers $0, \ldots, n-1$.

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- Good for answering "Does edge (x, y) exist?"
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Both implementations are useful: you must pick the one that best suits your situation.

Graph algorithms

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Dijkstra's algorithm finds the shortest paths from a given vertex to every other vertex in the graph.

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We'll assume weighted graphs; represent unweighted graphs by giving every edge weight 1.

We'll assume all weights are positive.

Dijkstra's algorithm: the key idea (1)

Dijkstra finds paths in order of their length.

Suppose we start at vertex s.

If (s, x, w) is the least-weight edge from s, then sx must be the shortest path to x.

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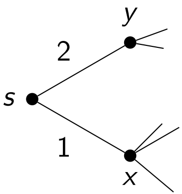
Dijkstra finds paths in order of their length.

Suppose we start at vertex s.

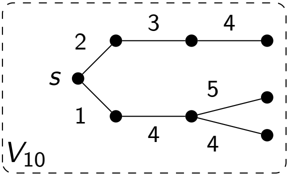
If (s, x, w) is the least-weight edge from s, then sx must be the shortest path to x.



Any path from s to x (or anywhere else) via y must have length at least 2.

Dijkstra's algorithm: the key idea (2)

Let V_{10} be the set of vertices with a path of length less than 10 from s and suppose we've found the shortest path to every vertex in V_{10}.



Dijkstra's algorithm: the key idea (2)

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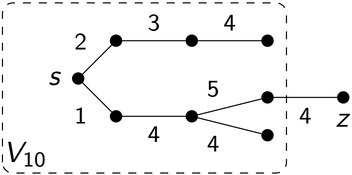
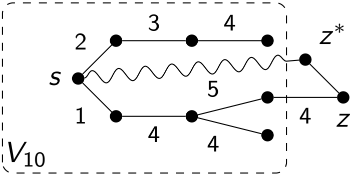


Let $z \notin V_{10}$ be the vertex that can be reached by the shortest path of the form "Take the shortest path to some vertex in V_{10}, then go along one more edge."

Dijkstra's algorithm: the key idea (2)

Let V_{10} be the set of vertices with a path of length less than 10 from s and suppose we've found the shortest path to every vertex in V_{10}.



Let $z \notin V_{10}$ be the vertex that can be reached by the shortest path of the form "Take the shortest path to some vertex in V_{10}, then go along one more edge."

This must be the shortest path to z. If there was a shorter path, it must contain some other vertex $z^* \notin V_{10}$, which is closer to s than z is. Contradiction – we would have chosen z^* instead of z!

Dijkstra's algorithm: data structures

Dijkstra uses the following data structures, where each array has length n.

- An array `boolean[] solved`: stores whether we've found the shortest path to each vertex.
- An array `double[] distance`: the length of shortest path found to each vertex so far.
- A priority queue `Q`: unsolved vertices x with priority equal to `distance[x]`.

Dijkstra's algorithm: pseudocode (1)

```
// Initialization
solved = {false, ..., false};
distance = {INFINITY, ..., INFINITY};
Q = new empty queue;

solved[source] = true;
distance[source] = 0;

Q.insert (0, x);

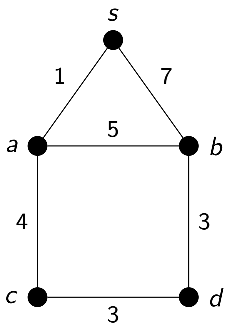
...
```

Dijkstra's algorithm: pseudocode (2)

```
// Main loop
while (!Q.isEmpty() {
    x = Q.next();
    solved[x] = true;

    foreach neighbour y of x
        if (!solved[y]) {
            double newDistance = distance[x] + weight(x,y);
            if (newDistance < distance[y]) {
                distance[y] = newDistance;
                if (Q.contains (y))
                    Q.setPriority (newDistance, y);
                else
                    Q.insert (newDistance, y);
            }
        }
}
```

Dijkstra's algorithm: example

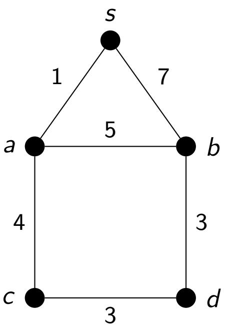


Queue:
dist:
Solved:

x =
y =
newDist =

```
initialize
while (queue not empty)
     x = next();
     solved[x] = true;
     for each unsolved neighbour y of x
         newDist = dist[x] + weight(x,y);
         if (newDist < dist[y])
             dist[y] = newDist;
             update queue;
```

Dijkstra's algorithm: example

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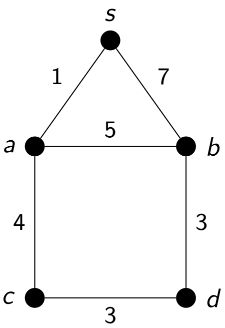
▷  initialize
   while (queue not empty)
       x = next();
       solved[x] = true;
       for each unsolved neighbour y of x
           newDist = dist[x] + weight(x,y);
           if (newDist < dist[y])
               dist[y] = newDist;
               update queue;
```

Queue: $s : 0$
dist: $s : 0$, $a : \infty$, $b : \infty$, $c : \infty$, $d : \infty$
Solved: none

x =
y =
newDist =

Dijkstra's algorithm: example

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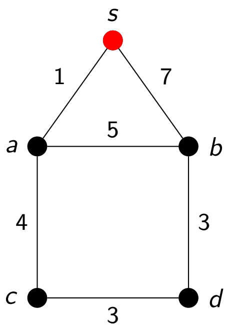
y =

newDist =

Dijkstra's algorithm: example

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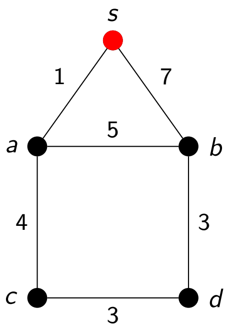
                update queue;
```

Queue: empty
dist: $s:0$, $a:\infty$, $b:\infty$, $c:\infty$, $d:\infty$
Solved: none

x = s
y =
newDist =

Dijkstra's algorithm: example

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Queue: empty
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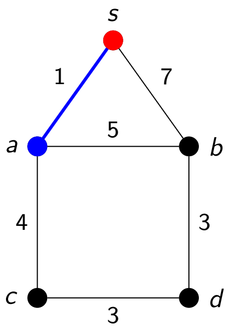
Solved: s

x = s
y =
newDist =

Dijkstra's algorithm: example

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Queue: empty

dist: $s : 0$, $a : \infty$, $b : \infty$, $c : \infty$, $d : \infty$

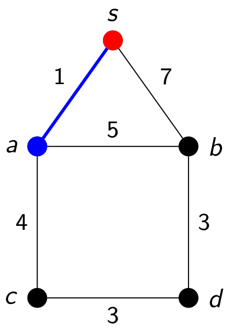
Solved: s

x = s

y = a

newDist =

Dijkstra's algorithm: example

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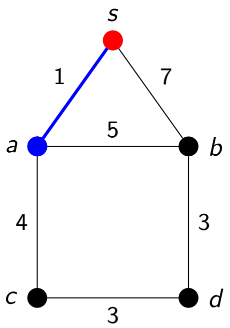
             update queue;
```

Queue: empty
dist: $s : 0,\ a : \infty,\ b : \infty,\ c : \infty,\ d : \infty$
Solved: s

x = s
y = a
newDist = $0 + 1 = 1$

Dijkstra's algorithm: example

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dist: $s:0,\ a:\infty,\ b:\infty,\ c:\infty,\ d:\infty$
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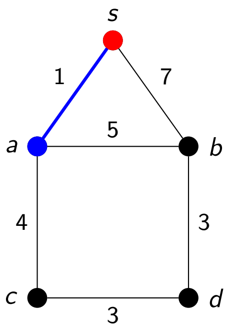


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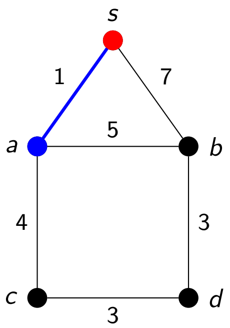
            update queue;
```

Queue: empty
dist: $s:0,\ a:1,\ b:\infty,\ c:\infty,\ d:\infty$
Solved: s

x $=$ s
y $=$ a
newDist $=$ $0+1=1$

Dijkstra's algorithm: example

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            dist[y] = newDist;
            update queue;
```

Queue: $a : 1$

dist: $s : 0,\ a : 1,\ b : \infty,\ c : \infty,\ d : \infty$

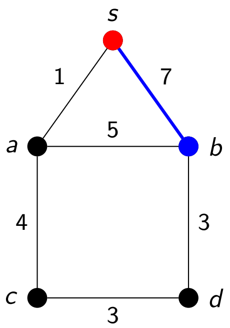
Solved: s

x = s

y = a

newDist = $0 + 1 = 1$

Dijkstra's algorithm: example

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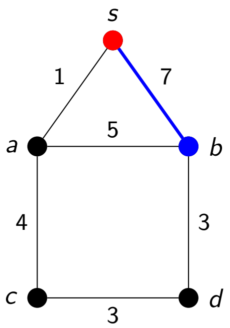
Queue: $a : 1$
dist: $s : 0$, $a : 1$, $b : \infty$, $c : \infty$, $d : \infty$
Solved: s

x $=$ s
y $=$ b
newDist $=$

Dijkstra's algorithm: example

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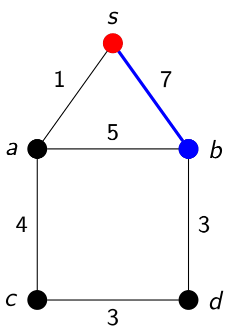
Queue: $a : 1$
dist: $s : 0,\ a : 1,\ b : \infty,\ c : \infty,\ d : \infty$
Solved: s

x $=$ s
y $=$ b
newDist $=$ $0 + 7 = 7$

Dijkstra's algorithm: example

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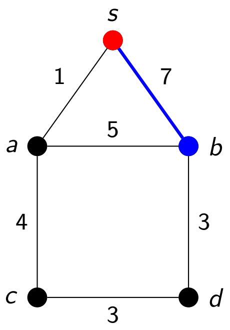
Dijkstra's algorithm: example

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Queue: $a : 1$

dist: $s : 0,\ a : 1,\ b : 7,\ c : \infty,\ d : \infty$

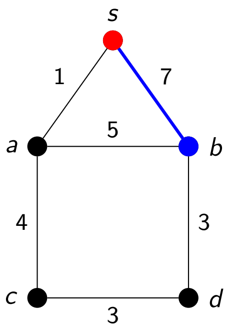
Solved: s

$\mathtt{x} =\ s$

$\mathtt{y} =\ b$

$\mathtt{newDist} =\ 0 + 7 = 7$

Dijkstra's algorithm: example

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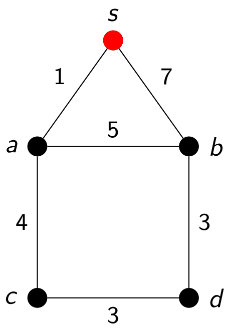
Dijkstra's algorithm: example

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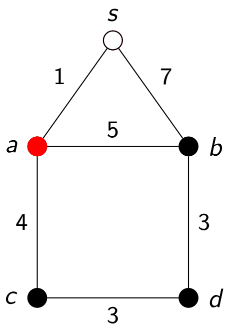
$\texttt{y} = b$

$\texttt{newDist} = 0 + 7 = 7$

Dijkstra's algorithm: example

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Queue: $b : 7$

dist: $s : 0,\ a : 1,\ b : 7,\ c : \infty,\ d : \infty$

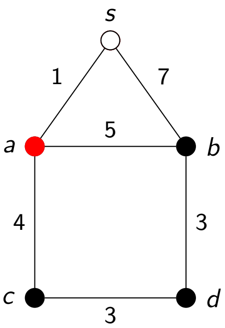
Solved: s

x = a

y =

newDist =

Dijkstra's algorithm: example

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Queue: $b : 7$

dist: $s : 0,\ a : 1,\ b : 7,\ c : \infty,\ d : \infty$

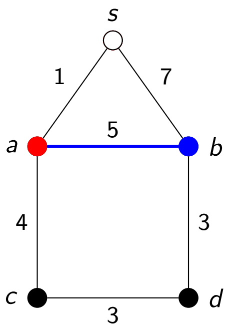
Solved: $s,\ a$

x = a

y =

newDist =

Dijkstra's algorithm: example

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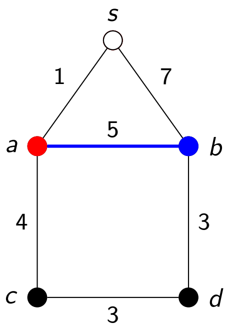
Queue: $b : 7$
dist: $s : 0,\ a : 1,\ b : 7,\ c : \infty,\ d : \infty$
Solved: $s,\ a$

x = a
y = b
newDist =

Dijkstra's algorithm: example

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             update queue;
```

Queue: $b : 7$

dist: $s : 0$, $a : 1$, $b : 7$, $c : \infty$, $d : \infty$

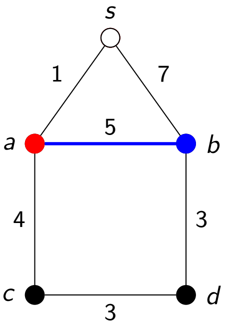
Solved: s, a

x = a

y = b

newDist = $1 + 5 = 6$

Dijkstra's algorithm: example

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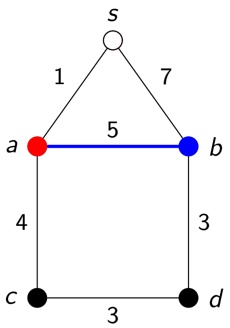
Queue: $b : 7$
dist: $s : 0$, $a : 1$, $b : 7$, $c : \infty$, $d : \infty$
Solved: s, a

x = a
y = b
newDist = $1 + 5 = 6$

Dijkstra's algorithm: example

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Queue: $b : 7$

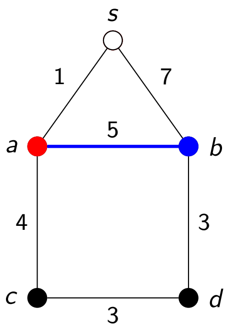
dist: $s : 0,\ a : 1,\ b : 6,\ c : \infty,\ d : \infty$

Solved: $s,\ a$

x $=$ a

y $=$ b

newDist $=$ $1 + 5 = 6$

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Queue: $b : 6$

dist: $s : 0,\ a : 1,\ b : 6,\ c : \infty,\ d : \infty$

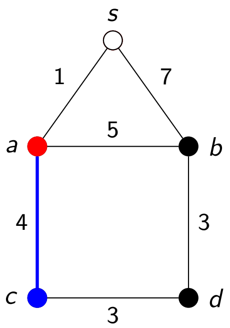
Solved: $s,\ a$

x = a

y = b

newDist = $1 + 5 = 6$

Dijkstra's algorithm: example

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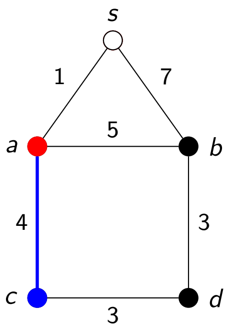
```

Queue: $b : 6$
dist: $s : 0,\ a : 1,\ b : 6,\ c : \infty,\ d : \infty$
Solved: $s,\ a$

x $=$ a
y $=$ c
newDist $=$

Dijkstra's algorithm: example

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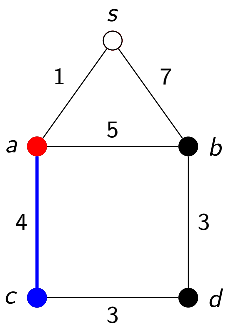
             dist[y] = newDist;
             update queue;
```

Queue: $b : 6$
dist: $s : 0,\ a : 1,\ b : 6,\ c : \infty,\ d : \infty$
Solved: $s,\ a$

$\mathtt{x} =\ a$
$\mathtt{y} =\ c$
$\mathtt{newDist} =\ 1 + 4 = 5$

Dijkstra's algorithm: example

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Queue: $b : 6$

dist: $s : 0,\ a : 1,\ b : 6,\ c : \infty,\ d : \infty$

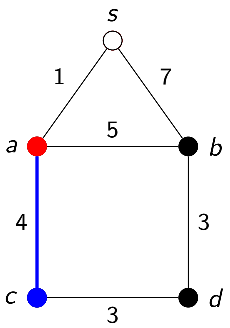
Solved: $s,\ a$

$\mathtt{x} =\ a$

$\mathtt{y} =\ c$

$\mathtt{newDist} =\ 1 + 4 = 5$

Dijkstra's algorithm: example

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Queue: $b : 6$

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Solved: s, a

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y = c

newDist = $1 + 4 = 5$

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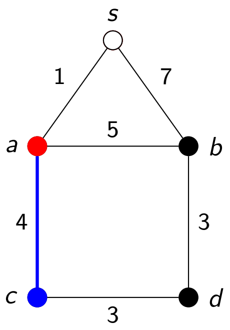


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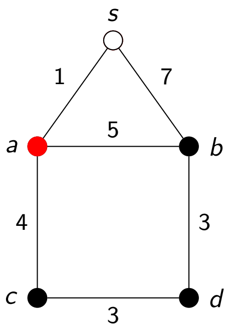


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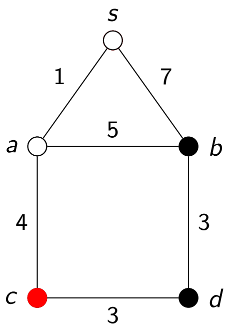


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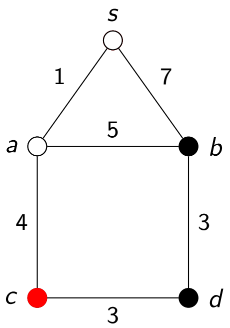


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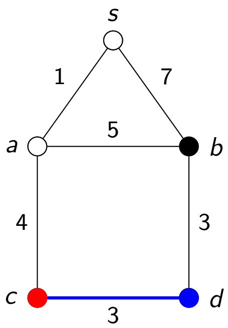


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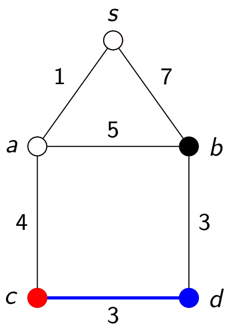


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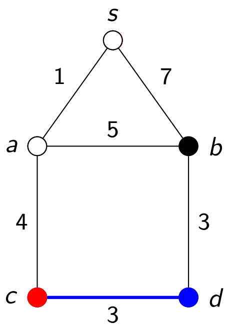


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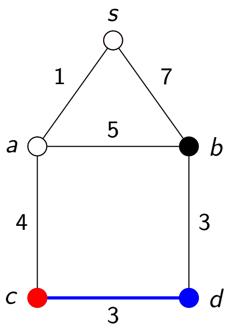


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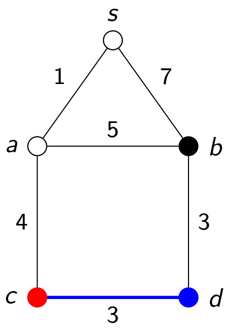


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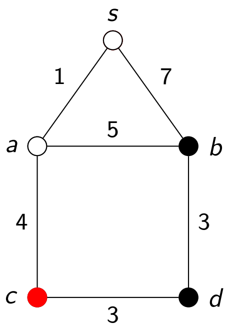


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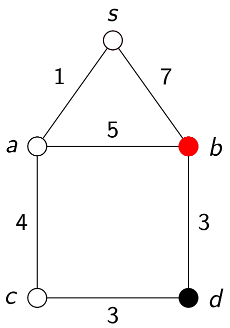


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x = b
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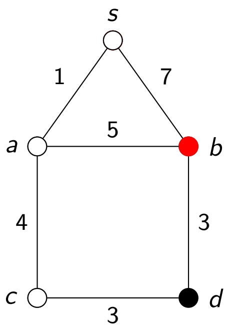


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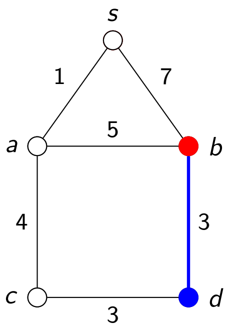


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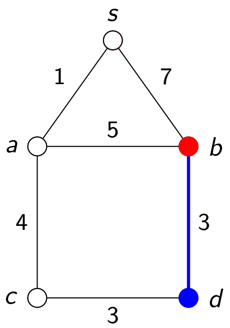


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Solved: $s,\ a,\ b,\ c$

x = b

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newDist = $6 + 3 = 9$

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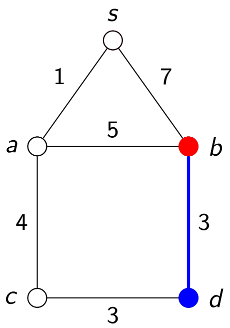


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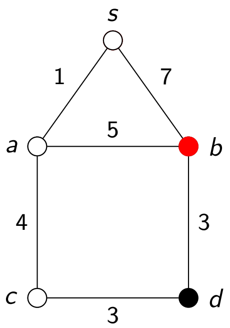


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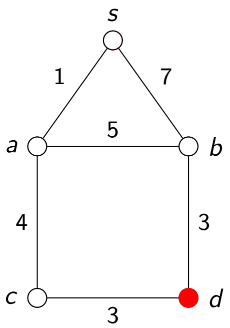


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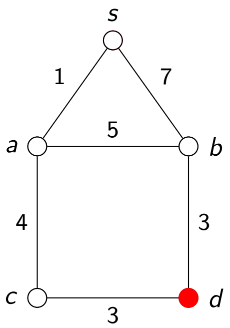


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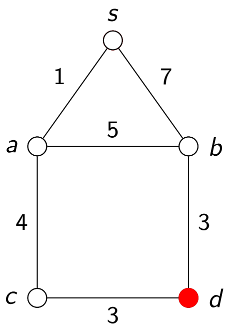


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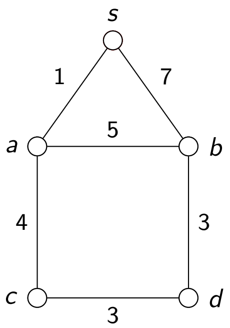


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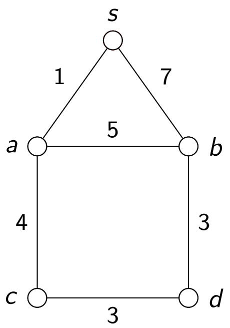


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Dijkstra's algorithm: finding the paths

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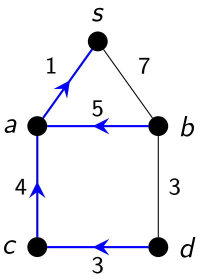

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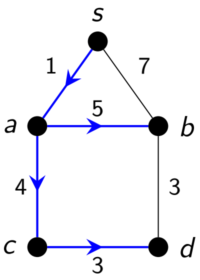

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- Total running time is $O((n + e) \log n)$.