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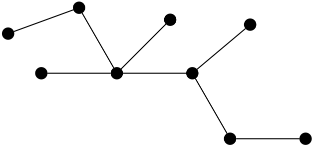
- Does so by exploring vertices in order of their distance from the source.
- Running time is $O((n + e)\log n)$ for a graph with n vertices and e edges.

Trees as graphs

- We have seen trees as data structures.
- A tree is also a graph: a connected graph with no cycles:

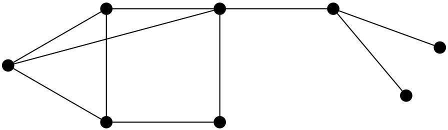


- Every tree with n vertices has $n-1$ edges (prove by induction).

Subgraphs

A **subgraph** of a graph G is a graph that can be made from G by deleting vertices and/or edges.

A **spanning subgraph** is one made by edge deletions only.



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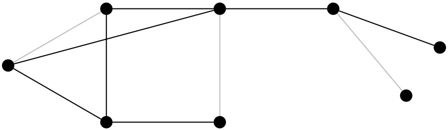

A spanning subgraph

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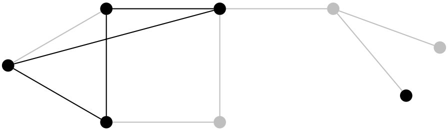
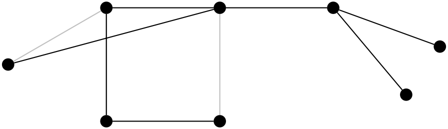


A non-spanning subgraph

Spanning trees

A **spanning tree** is a spanning subgraph that is a tree.



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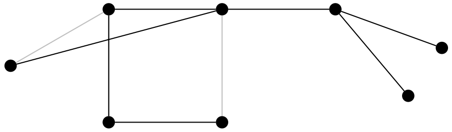

Spanning trees connect all the vertices using the least possible number of edges.

Every connected graph has at least one spanning tree.

Minimum spanning trees

The **minimum spanning tree** (MST) of a weighted graph is the spanning tree that has the least total edge weight.

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E.g., you're building a wind farm and you want to connect your turbines using the least amount of cable.

(Assuming cables must run from turbine to turbine with no junctions in the middle.)

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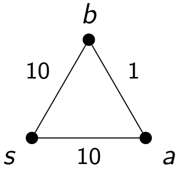
(Assuming cables must run from turbine to turbine with no junctions in the middle.)

Strictly, we should say "**An** MST is **a** spanning tree that has least possible weight" – a graph may have more than one MST.

Dijkstra doesn't compute MSTs

Dijkstra's algorithm computes a spanning tree, but not necessarily a minimal one.



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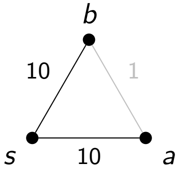

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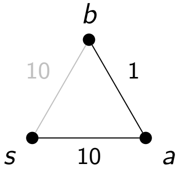

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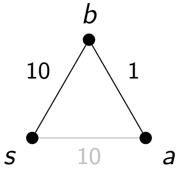

The other minimum spanning tree

Prim's algorithm

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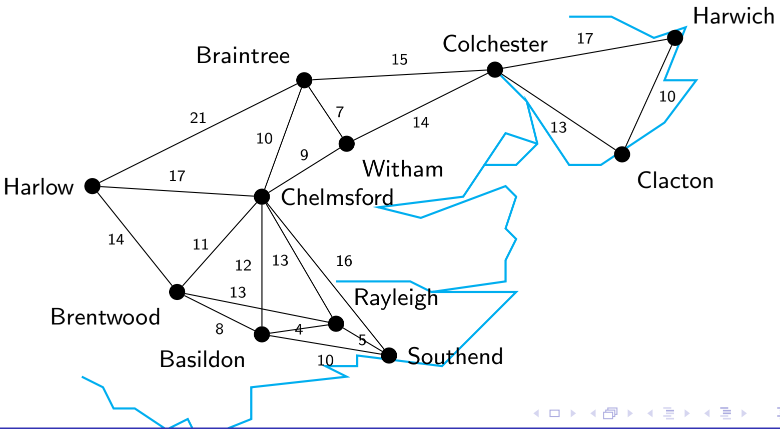

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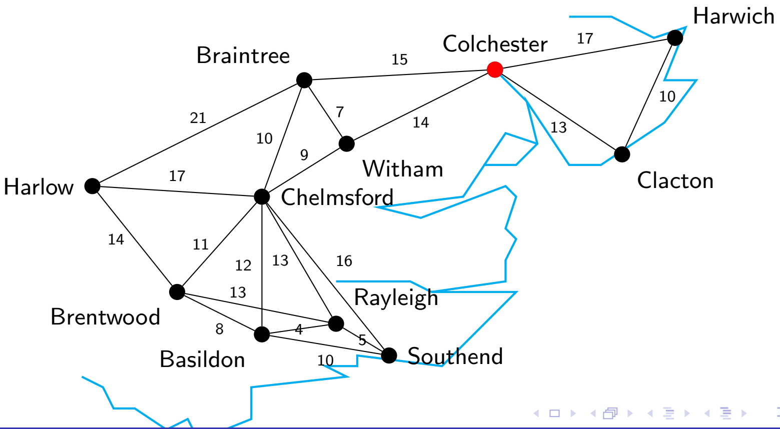

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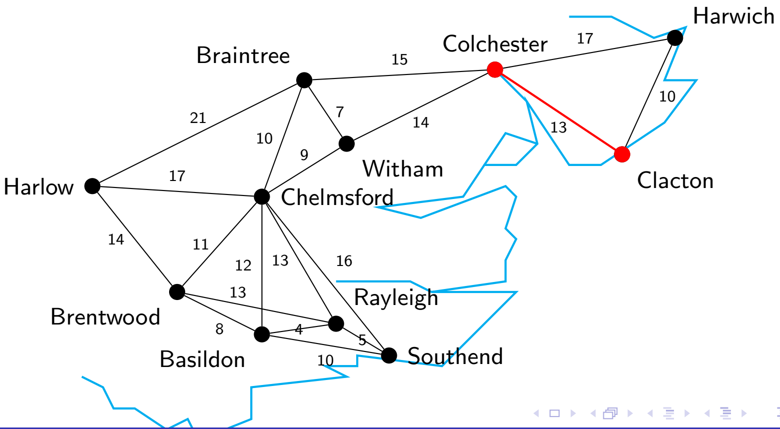

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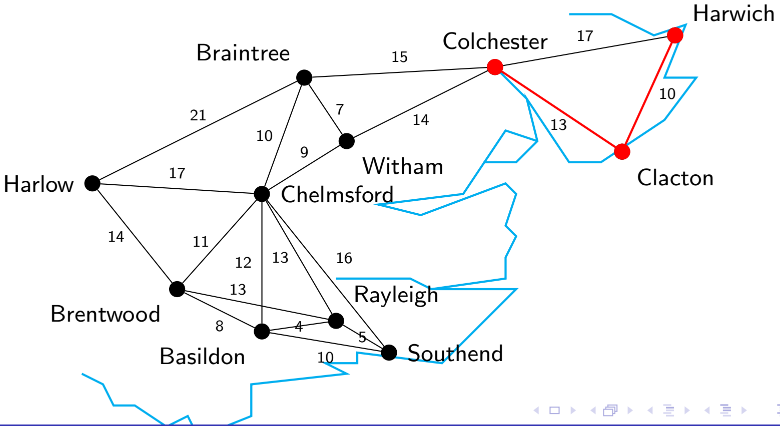

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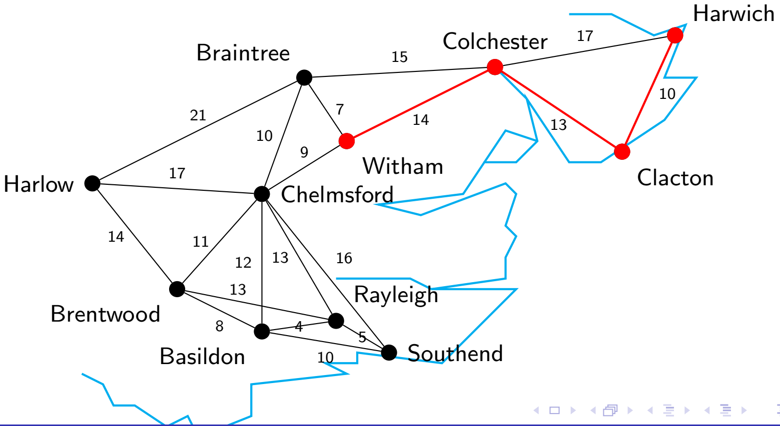

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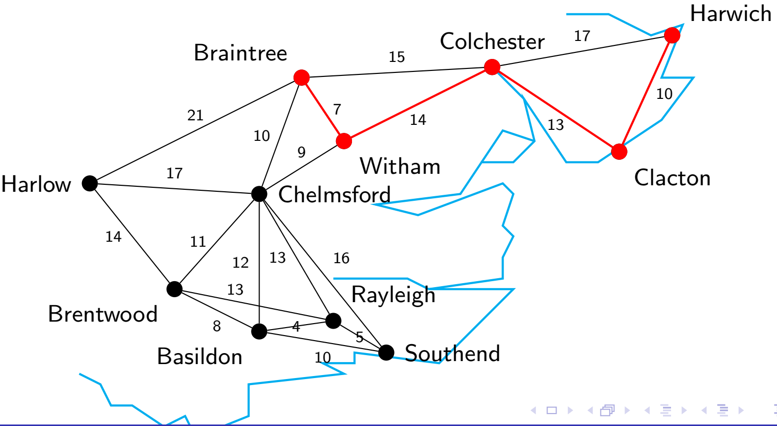

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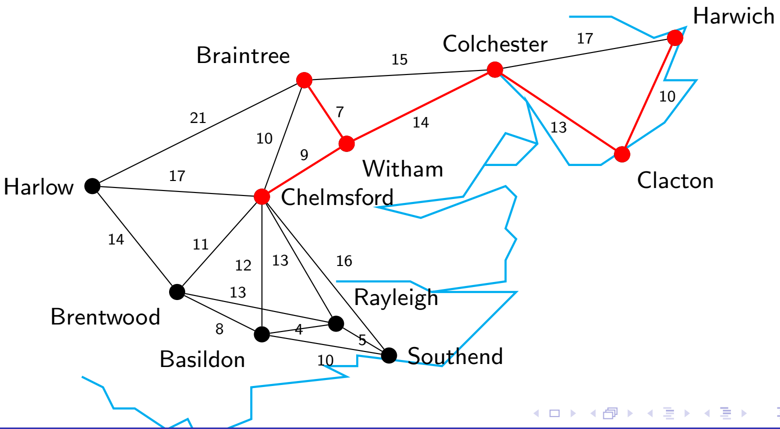

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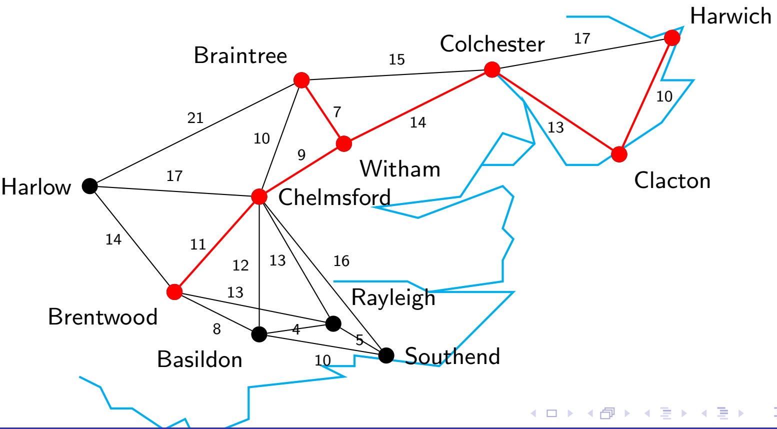

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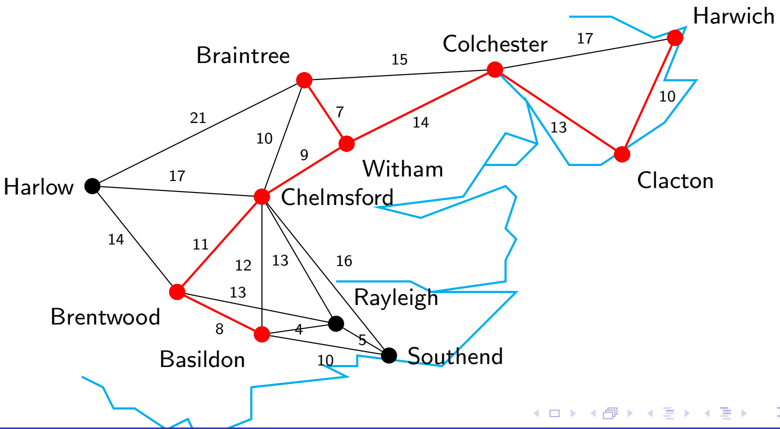

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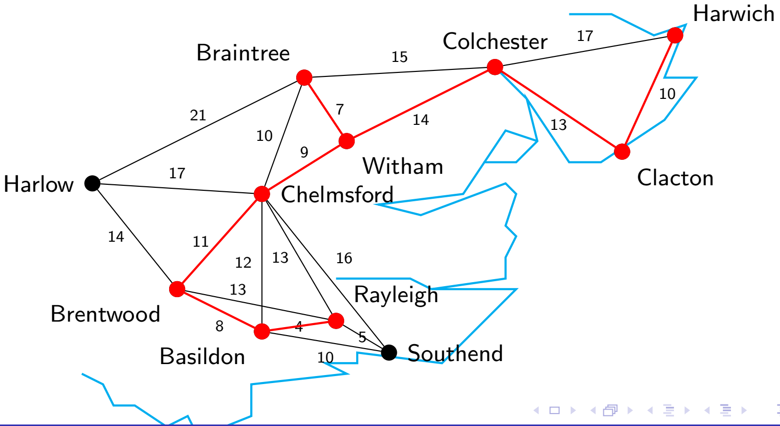

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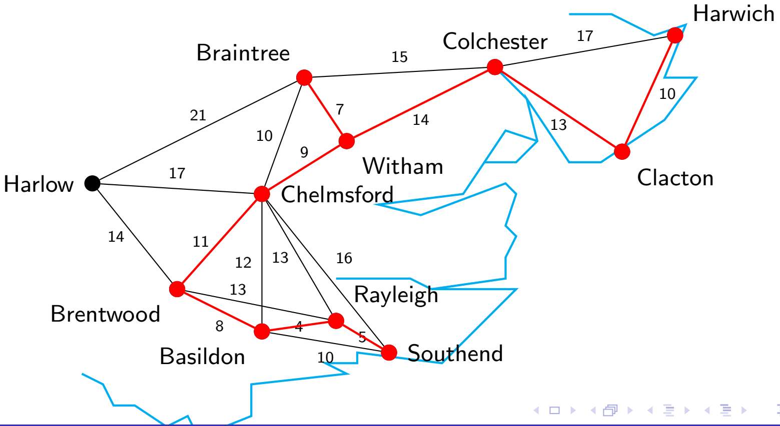

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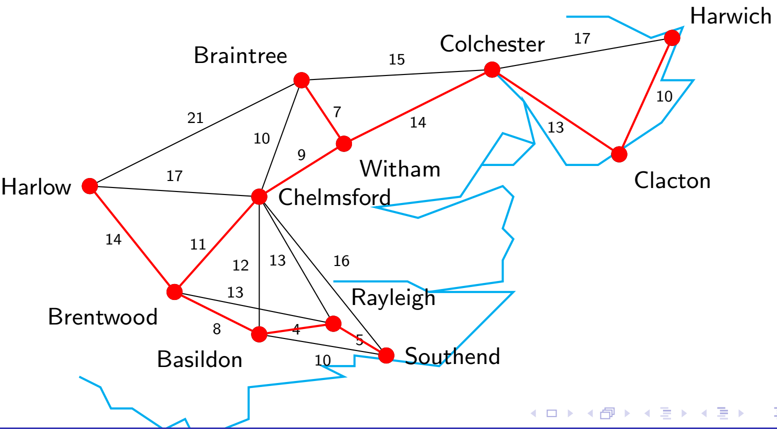

Prim's algorithm: implementation

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- `boolean[] inTree` tells us which vertices have been added to the tree so far.
- A priority queue of vertices that are adjacent to the tree built so far, with the weight of the shortest connecting edge as priority.
- `int[] treeNbr` stores, for each vertex `x` in the queue, its nearest neighbour in the tree built so far.

Prim's algorithm: pseudocode (1)

Prim's algorithm for a graph with n vertices.

```
// Initialization
inTree = {true, false, false, ..., false};
treeNbr = {0, ..., 0};
Q = new empty priority queue;
mst = new graph with n vertices;

for each nedge (0,y) in G
    Q.insert (weight (0,y), y);

...
```

Prim's algorithm: pseudocode (2)

```
// Main loop
while (Q not empty)
    // Add next vertex to tree
    x = Q.next();
    mst.addEdge (x, treeNbr[x], weight (x, treeNbr[x]));
    inTree[x] = true;

    // Process neighbours
    for each edge (x,y) in G
        if (!inTree[y])
            if (!Q.contains(y))
                Q.insert (weight(x, y), y);
                treeNbr[y] = x;
            else if (weight(x,y) < Q.priorityOf(y))
                Q.setPriority (weight(x,y), y);
                treeNbr[y] = x;
return mst;
```

A trick with priority queues

- Graph algorithms (Dijkstra, Prim, etc.) often need to find or update the priority of a specific item in a priority queue.
- A standard priority queue takes time $\Theta(n)$ to do this: you just have to search the array.

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- We can find items in time $O(1)$ by maintaining an array that tells us the index of each vertex in the priority queue's internal array.
- Allows find priority in time $O(1)$ (just look it up) and update priority in time $O(\log n)$ (because of bubble up/down).

Prim's algorithm: running time

For an input graph with n vertices and e edges, using adjacency lists.

- The main loop runs n times (once per vertex) and calls `next()` each time. Total time for `next()` is $O(n \log n)$.

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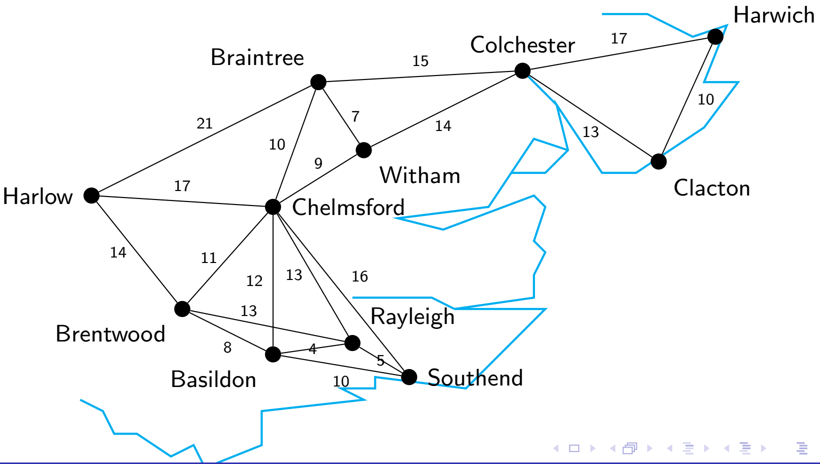
- For each edge, it can call `insert()` or `setPriority()`: each call takes time $O(\log n)$.
- Total running time is $O((n + e) \log n)$.
- This is $O(e \log n)$ if the input is connected (a connected graph must have at least $n - 1$ edges).

Kruskal's algorithm

1. Start with an empty subgraph.
2. Add to it the least-weight edge that doesn't create a cycle.
3. Repeat 2. until we have a spanning tree.



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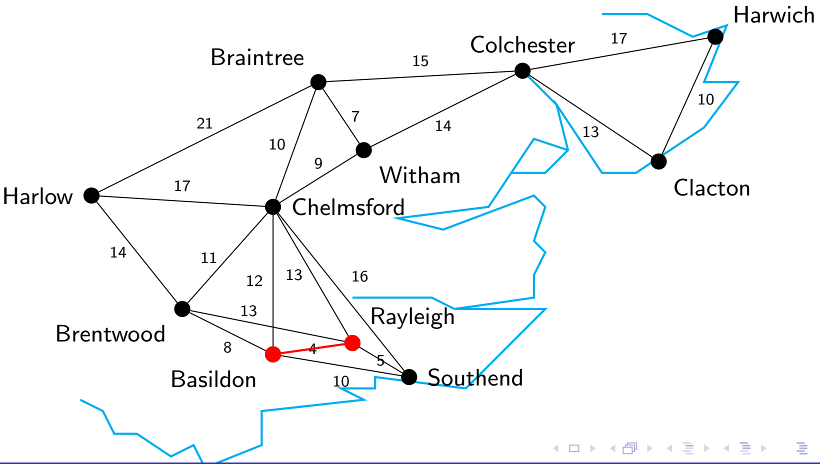

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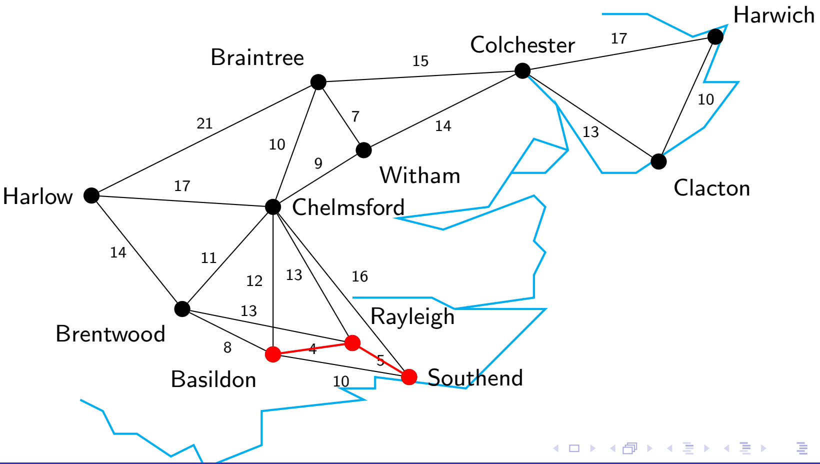

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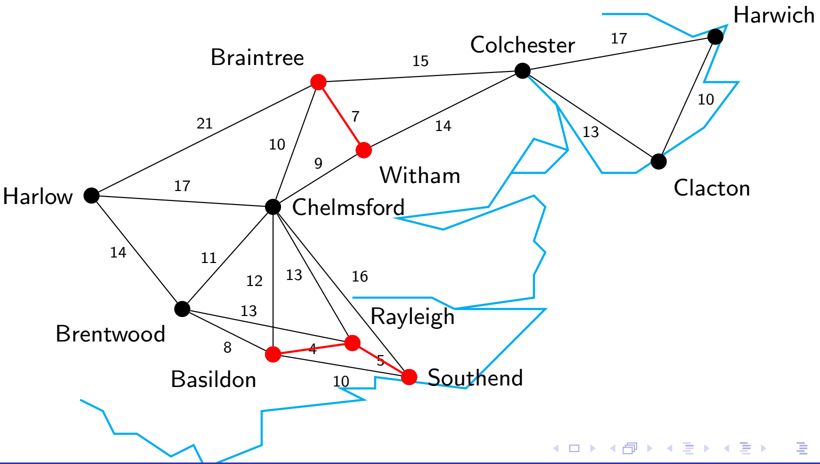

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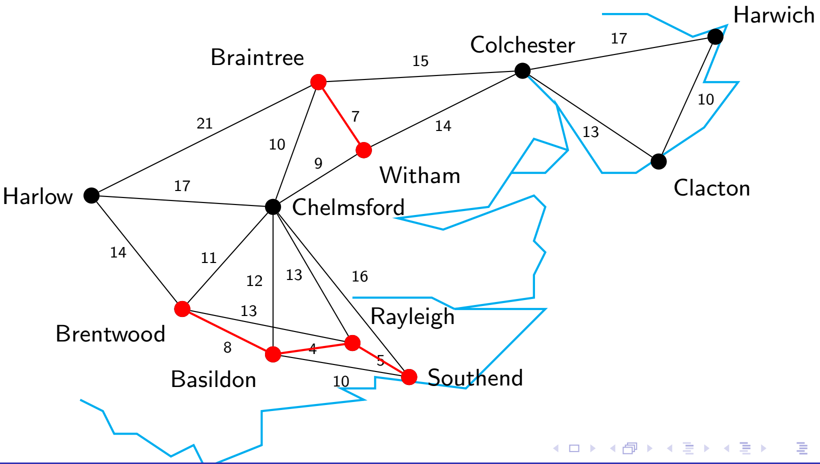

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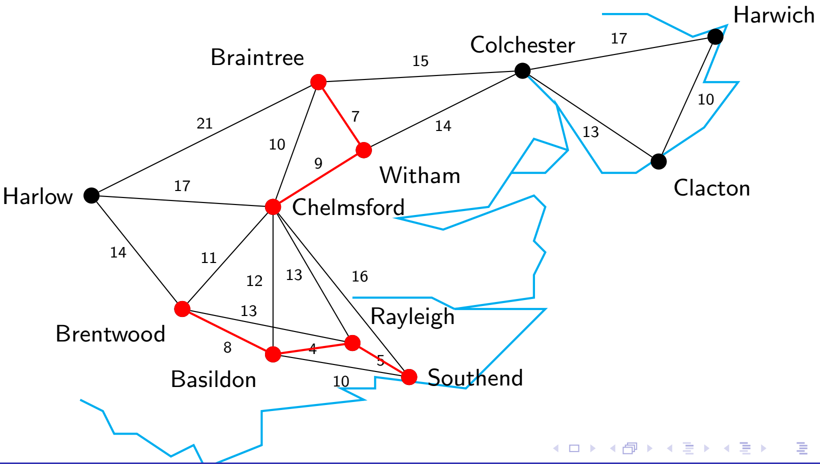

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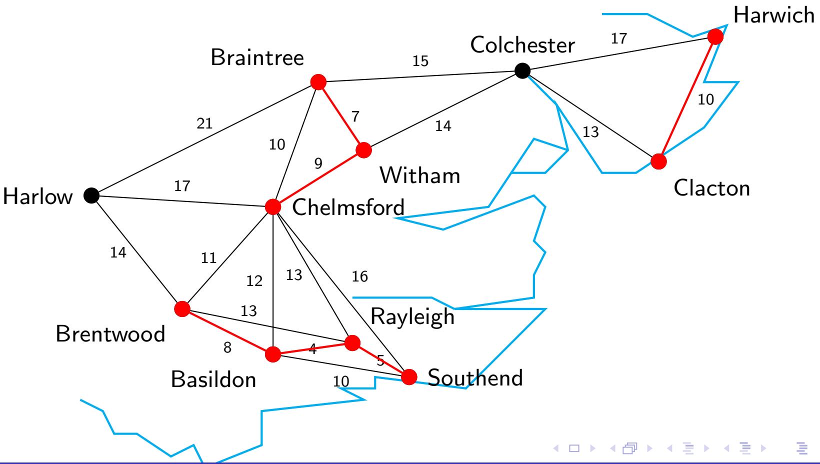

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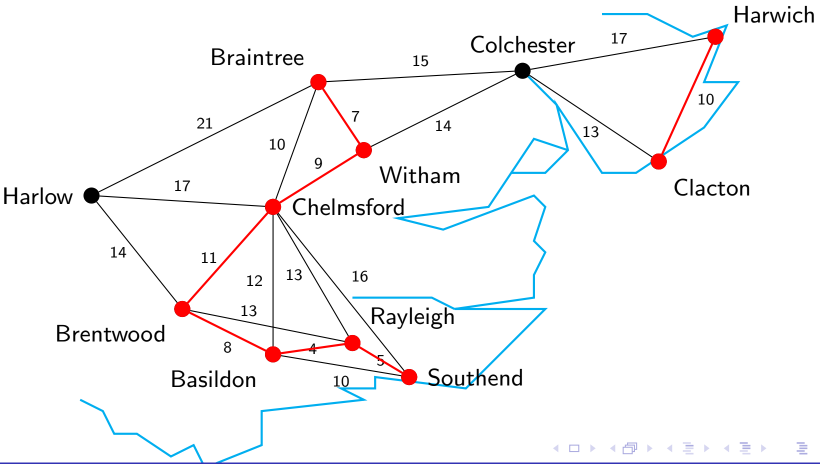

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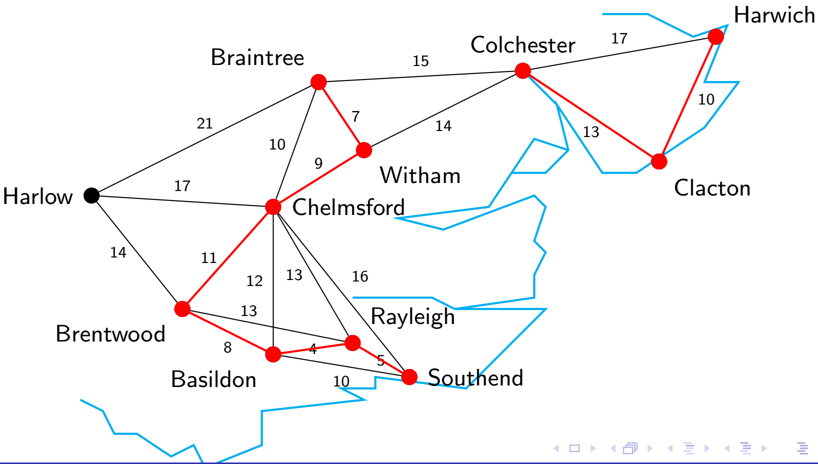

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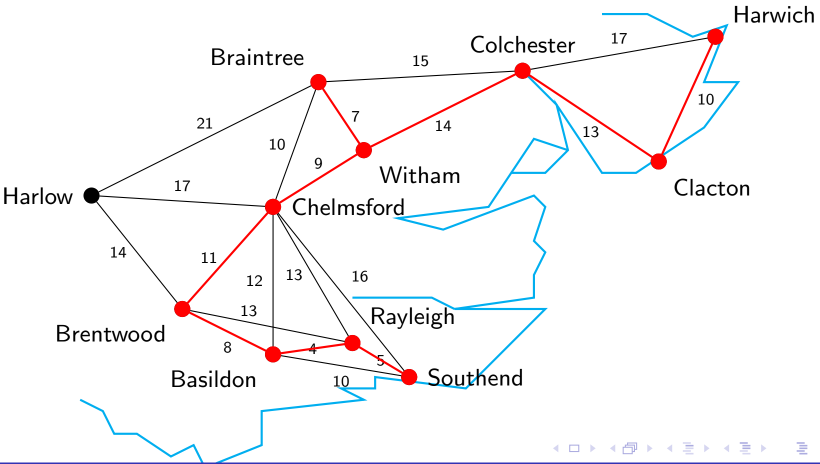

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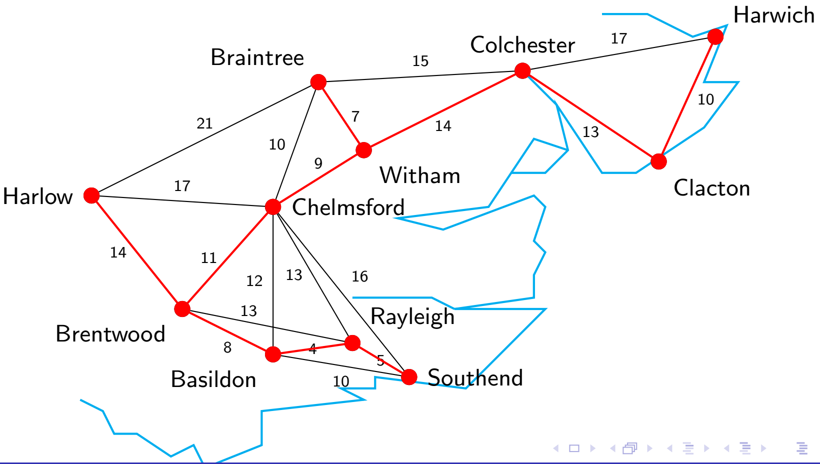

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To merge components, add the smaller list to the larger and update the references in the array.

Each time we merge, the shorter list doubles (or more) in length, so a vertex can't be moved more than $\log n$ times.

Kruskal's algorithm: pseudocode

For an input graph G with n edges.

```
mst = new graph with n vertices;
L = list of G's edges sorted by increasing weight;

for each edge (x, y, w) in L, in order
    if (componentOf(x) != componentOf(y))
        mst.addEdge (x, y, w);
        mergeComponents (x, y);

return mst;
```

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- Total time is $O((n + e) \log n) = O(e \log n)$.

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Conclusion:

- If there are only $O(e)$ possible edge-weights, use Kruskal.
- Otherwise, if $e > O(n)$ (usually true), use Prim.
- Otherwise, use either.

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Minimum spanning trees: summary

- A minimum spanning tree (MST) is a tree that includes all a graph's vertices and has the least possible total edge weight.
- Prim's algorithm builds an MST by adding the least-weight edge that extends the current tree.
- Kruskal's algorithm adds the least-weight edge from the graph that doesn't make a cycle.
- Both run in time $O(e \log n)$.
- Prim can be improved to $O(e + n \log n)$ for "most" graphs.
- Kruskal can be improved to almost $O(e)$ if few possible edge weights.