Chapter 2

8 / 1 / 2024

The Peano Axioms (God help us)

The natural numbers are the elements of the set $N := \{0, 1, 2, 3, 4, \dots\}$

The Axioms

1. 0 is a natural number
2. if n is a natural number, then $n++$ is also a natural number
3. $\forall n \in N, n++ \neq 0$
4. $\forall n, m \in N, n++ = m++ \rightarrow n = m$
5. *Induction:* $P(n)$ is a property of a natural number n, $P(0) \wedge P(n++) \rightarrow P(n)$ is true for all $n \in N$

Induction

At least while we only look at the natural numbers, proving $P(0)$, assuming $P(n) \forall n \in N$ and proving $P(n++)$ proves $P(n)$.

From what I get, we can do this because once $P(0) \wedge P(n++)$, $P(0++\text{indefinite times})$ becomes true.

Recursive definitions

Recursive definitions generate a sequence based on a function. An initial value, a_0 is defined, and the rest of the sequence is defined as $a_{n++} := f(a_n)$. Each element of the sequence will now have a unique value, which in turn produces a unique value, the output of f for the next element of the sequence.

Addition

- Only for $N \rightarrow N$, obviously.

$$m \in N$$

$$0 + m := m$$

Assume inductively that $n + m, n \in N$ is defined.

We now define $(n++) + m := (n + m)++$

We can now recursively ”count”.

Note that $3 + 5$ involves counting 5 times, while $5 + 3$ involves counting 3 times.

Lemma 2.2.2 $\forall n \in \mathbb{N} n + 0 = n$

We cannot just prove this from the definition of addition, as its in reverse.

Proof:

We use induction.

Base case: $0 + 0 = 0$, we *can* get this one by definition.

We now suppose inductively that $n + 0 = n$.

By definition. $n{+}{+} + 0 = (n + 0){+}{+}$, which $= n{+}{+}$ since we have assumed inductively that $(n + 0) = n$.

This concludes the proof.

Lemma 2.2.3 $\forall n, m \in \mathbb{N}, n + (m{+}{+}) = (n + m){+}{+}$

Proof:

We keep m fixed.

Consider $n = 0, \rightarrow 0 + (m{+}{+}) = (0 + m){+}{+}$. We know this is true by definition.

We assume what we are trying to prove is true. Now we must show that $(n{+}{+}) + (m{+}{+}) = ((n{+}{+}) + m){+}{+}$.

By definition of addition we have that the left side is $(n + (m{+}{+})){+}{+}$. By the inductive hypothesis we have assumed that this $= ((n + m){+}{+})\ \ {+}{+}$.

The right side then becomes $(n + m){+}{+}\ {+}{+}$, which $= ((n + m){+}{+}){+}{+}$. This concludes the induction.

Proposition 2.2.4 $\forall n, m \in \mathbb{N}, n + m = m + n$ **(Conmutativity)**

Proof:

We keep m fixed. This lets us prove the base case $(0 + m = m + 0)$ by definition. We now suppose that $n + m = m + n$ and have to prove $(n{+}{+}) + m = m + (n{+}{+})$.

Left side	By definition of addition we get $(n{+}{+}) + m =$ (n + m)++ .
Right side	By Lemna 2.2.3, we get $m + (n{+}{+}) = (m + n){+}{+}$, but by our previous assumption, $n + m = m + n$ and $\therefore$ $((m + n){+}{+} =$ (n + m)++

Proposition 2.2.5 $\forall a, b, c \in \mathbb{N}, (a + b) + c = a + (b + c)$ **(Asociativity)**

Proof: Exercise 2.2.1

Hint: Modify proofs of 2.2.2, 2.2.3 & 2.2.4

something something inductive assumption $\rightarrow$ both sides $= (a + b + c){+}{+}$

Let us induct on a.

Base case: $a = 0$

$$\begin{array}{ll} a = 0 \rightarrow & (0 + b) + c = 0 + (b + c) \\ \text{By definition of addition:} & \text{b + c} = b + c \quad \square \end{array}$$

Induction: Assume $(a + b) + c = a + (b + c)$

$$\begin{array}{ll} \text{Prove:} & (a++ \; + b) + c = a++ \; + (b + c) \\ \text{By definition of addition:} & (a + b)++ \; + c = a++ \; + (b + c) \\ \text{Same thing:} & (a + b)++ \; + c = (a + (b + c))++ \\ \text{One more time:} & ((a + b) + c)++ = (a + (b + c))++ \\ \text{We have already assumed asociativity.} & (a + b + c)++ = (a + b + c)++ \quad \square \end{array}$$

Proposition 2.2.6 (Cancellation law)

Let $a, b, c \in \mathbb{N} \ni a + b = a + c. \;\; \Rightarrow b = c$

- We don't have subtraction
- But this does the job and also helps define the integers later

Proof

We induct on a.

Base case: $a = 0$

$$\begin{array}{ll} a = 0 \rightarrow & 0 + b = 0 + c \\ \text{By definition of addition:} & \square \end{array}$$

Induction: Assume $a + b = a + c \rightarrow b = c$

$$\begin{array}{ll} \text{Prove:} & (a++) + b = (a++) + c \\ \text{By definition of addition:} & (a + b)++ = (a + c)++ \\ \text{By Axiom 4:} & (a + b) = (a + c) \\ \text{We assume cancelation} & b = c \quad \square \end{array}$$

Definition 2.2.7 (Positive natural numbers)

$n \in \mathbb{N} \iff n \neq 0$

Proposition 2.2.8

$a \in \mathbb{N} \ \wedge \ b \in \mathbb{N} \rightarrow (a + b) \in \mathbb{N}$

Proof:

We induct on b.

Base case: $b = 0$

$$\begin{array}{ll} b = 0 \rightarrow & a + 0 = a \\ a \in \mathbb{N} & \square \end{array}$$

Induction: Assume $a \in \mathbb{N} \ \wedge \ b \in \mathbb{N} \rightarrow (a + b) \in \mathbb{N}$

Prove:	$a + (b++) \in \mathbb{N}$
By Lemna 2.2.3	$a + (b++) = (a + b)++$
By Axiom 3:	$(a + b)++ \neq 0 \ \square$

Corollary 2.2.9

$\{a, b\} \in \mathbb{N} \wedge a + b = 0 \rightarrow a = 0 \wedge b = 0$

Proof

If a or $b \neq 0$ (are positive), by Proposition 2.2.8 (and conmutativity) $a+b \in \mathbb{N}$. $\therefore$, by contradiction, $a, b = 0$.

Lemma 2.2.10

$a \in \mathbb{N} \rightarrow \exists |\ b \in \mathbb{N}$ s.t. $(b++) = a$

Proof: Exercise 2.2.2

Hint: Use indiction. (The induction hypotesis is not actually used, but that doesn't undermine the validity of the argument)

We induct on b.

Base case: $b = 0$

$b = 0 \rightarrow$	$0++ = 1 \rightarrow a = 1$
By Axiom 3:	$a \in \mathbb{N} \ \square$

Induction: Assume $a \in \mathbb{N}$

$$\begin{array}{ll} \text{Prove:} & (b++)++ = a \rightarrow a \in \mathbb{N} \\ \text{By Axiom 3:} & (b++), (b++)++ \neq 0 \rightarrow (b++)++ \in \mathbb{N} \rightarrow a \in \mathbb{N} \ \ \square ?????? \end{array}$$

Definition 2.2.1: Ordering of $\mathbb{N}$

$n \geq m \iff \exists\, a \ s.t.\ n = m + a$

$n > m \iff n \geq m \wedge n \neq m$

Proposition 2.2.12: (Basic properties of the natural nubers)

(a). $x \geq x$

$$\begin{array}{ll} \text{Def. Addition:} & x + 0 = x \\ & \therefore\ a = 0 \rightarrow x \geq x \end{array}$$

(b). $a \geq b \wedge b \geq c \rightarrow a \geq c$

$$\begin{array}{l} a = b + n_1; b = c + n_2 \\ a = c + n_1 + n_2 \end{array}$$

(c). $a \geq b \wedge b \geq a \rightarrow a = b$

$$\begin{array}{ll} & a = b + n_1;\ b = a + n_2 \\ & a = a + n_2 + n_1 \\ \text{Asso. \& Def. Addition:} & a + 0 = a + (n_2 + n_1) \\ \text{Cancelation law:} & (n_2 + n_1) = 0 \\ \text{Collary 2.2.9:} & n_1, n_2 = 0 \\ \text{Def. Addition:} & a = b + 0 \rightarrow a = b \end{array}$$

(d). $a \geq b \iff a + c \geq b + c$

$$\begin{array}{ll} a + c \geq b + c \rightarrow & a + c = b + c + n_1 \\ \text{Cancellation law:} & a = b + n_1 \therefore a \geq b??? \end{array}$$

(e). $a < b \iff a++ \leq b$

$$\begin{array}{ll} & a++ \geq b \rightarrow a++\ + n_1 = b \\ \text{Def. Addition:} & a++ = a + 1 \\ & n_2 = n_1 + 1 \rightarrow a + n_2 = b \\ \text{Def} > 1 & n_2 > 0 \\ & a < b \end{array}$$

(f). $a < b \iff b = a + d \wedge d \in \mathbb{N}+$

Proposition 2.2.12 e:	$a++ \leq b$
Def. Addition	$a + 1 \leq b$
Def. $<$	$a + 1 + n_1 = b$
Def. Addition	$a + (n_1 ++) = b$
Axiom 2.3	$(n_1 ++) \neq 0$
Def. Positive	n_1 is positive

Proposition 2.2.13: Trichotomy of order for natural numbers

$a, b \in \mathbb{N} \rightarrow (a < b \vee a = b \vee a > b)$

1. $a = 0 \rightarrow a \leq b \ \forall b$ (why?) Induction.

$b = 0$	$0 + 0 = b$	
$b++$	$0 + n_1 = b++$	
	$\forall b++, n_1 = b++$	or something like that

2. $a > b \rightarrow a++ > b$ (why?)

Proposition 2.2.12 f:	$a = b + n_1, n_1$ is positive
Axiom 2.4:	$a++ = (b + n_1)++$
Associativiy	$a++ = b + (n_1 ++)$
Axiom 2.3:	$n_1 ++$ is positive
Proposition 2.2.12 f:	$a++ > b$

3. $a = b \rightarrow a++ > b$ (why?)

Def. Addition: $a++ = b++ = a + 1$

Proposition 2.2.14: Strong principle of induction Let m_0 be some natural number and $P(m)$ be a property of an arbetrary natural number m. Suppose that $\forall m \geq m_0 : \ P(m') \ \forall \ m_0 \leq m' < m \rightarrow P(m)$. It can then be concluded that $P(m) \ \forall \ m \geq m_0$.

Proof: Let $Q(n)$ be the property that $P(m) \ \forall \ m_0 \leq m < n$. The hypothesis can now be re-written as: $Q(n) \rightarrow P(n)$.

Base case: $n = 0$	
Im not proving this:	$\nexists m$ s.t. $m_0 \leq m < 0$
Assume $Q(n) \rightarrow P(n)$**. Prove** $Q(n++)$	
$Q(n) \wedge P(n) \rightarrow$	$P(m) \ \forall \ m_0 \leq m \leq n$
Proposition 2.2.12 e:	$P(m) \ \forall \ m_0 \leq m < n++$
Def. $Q(n)$	$Q(n++)$ $\square$