Graph Theory and Modeling

Chapter 1: Basic Notions of Graphs

Basic Terms

Graph

A graph is defined as a pair of sets: (V, E), where V is the set of nodes (vertices) and E is a set of pairs of members of V (edges). If the graph is directed, the edges are also known as arcs. An arc (v, u) is said to have a tail end at v and a head end at v. When $\exists$ an edge between nodes they are called adjacent.

Loop

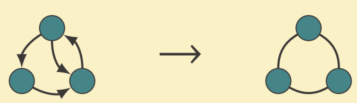
An edge (u, u) for some $u \in V$

Simple Graph

A graph without loops

Subadjacent graph

Let $G = (V, E)$ be a directed graph. Its subadjacent graph $\bar{G}$ is an undirected graph that has an edge (u, v) for every arc (u, v). In the case of G having arcs (u, v) and (v, u), both are replaced by a single edge (u, v).



Trail

A succession (node, edge, node, edge, ...)

Circuit (Closed Trail)

A trail in which the first and last nodes are the same.

Path

A trail where all nodes are distinct

Cycle (Simple circuit)

A circuit in which all nodes but the first and last are distinct.

Length

The number of edges in a trail

Power Set

The power set $\mathcal{P}(S)$ is the set of all subsets of S, including S

Adjacency Function

Let $G = (V, E)$ the adjacency function Γ is defined:
$\Gamma(u) : V \rightarrow \mathcal{P}(V) = \{v \in V : (u, v) \in E\} \forall u \in V$

Adjacency Matrix

Let $G = (V, E)$ s.t. $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix A is a square matrix of size $n \times n$ such that $\forall a_{ij}, a_{ij} = \begin{cases} 1 & (v_i, v_j) \in V \\ 0 & (v_i, v_j) \notin V \end{cases}$

Grade (not directed)

The grade of a node v, $d(v)$ is the number of edges that are incident with it. *A loop adds two units to the grade, not one.

If the grade of a node is 0, it is called isolated.

The maximum degree of a graph G is written $\Delta(G)$, while the minimum is written $\delta(G)$.

Grade (directed)

The indegree of a node v, $d^{-}(v)$, is the number of edge head ends adjacent to it.

The outdegree, $d^{-}(v)$, conversely, is the number of edge tail ends adjacent to it.

Obtaining degree from adjacency matrix

The degree of a node i from an undirected graph can be obtained by summing either the column or row i of the graph's adjacency matrix, and adding 1 if the node has a loop.

If the graph is directed, the outdegree can be obtained by summing the row i and the indegree can be obtained by summing the column i.

Handshaking Lemma

Let $G = (V, E)$ be an undirected graph. $\sum_{v \in V} d(v) = 2\#E$.

$\therefore$ the number of nodes with an uneven grade is even.

Let G be now be a directed graph. $\sum_{v \in V} d^{+}(v) = \sum_{v \in V} d^{-}(v) = \#E$

Subgraphs

Let G and H be graphs.

Subgraph

H is a subgraph of G $\iff$ $V(H) \subset V(G) \land E(H) \subset E(G)$

Spanning Subgraph

H is a spanning subgraph of G $\iff$ H is a subgraph of G $\land$ $V(H) = V(G)$

Induced subgraph

H is a subgraph of G induced by E' ($H = G[E']$) for some $E' \subset E(G)$ $\iff$ $E(H) = E' \land V(H) = \{v \in V(G) : v$ is and endpoint of some $e \in E\}$

H is a subgraph of G induced by V' ($H = G[V']$) for some $V' \subset V(G)$ $\iff$ $V(H) = V' \land E(H) = \{e \in E :$ the endpoints of $e\}$

Subtraction

Let $V' \subset V(G)$, $E' \subset E(G)$.

- $G - V' = (V - V', R - \{e \in E : e_1 \in V' \vee e_2 \in V'\})$
- $G - E' = (V, E - E')$
- Let $e \in E'$. $G - e = G - \{e\}$

Graphic Sequences

Graphic Sequence

A sequence of integers is considered a graphic sequence $\iff$ there exists an undirected graph such that the values of the sequence correspond to the degrees of its nodes.

Harvel-Hakimi Theorem

Let the following be a sequence of decreasing integers: $(s, t_1, t_2, \ldots, t_s, d_1, d_2, \ldots, d_r)$. $(s, t_1, t_2, \ldots, t_s, d_1, d_2, \ldots, d_r)$ is a graphic sequence $\iff$ $(t_1 - 1, t_2 - 1, \ldots, t_s - 1, d_1, d_2, \ldots, d_r)$ is also a graphic sequence.

Hakimi's Algorithm

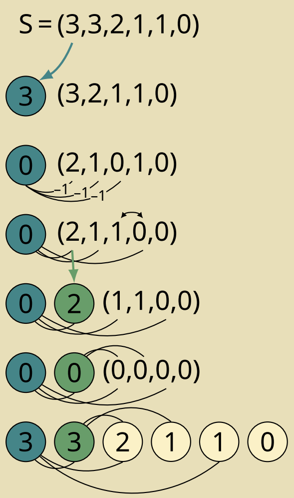
Let S be a decreasing sequence of integers: $(d_1, d_2, \ldots, d_n)$.

$(\sum_{i=1}^{n} d_i \text{ is odd } \vee d_1 > n - 1) \Rightarrow$ S is not a graphic sequence.

Breaks the Handshaking Lemma

The sequence has too few nodes

Otherwise:



Remove the first element of S (which we will call t).

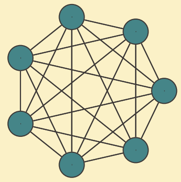
Subtract 1 from the first t elements of S.

If one of the elements of S is negative, the sequence is not graphic, if all the elements are equal to 0, the sequence is graphic. Otherwise, reorder the sequence and repeat.

Notable Graphs

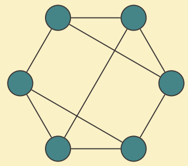
Complete Graph (K_n)

Undirected simple graph of n nodes where any pair of nodes are adjacent.



k-regular Graph

A graph where $\forall v \in V, d(v) = k$ or $\forall v \in V, d^{+}(v) = d^{-}(v) = k$.



Bipartite Graph

A graph $G = (V, E)$ is bipartite $\iff \exists V', V'' \subset V$ s.t. $\forall e \in E, e_1 \in V' \land e_2 \in V''$. This equates to the graph not having any odd-length cycles.