Chapter 3: Set Theory

Fundamentals

Definition 3.1.1 (Sets, "$\in$")

A set is any unordered collection of objects. If x is an object and A is a set, $x \in A \lor x \notin A$.

Axiom 3.1 Sets are objects

If A is a set, then A is also an object; A can belong to B, i.e. $\{3, \{3, 4\}, 4\}$

Axiom 3.2 (Equality of Sets)

$A = B \iff \forall x \in A x \in B \land \forall y \in B y \in A$

Axiom A.7 (Substitution)

If a and b are of the same type and $a = b$, then $f(a) = f(b)$ for all functions and operations f. The same applies to any property P.

It is important to note that the axiom of substitution applies to "$\in$". Therefore, it also applies to all operations we define exclusively in terms of it.

Axiom 3.3 (Empty Set)

There exists a set $\varnothing$ (or $\{\}$) $s.t. \forall x, x \notin \varnothing$.

Lemma 3.1.5 (Single Choice)

Let A be a non-empty set. $\Rightarrow \exists x \ s.t. \ x \in A$.

Proof: (By contradiction)
Suppose $\nexists x \in A$. $\rightarrow \forall x, x \notin A$. Also by axiom 3.3, $x \notin A$.

$$\text{"Thus, } x \in A \iff x \in \varnothing\text{."}$$

(Every element of A is in $\varnothing$ and vise versa; axiom 3.2.)
Thus, $A = \varnothing$, a contradiction. $\square$

Axiom 3.4 (Singleton and Pair Sets)

$\forall a \exists \{a\}; \forall y, y \in \{a\} \iff y = a$. We call $\{a\}$ the singleton set whose element is a.

$\forall a, b \exists \{a, b\}; \forall y, y \in \{a, b\} \iff (y = a \lor y = b)$

Because by axiom 3.2, $\{a, a\} = \{a\}$, the singleton set axiom is redundant.

Axiom 3.5 (Pairwise Union)

Let A and B be sets. $\exists A \cup B \ s.t. \ x \in A \cup B \iff (x \in A \lor x \in B)$.

Lemma 3.1.12

$\forall a, b \ \{a, b\} = \{a\} \cup \{b\}$.
$A \cup B = B \cup A$.
$(A \cup B) \cup C = A \cup (B \cup B)$.

Proof: I'm not going to write the proofs here, but the main idea seems to be to split the proof into two cases: for $A \cup B$, when $x \in A$ and when $x \in B$.

Definition 3.1.14 (Subsets)

Let A, B be sets.
$A \subseteq B \iff \forall x \in A, x \in B$. ($A$ is a subset of B)
$A \subsetneq B \iff (A \subseteq B \land A \neq B)$. ($A$ is a proper subset of B)

Because all of the above is defined in terms of $\in$, it follows the axiom of substitution.

Proposition 3.1.17 (Sets are partially ordered by set inclusion)

1. $(A \subseteq B \land B \subseteq C) \Rightarrow A \subseteq C$.
2. $(A \subseteq B \land B \subseteq A) \Rightarrow A = B$.
3. $(A \subsetneq B \land B \subsetneq C) \Rightarrow A \subsetneq C$.

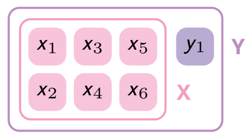
Proof (1): We **choose** an arbetrary element of A, x. Since $A \subseteq B$, $x \in B$, and since $B \subseteq C$, $x \in C$. Thus any element of A is an element of C. $\square$

Proof (3): Since $(X \subsetneq Y) \Rightarrow (X \subseteq Y), A \subseteq B \wedge B \subseteq C$. Therefore, by Tao's proof of **1**, $A \subseteq C$.

Thus, to show $A \subsetneq C$, we only need show $A \neq C$.

If $(X \subseteq Y \wedge X \neq Y) \Rightarrow$

$$\Big[\ \underbrace{(\forall x \in X, x \in Y)}_{\text{If this were false, } X \not\subseteq Y.} \quad \wedge \quad \sim \underbrace{(\forall y \in Y, y \in X)}_{\text{If this were true, } X = Y.}\ \Big]$$



$\therefore \exists$ some $y \in Y$ $s.t.$ $y \notin X$.

$\therefore (A \subsetneq B) \Rightarrow \exists b \in B \ s.t. \ b \notin A$
$\therefore (B \subsetneq C) \Rightarrow b \in C \wedge \exists c \in C \ s.t. \ (c \notin B \wedge c \notin A$ because $A \subseteq B)$
$\therefore b, c \notin A \wedge b, c \in C \therefore A \neq C$ $\square$

Axiom 3.6 (Specification)

Let A be a set, and $P(x)$ $s.t.$ $\forall x \in A$, $P(x)$ is true or false.
$\exists \{x \in A : P$ (is true)$\}$ s.t.

$$y \in \left\{x \in A : P(x)\right\} \iff (y \in A \wedge P(y))$$

Definition 3.1.22 (Intersections)

$$S_1 \cap S_2 := \{x \in S_1 : x \in S_2\};$$
$$x \in S_1 \cap S_2 \iff x \in S_1 \wedge x \in S_2.$$

Definition 3.1.26 (Difference sets)

$$A - B := \{x \in A : x \notin B\}$$

Proposition 3.1.27 (Sets for a bolean algebra)

a) **Minimal Element**
$A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$

b) **Maximal Element**
$A \cup X = X$ and $A \cap X = A$

c) **Identity**
$A \cap A = A$ and $A \cup A = A$

d) **Conmutativity**
$A \cup B = B \cup A$ and $A \cap B = B \cap A$

e) **Associativity**
$(A \cup B) \cup C = A \cup (B \cup B)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

f) **Distributivity**
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and
$A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$

g) **Partition**
$A \cup (X - A) = X$ and $A \cap (X - A) = \emptyset$

h) **De Morgan Laws**
$X - (A \cup B) = (X - A) \cap (X - B)$ and $X - (A \cap B) = (X - A) \cup (X - B)$

Axiom 3.7 (Replacement)

$$z \in \{y : P(x, y) \text{ is true for some } x \in A\} \iff P(x, z) \text{ is true for some } x \in A.$$

Example:
Let $P(x, y)$ be that $y = x{+}{+}$. Then, $\{y : P(x, y) \text{ for some } x \in \{1, 2, 3\}\} = \{2, 3, 4\}$.

The above could be re-written in the forms:
$f(x) = x++, \{y : y = f(x) \text{ for some } x \in \{1, 2, 3\}\}$
$\{f(x) : x \in \{1, 2, 3\}\}$

Axiom 3.8 (Infinity)

There exists a set $\mathbb{N}$, whose elements are called *natural numbers*, as well as an object 0 in $\mathbb{N}$, and an object $n++$ assigned to every natural number $n \in \mathbb{N}$, such that the *Peano axioms* hold.

Functions

Definition 3.3.1 (Functions)

Let X, Y be sets and $P(x, y)$ be a property of an objects $x \in X, y \in Y$ $s.t.$ $\forall x\ \exists! y$ $s.t.$ $P(x, y)$. We now define the function $f : X \to Y$, defined by P on the domain X and codomain Y to be the object which for any input $x \in X$, assigns an output $f(x) \in Y$.

In short:

$$y = f(x) \iff P(x, y)$$

Examples
One can simply derive a property from a function definition $f(x) = x++$ as $P(x, y) \iff y = x++$.

For $P(x, y) \iff y++ = x$, f must be defined $f : \mathbb{N} - \{0\} \to \mathbb{N}$, since $0 - 1 \notin \mathbb{N}$, leaving $f(0)$ undefined.

Functions can be defined explicitly by defining an expression for $f(x)$, or implicitly by only defining $P(x.y)$.

*For explicitness' sake, Tao's definition **does** imply that unequal inputs do not necessarily mean unequal outputs.*

Definition 3.3.8 (Equality of Functions)

Let $f : X \to Y$ and $g : X' \to Y'$.

$$f = g \iff (X = X' \wedge Y = Y') \wedge (\forall x \in X, f(x) = g(x))$$

Definition 3.3.13 (Composition)

Let $f : W \to X$ and $g : Y \to Z$.

$$(g \circ f)(x) := g(f(x)) \iff X = Y$$

Lemma 3.3.15 (Composition is Associative)

Let f, g, h be functions such that the following composition is possible.

$$\Rightarrow f \circ (g \circ h) = (f \circ g) \circ h$$

Definition 3.3.17 (One-to-one Functions)

If f is one-to-one (injective),

$$f(x) = f(x') \iff x = x'$$

Definition 3.3.21 (Onto Functions)

A function is onto (surjective) if

$$\forall y \in Y,\ \exists x \in X\ s.t.\ f(x) = y$$

Definition 3.3.21 (Bijective Functions)

A function is bijective (invertible) if it is both injective and surjective.

Definition 3.3.? (Inverse Functions)

If f if bijective, $\forall y \in Y,\ \exists! x \in X\ s.t.\ f(x) = y$. This x is given by the inverse function $f^{-1}(y)$, defined $: Y \to X$.