1 Field Theory

1.1 Definition

First,let us define scalar field φ as $\varphi = \varphi(\boldsymbol{r}, t) = \varphi(x, y, z, t)$ and vector field $\boldsymbol{a}$ as $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{r}, t) = \boldsymbol{a}(x, y, z, t)$.

Then grad φ of scalar φ would be

$$\operatorname{grad}\varphi = \frac{\partial\varphi}{\partial x}\boldsymbol{i} + \frac{\partial\varphi}{\partial y}\boldsymbol{j} + \frac{\partial\varphi}{\partial z}\boldsymbol{k} \tag{1.1.1}$$

Given a vector field $\boldsymbol{a}$,choose any point in the field and wrap it in volume V.We define divergence of the field as:

$$\operatorname{div}\boldsymbol{a} = \lim_{V\to 0}\frac{\oint_S \boldsymbol{a}\cdot \mathrm{d}\boldsymbol{S}}{V} \tag{1.1.2}$$

By Остроградский-Gauss formula,We can deduce:

$$\operatorname{div}\boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \tag{1.1.3}$$

We define projection along a direction $\boldsymbol{n}$ of curl of the vector field as:

$$\operatorname{rot}_n\boldsymbol{a} = \lim_{S\to 0}\frac{\oint_L \boldsymbol{a}\cdot \mathrm{d}\boldsymbol{r}}{S} \tag{1.1.4}$$

where we choose any point in the field and choose a infinitly small closed curve L on the curved surfuce S and direction of normal line of the curved surfuce is as same as $\boldsymbol{n}$.

By Stoke formula,we can de deduce:

$$\operatorname{rot}\boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \tag{1.1.5}$$

We define Hamiltonian as:

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{1.1.6}$$

and read it as nabla.

We define Laplace Operator as:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.1.7}$$

1.2 Basic formula

Let us deduce basic formula!

(1) Derivation formula

$$\operatorname{grad}(\varphi + \psi) = \operatorname{grad}\varphi + \operatorname{grad}\psi \tag{1.2.1}$$

Proof:It is as same as derivation.

$$\mathrm{grad}(\varphi\psi) = \psi\mathrm{grad}\varphi + \varphi\mathrm{grad}\psi \tag{1.2.2}$$

Proof:It is as same as derivation.

$$\mathrm{grad}F(\varphi) = F'(\varphi)\mathrm{grad}\varphi \tag{1.2.3}$$

Proof:It is as same as derivation.

$$\mathrm{div}(\boldsymbol{a} + \boldsymbol{b}) = \mathrm{div}\boldsymbol{a} + \mathrm{div}\boldsymbol{b} \tag{1.2.4}$$

Proof:It is as same as derivation.

$$\mathrm{div}(\varphi\boldsymbol{a}) = \varphi\mathrm{div}\boldsymbol{a} + \boldsymbol{a}\cdot\mathrm{grad}\varphi \tag{1.2.5}$$

Proof:

$$\mathrm{div}(\varphi\boldsymbol{a}) = \frac{\partial\varphi a_x}{\partial x} + \frac{\partial\varphi a_y}{\partial y} + \frac{\partial\varphi a_y}{\partial y} = \varphi\mathrm{div}\boldsymbol{a} + \boldsymbol{a}\cdot\mathrm{grad}\varphi$$

$$\mathrm{div}(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot\mathrm{rot}\boldsymbol{a} - \boldsymbol{a}\cdot\mathrm{rot}\boldsymbol{b} \tag{1.2.6}$$

Proof:

$$\begin{aligned}
\mathrm{div}(\boldsymbol{a}\times\boldsymbol{b}) &= \mathrm{div}\big((a_y b_z - b_y a_z)\boldsymbol{i} + (a_z b_x - a_x b_z)\boldsymbol{j} + \big(b_y a_x - a_y b_x\big)\boldsymbol{k}\big) \\
&= \left(a_y\frac{\partial b_z}{\partial x} + b_z\frac{\partial a_y}{\partial x} - b_y\frac{\partial a_z}{\partial x} - a_z\frac{\partial b_y}{\partial x}\right) \\
&+ \left(a_z\frac{\partial b_x}{\partial y} + b_x\frac{\partial a_z}{\partial y} - a_x\frac{\partial b_z}{\partial y} - b_z\frac{\partial a_x}{\partial y}\right) \\
&+ \left(b_y\frac{\partial a_x}{\partial z} + a_x\frac{\partial b_y}{\partial z} - a_y\frac{\partial b_x}{\partial z} - b_x\frac{\partial a_y}{\partial z}\right) \\
&= \boldsymbol{b}\cdot\mathrm{rot}\boldsymbol{a} - \boldsymbol{a}\cdot\mathrm{rot}\boldsymbol{b}
\end{aligned}$$

$$\mathrm{rot}(\boldsymbol{a} + \boldsymbol{b}) = \mathrm{rot}\boldsymbol{a} + \mathrm{rot}\boldsymbol{b} \tag{1.2.7}$$

Proof:It is eazy to prove.

$$\mathrm{rot}(\varphi\boldsymbol{a}) = \varphi\mathrm{rot}\boldsymbol{a} + \mathrm{grad}\varphi\times\mathrm{rot}\boldsymbol{a} \tag{1.2.8}$$

Proof:Expand it and prove it.

$$\mathrm{rot}(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + \boldsymbol{a}\mathrm{div}\boldsymbol{b} - \boldsymbol{b}\mathrm{div}\boldsymbol{a} \tag{1.2.9}$$

Proof:

$$\mathrm{rot}(\boldsymbol{a}\times\boldsymbol{b}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_y b_z - b_y a_z & a_z b_x - a_x b_z & b_y a_x - a_y b_x \end{vmatrix}$$

$$\mathrm{grad}(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + \boldsymbol{b}\times\mathrm{rot}\boldsymbol{a} + \boldsymbol{a}\times\mathrm{rot}\boldsymbol{b} \tag{1.2.10}$$

Proof:I have not prove yet.

$$\mathrm{grad}\left(\frac{a^2}{2}\right) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times\mathrm{rot}\boldsymbol{a} \tag{1.2.11}$$

Proof:I have not prove yet.

$$\mathrm{div}\ \mathrm{grad}\varphi = \Delta\varphi \tag{1.2.12}$$

Proof:I have not prove yet.

$$\mathrm{div}\ \mathrm{rot}\boldsymbol{a} = 0 \tag{1.2.13}$$

Proof:I have not prove yet.

$$\mathrm{rot}\ \mathrm{grad}\varphi = 0 \tag{1.2.14}$$

Proof:I have not prove yet.

$$\mathrm{rot}\ \mathrm{rot}\boldsymbol{a} = \mathrm{grad}\ \mathrm{div}\boldsymbol{a} - \Delta\boldsymbol{a} \tag{1.2.15}$$

Proof:I have not prove yet.

$$\mathrm{div}(\varphi\ \mathrm{grad}\psi) = \varphi\Delta\psi + \mathrm{grad}\varphi\cdot\mathrm{grad}\psi \tag{1.2.16}$$

Proof:I have not prove yet.

$$\Delta(\varphi\psi) = \psi\Delta\varphi + \varphi\Delta\psi + 2\mathrm{grad}\varphi\cdot\mathrm{grad}\psi \tag{1.2.17}$$

Proof:I have not prove yet.

(2) Integral formula

$$\int_V \mathrm{grad}\varphi\,\mathrm{d}V = \int_S \boldsymbol{n}\varphi\,\mathrm{d}S \tag{1.2.18}$$

Proof:I have not prove yet.

Остроградский-Gauss formula

$$\int_V \operatorname{div}\boldsymbol{a}\,\mathrm{d}V = \int_S \boldsymbol{n}\cdot\boldsymbol{a}\,\mathrm{d}S \tag{1.2.19}$$

Proof:I have not prove yet.

$$\int_V \operatorname{rot}\boldsymbol{a}\,\mathrm{d}V = \int_S \boldsymbol{n}\times\boldsymbol{a}\,\mathrm{d}S \tag{1.2.20}$$

Proof:I have not prove yet.

$$\int_V (\boldsymbol{v}\cdot\nabla)\boldsymbol{a}\,\mathrm{d}V = \int_S (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{a}\,\mathrm{d}S \tag{1.2.21}$$

$\boldsymbol{v}$ is a constant vector.

Proof:I have not prove yet.

$$\int_V \Delta\varphi\,\mathrm{d}V = \int_S \frac{\partial\varphi}{\partial n}\,\mathrm{d}S = \int_S \boldsymbol{n}\cdot\nabla\varphi\,\mathrm{d}S \tag{1.2.22}$$

Proof:I have not prove yet.

$$\int_V \Delta\boldsymbol{a}\,\mathrm{d}V = \int_S \frac{\partial\boldsymbol{a}}{\partial n}\,\mathrm{d}S = \int_S (\boldsymbol{n}\cdot\nabla)\boldsymbol{a}\,\mathrm{d}S \tag{1.2.23}$$

Proof:I have not prove yet.

First Green formula

$$\int_V (\varphi\Delta\psi + \operatorname{grad}\varphi\cdot\operatorname{grad}\psi)\,\mathrm{d}V = \int_S \varphi\frac{\partial\psi}{\partial n}\,\mathrm{d}S \tag{1.2.24}$$

and V is a simply connected domain.

Proof:I have not prove yet.

Second Green formula.

$$\int_V (\varphi\Delta\psi - \psi\Delta\varphi)\,\mathrm{d}V = \int_S \left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right)\mathrm{d}S \tag{1.2.25}$$

and V is a simply connected domain.

Proof:I have not prove yet.

$$\int_V (\mathrm{grad}\varphi)^2 \, \mathrm{d}V = \int_S \varphi \frac{\partial \varphi}{\partial n} \, \mathrm{d}S \tag{1.2.26}$$

and $\Delta\varphi = 0$,V is a simply connected domain.

Proof:I have not prove yet.