Note of Mathematical Physics Equations

Course Note

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Lawrence

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1 PDE

1.1 Concept of PDE

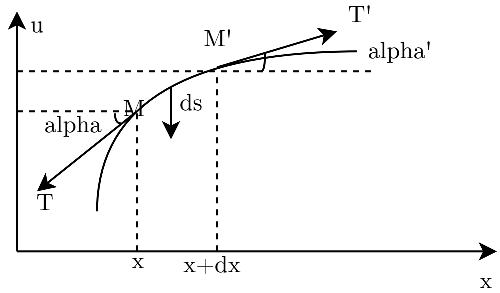
PDE includes relationship between variables more than 2.PDE can include integral.It is hard for to solve PDE. Example:

$$\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{1.1.1}$$

1.2 Conduct Basic equation

String vibration equation

Fig. 1 — String vibration



Let us do force analysis to this segment of the string.At x axis,we have:

$$-T\cos\alpha + T'\cos\alpha' = 0 \tag{1.2.1}$$

And the vibration of the string is very small,so we can assume that the angle is very small,so we have:

$$\begin{gathered} \cos\alpha \approx \cos\alpha' \approx 1 \\ T = T' \end{gathered} \tag{1.2.2}$$

At y axis,we have:

$$-T\sin\alpha + T'\sin\alpha' - mg = ma \tag{1.2.3}$$

And we have:

$$\sin\alpha \approx \tan\alpha = \frac{\partial u(x,t)}{\partial x} \tag{1.2.4}$$

For same reason,$\sin\alpha' \approx \tan\alpha' = \frac{\partial u(x+\mathrm{d}x)}{\partial x}$.And we have $m = \rho\,\mathrm{d}s = \rho\sqrt{1+\left[\frac{\partial u(x,t)}{\partial x}\right]^2}\,\mathrm{d}x \approx \rho\,\mathrm{d}x, a = \frac{\partial^2 u(x,t)}{\partial t^2}$.Thus we have the equation:

$$T\left[\frac{\partial u(x+\mathrm{d}x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] - \rho g\,\mathrm{d}x = \rho\frac{\partial^2 u(x,t)}{\partial t^2}\,\mathrm{d}x \tag{1.2.5}$$

By lagrange middle value theorem,we have:

$$\exists \xi \in (x, x + \mathrm{d}x), \frac{\partial u(x + \mathrm{d}x, t)}{\partial x} - \frac{\partial u(x, t)}{\partial x} = \frac{\partial^2 u(\xi, t)}{\partial x^2} \mathrm{d}x \tag{1.2.6}$$

And let $\mathrm{d}x \to 0, \xi \to x$,we can reform the equation as:

$$\frac{T}{\rho} \frac{\partial^2 u(x, t)}{\partial x^2} - g = \frac{\partial^2 u(x, t)}{\partial t^2} \tag{1.2.7}$$

Generally,when the tension of the string is big,the change of velocity of the string is much bigger than g.Thus we can ignore g,and we have:

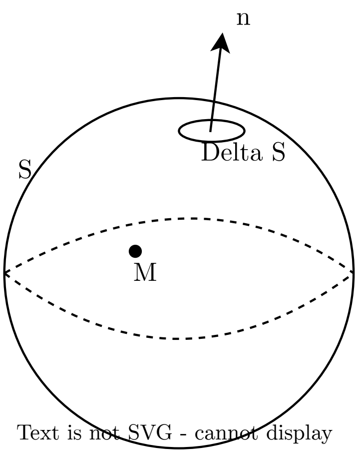
$$\frac{\partial^2 u(x, t)}{\partial t^2} = a^2 \frac{\partial^2 u(x, t)}{\partial x^2}, a^2 = \frac{T}{\rho} \tag{1.2.8}$$

It is easy to get the equation of the string vibration with external force:

$$\frac{\partial^2 u(x, t)}{\partial t^2} = a^2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t) \tag{1.2.9}$$

Heat Conduction

Fig. 2 — Heat Conduction



At moment t,the temperature of the point M is $u(x, y, z, t)$.$\boldsymbol{n}$ is the normal vector of the surface.By fourier's law,we have:

$$\mathrm{d}Q = -k \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}S \, \mathrm{d}t \tag{1.2.10}$$

Thus the total heat conduction that goes through the surface is:

$$Q = \int_{t_1}^{t_2} \left(\iint_S -k \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}S \right) \mathrm{d}t = \iiint_V c\rho[u(x, y, z, t_2) - u(x, y, z, t_1)] \, \mathrm{d}V \tag{1.2.11}$$

And LHS can be written as:

$$\int_{t_1}^{t_2}\left(\iint_S -k\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}S\right)\mathrm{d}t=\int_{t_1}^{t_2}\left(\iiint_V k\Delta u\,\mathrm{d}V\right) \tag{1.2.12}$$

And RHS can be written as:

$$\iiint_V c\rho[u(x,y,z,t_2)-u(x,y,z,t_1)]\,\mathrm{d}V=\int_{t_1}^{t_2}\left(\iiint_V c\rho\frac{\partial u}{\partial t}\,\mathrm{d}V\right)\mathrm{d}t \tag{1.2.13}$$

Because the time and space are arbitrary,we have:

$$\frac{\partial u}{\partial t}=a^2\Delta u=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), a=\frac{k}{c\rho} \tag{1.2.14}$$

If the field of the temperature is stable in other words $\frac{\partial u}{\partial t}=0$,the Équation (1.2.14) is called Laplace equation

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \tag{1.2.15}$$

If temperature is independent of time,we have Poisson equation:

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f(x,y,z) \tag{1.2.16}$$

1.3 Definite Condition

Initial condition For the string vibration,initial condition of string vibration is the initial position and velocity of the string.Let us note $\varphi(x)$ as initial position and $\psi(x)$ as initial velocity,we have:

$$\begin{cases} u|_{t=0}=\varphi(x) \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x) \end{cases} \tag{1.3.1}$$

For the heat conduction,initial condition is the initial temperature of any point M,we have:

$$u(x,y,z,t)|_{t=0}=\varphi(x,y,z) \tag{1.3.2}$$

Boundary condition There are three types of boundary condition.

1. Dirichlet boundary condition.If given the value of the function u on the boundary S,then the boundary condition is called Dirichlet boundary condition.

$$u|_S=f \tag{1.3.3}$$

2. Neumann boundary condition.If given the value of the normal derivative of the function u on the boundary S,then the boundary condition is called Neumann boundary condition.

$$\left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_S = f \tag{1.3.4}$$

3. Robin boundary condition.If given the value of the linear combination of the function u and the normal derivative of the function u on the boundary S,then the boundary condition is called Robin boundary condition.

$$\left.\left(u + \sigma\frac{\partial u}{\partial \boldsymbol{n}}\right)\right|_S = f \tag{1.3.5}$$

1.4 Definite Problem

A definite problem without initial condition is called boundary problem.A definite problem without boundary condition is called initial problem or Cauchy problem.A differential equation with initial condition and boundary condition is called mixed problem.

We judge whether a definite problem conforms to the actual situation by three dimension:

1. Existence of the solution.
2. Uniqueness of the solution.
3. Stability of the solution.

Formation of a general two order liner PDE with n variables should be:

$$Lu \equiv \sum_{i,k=1}^{n} A_{i,k}\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu = f \tag{1.4.1}$$

If we have 2 variables,we have:

$$a_{11}(x,y)\frac{\partial^2 u}{\partial x^2} + 2a_{12}(x,y)\frac{\partial^2 u}{\partial x \partial y} + a_{22}(x,y)\frac{\partial^2 u}{\partial y^2} + b_1(x,y)\frac{\partial u}{\partial x} + b_2(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y) \tag{1.4.2}$$

Linear PDE owns a important property that the sum of two solutions is still a solution which is called superposition principle.

1.5 Classification of two order liner PDE

Given a two order liner PDE:

$$a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + cu + f = 0 \tag{1.5.1}$$

We apply the following transformation on variables:

$$\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases} \text{also as} \begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases}, \text{where jacobian matrix } J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0 \tag{1.5.2}$$

We have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{1.5.3}$$

And second order partial derivative can be written as:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \\ &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}\end{aligned} \tag{1.5.4}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \\ &= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y \partial x} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y \partial x}\end{aligned} \tag{1.5.5}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right) \\ &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}\end{aligned} \tag{1.5.6}$$

Substitute the above equations into Équation (1.5.1) we have:

$$A_{11}\frac{\partial^2 u}{\partial \xi^2} + 2A_{12}\frac{\partial^2 u}{\partial \xi \partial \eta} + A_{22}\frac{\partial^2 u}{\partial \eta^2} + B_1\frac{\partial u}{\partial \xi} + B_2\frac{\partial u}{\partial \eta} + Cu + F = 0 \tag{1.5.7}$$

where:

$$\begin{cases} A_{11} = a_{11}\left(\dfrac{\partial \xi}{\partial x}\right)^2 + 2a_{12}\dfrac{\partial \xi}{\partial x}\dfrac{\partial \xi}{\partial y} + a_{22}\left(\dfrac{\partial \xi}{\partial x}\right)^2 \\ A_{12} = a_{11}\dfrac{\partial \xi}{\partial x}\dfrac{\partial \eta}{\partial x} + a_{12}\left(\dfrac{\partial \xi}{\partial x}\dfrac{\partial \eta}{\partial y} + \dfrac{\partial \xi}{\partial y}\dfrac{\partial \eta}{\partial x}\right) + a_{22}\dfrac{\partial \xi}{\partial y}\dfrac{\partial \eta}{\partial y} \\ A_{22} = a_{11}\left(\dfrac{\partial \eta}{\partial x}\right)^2 + 2a_{12}\dfrac{\partial \eta}{\partial x}\dfrac{\partial \eta}{\partial y} + a_{22}\left(\dfrac{\partial \eta}{\partial y}\right)^2 \\ B_1 = a_{11}\dfrac{\partial^2 \xi}{\partial x^2} + 2a_{12}\dfrac{\partial^2 \xi}{\partial x \partial y} + a_{22}\dfrac{\partial^2 \xi}{\partial y^2} + b_1\dfrac{\partial \xi}{\partial x} + b_2\dfrac{\partial \xi}{\partial y} \\ B_2 = a_{11}\dfrac{\partial^2 \eta}{\partial x^2} + 2a_{12}\dfrac{\partial^2 \eta}{\partial x \partial y} + a_{22}\dfrac{\partial^2 \eta}{\partial y^2} + b_1\dfrac{\partial \eta}{\partial x} + b_2\dfrac{\partial \eta}{\partial y} \\ C = c \\ F = f \end{cases} \tag{1.5.8}$$

If we let $A_{11} = 0$,we have:

$$a_{11}\left(\frac{\partial z}{\partial x}\right)^2 + 2a_{12}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + a_{22}\left(\frac{\partial y}{\partial x}\right)^2 = 0 \tag{1.5.9}$$

If the above equation have a solution $z = \varphi(x, y)$,then let $\xi = \varphi(x, y)$ we have $A_{11} = 0$.If the above equation have another linearly independent solution $z = \psi(x, y)$,then let $\xi = \psi(x, y)$ we have $A_{22} = 0$.

To solve Équation (1.5.9) we can use the following method.Transform the equation into:

$$a_{11}\left(-\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}\right) - 2a_{12}\left(-\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}\right) + a_{22} = 0 \tag{1.5.10}$$

By implicit function theorem,implicit function $z(x, y(x)) = C$ satisfies:

$$a_{11}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2a_{12}\frac{\mathrm{d}y}{\mathrm{d}x} + a_{22} = 0 \tag{1.5.11}$$

The above equation is called characteristic equation of Équation (1.5.1). Its solution is called characteristic curve of Équation (1.5.1).

$\Delta(x, y) = a_{12}^2(x, y) - a_{11}(x, y)a_{22}(x, y)$.We classify PDE by the value of $\Delta(x, y)$.If $\Delta(x, y) > 0$,we have hyperbolic PDE.If $\Delta(x, y) = 0$,we have parabolic PDE.If $\Delta(x, y) < 0$,we have elliptic PDE.

1.6 Simplify PDE

We can simplify PDE by characteristic equation.

Hyperbolic PDE

If $\Delta(x,y) > 0$,the PDE a hyperbolic PDE.It has two linearly independent characteristic curves $z = \varphi(x,y), z = \psi(x,y)$.We can transform the PDE into:

$$\begin{cases} \xi = \varphi(x,y) \\ \eta = \psi(x,y) \end{cases} \tag{1.6.1}$$

And we have:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = -\frac{1}{2A_{12}} \left(B_1 \frac{\partial u}{\partial \xi} + B_2 \frac{\partial u}{\partial \eta} + Cu + F \right) \tag{1.6.2}$$

We can do such transformation on above equation to eliminate the cross term $\frac{\partial^2 u}{\partial \xi \partial \eta}$.

$$\begin{cases} \xi = \alpha + \beta \\ \eta = \alpha - \beta \end{cases} \text{also as} \begin{cases} \alpha = \dfrac{\xi + \eta}{2} \\ \beta = \dfrac{\xi - \eta}{2} \end{cases} \tag{1.6.3}$$

And we have:

$$\frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial \beta^2} = -\frac{1}{2A_{12}} \left[(B_1 + B_2) \frac{\partial u}{\partial \alpha} + (B_1 - B_2) \frac{\partial u}{\partial \beta} + 2Cu + 2F \right] \tag{1.6.4}$$

Both Équation (1.6.2) and Équation (1.6.4) are canonical form of hyperbolic PDE.

Parabolic PDE

If $\Delta(x,y) = 0$,the PDE a parabolic PDE.It has one linearly independent characteristic curve $z = \varphi(x,y)$.We can transform the PDE into:

$$\begin{cases} \xi = \varphi(x,y) \\ \eta = \eta(x,y) \end{cases} \tag{1.6.5}$$

where $\eta(x,y)$ is any function satisfies $J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$

And we have:

$$\frac{\partial^2 u}{\partial \eta^2} = -\frac{1}{A_{22}} \left(B_1 \frac{\partial u}{\partial \xi} + B_2 \frac{\partial u}{\partial \eta} + Cu + F \right) \tag{1.6.6}$$

Elliptic PDE

If $\Delta(x,y) < 0$,the PDE a elliptic PDE.It has two conjugate characteristic curve.

$$\varphi(x,y) = c, \overline{\varphi}(x,y) = c \tag{1.6.7}$$

We can transform the PDE into:

$$\begin{cases} \xi = \varphi(x, y) \\ \eta = \overline{\varphi}(x, y) \end{cases} \tag{1.6.8}$$

And we have:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = -\frac{1}{2A_{12}}\left(B_1\frac{\partial u}{\partial \xi} + B_2\frac{\partial u}{\partial \eta} + Cu + F\right) \tag{1.6.9}$$

where ξ, η are complex function.Furthermore,we do such transformation for convenience:

$$\begin{cases} \xi = \mathrm{Re}(\varphi(x, y)) \\ \eta = \mathrm{Im}(\varphi(x, y)) \end{cases} \tag{1.6.10}$$

And we have:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = -\frac{1}{A_{12}}\left[(B_1 + B_2)\frac{\partial u}{\partial \xi} + i(B_1 - B_2)\frac{\partial u}{\partial \eta} + 2Cu + 2F\right] \tag{1.6.11}$$

2 Method of Separation of Variables

2.1 String Vibration Equation

Given a string vibration equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u|_{x=0} = 0, u|_{x=l} = 0 \\ u|_{t=0} = \varphi(x), \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \psi(x), 0 \le x \le l \end{cases} \tag{2.1.1}$$

We assume the solution to be of the form:

$$u_n(x,t) = X_n(x)T_n(t) \tag{2.1.2}$$

And we superimpose the solutions to get the general solution:

$$u(x,t) = \sum_{n=1}^{\infty} C_n X_n(x) T_n(t) \tag{2.1.3}$$

We substitute $u_n(x,t) = X_n(x)T_n(t)$ into the string vibration equation to get:

$$\frac{X''(x)}{X(x)} = \left(\frac{1}{a^2}\right)\frac{T''(t)}{T(t)} = -\lambda \tag{2.1.4}$$

Where λ is a constant.(LHS is independent of t and RHS is independent of x thus both should be equal to a constant otherwise the equation will not hold true for all x and t).We divide a^2 to the RHS to make $X(x)$ part does not contain a so we can solve it with the boundary conditions in next step.

We have two ODEs:

$$T''(t) + \lambda a^2 T(t) = 0 \tag{2.1.5}$$

$$X''(x) + \lambda X(x) = 0 \tag{2.1.6}$$

We solve the $X(x)$ ODE with the boundary conditions:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, X(l) = 0 \end{cases} \tag{2.1.7}$$

Then we discuss the different cases of λ.

(1) $\lambda < 0$

The general solution is:

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} \tag{2.1.8}$$

And the boundary conditions give:

$$\begin{cases} A + B = 0 \\ Ae^{\sqrt{-\lambda}l} + Be^{-\sqrt{-\lambda}l} = 0 \end{cases} \tag{2.1.9}$$

Because the determinant of the coefficient matrix is not zero, the only solution is $A = B = 0$ which is $X(x) \equiv 0$.This is not a solution to the problem.

(2) $\lambda = 0$

The general solution is:

$$X(x) = Ax + B \tag{2.1.10}$$

To satisfy the boundary conditions we have:

$$A = B = 0 \tag{2.1.11}$$

Which is not a solution to the problem.

(3) $\lambda > 0$

Let $\lambda = \beta^2 (\beta > 0)$.The general solution is:

$$X(x) = A\cos(\beta x) + B\sin(\beta x) \tag{2.1.12}$$

And the boundary conditions give:

$$\begin{cases} X(0) = A = 0 \\ X(l) = A\cos(\beta x) + B\sin(\beta l) = 0 \end{cases} \tag{2.1.13}$$

We have $A = 0, \beta = \frac{n\pi}{l}, n = 1, 2, 3....$(We don't consider the situation where $n = -1, -2...$ because the coefficient B can be negative)Thus we have:

$$\lambda_n = \frac{n^2\pi^2}{l^2} \tag{2.1.14}$$

$$X_{n(x)} = B_n \sin\left(\frac{n\pi}{l}x\right) \tag{2.1.15}$$

We substitute $\lambda = \frac{n^2\pi^2}{l^2}$ into the $T(t)$ ODE to get:

$$T_n''(t) + \frac{n^2\pi^2a^2}{l^2}T_n(t) = 0 \tag{2.1.16}$$

We have:

$$T_n(t) = C_n' \cos\left(\frac{n\pi a}{l}t\right) + D_n' \sin\left(\frac{n\pi a}{l}t\right) \tag{2.1.17}$$

Thus we can get the general solution:

$$u_n(x,t) = \left(C_n \cos\left(\frac{n\pi a}{l}t\right) + D_n \sin\left(\frac{n\pi a}{l}t\right)\right) \sin\left(\frac{n\pi}{l}x\right) \tag{2.1.18}$$

where $C_n = B_n C'_n, D_n = B_n D'_n$ And we superimpose the solutions to get solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{n\pi a}{l} t\right) + D_n \sin\left(\frac{n\pi a}{l} t\right) \right) \sin\left(\frac{n\pi}{l} x\right) \tag{2.1.19}$$

We substitute the initial conditions to get the final solution(if RHS is convergent and can be differentiated term by term)

$$u(x,t)|_{t=0} = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{l} x\right) = \varphi(x) \tag{2.1.20}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \left(D_n \frac{n\pi a}{l} \right) \sin\left(\frac{n\pi}{l} x\right) = \psi(x) \tag{2.1.21}$$

Just in case you've forgotten, fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$
$$a_n = \left(\frac{2}{T}\right) \int_{t_0}^{t_0+T} f(t) \cos(n\omega t) \,\mathrm{d}t$$
$$b_n = \left(\frac{2}{T}\right) \int_{t_0}^{t_0+T} f(t) \sin(n\omega t) \,\mathrm{d}t$$

Thus we have;

$$\begin{cases} C_n = \left(\frac{2}{l}\right) \int_0^l \varphi(x) \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x \\ D_n = \left(\frac{2}{n\pi a}\right) \int_0^l \psi(x) \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x \end{cases} \tag{2.1.22}$$

It works same for the second boundary condition.

2.2 Heat Conduct Equation

Given a heat conduct equation:

$$\begin{cases} \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}, 0 < x < l, t > 0 \\ \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0, t > 0 \\ u|_{t=0} = \varphi(x), 0 \le x \le l \end{cases} \tag{2.2.1}$$

We assume the solution to be of the form:

$$u(x,t) = X(x)T(t) \tag{2.2.2}$$

Substitute $u(x,t) = X(x)T(t)$ into the heat conduct equation to get:

$$\frac{T'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{2.2.3}$$

We use boundary conditions to solve the $X(x)$ ODE:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = 0, X'(l) = 0 \end{cases} \tag{2.2.4}$$

Then we discuss the different cases of λ.

(1) $\lambda < 0$ The general solution is:

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} \tag{2.2.5}$$

And we have:

$$\begin{cases} \sqrt{-\lambda}A - \sqrt{-\lambda}B = 0 \\ \sqrt{-\lambda}Ae^{\sqrt{-\lambda}l} - \sqrt{-\lambda}Be^{-\sqrt{-\lambda}l} = 0 \end{cases} \tag{2.2.6}$$

The determinant of the coefficient matrix is:

$$\begin{vmatrix} \sqrt{-\lambda} & -\sqrt{-\lambda} \\ \sqrt{-\lambda}e^{\sqrt{-\lambda}l} & -\sqrt{-\lambda}e^{-\sqrt{-\lambda}l} \end{vmatrix} \neq 0 \tag{2.2.7}$$

Thus the only solution is $A = B = 0$ which is $X(x) \equiv 0$.This is not a solution to the problem.

(2) $\lambda = 0$

The general solution is:

$$X(x) = Ax + B \tag{2.2.8}$$

And we have:

$$A = 0 \tag{2.2.9}$$

Thus we have:$X(x) = B, B \neq 0$

(3) $\lambda > 0$

Let $\lambda = \beta^2(\beta > 0)$.The general solution is:

$$X(x) = A\cos(\beta x) + B\sin(\beta x) \tag{2.2.10}$$

And we have:

$$\begin{cases} B\beta = 0 \\ -A\beta\sin(\beta l) + B\beta\cos(\beta l) = 0 \end{cases} \tag{2.2.11}$$

Thus we have:

$$\beta = \frac{n\pi}{l}, n = 1, 2, 3...$$
$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \tag{2.2.12}$$

$$X_n(x) = A_n \cos\left(\frac{n\pi}{l}x\right) \tag{2.2.13}$$

Form discussion above we know that $\lambda = 0$ can be also considered as a special case of $\lambda > 0$.Thus we have the general solution:

$$X_n(x) = A_n \cos\left(\frac{n\pi}{l}x\right), n = 0, 1, 2, 3... \tag{2.2.14}$$

We substitute $\lambda_n = \left(\frac{n\pi}{l}\right)^2, n = 0, 1, 2, 3...$ into the $T(t)$ ODE:

$$T'(t) + \frac{n^2\pi^2a^2}{l^2}T(t) = 0 \tag{2.2.15}$$

The general solution is:

$$T_n(t) = C_n e^{-\frac{n^2\pi^2a^2}{l^2}t} \tag{2.2.16}$$

So we have gotten the general solution:

$$u_n(x,t) = a_n \cos\left(\frac{n\pi}{l}x\right)e^{-\frac{n^2\pi^2a^2}{l^2}t}, a_n = A_nC_n \tag{2.2.17}$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)e^{-\frac{n^2\pi^2a^2}{l^2}t} \tag{2.2.18}$$

We have initial condition to get the coefficient a_n:

$$u(t,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) = \varphi(x) \tag{2.2.19}$$

By fourier transformation we have:

$$a_n = \left(\frac{2}{l}\right)\int_0^l \varphi(x)\cos\left(\frac{n\pi}{l}x\right)\mathrm{d}x \tag{2.2.20}$$

2.3 Laplace Equation in a Rectangular Region

Given a Laplace equation in a rectangular region:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u(0,y) = f_1(y), u(a,y) = f_2(y) \\ u(x,0) = g_1(x), u(x,b) = g_2(x) \end{cases} \tag{2.3.1}$$

We notice that in this equation the boundary conditions are not homogeneous, so we can't use the method of separation of variables directly. We can use the method of superposition to solve this problem.We assume the solution to be of the form:$u = u_1 + u_2$ where u_1, u_2 are solutions to the following equations:

u_1 is a solution to the following problem:

$$\begin{cases} \dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} = 0 \\ u_1(0,y) = 0, u_1(a,y) = 0 \\ u_1(x,0) = g_1(x), u_1(x,b) = g_2(x) \end{cases} \tag{2.3.2}$$

And u_2 is a solution to the following problem:

$$\begin{cases} \dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2} = 0 \\ u_2(0,y) = f_1(y), u_2(a,y) = f_2(y) \\ u_2(x,0) = 0, u_2(x,b) = 0 \end{cases} \tag{2.3.3}$$

In this way we have transformed a non-homogeneous boundary condition problem into two homogeneous boundary condition problems. We can use the method of separation of variables to solve these two problems respectively.We only solve the u_1 problem here, the u_2 problem is similar.

$$\begin{cases} \dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2} = 0 \\ u_1(0,y) = 0, u_1(a,y) = 0 \\ u_1(x,0) = g_1(x), u_1(x,b) = g_2(x) \end{cases} \tag{2.3.4}$$

We assume the solution to be of the form:

$$u(x,y) = X(x)Y(y) \tag{2.3.5}$$

We substitute $u(x,y) = X(x)Y(y)$ into the equation to get:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda \tag{2.3.6}$$

Thus we have two ODEs:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ Y''(y) - \lambda Y(y) = 0 \end{cases} \tag{2.3.7}$$

We solve the $X(x)$ ODE with the boundary conditions:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, X(a) = 0 \end{cases} \tag{2.3.8}$$

It is as same as Équation (2.1.7) thus we have:

$$\lambda_n = \frac{n^2\pi^2}{a^2}, X_n(x) = C_n \sin\left(\frac{n\pi}{a}x\right), n = 1, 2, 3... \tag{2.3.9}$$

We substitute $\lambda_n = \frac{n^2\pi^2}{a^2}$ into the $Y(y)$ ODE to get:

$$Y''(y) - \frac{n^2\pi^2}{a^2}Y(y) = 0 \tag{2.3.10}$$

The general solution is:

$$Y_n(y) = D_n e^{\frac{n\pi}{a}y} + E_n e^{-\frac{n\pi}{a}y}, n = 1, 2, 3... \tag{2.3.11}$$

Thus we have the general solution:

$$u_n(x,y) = \left(A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}\right) \sin\left(\frac{n\pi}{a}x\right), A_n = C_n D_n, B_n = C_n E_n \tag{2.3.12}$$

$$u(x,y) = \sum_{n=1}^{\infty} \left(A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}\right) \sin\left(\frac{n\pi}{a}x\right) \tag{2.3.13}$$

And we have:

$$\begin{cases} u(x,0) = (A_n + B_n) \sin\left(\frac{n\pi}{a}x\right) = g_1(x) \\ u(x,b) = \left(A_n e^{\frac{n\pi}{a}b} + B_n e^{-\frac{n\pi}{a}b}\right) \sin\left(\frac{n\pi}{a}x\right) = g_2(x) \end{cases} \tag{2.3.14}$$

We can get the coefficients A_n, B_n by fourier transformation of $g_1(x), g_2(x)$.

$$\begin{cases} A_n + B_n = \left(\frac{2}{a}\right) \int_0^a g_1(x) \sin\left(\frac{n\pi}{a}x\right) \mathrm{d}x \\ A_n e^{\frac{n\pi}{a}b} + B_n e^{-\frac{n\pi}{a}b} = \left(\frac{2}{a}\right) \int_0^a g_2(x) \sin\left(\frac{n\pi}{a}x\right) \mathrm{d}x \end{cases} \tag{2.3.15}$$

2.4 Laplace Equation in a Circular Region

Given a Laplace equation in a circular region:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u(x,y)|_{x^2+y^2=r_0^2} = f(x,y) \end{cases} \tag{2.4.1}$$

Because the boundary condition is a circle,we consider this problem in polar coordinates:

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{r}\right)\frac{\partial u}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2 u}{\partial \theta^2} = 0 \\ u(r,\theta)|_{r=r_0} = f(\theta) \\ |u(0,\theta)| < \infty \\ u(r,\theta) = u(r,\theta+2\pi) \end{cases} \tag{2.4.2}$$

The third condition is natural condition.Why we add this condition is that we need more condition to determine a coefficient in later discussion.The fourth condition is periodic condition.

Now we solve this equation with the method of separation of variables.We assume the solution to be of the form:

$$u(r,\theta) = R(r)\Theta(\theta) \tag{2.4.3}$$

We substitute $u(r,\theta) = R(r)\Theta(\theta)$ into the equation to get:

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda \tag{2.4.4}$$

Thus we have two ODEs:

$$\begin{cases} r^2R''(r) + rR'(r) - \lambda R(r) = 0 \\ |R(0)| < \infty \end{cases} \tag{2.4.5}$$

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ \Theta(\theta) = \Theta(\theta + 2\pi) \end{cases} \tag{2.4.6}$$

We solve the $\Theta(\theta)$ ODE first because it is easier to solve.We discuss the different cases of λ.

(1) $\lambda < 0$ The general solution is:

$$\Theta(\theta) = Ae^{\sqrt{-\lambda}\theta} + Be^{-\sqrt{-\lambda}\theta} \tag{2.4.7}$$

It don't satisfy the periodic condition,thus $\lambda < 0$ is not a solution to the problem.

(2) $\lambda = 0$

The general solution is:

$$\Theta(\theta) = A\theta + B \tag{2.4.8}$$

It satisfy the periodic condition when $\Theta(\theta) = B, B \neq 0$.

(3) $\lambda > 0$

Let $\lambda = \beta^2$.The general solution is:

$$\Theta(\theta) = A\cos(\beta\theta) + B\sin(\beta\theta) \tag{2.4.9}$$

To satisfy the periodic condition β is a integer $n, n = 1, 2, 3$....Thus we have:

$$\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \lambda = n^2, n = 1, 2, 3... \tag{2.4.10}$$

Let us then solve the $R(r)$ ODE.

And we substitute $\lambda = n^2$ into the $R(r)$ ODE to get:

$$\begin{cases} r^2R''(r)+rR'(r)-n^2R(r)=0 \\ |R(0)|<\infty \end{cases} \tag{2.4.11}$$

This is a Euler equation.Because of $r>0$,we assume $r=e^t$.We have:

$$\frac{\mathrm{d}R}{\mathrm{d}r}=\frac{\mathrm{d}R}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}t}$$
$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}=\frac{\mathrm{d}\frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}t}}{\mathrm{d}r}=-\frac{1}{r^2}\frac{\mathrm{d}R}{\mathrm{d}t}+\frac{1}{r^2}\frac{\mathrm{d}^2R}{\mathrm{d}t^2}$$

We substitute them into Équation (2.4.11):

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2}-n^2\frac{\mathrm{d}R}{\mathrm{d}t}=0 \tag{2.4.12}$$

We get the solution:

$$R_n(t)=C_ne^{nt}+D_ne^{-nt} \tag{2.4.13}$$

Thus

$$R_n(r)=\begin{cases} C_0+D_0\ln r, n=0 \\ C_nr^n+D_nr^{-n}, n=1,2,3... \end{cases} \tag{2.4.14}$$

And we have natural condition:

$$|R(0)|<\infty \tag{2.4.15}$$

Thus $D_n=0$

So we get the general solution:

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n(a_n\cos(n\theta)+b_n\sin(n\theta)) \tag{2.4.16}$$

And we substitute it into boundary condition:

$$u(r_0,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r_0^n(a_n\cos(n\theta)+b_n\sin(n\theta))=f(\theta) \tag{2.4.17}$$

By comparing fourier coefficient we have:

$$\begin{cases} a_n=\dfrac{1}{r_0^n\pi}\displaystyle\int_0^{2\pi}f(\theta)\cos(n\theta)\,\mathrm{d}\theta \\ b_n=\dfrac{1}{r_0^n\pi}\displaystyle\int_0^{2\pi}f(\theta)\sin(n\theta)\,\mathrm{d}\theta \end{cases} \tag{2.4.18}$$

We substitute these coefficients into the general solution:

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(t)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n \cos(n(\theta - t))\right]\mathrm{d}t \tag{2.4.19}$$

3 Method of Traveling Waves

This chapter will discuss the method of solving a three dimension wave equation with initial condition.To achieve this we will follow the following steps:

1. Solve the one dimensional wave equation.
2. Solve the special case of the three dimensional wave equation.(Radially symmetric case)
3. Solve the general case of the three dimensional wave equation.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} \right) \\ u|_{t=0} = \varphi(x, y, z) \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \psi(x, y, z) \end{cases} \tag{3.0.1}$$

3.1 Initial Problem of One Dimensional Wave Equation

Given a one dimensional wave equation with initial conditions:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u|_{t=0} = \varphi(x) \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \end{cases} \tag{3.1.1}$$

First we try to simplify it.This equation is a hyperbolic partial differential equation.Its characteristic equation is:

$$(\mathrm{d}x)^2 - a^2(\mathrm{d}t)^2 = 0 \tag{3.1.2}$$

We have two characteristic lines:

$$x + at = C, x - at = C \tag{3.1.3}$$

We do such transformation:

$$\begin{cases} \xi = x + at \\ \eta = x - at \end{cases} \tag{3.1.4}$$

By differentiating we get:

$$\begin{gathered} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right) \end{gathered} \tag{3.1.5}$$

Substitute these into the wave equation we get canonical form

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0 \tag{3.1.6}$$

We integrate Équation (3.1.6) with respect to η:

$$\frac{\partial u}{\partial\xi} = f(\xi) \tag{3.1.7}$$

Then we integrate it with respect to ξ:

$$u(x,t) = \int f(\xi)\,\mathrm{d}\xi + f_2(\eta) = f_1(x+at) + f_2(x-at) \tag{3.1.8}$$

Let us consider the initial conditions:

$$\begin{cases} u|_{t=0} = \varphi(x) \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \end{cases} \tag{3.1.9}$$

We substitute Équation (3.1.8) into Équation (3.1.9):

$$\begin{cases} f_1(x) + f_2(x) = \varphi(x) \\ af_1'(x) - af_2'(x) = \psi(x) \end{cases} \tag{3.1.10}$$

We integrate the second equation with respect to x(**Here we do definite integral instead of indefinite integral because we do this way to get rid of the constant of integration**):

$$f_1(x) - f_2(x) = \frac{1}{a}\int_0^x \psi(\xi)\,\mathrm{d}\xi + C \tag{3.1.11}$$

Thus we can solve these two equations:

$$\begin{cases} f_1(x) = \dfrac{1}{2}\varphi(x) + \dfrac{1}{2a}\displaystyle\int_0^x \psi(\xi)\,\mathrm{d}\xi + \dfrac{C}{2} \\ f_2(x) = \dfrac{1}{2}\varphi(x) - \dfrac{1}{2a}\displaystyle\int_0^x \psi(\xi)\,\mathrm{d}\xi - \dfrac{C}{2} \end{cases} \tag{3.1.12}$$

Then we get the final solution also called D'Alembert's formula:

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi \tag{3.1.13}$$

3.2 Special Case of Three Dimensional Wave Equation

If the three dimensional wave equation is radially symmetric.Then we can use the spherical coordinate system:

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 u}{\partial \theta^2}\right) \quad (3.2.1)$$

Because the equation is radially symmetric,u is independent of θ and φ in other words $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \varphi} = 0$.Thus the equation becomes:

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)\right) \quad (3.2.2)$$

Équation (3.2.2) is called the radical symmetric wave equation.We can expand it:

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) \quad (3.2.3)$$

We multiply r to both sides of Équation (3.2.2):

$$\frac{\partial^2 (ru)}{\partial t^2} = r\frac{\partial^2 u}{\partial t^2} = a^2\left(r\frac{\partial^2 u}{\partial r^2} + 2\frac{\partial u}{\partial r}\right) = a^2\left(\frac{\partial^2 (ru)}{\partial r^2}\right) \quad (3.2.4)$$

This is a one dimensional wave equation with initial conditions with respect to ru.General Solution of it is:

$$ru = f_1(r+at) + f_2(r-at) \quad (3.2.5)$$

Also as:

$$u(r,t) = \frac{f_1(r+at) + f_2(r-at)}{r} \quad (3.2.6)$$

By substituting the initial conditions we get the final solution.

3.3 General Case of Three Dimensional Wave Equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) \\ u\big|_{t=0} = \varphi(x,y,z) \\ \dfrac{\partial u}{\partial t}\bigg|_{t=0} = \psi(x,y,z) \end{cases} \quad (3.3.1)$$

Let us consider the general case of the three dimensional wave equation.Because u is not radically symmetric,u is a function of x, y, z and t.But if a given point $M(x,y,z)$,we consider the average of the values of u in the sphere with radius r and center M.This value is independent of x, y, z since the point M is given in other words this value is a function of r and t.We denote this value as $\overline{u}(r,t)$.Thus we have:

$$\overline{u}(r,t) = \frac{1}{4\pi r^2}\iint_{S_r^M} u(\xi,\eta,\zeta,t)\,\mathrm{d}S \quad (3.3.2)$$

where $\xi = x + r\sin(\theta)\cos(\varphi), \eta = y + r\sin(\theta)\sin(\varphi), \zeta = z + r\cos(\theta)$.

It is easy to see that

$$\lim_{r\to 0} \overline{u}(r,t) = u(x,y,z,t) \tag{3.3.3}$$

Because u is continuous.Thus we have:

$$\overline{u}(0,t) = u(x,y,z,t) \tag{3.3.4}$$

We integrate Équation (3.3.1) over the sphere with radius r and center M:

$$\frac{1}{4\pi r^2}\iint_{S_r^M} \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}S = \frac{a^2}{4\pi r^2}\iint_{S_r^M} \Delta u\,\mathrm{d}S \tag{3.3.5}$$

Also

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{4\pi r^2}\iint_{S_r^M} u\,\mathrm{d}S\right) = a^2\Delta\iint_{S_r^M} u\,\mathrm{d}S \tag{3.3.6}$$

We substitute Équation (3.3.2) into the above equation:

$$\frac{\partial^2 \overline{u}}{\partial t^2} = a^2\Delta\overline{u} \tag{3.3.7}$$

Thus let us conduct the following transformation.We have:

$$(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = r^2 \tag{3.3.8}$$

$$\begin{gathered}\frac{\partial\overline{u}}{\partial x} = \frac{\partial\overline{u}}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial\overline{u}}{\partial r}\frac{x-\xi}{r}\\ \frac{\partial^2\overline{u}}{\partial x^2} = \frac{\partial^2\overline{u}}{\partial r^2}\left(\frac{x-\xi}{r}\right)^2 + \frac{\partial\overline{u}}{\partial r}\frac{r^2-(x-\xi)^2}{r^3}\end{gathered} \tag{3.3.9}$$

Similarly we obtain:

$$\frac{\partial^2\overline{u}}{\partial y^2} = \frac{\partial^2\overline{u}}{\partial r^2}\left(\frac{y-\eta}{r}\right)^2 + \frac{\partial\overline{u}}{\partial r}\frac{r^2-(y-\eta)^2}{r^3} \tag{3.3.10}$$

$$\frac{\partial^2\overline{u}}{\partial z^2} = \frac{\partial^2\overline{u}}{\partial r^2}\left(\frac{z-\zeta}{r}\right)^2 + \frac{\partial\overline{u}}{\partial r}\frac{r^2-(z-\zeta)^2}{r^3} \tag{3.3.11}$$

Thus we have:

$$\Delta\overline{u} = \frac{\partial^2\overline{u}}{\partial x^2} + \frac{\partial^2\overline{u}}{\partial y^2} + \frac{\partial^2\overline{u}}{\partial z^2} = \frac{\partial^2\overline{u}}{\partial r^2} + \frac{2}{r}\frac{\partial\overline{u}}{\partial r} \tag{3.3.12}$$

We substitute this into Équation (3.3.7):

$$\frac{\partial^2\overline{u}}{\partial t^2} = a^2\left(\frac{\partial^2\overline{u}}{\partial r^2} + \frac{2}{r}\frac{\partial\overline{u}}{\partial r}\right) \tag{3.3.13}$$

This is a radical symmetric wave equation.Thus we get the general solution:

$$r\overline{u}(r,t) = f(r+at) + g(r-at) \tag{3.3.14}$$

let $r = 0$ we have:

$$\begin{aligned} f(at) &= -g(-at) \\ f'(at) &= g'(-at) \end{aligned} \tag{3.3.15}$$

We take the partial derivative of Équation (3.3.14) with respect to r:

$$\overline{u} + r\frac{\partial\overline{u}}{\partial r} = f'(r+at) + g'(r-at) \tag{3.3.16}$$

Let $r = 0$ We have:

$$\overline{u}(0,t) = f'(at) + g'(-at) = 2f'(at) \tag{3.3.17}$$

To obtain $f'(at)$,we take the partial derivative of Équation (3.3.14) with respect to t:

$$\frac{\partial}{\partial t}(r\overline{u}) = af'(r+at) - ag'(r-at) \tag{3.3.18}$$

We have:

$$a\frac{\partial}{\partial r}(r,\overline{u}) + \frac{\partial}{\partial t}(r\overline{u}) = 2af'(r+at) \tag{3.3.19}$$