2 Description of State Space

2.1 Definition

1. **Input variables**

We usually use $\boldsymbol{u}_t = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n(t)} \end{bmatrix}$ to represent input variables.

2. **State variables**

We usually use $\boldsymbol{x}_t = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n(t)} \end{bmatrix}$ to represent state variables.It is a least set to describe state of system.

3. **Output variables**

We usually use $\boldsymbol{y}_t = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n(t)} \end{bmatrix}$ to represent output variables.

4. **State equation**

State equation is a first order differential equation that describe relationship between input variables and state variables. We can write it as:

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ \dot{x}_2 &= f_2(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \end{cases} \tag{2.1.1}$$

Rewrite it as vector form:

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{2.1.2}$$

5. **Output equation**

Output equation is a equation that describe relationship between state variables and output variables. We can write it as:

$$\begin{cases} y_1 &= g_1(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ y_2 &= g_2(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ &\vdots \\ y_n &= g_n(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \end{cases} \tag{2.1.3}$$

Rewrite it as vector form:

$$\boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{2.1.4}$$

6. **Description of State space of System**

We can describe state space of system by equations as:

$$\begin{cases} \dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \\ \boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \end{cases} \tag{2.1.5}$$

When the system is linear, we can write it as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}(t)\boldsymbol{x} + \boldsymbol{D}(t)\boldsymbol{u} \end{cases} \tag{2.1.6}$$

2.2 Transfer function

Transfer function is a function that describe relationship between input and output of system.Given a system with different state,the transfer function is still the same which means it is not related to state of system in other words state variables.

Single input – Single output system

Given a linear single input-single output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \\ y = \boldsymbol{C}\boldsymbol{x} + Du \end{cases} \tag{2.2.1}$$

To get transfer function,we can use Laplace transform to get:

$$s\boldsymbol{X} - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}U(s)$$
$$Y(s) = \boldsymbol{C}\boldsymbol{X}(s) + DU(s)$$

Laplace transfer:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

$$\mathcal{L}[kf(t)] = kF(s)$$

$$\mathcal{L}[f(t) + g(t)] = F(s) + G(s)$$

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

$$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$$

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right] = s^2F(s) - sf(0) - f'^{(0)}$$

$$\mathcal{L}\left[\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'^{(0)} - \ldots - f^{n-1}(0)$$

$$\mathcal{L}\left[\int_0^t f(t)\,\mathrm{d}t\right] = \frac{F(s)}{s}$$

$$f(\infty) = \lim_{s \to 0} sF(s)$$

$$f(0) = \lim_{s \to \infty} sF(s)$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$s + a$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u(t)$	$\frac{1}{s}$
$\delta(t)$	1

The equations are organized as follows:

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)]$$
$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)] + DU(s)$$

Let initial condition be zero($\boldsymbol{x}(0) = 0$),we can get:

$$Y(s) = \left[\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D\right]U(s)$$

Thus,we can get transfer function as:

$$g(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D \tag{2.2.2}$$

Let $D = 0$,we can get:

$$g(s) = \frac{\boldsymbol{C}\ \mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B}}{\det(s\boldsymbol{I} - \boldsymbol{A})} \tag{2.2.3}$$

Multi input – Multi output system

Given a multi input-multi output system,we define transfer function between i-th out y_i and j-th input u_j as:

$$g_{ij}(s) = \frac{Y_i(s)}{U_j(s)} \tag{2.2.4}$$

where $Y_i(s)$ is Laplace transform of $y_i(t)$ and $U_j(s)$ is Laplace transform of $u_j(t)$. Must mention that if we define transfer function in this way,we assume that all other inputs are zero.Because linear system satisfies the principle of superposition,so when we plus all inputs $U_1, U_2, ..., U_p$,we can get the i-th output Y_i as:

$$Y_i = \sum_{j=1}^{p} g_{ij}U_j \tag{2.2.5}$$

We can write it as matrix form:

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\boldsymbol{U}(s) \tag{2.2.6}$$

Thus given a linear multi input-multi output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{cases} \tag{2.2.7}$$

We can conduct as before to get transfer function as:

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \frac{\boldsymbol{C} \operatorname{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \boldsymbol{D} \det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} \tag{2.2.8}$$

Regular

We say a transfer function is regular if and only if when

$$\lim_{s \to \infty} g(s) = c \tag{2.2.9}$$

where c is a constant. And a transfer function is strictly regular if and only if when

$$\lim_{s \to \infty} g(s) = 0 \tag{2.2.10}$$

2.3 Establishing State Space Model by Differential Equation

Given a single input and single output system,if we have differential equation as:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \ldots + a_0 y = b_n u^{(n)} + b_{n-1}u^{(n-1)} + \ldots + b_0 u \tag{2.3.1}$$

where $m \leq n$.

Condition 1: $m = 0$

We have:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \ldots + a_0 y = b_0 u \tag{2.3.2}$$

We can define state variables as:

$$\begin{cases} x_1 &= y \\ x_2 &= y^{(1)} \\ x_3 &= y^{(2)} \\ &\vdots \\ x_n &= y^{(n-1)} \end{cases} \tag{2.3.3}$$

We can get state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + b_0 u \end{cases} \tag{2.3.4}$$

We can rewrite it as vector form:

$$
\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} \tag{2.3.5}
$$
$$
y = [1 \;\; 0 \;\; \dots \;\; 0] \boldsymbol{x}
$$

Condition 2:$m \neq n$

Controllable Canonical Form Method:

Let us note D as $\frac{\mathrm{d}}{\mathrm{d}t}$,we can rewrite Equation (2.3.1) as:

$$
y = \frac{b_m D^m + b_{m-1} D^{m-1} + b_{m-2} D^{m-2} + \dots + b_0}{D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \dots + a_0} u \tag{2.3.6}
$$

Let us discuss the case when $m < n$

Let

$$
\tilde{y}^{(n)} + a_{n-1} \tilde{y}^{(n-1)} + a_{n-2} \tilde{y}^{(n-2)} + \dots + a_1 \tilde{y}^{(1)} + a_0 \tilde{y} = u \tag{2.3.7}
$$

Also as

$$
\tilde{y} = \frac{1}{D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \dots + a_0} u \tag{2.3.8}
$$

we can get:

$$
y = b_m \tilde{y}^{(m)} + b_{m-1} \tilde{y}^{(m-1)} + b_{m-2} \tilde{y}^{(m-2)} + \dots + b_0 \tilde{y} \tag{2.3.9}
$$

We choose state variables as $x_1 = \tilde{y}, x_2 = \tilde{y}^{(1)}, \dots, x_n = \tilde{y}^{(n-1)}$.We can get state equation as:

$$
\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n + u \end{cases} \tag{2.3.10}
$$

and output equation as:

$$
y = b_0 x_1 + b_1 x_2 + \dots + b_m x_{m+1} \tag{2.3.11}
$$

We can rewrite it as vector form:

$$
\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \tag{2.3.12}
$$
$$
y = [b_0, \dots, b_m, 0, \dots, 0] \boldsymbol{x}
$$

Let us discuss the case when $m = n$,we can rewrite Equation (2.3.6) as:

$$
y = \left[b_n + \frac{(b_{n-1} - b_n a_{n-1}) D^{n-1} + \dots + (b_0 - b_n a_0)}{D^n + a_{n-1} D^{n-1} + \dots + a_0} \right] u \tag{2.3.13}
$$

Also let

$$\tilde{y}^{(n)} + a_{n-1}\tilde{y}^{(n-1)} + a_{n-2}\tilde{y}^{(n-2)} + \dots + a_1\tilde{y}^{(1)} + a_0\tilde{y} = u \tag{2.3.14}$$

We can get:

$$y = (b_{n-1} - b_n a_{n-1})\tilde{y}^{(n-1)} + (b_{n-2} - b_n a_{n-2})\tilde{y}^{(n-2)} + \dots + (b_0 - b_n a_0)\tilde{y} + b_n u \tag{2.3.15}$$

Thus we can write state equation in vector form in familiar way as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \\ y &= [b_0 - b_n a_0, b_1 - b_n a_1, \dots, b_{n-1} - b_n a_{n-1}]\boldsymbol{x} + b_n u \end{aligned} \tag{2.3.16}$$

Undetermined Canonical Form Method: W.l.o.g,we assume that the equation is in the form of:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = b_n u^{(n)} + b_{n-1}u^{(n-1)} + \dots + b_0 u \tag{2.3.17}$$

We can define state variables as:

$$\begin{cases} x_1 &= y - \beta_0 u \\ x_2 &= \dot{x}_1 - \beta_1 u = \dot{y} - \beta_0 \dot{u} - \beta_1 u \\ x_3 &= \dot{x}_2 - \beta_2 u = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u \\ \vdots \\ x_n &= \dot{x}_{n-1} - \beta_{n-1} u = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \dots - \beta_{n-1} u \end{cases} \tag{2.3.18}$$

Thus we have:

$$\begin{cases} y = x_1 + \beta_0 u \\ \dot{y} = x_2 + \beta_0 \dot{u} + \beta_1 u \\ \ddot{y} = x_3 + \beta_0 \ddot{u} + \beta_1 \dot{u} + \beta_2 u \\ \vdots \\ y^{(n-1)} = x_n + \beta_0 u^{(n-1)} + \beta_1 u^{(n-2)} + \dots + \beta_{n-1} u \end{cases} \tag{2.3.19}$$

Let us introduce a new variables $x_{n+1} = \dot{x}_n - \beta_n u = \dot{x}_{n-1} - \beta_{n-1} u = y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \dots - \beta_n u$.Thus we have:

$$y^{(n)} = x_{n+1} + \beta_0 u^{(n)} + \beta_1 u^{(n-1)} + \dots + \beta_n u \tag{2.3.20}$$

Substitute $y, \dot{y}, \dots, y^{(n)}$ into Equation (2.3.17),we can get:

$$\begin{gathered} (x_{n+1} + a_1 x_n + \dots + a_n x_1) + \beta_0 u^{(n)} + (\beta_1 + a_1 \beta_0) u^{(n-1)} + \\ (\beta_2 + a_1 \beta_1 + a_2 \beta_0) u^{(n-2)} + \dots + (\beta_n + a_1 \beta_{n-1} + a_2 \beta_{n-2} + \dots + a_n \beta_0) u \\ = b_0 u^{(n)} + b_1 u^{(n-1)} + b_2 u^{(n-2)} + \dots + b_n u \end{gathered} \tag{2.3.21}$$

Compare the coefficients of $u^{(n)}, u^{(n-1)}, \dots, u$,we can get:

$$
\begin{cases}
x_{n+1} + a_1 x_n + \ldots + a_n x_1 & = 0 \\
\beta_0 & = b_0 \\
\beta_1 + a_1 \beta_0 & = b_1 \\
\beta_2 + a_1 \beta_1 + a_2 \beta_0 & = b_2 \\
\vdots \\
\beta_n + a_1 \beta_{n-1} + a_2 \beta_{n-2} + \ldots + a_n \beta_0 & = b_n
\end{cases} \tag{2.3.22}
$$

we can rewrite it as matrix form:

$$
\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_1 & 1 & 0 & \ldots & 0 \\ a_2 & a_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \ldots & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \tag{2.3.23}
$$

In summary,we can get state equation as:

$$
\begin{cases}
\dot{x}_1 = \dot{y} - \beta_0 \dot{u} = x_2 + \beta_1 u \\
\dot{x}_2 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} = x_3 + \beta_2 u \\
\quad \vdots \\
\dot{x}_{n-1} = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \ldots - \beta_{n-2} \dot{u} = x_n + \beta_{n-1} u \\
\dot{x}_n = y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \ldots - \beta_{n-1} \dot{u} = -a_n x_1 - a_{n-1} x_2 - \ldots - a_1 x_n + \beta_n u
\end{cases} \tag{2.3.24}
$$

and output equation as:

$$
y = x_1 + \beta_0 u \tag{2.3.25}
$$

We can rewrite it as vector form:

$$
\begin{gathered}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u \\
y = [1, 0, 0, \ldots, 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 u
\end{gathered} \tag{2.3.26}
$$

2.4 Establishing State Space Model by Transfer Function

For a actual physical system,the transfer function of the system is always regular.

First,let us discuss the situation where the system is restrict regular.If we have a differential equation of system as:

$$
y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \ldots + a_1 \dot{y} + a_0 y = b_{n-1} u^{(n-1)} + \ldots + b_1 \dot{u} + b_0 u \tag{2.4.1}
$$

Then we have transfer function as:

$$
g(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \tag{2.4.2}
$$

Introduce a intermediate variables $Z(s)$ We have:

$$g(s) = \frac{Y(s)}{Z(s)}\frac{Z(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0}\frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + ... + b_0}{1} \tag{2.4.3}$$

Let us do inverse Laplace transform of $Z(s)$,we can get:

$$\begin{cases} y = z^{(n)} + a_{n-1}z^{(n-1)} + a_{n-2}z^{(n-2)} + ... + a_1\dot{z} + a_0 z \\ b_{n-1}z^{(n-1)} + b_{n-2}z^{(n-2)} + ... + b_1\dot{z} + b_0 z = u \end{cases} \tag{2.4.4}$$

To be continued...

2.5 Linear Transformation

Given a state variable vector $\boldsymbol{x}$,the linear combination of the state variable vector is also a state variable vector $\bar{\boldsymbol{x}}$ if and only if the linear transformation matrix $\boldsymbol{P}$ is invertible.

$$\boldsymbol{x} = \boldsymbol{P}\bar{\boldsymbol{x}} \tag{2.5.1}$$

In other words:

$$\bar{\boldsymbol{x}} = \boldsymbol{P}^{-1}\boldsymbol{x} \tag{2.5.2}$$