Note of Modern Control Theory

Course Note

2024-10-08

Lawrence

2024 Year

Fall Season Class

2 Description of State Space

2.1 Definition

1. **Input variables**

We usually use $\boldsymbol{u}_t = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n(t)} \end{bmatrix}$ to represent input variables.

2. **State variables**

We usually use $\boldsymbol{x}_t = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n(t)} \end{bmatrix}$ to represent state variables.It is a least set to describe state of system.

3. **Output variables**

We usually use $\boldsymbol{y}_t = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n(t)} \end{bmatrix}$ to represent output variables.

4. **State equation**

State equation is a first order differential equation that describe relationship between input variables and state variables. We can write it as:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ \dot{x}_2 = f_2(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ \quad\vdots \\ \dot{x}_n = f_n(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \end{cases} \tag{2.1.1}$$

Rewrite it as vector form:

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{2.1.2}$$

5. **Output equation**

Output equation is a equation that describe relationship between state variables and output variables. We can write it as:

$$\begin{cases} y_1 = g_1(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ y_2 = g_2(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ \quad\vdots \\ y_n = g_n(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \end{cases} \tag{2.1.3}$$

Rewrite it as vector form:

$$\boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{2.1.4}$$

6. **Description of State space of System**

We can describe state space of system by equations as:

$$\begin{cases} \dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \\ \boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \end{cases} \tag{2.1.5}$$

When the system is linear, we can write it as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}(t)\boldsymbol{x} + \boldsymbol{D}(t)\boldsymbol{u} \end{cases} \tag{2.1.6}$$

2.2 Transfer function

Transfer function is a function that describe relationship between input and output of system.Given a system with different state,the transfer function is still the same which means it is not related to state of system in other words state variables.

Single input – Single output system

Given a linear single input-single output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \\ y = \boldsymbol{C}\boldsymbol{x} + Du \end{cases} \tag{2.2.1}$$

To get transfer function,we can use Laplace transform to get:

$$s\boldsymbol{X} - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}U(s)$$
$$Y(s) = \boldsymbol{C}\boldsymbol{X}(s) + DU(s)$$

Laplace transfer:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

$$\mathcal{L}[kf(t)] = kF(s)$$

$$\mathcal{L}[f(t) + g(t)] = F(s) + G(s)$$

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

$$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$$

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$\mathcal{L}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right] = s^2F(s) - sf(0) - f'^{(0)}$$

$$\mathcal{L}\left[\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'^{(0)} - \dots - f^{n-1}(0)$$

$$\mathcal{L}\left[\int_0^t f(t)\,\mathrm{d}t\right] = \frac{F(s)}{s}$$

$$f(\infty) = \lim_{s\to 0} sF(s)$$

$$f(0) = \lim_{s\to\infty} sF(s)$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$s+a$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u(t)$	$\frac{1}{s}$
$\delta(t)$	1

The equations are organized as follows:

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)]$$
$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)] + DU(s)$$

Let initial condition be zero($\boldsymbol{x}(0) = 0$),we can get:

$$Y(s) = \left[\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D\right]U(s)$$

Thus,we can get transfer function as:

$$g(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D \tag{2.2.2}$$

Let $D = 0$,we can get:

$$g(s) = \frac{\boldsymbol{C}\ \mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B}}{\det(s\boldsymbol{I} - \boldsymbol{A})} \tag{2.2.3}$$

Multi input − Multi output system

Given a multi input-multi output system,we define transfer function between i-th out y_i and j-th input u_j as:

$$g_{ij}(s) = \frac{Y_i(s)}{U_j(s)} \tag{2.2.4}$$

where $Y_i(s)$ is Laplace transform of $y_i(t)$ and $U_j(s)$ is Laplace transform of $u_j(t)$. Must mention that if we define transfer function in this way,we assume that all other inputs are zero.Because linear system satisfies the principle of superposition,so when we plus all inputs $U_1, U_2, ..., U_p$,we can get the i-th output Y_i as:

$$Y_i = \sum_{j=1}^{p} g_{ij} U_j \tag{2.2.5}$$

We can write it as matrix form:

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\boldsymbol{U}(s) \tag{2.2.6}$$

Thus given a linear multi input-multi output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{cases} \tag{2.2.7}$$

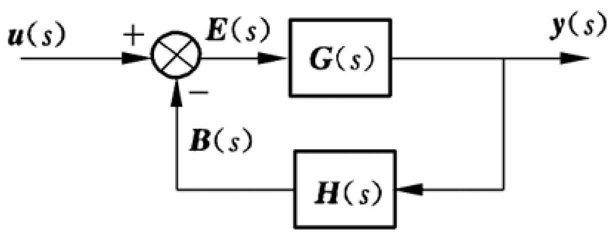
We can conduct as before to get transfer function as:

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \frac{\boldsymbol{C}\ \mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \boldsymbol{D}\ \det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} \tag{2.2.8}$$

Closed-loop System

We have a closed-loop system as figure below:

Fig. 1 — Closed-loop System



We have:

$$\boldsymbol{E}(s) = \boldsymbol{u}(s) - \boldsymbol{B}(s)$$
$$\boldsymbol{B}(s) = \boldsymbol{H}(s)\boldsymbol{y}(s) = \boldsymbol{H}(s)\boldsymbol{G}(s)\boldsymbol{E}(s)$$
$$\boldsymbol{y}(s) = [\boldsymbol{I} + \boldsymbol{H}(s)\boldsymbol{G}(s)]^{-1}\boldsymbol{G}(s)\boldsymbol{u}(s)$$

Thus the transfer function of closed-loop system is:

$$\boldsymbol{G}_{\boldsymbol{H}}(s) = [\boldsymbol{I} + \boldsymbol{H}(s)\boldsymbol{G}(s)]^{-1}\boldsymbol{G}(s) \tag{2.2.9}$$

Regular

We say a transfer function is regular if and only if when

$$\lim_{s\to\infty} g(s) = c \tag{2.2.10}$$

where c is a constant. And a transfer function is strictly regular if and only if when

$$\lim_{s\to\infty} g(s) = 0 \tag{2.2.11}$$

2.3 Establishing State Space Model by Differential Equation

Given a single input and single output system,if we have differential equation as:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + ... + a_0 y = b_n u^{(n)} + b_{n-1}u^{(n-1)} + ... + b_0 u \tag{2.3.1}$$

where $m \leq n$.

Condition 1: $m = 0$

We have:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + ... + a_0 y = b_0 u \tag{2.3.2}$$

We can define state variables as:

$$\begin{cases} x_1 &= y \\ x_2 &= y^{(1)} \\ x_3 &= y^{(2)} \\ &\vdots \\ x_n &= y^{(n-1)} \end{cases} \tag{2.3.3}$$

We can get state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0 x_1 - a_1 x_2 - ... - a_{n-1} x_n + b_0 u \end{cases} \tag{2.3.4}$$

We can rewrite it as vector form:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \\ -a_0 & -a_1 & ... & -a_{n-1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} \tag{2.3.5}$$
$$y = [1 \ \ 0 \ \ ... \ \ 0]\boldsymbol{x}$$

Condition 2:$m \neq n$

Controllable Canonical Form Method:

Let us note D as $\frac{\mathrm{d}}{\mathrm{d}t}$,we can rewrite Équation (2.3.1) as:

$$y = \frac{b_m D^m + b_{m-1} D^{m-1} + b_{m-2} D^{m-2} + ... + b_0}{D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + ... + a_0} u \tag{2.3.6}$$

Let us discuss the case when $m < n$

Let

$$\tilde{y}^{(n)} + a_{n-1}\tilde{y}^{(n-1)} + a_{n-2}\tilde{y}^{(n-2)} + ... + a_1\tilde{y}^{(1)} + a_0\tilde{y} = u \tag{2.3.7}$$

Also as

$$\tilde{y} = \frac{1}{D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + ... + a_0} u \tag{2.3.8}$$

we can get:

$$y = b_m\tilde{y}^{(m)} + b_{m-1}\tilde{y}^{(m-1)} + b_{m-2}\tilde{y}^{(m-2)} + ... + b_0\tilde{y} \tag{2.3.9}$$

We choose state variables as $x_1 = \tilde{y}, x_2 = \tilde{y}^{(1)}, ..., x_n = \tilde{y}^{(n-1)}$.We can get state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0 x_1 - a_1 x_2 - ... - a_{n-1} x_n + u \end{cases} \tag{2.3.10}$$

and output equation as:

$$y = b_0 x_1 + b_1 x_2 + ... + b_m x_{m+1} \tag{2.3.11}$$

We can rewrite it as vector form:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \\ -a_0 & -a_1 & ... & -a_{n-1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \tag{2.3.12}$$
$$y = [b_0, ..., b_m, 0, ..., 0]\boldsymbol{x}$$

Let us discuss the case when $m = n$,we can rewrite Équation (2.3.6) as:

$$y = \left[b_n + \frac{(b_{n-1} - b_n a_{n-1})D^{n-1} + ... + (b_0 - b_n a_0)}{D^n + a_{n-1}D^{n-1} + ... + a_0} \right] u \tag{2.3.13}$$

Also let

$$\tilde{y}^{(n)} + a_{n-1}\tilde{y}^{(n-1)} + a_{n-2}\tilde{y}^{(n-2)} + ... + a_1\tilde{y}^{(1)} + a_0\tilde{y} = u \tag{2.3.14}$$

We can get:

$$y = (b_{n-1} - b_n a_{n-1})\tilde{y}^{(n-1)} + (b_{n-2} - b_n a_{n-2})\tilde{y}^{(n-2)} + ... + (b_0 - b_n a_0)\tilde{y} + b_n u \tag{2.3.15}$$

Thus we can write state equation in vector form in familiar way as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \\ -a_0 & -a_1 & ... & -a_{n-1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \\ y &= [b_0 - b_n a_0, b_1 - b_n a_1, ..., b_{n-1} - b_n a_{n-1}]\boldsymbol{x} + b_n u \end{aligned} \tag{2.3.16}$$

Undetermined Canonical Form Method: W.l.o.g,we assume that the equation is in the form of:

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_0 y = b_n u^{(n)} + b_{n-1}u^{(n-1)} + ... + b_0 u \tag{2.3.17}$$

We can define state variables as:

$$\begin{cases} x_1 &= y - \beta_0 u \\ x_2 &= \dot{x}_1 - \beta_1 u = \dot{y} - \beta_0 \dot{u} - \beta_1 u \\ x_3 &= \dot{x}_2 - \beta_2 u = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u \\ &\vdots \\ x_n &= \dot{x}_{n-1} - \beta_{n-1} u = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - ... - \beta_{n-1} u \end{cases} \tag{2.3.18}$$

Thus we have:

$$\begin{cases} y = x_1 + \beta_0 u \\ \dot{y} = x_2 + \beta_0 \dot{u} + \beta_1 u \\ \ddot{y} = x_3 + \beta_0 \ddot{u} + \beta_1 \dot{u} + \beta_2 u \\ \quad \vdots \\ y^{(n-1)} = x_n + \beta_0 u^{(n-1)} + \beta_1 u^{(n-2)} + ... + \beta_{n-1} u \end{cases} \tag{2.3.19}$$

Let us introduce a new variables $x_{n+1} = \dot{x}_n - \beta_n u = \dot{x}_{n-1} - \beta_{n-1} u = y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - ... - \beta_n u$.Thus we have:

$$y^{(n)} = x_{n+1} + \beta_0 u^{(n)} + \beta_1 u^{(n-1)} + ... + \beta_n u \tag{2.3.20}$$

Substitute $y, \dot{y}, ..., y^{(n)}$ into Équation (2.3.17),we can get:

$$\begin{aligned}&(x_{n+1}+a_{n-1}x_n+...+a_0x_1)+\beta_0u^{(n)}+(\beta_1+a_{n-1}\beta_0)u^{(n-1)}+\\&(\beta_2+a_{n-1}\beta_1+a_{n-2}\beta_0)u^{(n-2)}+...+(\beta_n+a_{n-1}\beta_{n-1}+a_{n-2}\beta_{n-2}+...+a_0\beta_0)u\\&=b_nu^{(n)}+b_{n-1}u^{(n-1)}+b_{n-2}u^{(n-2)}+...+b_0u\end{aligned} \tag{2.3.21}$$

Compare the coefficients of $u^{(n)}, u^{(n-1)}, ..., u$,we can get:

$$\begin{cases}x_{n+1}+a_{n-1}x_n+...+a_0x_1 &= 0\\ \beta_0 &= b_n\\ \beta_1+a_{n-1}\beta_0 &= b_{n-1}\\ \beta_2+a_{n-1}\beta_1+a_{n-2}\beta_0 &= b_{n-2}\\ \vdots\\ \beta_n+a_{n-1}\beta_{n-1}+a_{n-2}\beta_{n-2}+...+a_0\beta_0 &= b_0\end{cases} \tag{2.3.22}$$

In summary,we can get state equation as:

$$\begin{cases}\dot{x}_1=\dot{y}-\beta_0\dot{u}=x_2+\beta_1u\\ \dot{x}_2=\ddot{y}-\beta_0\ddot{u}-\beta_1\dot{u}=x_3+\beta_2u\\ \quad\vdots\\ \dot{x}_{n-1}=y^{(n-1)}-\beta_0u^{(n-1)}-\beta_1u^{(n-2)}-...-\beta_{n-2}\dot{u}=x_n+\beta_{n-1}u\\ \dot{x}_n=y^{(n)}-\beta_0u^{(n)}-\beta_1u^{(n-1)}-...-\beta_{n-1}\dot{u}=-a_0x_1-a_1x_2-...-a_{n-1}x_n+\beta_nu\end{cases} \tag{2.3.23}$$

and output equation as:

$$y=x_1+\beta_0u \tag{2.3.24}$$

We can rewrite it as vector form:

$$\begin{gathered}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0&1&0&...&0\\0&0&1&...&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&...&1\\-a_0&-a_1&-a_2&...&-a_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\ \vdots\\x_n\end{bmatrix}+\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{bmatrix}u\\ y=[1,0,0,...,0]\begin{bmatrix}x_1\\x_2\\ \vdots\\x_n\end{bmatrix}+\beta_0u\end{gathered} \tag{2.3.25}$$

2.4 Establishing State Space Model by Transfer Function

For a actual physical system,the transfer function of the system is always regular.

First,let us discuss the situation where the system is restrict regular in other words order of numerator of the transfer function is less than denominator of the transfer function.If we have a differential equation of system as:

$$y^{(n)}+a_{n-1}y^{(n-1)}+a_{n-2}y^{(n-2)}+...+a_1\dot{y}+a_0y=b_{n-1}u^{(n-1)}+...+b_1\dot{u}+b_0u \tag{2.4.1}$$

Then we have transfer function as:

$$g(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+...+b_0}{s^n+a_{n-1}s^{n-1}+...+a_1s+a_0} \tag{2.4.2}$$

Introduce a intermediate variables $Z(s)$ We have:

$$g(s)=\frac{Y(s)}{Z(s)}\frac{Z(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+...+b_0}{1}\frac{1}{s^n+a_{n-1}s^{n-1}+...+a_1s+a_0} \tag{2.4.3}$$

Let us do inverse Laplace transform of $Z(s)$,we can get:

$$\begin{cases} y=b_{n-1}z^{(n-1)}+b_{n-2}z^{(n-2)}+...+b_1\dot{z}+b_0z \\ z^{(n)}+a_{n-1}z^{(n-1)}+a_{n-2}z^{(n-2)}+...+a_1\dot{z}+a_0z=u \end{cases} \tag{2.4.4}$$

We can define state variables as $x_1=z,x_2=\dot{z},x_3=\ddot{z},x_n=z^{(n-1)}$.We have state equation as:

$$\begin{cases} \dot{x}_1=x_2 \\ \dot{x}_2=x_3 \\ \vdots \\ \dot{x}_{n-1}=x_n \\ \dot{x}_n=-a_0x_1-a_1x_2-...-a_{n-1}x_n+u \end{cases} \tag{2.4.5}$$

And output equation as:

$$y=b_0x_1+b_1x_2+...+b_{n-1}x_n \tag{2.4.6}$$

Let us discuss when the order of numerator of transfer function is as same as order of denominator of transfer function.We have transfer function as:

$$\begin{aligned} g(s)=\frac{Y(s)}{U(s)}&=\frac{b_ns^n+b_{n-1}s^{n-1}+...+b_0}{s^n+a_{n-1}s^{n-1}+...+a_1s+a_0} \\ =b_n+&\frac{(b_{n-1}-b_na_{n-1})s^{n-1}+(b_{n-2}-b_na_{n-2})s^{n-2}+...+(b_0-b_na_0)}{s^n+a_{n-1}s^{n-1}+...+a_1s+a_0} \end{aligned} \tag{2.4.7}$$

Let us note $h(s)$ as intermediate transfer function:

$$h(s)=\frac{\beta_{n-1}s^{n-1}+\beta_{n-2}s^{n-2}+...+\beta_0}{s^n+a_{n-1}s^{n-1}+...+a_1s+a_0} \tag{2.4.8}$$

where $\beta_i=b_i-b_na_i$ for $i=0,1,...,n-1$. We do the same thing as before,we can get:

$$\begin{cases} y=\beta_{n-1}z^{(n-1)}+\beta_{n-2}z^{(n-2)}+...+\beta_1\dot{z}+\beta_0z \\ z^{(n)}+a_{n-1}z^{(n-1)}+a_{n-2}z^{(n-2)}+...+a_1\dot{z}+a_0z=u \end{cases} \tag{2.4.9}$$

And

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0 x_1 - a_1 x_2 - ... - a_{n-1} x_n + u \end{cases} \tag{2.4.10}$$

For output equation,all we need is to add a $b_n u$ term.

$$y = \beta_0 x_1 + \beta_1 x_2 + ... + \beta_{n-1} x_n + b_n u \tag{2.4.11}$$

2.5 Linear Transformation

Given a state variable vector $\boldsymbol{x}$,the linear combination of the state variable vector is also a state variable vector $\bar{\boldsymbol{x}}$ if and only if the linear transformation matrix $\boldsymbol{P}$ is invertible.

$$\boldsymbol{x} = \boldsymbol{P}\bar{\boldsymbol{x}} \tag{2.5.1}$$

In other words:

$$\bar{\boldsymbol{x}} = \boldsymbol{P}^{-1}\boldsymbol{x} \tag{2.5.2}$$

Let us discuss what would happen if we apply liner transformation to a **liner system**.

Given a linear system as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{cases} \tag{2.5.3}$$

Let $\boldsymbol{x} = \boldsymbol{P}\bar{\boldsymbol{x}}$,we have:

$$\begin{cases} \dot{\bar{\boldsymbol{x}}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\bar{\boldsymbol{x}} + \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{P}\bar{\boldsymbol{x}} + \boldsymbol{D}\boldsymbol{u} \end{cases} \tag{2.5.4}$$

We have:

$$\bar{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \tag{2.5.5}$$

$$\bar{\boldsymbol{B}} = \boldsymbol{P}^{-1}\boldsymbol{B} \tag{2.5.6}$$

$$\bar{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{P} \tag{2.5.7}$$

$$\bar{\boldsymbol{D}} = \boldsymbol{D} \tag{2.5.8}$$

Let us try to transform state equations to **diagonal canonical form**.

Given a state equation as:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{2.5.9}$$

The eigenvalues of the system is defined as:

$$\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0 \tag{2.5.10}$$

Diagonal Canonical Form

If the geometric multiplicity of the system is equal to the order of the system,we can transform the state equation to diagonal canonical form by linear transformation.

Let $\boldsymbol{P} = [\boldsymbol{v_1} \;\; \boldsymbol{v_2} \;\; \cdots \;\; \boldsymbol{v_n}]^{-1}$.Then the state equation can be transformed to diagonal form as:

$$\dot{\bar{\boldsymbol{x}}} = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_n \end{bmatrix} \bar{\boldsymbol{x}} + \bar{\boldsymbol{B}}\boldsymbol{u} \tag{2.5.11}$$

where $\bar{\boldsymbol{B}} = \boldsymbol{P}^{-1}\boldsymbol{B}$.

Trick

If A is a companion matrix,then the state equation can be transformed to diagonal canonical form by transformation matrix $\boldsymbol{P}$ where $\boldsymbol{P}$ is a inverse vandermonde matrix.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \\ -a_0 & -a_1 & -a_2 & ... & -a_{n-1} \end{bmatrix}, \boldsymbol{P} = \begin{bmatrix} 1 & 1 & ... & 1 \\ \lambda_1 & \lambda_2 & ... & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & ... & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & ... & \lambda_n^{n-1} \end{bmatrix}^{-1} \tag{2.5.12}$$

Jordan Canonical Form

If the geometric multiplicity of the system is less than the order of the system,we can transform the state equation to jordan canonical form by linear transformation.

For those eigenvalues with geometric multiplicity less than the order of the system and let $\boldsymbol{v_i}$ be their corresponding eigenvectors($\lambda_i \boldsymbol{v_i} = \boldsymbol{A}\boldsymbol{v_i}$),we define generalized eigenvectors as:

$$\begin{cases} (\lambda_i \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v_i} &= 0 \\ (\lambda_i \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v_i'} &= -\boldsymbol{v_i} \\ (\lambda_i \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v_i''} &= -\boldsymbol{v_i'} \\ \qquad\vdots & \\ (\lambda_i \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v_i^{\sigma_i}} &= -\boldsymbol{v_i^{\sigma_{i-1}}} \end{cases} \tag{2.5.13}$$

Then let $\boldsymbol{P} = [\boldsymbol{v_1} \;\; \boldsymbol{v_1'} \;\; \cdots \;\; \boldsymbol{v_1^{\sigma_1}} \;\; \boldsymbol{v_2} \;\; \boldsymbol{v_2'} \;\; \cdots \;\; \boldsymbol{v_2^{\sigma_2}} \;\; \ldots]^{-1}$.Then the state equation can be transformed to jordan canonical form as:

$$\dot{\bar{\boldsymbol{x}}} = \begin{bmatrix} \boldsymbol{J}_1 & 0 & ... & 0 \\ 0 & \boldsymbol{J}_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \boldsymbol{J}_n \end{bmatrix} \bar{\boldsymbol{x}} + \bar{\boldsymbol{B}}\boldsymbol{u} \tag{2.5.14}$$

where $\boldsymbol{J}_i$ is a jordan block corresponding to eigenvalue λ_i and $\bar{\boldsymbol{B}} = \boldsymbol{P}^{-1}\boldsymbol{B}$.

Modal Form If the eigenvalues of the system are complex numbers,we can transform the state equation to modal form.

Let

$$\lambda_1 = \sigma + \omega i, \lambda_2 = \sigma - \omega i \tag{2.5.15}$$

In this situation,the modal form of A is

$$\boldsymbol{M} = \left[\begin{smallmatrix} \sigma & \omega \\ -\omega & \sigma \end{smallmatrix}\right] \tag{2.5.16}$$

Let $\boldsymbol{v_1}$ be the eigenvector of λ_1 ($\lambda_1\boldsymbol{v_1} = \boldsymbol{A}\boldsymbol{v_1}$).

$$\boldsymbol{v_1} = \boldsymbol{\alpha} + \beta i \tag{2.5.17}$$

The the transformation matrix $\boldsymbol{P}$ is $\left[\begin{smallmatrix} \boldsymbol{\alpha} & \beta \end{smallmatrix}\right]^{-1}$.

3 Solution of State Equations

3.1 Solution of Liner Time Invariant Homogeneous State Equations

Given a liner time invariant homogeneous state equation:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \tag{3.1.1}$$

If the input is zero and $t_0 = 0$,it is easy to see that the solution is:

$$\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{x}(0) \tag{3.1.2}$$

If the $t_0 \neq 0$, the solution is:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) \tag{3.1.3}$$

We call $e^{\boldsymbol{A}(t-t_0)}$ as matrix exponential or state transition matrix and note it as $\Phi(t)$.

3.2 Matrix Exponential

Definition:

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \boldsymbol{A}^2\frac{t^2}{2!} + \boldsymbol{A}^3\frac{t^3}{3!} + ... \tag{3.2.1}$$

(1)

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A} \\ \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = \boldsymbol{A}\Phi(t) = \Phi(t)\boldsymbol{A} \end{gathered} \tag{3.2.2}$$

(2)

$$\begin{gathered} e^{\boldsymbol{A}\cdot 0} = \boldsymbol{I} \\ \Phi(0) = \boldsymbol{I} \end{gathered} \tag{3.2.3}$$

(3)

$$\begin{gathered} \left[e^{\boldsymbol{A}t}\right]^{-1} = e^{-\boldsymbol{A}t} \\ [\Phi(t)]^{-1} = \Phi^{-1}(t) = \Phi(-t) \end{gathered} \tag{3.2.4}$$

(4)

$$e^{\boldsymbol{A}(t_2-t_0)}e^{\boldsymbol{A}(t_0-t_1)} = e^{\boldsymbol{A}(t_2-t_1)} \tag{3.2.5}$$

(5)If and only if $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$

$$e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{\boldsymbol{A}t}e^{\boldsymbol{B}t} \tag{3.2.6}$$

Some special matrix exponential:

(1) Diagonal matrix:It is easy to see that:

$$\begin{aligned} A &= \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n) \\ e^{\boldsymbol{A}t} &= \operatorname{diag}\left(e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t}\right) \end{aligned} \tag{3.2.7}$$

(2) Jordan block matrix:

$$A = \begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & \ddots & & \\ & & \ddots & & \\ & & & \lambda & 1 \\ 0 & & & & \lambda \end{bmatrix} \tag{3.2.8}$$

We have:

$$\boldsymbol{A}^2 = \begin{bmatrix} \lambda^2 & 2\lambda & 1 & & 0 \\ & \lambda^2 & 2\lambda & \ddots & \\ & & \lambda^2 & \ddots & \\ & & & & \ddots \end{bmatrix}, \boldsymbol{A}^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 & & 0 \\ & \lambda^3 & 3\lambda^2 & \ddots & & \\ & & \lambda^3 & \ddots & & \\ & & & & \ddots & \end{bmatrix} \tag{3.2.9}$$

We can induct that:

$$\boldsymbol{f}(\boldsymbol{A}) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \frac{f'''(\lambda)}{3!} & \cdots & \cdots \\ & f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \cdots & \cdots \\ & & \ddots & \ddots & & \\ & & & & \ddots & \end{bmatrix} \tag{3.2.10}$$

where $\boldsymbol{f}$ stands for a power function.

Let us plus them by definition of matrix exponential:

$$e^{\boldsymbol{A}t} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{m-1}}{(m-1)!} \\ & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m-2}}{(m-2)!} \\ & & \ddots & \ddots & & \vdots \\ & & & & & 1 \end{bmatrix} \tag{3.2.11}$$

(3) Jordan form matrix Let $\boldsymbol{A}$ be a matrix with Jordan form:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_2 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{A}_n \end{bmatrix} \tag{3.2.12}$$

where $\boldsymbol{A}_i$ is a Jordan block matrix. We have:

$$e^{\boldsymbol{A}t} = \begin{bmatrix} e^{\boldsymbol{A}_1 t} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & e^{\boldsymbol{A}_2 t} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & e^{\boldsymbol{A}_n t} \end{bmatrix} \tag{3.2.13}$$

Method of Calculate Matrix Exponential

(1) **Laplace transformation method**

Let us consider a liner time invariant homogeneous state equation:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t), \boldsymbol{x}(0) = \boldsymbol{x}_0, t \geq t_0 \tag{3.2.14}$$

We do laplace transformation on both sides:

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s) \tag{3.2.15}$$

We can get:

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}\left[(s\boldsymbol{I} - \boldsymbol{A})^{-1}\right]\boldsymbol{x}(0) \tag{3.2.16}$$

By uniqueness of solution of differential equation, we can get:

$$e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\left[(s\boldsymbol{I} - \boldsymbol{A})^{-1}\right] \tag{3.2.17}$$

(2) **Polynomial method**

Cayley-Hamilton theorem tells us that:If matrix $\boldsymbol{A}$ is a $n \times n$ square matrix, then:

$$f(\lambda) = \det(\lambda\boldsymbol{I} - \boldsymbol{A}) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n = 0 \tag{3.2.18}$$

$$f(\boldsymbol{A}) = \boldsymbol{A}^n + a_1\boldsymbol{A}^{n-1} + a_2\boldsymbol{A}^{n-2} + ... + a_n\boldsymbol{I} = \boldsymbol{0} \tag{3.2.19}$$

By Cayley-Hamilton theorem, we get that $\boldsymbol{A}^n$ is actually a linear combination of $\boldsymbol{A}^{n-1}, \boldsymbol{A}^{n-2}, ..., \boldsymbol{I}$.It is similar to $\boldsymbol{A}^{n+1}, \boldsymbol{A}^{n+2}, ...$ which means we can write $e^{\boldsymbol{A}t}$ as a linear combination of $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^{n-1}$.

When **eigenvalues of $\boldsymbol{A}$ are all different**,let us consider the following equation to get $e^{\lambda t}$($e^{\boldsymbol{A}t}$ and $e^{\lambda t}$ should have the same coefficient $a_i(t)$):

$$\begin{cases} e^{\lambda_1 t} = a_0(t) + a_1(t)\lambda_1 + a_2(t)\lambda_1^2 + ... + a_{n-2}(t)\lambda_1^{n-2} + a_{n-1}(t)\lambda_1^{n-1} \\ e^{\lambda_2 t} = a_0(t) + a_1(t)\lambda_2 + a_2(t)\lambda_2^2 + ... + a_{n-2}(t)\lambda_2^{n-2} + a_{n-1}(t)\lambda_2^{n-1} \\ \quad\vdots \\ e^{\lambda_n t} = a_0(t) + a_1(t)\lambda_n + a_2(t)\lambda_n^2 + ... + a_{n-2}(t)\lambda_n^{n-2} + a_{n-1}(t)\lambda_n^{n-1} \end{cases} \tag{3.2.20}$$

Thus we have:

$$\begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & ... & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & ... & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & ... & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} \tag{3.2.21}$$

When **eigenvalues of $\boldsymbol{A}$ are not all different**,let us consider the following equation to get $e^{\lambda t}$.We still have:

$$\begin{cases} e^{\lambda_1 t} = a_0(t) + a_1(t)\lambda_1 + a_2(t)\lambda_1^2 + ... + a_{n-2}(t)\lambda_1^{n-2} + a_{n-1}(t)\lambda_1^{n-1} \\ e^{\lambda_2 t} = a_0(t) + a_1(t)\lambda_2 + a_2(t)\lambda_2^2 + ... + a_{n-2}(t)\lambda_2^{n-2} + a_{n-1}(t)\lambda_2^{n-1} \\ \quad\vdots \\ e^{\lambda_m t} = a_0(t) + a_1(t)\lambda_m + a_2(t)\lambda_m^2 + ... + a_{n-2}(t)\lambda_m^{n-2} + a_{n-1}(t)\lambda_m^{n-1} \end{cases} \tag{3.2.22}$$

where $m < n$.To get more equations to get a_i,we do differential on both sides of $e^{\lambda_i t}$ to λ_i until $(n-1)$ times.For example,we have a n-fold eigenvalue λ_k:

$$\begin{cases} e^{\lambda_k t} = a_0(t) + a_1(t)\lambda_k + a_2(t)\lambda_k^2 + ... + a_{n-2}(t)\lambda_k^{n-2} + a_{n-1}(t)\lambda_k^{n-1} \\ \frac{\lambda_k}{1!}e^{\lambda_k t} = a_1(t) + 2a_2(t)\lambda_k + ... + (n-1)a_{n-1}(t)\lambda_k^{n-2} \\ \frac{\lambda_k^2}{2!}e^{\lambda_k t} = a_2(t) + 3a_3(t)\lambda_k + ... + \frac{(n-1)(n-2)}{2!}a_{n-1}(t)\lambda_k^{n-3} \\ \quad\vdots \\ \frac{\lambda_k^{n-1}}{(n-1)!}e^{\lambda_k t} = a_{n-1}(t) \end{cases} \tag{3.2.23}$$

Thus we have enough equations to get $a_i(t)$ for example.

$$\begin{cases} e^{\lambda_1 t} = a_0(t) + a_1(t)\lambda_1 + a_2(t)\lambda_1^2 + ... + a_{n-2}(t)\lambda_1^{n-2} + a_{n-1}(t)\lambda_1^{n-1} \\ \frac{\lambda_1}{1!}e^{\lambda_1 t} = a_1(t) + 2a_2(t)\lambda_1 + ... + (n-1)a_{n-1}(t)\lambda_1^{n-2} \\ \frac{\lambda_1^2}{2!}e^{\lambda_1 t} = a_2(t) + 3a_3(t)\lambda_1 + ... + \frac{(n-1)(n-2)}{2!}a_{n-1}(t)\lambda_1^{n-3} \\ e^{\lambda_2 t} = a_0(t) + a_1(t)\lambda_2 + a_2(t)\lambda_2^2 + ... + a_{n-2}(t)\lambda_2^{n-2} + a_{n-1}(t)\lambda_2^{n-1} \\ \frac{\lambda_2}{1!}e^{\lambda_2 t} = a_1(t) + 2a_2(t)\lambda_2 + ... + (n-1)a_{n-1}(t)\lambda_2^{n-2} \\ e^{\lambda_3 t} = a_0(t) + a_1(t)\lambda_3 + a_2(t)\lambda_3^2 + ... + a_{n-2}(t)\lambda_3^{n-2} + a_{n-1}(t)\lambda_3^{n-1} \\ \quad\vdots \\ e^{\lambda_m t} = a_0(t) + a_1(t)\lambda_m + a_2(t)\lambda_m^2 + ... + a_{n-2}(t)\lambda_m^{n-2} + a_{n-1}(t)\lambda_m^{n-1} \end{cases} \tag{3.2.24}$$

We can write it as matrix form for example:

$$\begin{bmatrix} a_0(t) \\ a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 3\lambda_1 & ... & \frac{(n-1)(n-2)}{2!}\lambda_1^{n-3} \\ 0 & 1 & 2\lambda_1 & 3\lambda^2 & ... & \frac{(n-1)}{1!}\lambda_1^{n-2} \\ 1 & \lambda_1 & \lambda_1^2 & \lambda^3 & ... & \lambda_1^{n-1} \\ 0 & 1 & 2\lambda_2 & 3\lambda_2^2 & ... & \frac{(n-1)}{1!}\lambda_2^{n-2} \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & ... & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 & ... & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \lambda_m^3 & ... & \lambda_m^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2!}t^2e^{\lambda_1 t} \\ \frac{1}{1!}te^{\lambda_1 t} \\ e^{\lambda_1 t} \\ \frac{1}{1!}te^{\lambda_2 t} \\ e^{\lambda_2 t} \\ e^{\lambda_3 t} \\ \vdots \\ e^{\lambda_m t} \end{bmatrix} \tag{3.2.25}$$

(3) **Liner transformation method**

If matrix $\boldsymbol{A}$ can be transformed to a diagonal matrix($\Lambda = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}$),in other words,all eigenvalues are different,we can get $e^{\boldsymbol{A}t}$ easily by the following equation:

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2t^2 + \frac{1}{3!}\boldsymbol{A}^3t^3 + ... \\ &= \boldsymbol{I} + \boldsymbol{P}^{-1}\Lambda\boldsymbol{P}t + \frac{1}{2!}\boldsymbol{P}^{-1}\Lambda\boldsymbol{P}\boldsymbol{P}^{-1}\Lambda\boldsymbol{P}t^2 + \frac{1}{3!}\boldsymbol{P}^{-1}\Lambda\boldsymbol{P}\boldsymbol{P}^{-1}\Lambda\boldsymbol{P}\boldsymbol{P}^{-1}\Lambda\boldsymbol{P}t^3 + ... \\ &= \boldsymbol{I} + \boldsymbol{P}^{-1}\Lambda\boldsymbol{P}t + \frac{1}{2!}\boldsymbol{P}^{-1}\Lambda^2\boldsymbol{P}t^2 + \frac{1}{3!}\boldsymbol{P}^{-1}\Lambda^3\boldsymbol{P}t^3 + ... \\ &= \boldsymbol{P}^{-1}\left(\boldsymbol{I} + \boldsymbol{\Lambda}t + \frac{1}{2!}\boldsymbol{\Lambda}^2t^2 + \frac{1}{3!}\boldsymbol{\Lambda}^3t^3 + ...\right)\boldsymbol{P} \\ &= \boldsymbol{P}^{-1}e^{\Lambda t}\boldsymbol{P} \end{aligned} \tag{3.2.26}$$

If matrix $\boldsymbol{A}$ can't be transformed to a diagonal matrix,in other words,some eigenvalues are the same,we can use Jordan form matrix to get $e^{\boldsymbol{A}t}$.We have $\boldsymbol{J} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}$ where $\boldsymbol{J}$ is a Jordan form matrix.

We have

$$e^{\boldsymbol{A}t} = \boldsymbol{P}^{-1}e^{\boldsymbol{J}t}\boldsymbol{P} \tag{3.2.27}$$

If matrix $\boldsymbol{A}$ have complex eigenvalues,we can use the following equation to get $e^{\boldsymbol{A}t}$.The eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$.We have:

$$\boldsymbol{M} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \tag{3.2.28}$$

And

$$e^{\boldsymbol{M}t} = e^{\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}t} e^{\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}t} \tag{3.2.29}$$

where

$$\begin{aligned} e^{\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}t} &= \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha t} \end{bmatrix} \\ e^{\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}t} &= \begin{bmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{bmatrix} \end{aligned} \tag{3.2.30}$$

Thus we have:

$$e^{\boldsymbol{A}t} = \boldsymbol{P}^{-1}e^{\boldsymbol{M}t}\boldsymbol{P} \tag{3.2.31}$$

3.3 Solution of Liner Time Invariant Nonhomogeneous State Equations

Given a liner time invariant nonhomogeneous state equation:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{3.3.1}$$

We can conduct the following steps to get the solution:

$$\begin{gathered}\dot{\boldsymbol{x}}(t)-\boldsymbol{A}\boldsymbol{x}(t)=\boldsymbol{B}\boldsymbol{u}(t)\\ e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t)-e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t)=e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\big)=e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)\end{gathered} \tag{3.3.2}$$

Thus we do intergration on both sides from 0 to t:

$$e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\ |_0^t=\int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau \tag{3.3.3}$$

So we have:

$$\boldsymbol{x}(t)=e^{\boldsymbol{A}t}\boldsymbol{x}(0)+\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau \tag{3.3.4}$$

Or we can write it as:

$$\boldsymbol{x}(t)=\Phi(t)\boldsymbol{x}(0)+\int_0^t \Phi(t-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau \tag{3.3.5}$$

And it is easy to see output is:

$$\begin{aligned}\boldsymbol{y}(t)&=\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{D}\boldsymbol{u}(t)\\ &=\boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0)+\boldsymbol{C}\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau+\boldsymbol{D}\boldsymbol{u}(t)\end{aligned} \tag{3.3.6}$$

4 Controllability and Observability of Linear Time Invariant Systems

4.1 Definition of Controllability of Liner Time Invariant Systems

Controllability is a property to determine whether a system can be driven from any initial state to any desired final state in finite time by applying a finite control input.

Given a linear time-invariant state equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{4.1.1}$$

Given any initial state $\boldsymbol{x}(t_0)$,if there exists a finite time t_1 and a admissible control input $\boldsymbol{u}(t)$ such that the system can be driven from $\boldsymbol{x}(t_0)$ to zero($\boldsymbol{x}(t_1) = \boldsymbol{0}$),then the system is said to be controllable.

Given any initial state $\boldsymbol{x}(t_0)$,if there exists a finite time t_1 and a admissible control input $\boldsymbol{u}(t)$ such that the system can be driven from $\boldsymbol{x}(t_0)$ to any state $\boldsymbol{x}(t_1)$,then the system is said to be reachable.

In linear time-invariant systems,controllability and reachability are equivalent.

Proof: Let us prove the necessity($\Rightarrow$) of the above theorem. Given any end state $\boldsymbol{x}(t_1)$,we do linear transformation as below:

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \boldsymbol{x}(t_1) \tag{4.1.2}$$

Équation (4.1.1) would be transformed as:

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}(\tilde{\boldsymbol{x}}(t) + \boldsymbol{x}(t_1)) + \boldsymbol{B}\boldsymbol{u}(t) \tag{4.1.3}$$

We can solve this equation:

$$\begin{aligned}\tilde{\boldsymbol{x}}(t) &= e^{\boldsymbol{A}t}\tilde{\boldsymbol{x}}(t_0) + e^{\boldsymbol{A}t}\int_{t_0}^{t} e^{-\boldsymbol{A}\tau}\boldsymbol{A}\boldsymbol{x}(t_1)\,\mathrm{d}\tau + e^{\boldsymbol{A}t}\int_{t_0}^{t} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau \\ &= e^{\boldsymbol{A}t}\left(\tilde{\boldsymbol{x}}(t_0) + \int_{t_0}^{t} e^{-\boldsymbol{A}\tau}\boldsymbol{A}\boldsymbol{x}(t_1)\,\mathrm{d}\tau\right) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau\end{aligned} \tag{4.1.4}$$

Because given any initial state $\boldsymbol{x}(t_0)$ there exists a finite time t_1 and a admissible control input $u(t)$ such that the system can be driven from $\boldsymbol{x}(t_0)$ to zero($\boldsymbol{x}(t_1) = 0$),we have:

$$\begin{gathered}\text{let } \boldsymbol{x}(t_0) = \tilde{\boldsymbol{x}}(t_0) + \int_{t_0}^{t} e^{-\boldsymbol{A}\tau}\boldsymbol{A}\boldsymbol{x}(t_1)\,\mathrm{d}\tau, \exists t, \boldsymbol{u}, \text{such that} \\ \tilde{\boldsymbol{x}}(t) = e^{\boldsymbol{A}t}\left(\tilde{\boldsymbol{x}}(t_0) + \int_{t_0}^{t} e^{-\boldsymbol{A}\tau}\boldsymbol{A}\boldsymbol{x}(t_1)\,\mathrm{d}\tau\right) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = 0\end{gathered} \tag{4.1.5}$$

Sufficiency($\Leftarrow$) is obvious.

If the system have determined disturbance($\boldsymbol{f}(t)$) and the disturbance is independent of $\boldsymbol{u}$,the disturbance will not affect the controllability of the system.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{f}(t) \tag{4.1.6}$$

It is easy to prove similarly as above.

4.2 Criteria of Controllability of Liner Time Invariant Systems

1. System like $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is controllable if and only if the $n \times n$ gram matrix is full rank.

$$\boldsymbol{W}_c(0, t_1) = \int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{B}^T e^{-\boldsymbol{A}^T\tau}\,\mathrm{d}\tau \tag{4.2.1}$$

Proof:Let us prove the sufficiency ($\Leftarrow$).We have the following equation:

$$\begin{aligned}\boldsymbol{x}(t_1) &= e^{\boldsymbol{A}t_1}\boldsymbol{x}(0) + \int_0^{t_1} e^{\boldsymbol{A}(t_1-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau \\ &= e^{\boldsymbol{A}t_1}\left(\boldsymbol{x}(0) + \int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau\right)\end{aligned} \tag{4.2.2}$$

To let the system be controllable,we let

$$\boldsymbol{x}(0) + \int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = 0 \tag{4.2.3}$$

Because the gram matrix is full rank,we let

$$\boldsymbol{u}(\tau) = -\boldsymbol{B}^T e^{-\boldsymbol{A}^T\tau}\boldsymbol{W}_c^{-1}(0, t_1)\boldsymbol{x}(0) \tag{4.2.4}$$

Thus we have $\boldsymbol{x}(t_1) = 0$ and the system is controllable.

Let us prove the necessity($\Rightarrow$).We prove it by contradiction.Suppose the gram matrix is not full rank,then there exists a constant non-zero vector $\boldsymbol{a}$ such that $\boldsymbol{a}\boldsymbol{W}_c(0, t_1)\boldsymbol{a}^T = \boldsymbol{0}$

Thus we have:

$$\begin{gathered}\boldsymbol{a}\boldsymbol{W}_c(0, t_1)\boldsymbol{a}^T = \int_0^{t_1} \boldsymbol{a}e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{B}^T e^{-\boldsymbol{A}^T\tau}\boldsymbol{a}^T\,\mathrm{d}\tau = \boldsymbol{0} \\ \boldsymbol{a}e^{-\boldsymbol{A}\tau}\boldsymbol{B} = \boldsymbol{0}\end{gathered} \tag{4.2.5}$$

Let the initial state be $\boldsymbol{x}(0) = \boldsymbol{a}^T$,then we have:

$$\begin{aligned} \boldsymbol{x}(t_1) &= e^{\boldsymbol{A}t_1}\boldsymbol{a}^T + \int_0^{t_1} e^{\boldsymbol{A}(t_1-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau \\ &= e^{\boldsymbol{A}t_1}\left(\boldsymbol{a}^T + \int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau\right) = \boldsymbol{0} \\ &\text{We have } \boldsymbol{a}^T + \int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = \boldsymbol{0} \end{aligned} \tag{4.2.6}$$

We multiply $\boldsymbol{a}$ on both sides:

$$\begin{gathered} \boldsymbol{a}\boldsymbol{a}^T + \int_0^{t_1} \boldsymbol{a}e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = \boldsymbol{0} \\ \text{We have } \boldsymbol{a}\boldsymbol{a}^T = \boldsymbol{0} \end{gathered} \tag{4.2.7}$$

This is a contradiction,thus the system is controllable.

2. System like $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is controllable if and only if the controllability matrix below is full rank.

$$\boldsymbol{Q}_c = [\boldsymbol{B}\ \ \boldsymbol{A}\boldsymbol{B}\ \ \boldsymbol{A}^2\boldsymbol{B}\ \ ...\ \ \boldsymbol{A}^{n-1}\boldsymbol{B}] \tag{4.2.8}$$

Proof:Let us prove the sufficiency($\Leftarrow$).We prove it by contradiction.Suppose the system is not controllable in other words the gram matrix is not full rank,then there exists a constant non-zero vector $\boldsymbol{a}$ such that $\boldsymbol{a}\boldsymbol{W}_c(0,t_1)\boldsymbol{a}^T = \boldsymbol{0}$.We have:

$$\boldsymbol{a}e^{-\boldsymbol{A}\tau}\boldsymbol{B} = \boldsymbol{0}, \tau \in [0, t_1] \tag{4.2.9}$$

Because $\forall\tau \in [0, t_1]$ the equation above holds,we have:

$$\boldsymbol{a}\boldsymbol{B} = \boldsymbol{0} \tag{4.2.10}$$

And do differentiation on both sides to τ,we have:

$$\begin{cases} \boldsymbol{a}\boldsymbol{B} = \boldsymbol{0} \\ \boldsymbol{a}\boldsymbol{A}\boldsymbol{B} = \boldsymbol{0} \\ \boldsymbol{a}\boldsymbol{A}^2\boldsymbol{B} = \boldsymbol{0} \\ \quad\vdots \\ \boldsymbol{a}\boldsymbol{A}^{n-1}\boldsymbol{B} = \boldsymbol{0} \end{cases} \tag{4.2.11}$$

then we have $\boldsymbol{Q_c}$ is not full rank,which is a contradiction.(Because we have non-zero solution of homogeneous equations).to be continued...

Proof:Let $\boldsymbol{x}(t_1) = 0$,we have:

$$\boldsymbol{x}(t_1) = e^{\boldsymbol{A}t_1}\boldsymbol{x}(0) + \int_0^{t_1} e^{\boldsymbol{A}(t_1-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau = 0 \tag{4.2.12}$$

Thus we have:

$$\boldsymbol{x}(0) = -\int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau \tag{4.2.13}$$

We use Cayley-Hamilton theorem to get:

$$e^{-\boldsymbol{A}\tau} = \sum_{i=0}^{n-1} a_i(\tau)\boldsymbol{A}^i \tag{4.2.14}$$

Thus we have:

$$\begin{aligned}\boldsymbol{x}(0) &= -\sum_{i=0}^{n-1}\boldsymbol{A}^i\boldsymbol{B}\int_0^{t_1} a_i(\tau)\boldsymbol{u}(\tau)\,\mathrm{d}\tau \\ &= [\boldsymbol{B}\ \ \boldsymbol{A}\boldsymbol{B}\ \ ...\ \ \boldsymbol{A}^{n-1}\boldsymbol{B}]\begin{bmatrix} -\int_0^{t_1} a_0(\tau)\boldsymbol{u}(\tau)\,\mathrm{d}\tau \\ -\int_0^{t_1} a_1(\tau)\boldsymbol{u}(\tau)\,\mathrm{d}\tau \\ \vdots \\ -\int_0^{t_1} a_{n-1}(\tau)\boldsymbol{u}(\tau)\,\mathrm{d}\tau \end{bmatrix}\end{aligned} \tag{4.2.15}$$

From the equation above,we can see that only when $\boldsymbol{Q}_c$ is full rank,we can get the input $\boldsymbol{u}(\tau)$.

3. PBH Criterion.A liner time-invariant system is controllable is equivalent to for all eigenvalues of the system,we have:

$$\forall\lambda_i, \operatorname{rank}([\lambda_i\boldsymbol{I} - \boldsymbol{A}\ \ \boldsymbol{B}]) = n \tag{4.2.16}$$

Proof:Let us prove the necessity($\Rightarrow$) first.Prove it by contradiction.Suppose $\exists\lambda_i, \operatorname{rank}([\lambda_i\boldsymbol{I}-\boldsymbol{A}\ \ \boldsymbol{B}]) < n$.Then we have:

$$\exists\boldsymbol{a} \neq \boldsymbol{0}, \boldsymbol{a}[\lambda_i\boldsymbol{I} - \boldsymbol{A}\ \ \boldsymbol{B}] = \boldsymbol{0} \tag{4.2.17}$$

We can get:

$$\boldsymbol{a}\boldsymbol{A} = \lambda_i\boldsymbol{a}, \boldsymbol{a}\boldsymbol{B} = \boldsymbol{0} \tag{4.2.18}$$

Thus $\boldsymbol{a}\boldsymbol{A}^m\boldsymbol{B} = \lambda_i^m\boldsymbol{a}\boldsymbol{B} = \boldsymbol{0}$

$$\boldsymbol{a}[\boldsymbol{B}\ \ \boldsymbol{A}\boldsymbol{B}\ \ ...\ \ \boldsymbol{A}^{n-1}\boldsymbol{B}] = \boldsymbol{0} \tag{4.2.19}$$

So the matrix $\boldsymbol{Q}_c$ is not full rank,which is a contradiction.