1. Given differential equations of system,try to write down their state space representation.

   (1) $\dddot{y} + \ddot{y} + 4\dot{y} + 5y = 3u$

   (2) $2\dddot{y} + 3\dot{y} = \ddot{u} - u$

   Solution:(1)let us choose $x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y}$ as state variables.We have state equations:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = x_3 \\ \dot{x_3} = -5x_1 - 3x_2 - 2x_3 + 7u \end{cases}$$

   And output equation:

$$y = x_1$$

   (2)Let $2\tilde{y}^{(3)} + 3\tilde{y}^{(1)} = u$ We have:

$$y = \tilde{y}^{(2)} - \tilde{y}^{(1)}$$

   Let us choose $x_1 = \tilde{y}, x_2 = \tilde{y}^{(1)}, x_3 = \tilde{y}^{(2)}$ We have state equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\frac{3}{2}x_2 + \frac{1}{2}u \end{cases}$$

   And output equation:

$$y = x_3 - x_2$$

2. Given transfer function of system.try to establish its state space representation.

   (1)$g(s) = \frac{s^3+s+1}{s^3+6s^2+11s+6}$

   Solution:

$$g(s) = \frac{s^3 + s + 1}{s^3 + 6s^2 + 11s + 6} = 1 + \frac{-6s^2 - 10s - 5}{s^3 + 6s^2 + 11s + 6}$$

   Let $h(s) = \frac{-6s^2-10s-5}{s^3+6s^2+11s+6}$ Introduce intermediate variable $Z(s)$

$$h(s) = \frac{Y(s)}{Z(s)}\frac{Z(s)}{U(s)} = \frac{-6s^2 - 10s - 5}{1}\frac{1}{s^3 + 6s^2 + 11s + 6}$$

   Thus we have:

$$\begin{cases} y = -6z^{(2)} - 10z^{(1)} - 5z \\ z^{(3)} + 6z^{(2)} + 11z^{(1)} + 6z = u \end{cases}$$

   Let us choose $x_1 = z, x_2 = z^{(1)}, x_3 = z^{(2)}$ We have state equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -6x_3 - 11x_2 - 6x_1 + u \end{cases}$$

   And output equation:

$$y = -6x_3 - 10x_2 - 5x_1 + u$$

3. Try to transform the state matrix into diagonal canonical form.

(1) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Solution: Let $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Let us find the eigenvalues vectors of $\boldsymbol{A}$

$$|\lambda \boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda & -1 \\ 5 & \lambda + 6 \end{vmatrix} = \lambda^2 + 6\lambda + 5 = 0$$
$$\lambda_1 = -1, \lambda_2 = -5$$

Let

$$(-\boldsymbol{I} - \boldsymbol{A})\boldsymbol{v_1} = \begin{bmatrix} -1 & -1 \\ 5 & 5 \end{bmatrix} \boldsymbol{v_1} = 0$$
$$(-5\boldsymbol{I} - \boldsymbol{A})\boldsymbol{v_2} = \begin{bmatrix} -5 & -1 \\ 5 & 1 \end{bmatrix} \boldsymbol{v_2} = 0$$

Take the basic solution$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

Thus we have transformation matrix

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix} \text{ and } \boldsymbol{P}^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

So we have:

$$\bar{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}$$
$$\bar{\boldsymbol{B}} = \boldsymbol{P}^{-1}\boldsymbol{B} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

Thus the diagonal canonical form is:

$$\dot{\bar{\boldsymbol{x}}} = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \bar{\boldsymbol{x}} + \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} u$$

4. Try to transform the state matrix into Jordan canonical form.

(1) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Solution:Let $\boldsymbol{A} = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 3 & 1 \\ 2 & 7 \\ 5 & 3 \end{bmatrix}$

Let us find the eigenvalues vectors of $\boldsymbol{A}$

$$|\lambda \boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda - 4 & -1 & 2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda - 3 \end{vmatrix} = \lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$
$$\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 3$$

Let

$$|\boldsymbol{I}-\boldsymbol{A}|\ \boldsymbol{v} = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix} \boldsymbol{v} = 0$$

$$\boldsymbol{v_1} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$|3\boldsymbol{I}-\boldsymbol{A}|\ \boldsymbol{v_1} = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix} \boldsymbol{v} = 0$$

$$\boldsymbol{v_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let

$$|3\boldsymbol{I}-\boldsymbol{A}|\ \boldsymbol{v_3} = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix} \boldsymbol{v_3} = -\boldsymbol{v_2} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\boldsymbol{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus we have transformation matrix:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \boldsymbol{P}^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 2 \\ 1 & 1 & -2 \end{bmatrix}$$

So we have:

$$\bar{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\bar{\boldsymbol{B}} = \boldsymbol{P}^{-1}\boldsymbol{B} = \begin{bmatrix} -3 & 4 \\ 8 & -1 \\ 15 & 14 \end{bmatrix}$$

Thus the Jordan canonical form is:

$$\dot{\bar{\boldsymbol{x}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \bar{\boldsymbol{x}} + \begin{bmatrix} -3 & 4 \\ 8 & -1 \\ 15 & 14 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

5. Let the forward path transfer function matrix and the feedback path transfer function matrix of the system be matrix below. Find the closed-loop transfer function matrix.

$$\boldsymbol{G} = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{bmatrix}, \boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution:

$$
\begin{aligned}
\boldsymbol{G_H}(s) = (\boldsymbol{I} + \boldsymbol{G}(s)\boldsymbol{H}(s))^{-1}\boldsymbol{G}(s) &= \begin{bmatrix} \frac{s+2}{s+1} & -\frac{1}{s} \\ 2 & \frac{s+3}{s+2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{s(s+1)(s+3)}{(s+2)(s^2+5s+2)} & \frac{s+1}{s^2+5s+2} \\ \frac{-2s(s+1)}{s^2+5s+2} & \frac{s(s+2)}{s^2+5s+2} \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3s^2+9s+4}{s^2+5s+2} & \frac{-s-1}{s^2+5s+2} \\ \frac{2s(s+1)}{s^2+5s+2} & \frac{3s+2}{s^2+5s+2} \end{bmatrix}
\end{aligned}
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