1 Description of State Space

1.1 Definition

1. **Input variables**

We usually use $\boldsymbol{u}_t = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n(t)} \end{bmatrix}$ to represent input variables.

2. **State variables**

We usually use $\boldsymbol{x}_t = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n(t)} \end{bmatrix}$ to represent state variables.It is a least set to describe state of system.

3. **Output variables**

We usually use $\boldsymbol{y}_t = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n(t)} \end{bmatrix}$ to represent output variables.

4. **State equation**

State equation is a first order differential equation that describe relationship between input variables and state variables. We can write it as:

$$\begin{cases} \dot{x}_1 &= f_1\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \\ \dot{x}_2 &= f_2\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \\ &\vdots \\ \dot{x}_n &= f_n\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \end{cases} \tag{1.1.1}$$

Rewrite it as vector form:

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{1.1.2}$$

5. **Output equation**

Output equation is a equation that describe relationship between state variables and output variables. We can write it as:

$$\begin{cases} y_1 &= g_1\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \\ y_2 &= g_2\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \\ &\vdots \\ y_n &= g_n\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \end{cases} \tag{1.1.3}$$

Rewrite it as vector form:

$$\boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{1.1.4}$$

6. **Description of State space of System**

We can describe state space of system by equations as:

$$\begin{cases} \dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \\ \boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \end{cases} \tag{1.1.5}$$

When the system is linear, we can write it as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}(t)\boldsymbol{x} + \boldsymbol{D}(t)\boldsymbol{u} \end{cases} \tag{1.1.6}$$

1.2 Transfer function

Transfer function is a function that describe relationship between input and output of system.Given a system with different state,the transfer function is still the same which means it is not related to state of system in other words state variables.

Single input – Single output system

Given a linear single input-single output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \\ y = \boldsymbol{C}\boldsymbol{x} + Du \end{cases} \tag{1.2.1}$$

To get transfer function,we can use Laplace transform to get:

$$s\boldsymbol{X} - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}U(s)$$
$$Y(s) = \boldsymbol{C}\boldsymbol{X}(s) + DU(s)$$

Laplace transfer:

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

$$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{\{n-1\}}f(0) - s^{\{n-2\}}f^{\{(1)\}}(0) - \ldots - f^{\{(n-1)\}}(0)$$

$$\mathcal{L}\left[\int_0^t \mathrm{d}t \int_0^t \mathrm{d}t \ldots \int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s^n}\mathcal{L}[f(t)]$$

To be continued...

The equations are organized as follows:

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)]$$
$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)] + DU(s)$$

Let initial condition be zero($\boldsymbol{x}(0) = 0$),we can get:

$$Y(s) = \left[\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D\right]U(s)$$

Thus,we can get transfer function as:

$$g(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D \tag{1.2.2}$$

Let $D = 0$,we can get:

$$g(s) = \frac{\boldsymbol{C}\ \mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B}}{\det(s\boldsymbol{I} - \boldsymbol{A})} \tag{1.2.3}$$

Multi input – Multi output system

Given a multi input-multi output system,we define transfer function between i-th out y_i and j-th input u_j as:

$$g_{ij}(s) = \frac{Y_i(s)}{U_j(s)} \tag{1.2.4}$$

where $Y_i(s)$ is Laplace transform of $y_i(t)$ and $U_j(s)$ is Laplace transform of $u_j(t)$. Must mention that if we define transfer function in this way,we assume that all other inputs are zero.Because linear system satisfies the principle of superposition,so when we plus all inputs $U_1, U_2, ..., U_p$,we can get the i-th output Y_i as:

$$Y_i = \sum_{j=1}^{p} g_{ij} U_j \tag{1.2.5}$$

We can write it as matrix form:

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\boldsymbol{U}(s) \tag{1.2.6}$$

Thus given a linear multi input-multi output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{cases} \tag{1.2.7}$$

We can conduct as before to get transfer function as:

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \frac{\boldsymbol{C} \ \mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \boldsymbol{D} \ \det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} \tag{1.2.8}$$