1. Determine the observability of the following systems:

(1) $\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{x}, y = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}$

(2) $\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \boldsymbol{x}, y = \begin{bmatrix} -1 & 3 & 0 \end{bmatrix} \boldsymbol{x}$

(3) $\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}, y = \begin{bmatrix} 1 & 1 & 4 \end{bmatrix} \boldsymbol{x}$

Solution:(1)We use Kalman's observability criterion to determine the observability of the system.

$$\boldsymbol{Q}_o = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$
$$\operatorname{rank}(\boldsymbol{Q}_o) = 2 = n$$

Thus the system is observable.

(2)We use Kalman's observability criterion to determine the observability of the system.

$$\boldsymbol{Q}_o = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \boldsymbol{CA^2} \end{bmatrix} = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 56 & 45 \\ 0 & -5 & -4 \end{bmatrix}$$
$$\operatorname{rank}(\boldsymbol{Q}_o) = 3 = n$$

Thus the system is observable.

(3)The matrix $\boldsymbol{A}$ is in diagonal form.Thus if matrix $\boldsymbol{C}$ does not have a zero column,then the system is observable.By this criterion the system is observable.

2. Try to determine for which values of p, q the following systems are not controllable and for which values they are not observable.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 12 \\ 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} p \\ -1 \end{bmatrix} u$$
$$y = \begin{bmatrix} q & 1 \end{bmatrix} \boldsymbol{x}$$

Solution:By Kalman's criterion,we have:

$$\boldsymbol{Q}_c = [\boldsymbol{B} \ \ \boldsymbol{AB}] = \begin{bmatrix} p & p-12 \\ -1 & p \end{bmatrix}$$
$$\boldsymbol{Q}_o = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix} = \begin{bmatrix} q & 1 \\ q+1 & 12q \end{bmatrix}$$
$$\text{let } \det(\boldsymbol{Q}_c) = p^2 + p - 12 = 0, p = -4, 3$$
$$\text{let } \det(\boldsymbol{Q}_o) = 12q^2 - q - 1 = 0, q = -\frac{1}{4}, \frac{1}{3}$$

Thus the system is not controllable for $p = -4, 3$ and not observable for $q = -\frac{1}{4}, \frac{1}{3}$.

3. Try to prove the following system is not controllable in any condition of a, b, c.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 20 & -1 & 0 \\ 4 & 16 & 0 \\ 12 & 0 & 18 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} u$$

Solution:We calculate the eigenvalues of the matrix $\boldsymbol{A}$.

$$\text{let } \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = -(\lambda - 18)^3 = 0$$
$$\lambda_{1,2,3} = 18$$

We apply jordan transformation to system. We have:

$$\text{Let } \bar{\boldsymbol{A}} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1} = \begin{bmatrix} 18 & 1 & 0 \\ 0 & 18 & 1 \\ 0 & 0 & 18 \end{bmatrix}$$

$$\boldsymbol{P} = \begin{bmatrix} 0 & -\frac{1}{8} & \frac{1}{24} \\ 0 & \frac{1}{4} & 0 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

$$\bar{\boldsymbol{B}} = \boldsymbol{P}\boldsymbol{B} = \begin{bmatrix} -\frac{1}{8}b + \frac{1}{24}c \\ \frac{1}{4}b \\ a - \frac{1}{2}b \end{bmatrix}$$

$$\bar{\boldsymbol{x}} = \boldsymbol{P}\boldsymbol{x}$$

Thus we have:

$$\dot{\bar{\boldsymbol{x}}} = \begin{bmatrix} 18 & 1 & 0 \\ 0 & 18 & 1 \\ 0 & 0 & 18 \end{bmatrix} \bar{\boldsymbol{x}} + \begin{bmatrix} -\frac{1}{8}b + \frac{1}{24}c \\ \frac{1}{4}b \\ a - \frac{1}{2}b \end{bmatrix} u$$

Thus if $a \neq \frac{1}{2}b$ then the system is controllable.

4. Given a linear time-invariant system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$.If $\boldsymbol{x}_a, \boldsymbol{x}_b$ is controllable states of the system,try to prove that $\alpha\boldsymbol{x}_a + \beta\boldsymbol{x}_b$ state is also a controllable state of the system.

   Solution:$\boldsymbol{x}_a, \boldsymbol{x}_b$ is controllable states of the system means

$$\exists t_1, t_2, u_1, u_2 \text{ such that } \boldsymbol{x}(t_{1,2}) = e^{\boldsymbol{A}t_{1,2}}\boldsymbol{x}_{a,b} + \int_0^{t_{1,2}} e^{\boldsymbol{A}(t_{1,2}-\tau)}\boldsymbol{B}u_{1,2}(\tau)\,\mathrm{d}\tau = 0$$

   W.L.O.G we assume $t_1 < t_2$.

$$\text{Let } u_3(t) = \begin{cases} \alpha u_1(t) + \beta u_2(t) \text{ if } 0 \le t \le t_1 \\ \beta u_2(t) \text{ if } t_1 < t \le t_2 \end{cases}$$

$$\text{And we have: } \boldsymbol{x}_\alpha = -\int_0^{t_1} e^{\boldsymbol{A}\tau}\boldsymbol{B}u_1(\tau)\,\mathrm{d}(\tau)$$

$$\boldsymbol{x}_\beta = -\int_0^{t_2} e^{\boldsymbol{A}\tau}\boldsymbol{B}u_2(\tau)\,\mathrm{d}(\tau)$$

$$\text{Thus } \boldsymbol{x}(t) = e^{\boldsymbol{A}t}(\alpha\boldsymbol{x}_a + \beta\boldsymbol{x}_b) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u_3(\tau)\,\mathrm{d}\tau$$

$$= e^{\boldsymbol{A}t}\left(\int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}u_3(\tau)\,\mathrm{d}\tau - \int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\alpha u_1(\tau)\,\mathrm{d}\tau - \int_0^{t_2} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\beta u_2(\tau)\,\mathrm{d}\tau\right)$$

$$\text{Let } t = t_2, \boldsymbol{x}(t_2) = 0$$

Thus the state $\alpha \boldsymbol{x}_a + \beta \boldsymbol{x}_b$ is also a controllable state of the system.

5. Transform the following system to controllable canonical form.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Solution:

$$\boldsymbol{Q}_c = [\boldsymbol{B} \ \ \boldsymbol{AB}] = \begin{bmatrix} 1 & -1 \\ 1 & 7 \end{bmatrix}$$

$$\mathrm{rank}(\boldsymbol{Q}_c) = 2 = n$$

$$\text{Let } \boldsymbol{P} = [\boldsymbol{B} \ \ \boldsymbol{AB}]^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

$$\bar{\boldsymbol{x}} = \boldsymbol{P}\boldsymbol{x}$$

$$\bar{\boldsymbol{A}} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1} = \begin{bmatrix} 0 & -10 \\ 1 & 5 \end{bmatrix}$$

$$\bar{\boldsymbol{B}} = \boldsymbol{P}\boldsymbol{B} = [1 \ \ 0]$$

$$\text{Controllable canonical form } \dot{\bar{\boldsymbol{x}}} = \begin{bmatrix} 0 & -10 \\ 1 & 5 \end{bmatrix} \bar{\boldsymbol{x}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$