1. Given a system below:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & -2 & -3 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u}$$

$$y = [1 \ \ 1 \ \ 0]\boldsymbol{x}$$

Design a full-order state observer with eigenvalues −3, −4, −5.

Solution:

$$\det(s\boldsymbol{I} - \boldsymbol{A}^T) = \begin{vmatrix} s+1 & 0 & -1 \\ 2 & s+1 & 0 \\ 3 & -1 & s+1 \end{vmatrix} = s^3 + 3s^2 + 6s + 6$$

$$a_0 = 6, a_1 = 6, a_2 = 3$$

$$\alpha^*(s) = (s+3)(s+4)(s+5) = s^3 + 12s^2 + 47s + 60$$

$$\tilde{\boldsymbol{E}} = [a_3]$$

$$a_0^* = 60, a_1^* = 47, a_2^* = 12$$

$$\tilde{\boldsymbol{E}} = [a_0^* - a_0 \ \ a_1^* - a_1 \ \ a_2^* - a_2] = [54 \ \ 41 \ \ 9]$$

$$\boldsymbol{Q} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -3 & 5 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 6 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 1 \\ -4 & -2 & 0 \end{bmatrix}$$

$$\boldsymbol{P} = \boldsymbol{Q}^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\boldsymbol{E}^T = \tilde{\boldsymbol{E}}^T P = \begin{bmatrix} \frac{23}{2} & -\frac{5}{2} & -9 \end{bmatrix}$$

2. Given a system below:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u$$

$$y = [1 \ \ 0 \ \ 0 \ \ 0]\boldsymbol{x}$$

- Design a state feedback gain $\boldsymbol{K}$ such that $\lambda_1^* = -1, \lambda_{2,3}^* = -1 \pm j, \lambda_4^* = -2$.
- Design a reduced-order state observer such that eigenvalues are $\lambda_1 = -3, \lambda_{2,3} = -3 \pm 2j$
- Determine the reconstructed state $\hat{\boldsymbol{x}}$ and the state feedback law composed of $\hat{\boldsymbol{x}}$.

Solution: (1)$\alpha^*(s) = (s+1)(s+1-j)(s+1+j)(s+2) = s^4 + 5s^3 + 10s^2 + 10s + 4$

Let $K = -\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$

$$\boldsymbol{A} - \boldsymbol{B}K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_1 & k_2 & k_3 - 2 & k_4 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & 4 - k_3 & -k_4 \end{bmatrix}$$

$$\alpha(s) = s^4 + (k_4 - k_2)s^3 + (k_3 - k_1 - 4)s^2 + 2k_2 s + 2k_1$$

Compare the coefficients of $\alpha(s)$ and $\alpha^*(s)$, we have:

$$K = -[2 \;\; 5 \;\; 16 \;\; 10]$$

(2)We note:

$$A_{11} = 0, A_{12} = [1 \;\; 0 \;\; 0], A_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

$$B_1 = 0, B_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\alpha^*(s) = (s+3)(s+3-2j)(s+3+2j) = s^3 + 9s^2 + 31s + 39$$

We let:$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$,then we have:

$$A_{22} - LA_{12} = \begin{bmatrix} -l_1 & -2 & 0 \\ -l_2 & 0 & 1 \\ -l_3 & 4 & 0 \end{bmatrix}$$

$$\alpha(s) = s^3 + l_1 s^2 - (2l_1 + 4)s - (2l_3 + 4l_1)$$

Compare the coefficients of $\alpha(s)$ and $\alpha^*(s)$, we have:

$$L = \begin{bmatrix} 9 \\ -\frac{35}{2} \\ -\frac{75}{2} \end{bmatrix}$$

By substituting L into system, we have:

$$\dot{\boldsymbol{z}} = \begin{bmatrix} -9 & -2 & 0 \\ \frac{35}{2} & 0 & 1 \\ \frac{75}{2} & 4 & 0 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} -46 \\ 120 \\ \frac{535}{2} \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \boldsymbol{u}$$

(3) Substituting equation above into system, we have:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ -\frac{35}{2} & 0 & 1 & 0 \\ -\frac{75}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\boldsymbol{u} = v + [2 \;\; 5 \;\; 16 \;\; 10]\dot{\boldsymbol{x}}$$

3. A particle moves along the curve $y = f(x)$ from point (0,8) to (4,0). Assuming the particle's speed is x, what shape should the curve take to minimize the travel time?

   Solution: We have:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{1 + (f'(x))^2} \frac{\mathrm{d}x}{\mathrm{d}t} = x$$

$$\mathrm{d}t = \frac{\sqrt{1 + (f'(x))^2}}{x} \mathrm{d}x$$

Thus we need to find a functional J such that minimize:

$$J(x) = \int_0^4 \frac{\sqrt{1 + (f'(x))^2}}{x} \, \mathrm{d}x$$

$$f(0) = 8, f(4) = 0$$

By using Euler equation, note $y = f(x)$ and we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial}{\partial y'}\left(\frac{\sqrt{1+y'^2}}{x}\right)\right)$$

We have:

$$\frac{xy''}{1+y'^2} = y'$$

Let $y' = \tan\theta$,$y'' = \frac{\mathrm{d}y'}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{\cos^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x}$.Substituting y' and y'' into equation above, we have:

$$x\frac{\mathrm{d}\theta}{\mathrm{d}x} = \tan\theta$$

Solving the equation above, we have:

$$\begin{cases} x = c_1 \sin\theta \\ y' = \tan\theta \end{cases}$$

Thus we have:

$$\mathrm{d}y = \tan\theta \, \mathrm{d}x = c_1 \sin\theta \, \mathrm{d}\theta$$

By integrating the equation above, we have:

$$\begin{cases} x = c_1 \sin\theta \\ y = -c_1 \cos\theta + c_2 \end{cases}$$

By substituting $f(0) = 8, f(4) = 0$, we have:

$$c_1 = 5, c_2 = 3$$

Thus the shape of curve should be:

$$x^2 + (y-3)^2 = 25$$

4. Assume a simplified control system of satellite is given by:

$$\dot{\boldsymbol{x}} = [0 \;\; 1 \;\; 0 \;\; 0]\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\boldsymbol{u}$$

Target functional is:

$$J = \frac{1}{2}\int_0^2 u^2(t) \, \mathrm{d}t$$

And boundary conditions are:

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{x}(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Try to find the optimal control $\boldsymbol{u}$ and curve $\boldsymbol{x}^*(t)$.

Solution: We have:

$$L = \frac{1}{2}u^2, \boldsymbol{\lambda}^T = [\lambda_1 \ \ \lambda_2], \boldsymbol{f} = [x_2 \ \ u]$$

$$H = L + \boldsymbol{\lambda}^T \boldsymbol{f} = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$$

$$G = H - \boldsymbol{\lambda}^T \dot{\boldsymbol{x}} = \frac{1}{2}u^2 + \lambda_1(x_2 - \dot{x}_1) + \lambda_2(u - \dot{x}_2)$$

Using Euler equation, we have:

$$\frac{\partial L}{\partial x_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_1} = \dot{\lambda}_1 = 0, \lambda_1 = a$$

$$\frac{\partial L}{\partial x_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_2} = \lambda_1 + \dot{\lambda}_2 = 0, \lambda_2 = -at + b$$

$$\frac{\partial L}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{u}} = u + \lambda_2 = 0, u = at - b$$

And we have:

$$\dot{x}_2 = at - b, x_2 = \frac{1}{2}at^2 - bt + c$$

$$\dot{x}_1 = x_2 = \frac{1}{2}at^2 - bt + c, x_1 = \frac{1}{6}at^3 - \frac{1}{2}bt^2 + ct + d$$

By substituting boundary conditions, we have:

$$a = 3, b = \frac{7}{2}, c = 1, d = 1$$

Thus the optimal curve and control are:

$$x_1^*(t) = \frac{1}{2}t^3 - \frac{7}{4}t^2 + t + 1$$

$$x_2^*(t) = \frac{3}{2}t^2 - \frac{7}{2}t + 1$$

$$u^*(t) = 3t - \frac{7}{2}$$

5. Given a system below:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, \frac{\mathrm{d}x_2}{\mathrm{d}t} = u(t)$$

Try to transfer the system from $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ within 2 seconds and minimize the functional:

$$J = \frac{1}{2}\int_0^2 u^2(t)\,\mathrm{d}t$$

Try to find the optimal control $\boldsymbol{u}$.

Solution: The system is given by:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u}$$

The Hamiltonian is given by:

$$\begin{aligned} H &= \frac{1}{2}u^2 + \boldsymbol{\lambda}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u} \right) \\ &= \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u \end{aligned}$$

Let $\frac{\partial H}{\partial u} = 0$, we have:

$$u = -\lambda_2$$

Using Hamiltonian companion equations, we have:

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = -\frac{\partial H}{\partial x} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix}$$

Thus we have:

$$\begin{cases} \lambda_1 = a \\ \lambda_2 = -at + b \\ u = at - b \\ x_1 = \frac{1}{6}at^3 - \frac{1}{2}bt^2 + ct + d \\ x_2 = \frac{1}{2}at^2 - bt + c \end{cases}$$

By substituting boundary conditions, we have:

$$a = 3, b = \frac{7}{2}, c = 1, d = 1$$

Thus the optimal control is:

$$u^*(t) = 3t - \frac{7}{2}$$

The optimal curve is:

$$\begin{aligned} x_1^*(t) &= \frac{1}{2}t^3 - \frac{7}{4}t^2 + t + 1 \\ x_2^*(t) &= \frac{3}{2}t^2 - \frac{7}{2}t + 1 \end{aligned}$$