2 Description of State Space

2.1 Definition

1. **Input variables**

We usually use $\boldsymbol{u}_t = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n(t)} \end{bmatrix}$ to represent input variables.

2. **State variables**

We usually use $\boldsymbol{x}_t = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n(t)} \end{bmatrix}$ to represent state variables.It is a least set to describe state of system.

3. **Output variables**

We usually use $\boldsymbol{y}_t = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n(t)} \end{bmatrix}$ to represent output variables.

4. **State equation**

State equation is a first order differential equation that describe relationship between input variables and state variables. We can write it as:

$$\begin{cases} \dot{x}_1 &= f_1\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \\ \dot{x}_2 &= f_2\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \\ \quad\vdots \\ \dot{x}_n &= f_n\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \end{cases} \tag{2.1.1}$$

Rewrite it as vector form:

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{2.1.2}$$

5. **Output equation**

Output equation is a equation that describe relationship between state variables and output variables. We can write it as:

$$\begin{cases} y_1 &= g_1\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \\ y_2 &= g_2\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \\ \quad\vdots \\ y_n &= g_n\left(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t\right) \end{cases} \tag{2.1.3}$$

Rewrite it as vector form:

$$\boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{2.1.4}$$

6. **Description of State space of System**

We can describe state space of system by equations as:

$$\begin{cases} \dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \\ \boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \end{cases} \tag{2.1.5}$$

When the system is linear, we can write it as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}(t)\boldsymbol{x} + \boldsymbol{D}(t)\boldsymbol{u} \end{cases} \quad (2.1.6)$$

2.2 Transfer function

Transfer function is a function that describe relationship between input and output of system.Given a system with different state,the transfer function is still the same which means it is not related to state of system in other words state variables.

Single input – Single output system

Given a linear single input-single output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \\ y = \boldsymbol{C}\boldsymbol{x} + Du \end{cases} \quad (2.2.1)$$

To get transfer function,we can use Laplace transform to get:

$$s\boldsymbol{X} - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}U(s)$$
$$Y(s) = \boldsymbol{C}\boldsymbol{X}(s) + DU(s)$$

Laplace transfer:

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

$$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{\{n-1\}}f(0) - s^{\{n-2\}}f^{\{(1)\}}(0) - \dots - f^{\{(n-1)\}}(0)$$

$$\mathcal{L}\left[\int_0^t \mathrm{d}t \int_0^t \mathrm{d}t \dots \int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s^n}\mathcal{L}[f(t)]$$

To be continued...

The equations are organized as follows:

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)]$$
$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)] + DU(s)$$

Let initial condition be zero($\boldsymbol{x}(0) = 0$),we can get:

$$Y(s) = \left[\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D\right]U(s)$$

Thus,we can get transfer function as:

$$g(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D \quad (2.2.2)$$

Let $D = 0$,we can get:

$$g(s) = \frac{\boldsymbol{C}\ \mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B}}{\det(s\boldsymbol{I} - \boldsymbol{A})} \quad (2.2.3)$$

Multi input – Multi output system

Given a multi input-multi output system,we define transfer function between i-th out y_i and j-th input u_j as:

$$g_{ij}(s) = \frac{Y_i(s)}{U_j(s)} \tag{2.2.4}$$

where $Y_i(s)$ is Laplace transform of $y_i(t)$ and $U_j(s)$ is Laplace transform of $u_j(t)$. Must mention that if we define transfer function in this way,we assume that all other inputs are zero.Because linear system satisfies the principle of superposition,so when we plus all inputs $U_1, U_2, ..., U_p$,we can get the i-th output Y_i as:

$$Y_i = \sum_{j=1}^{p} g_{ij} U_j \tag{2.2.5}$$

We can write it as matrix form:

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\boldsymbol{U}(s) \tag{2.2.6}$$

Thus given a linear multi input-multi output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{cases} \tag{2.2.7}$$

We can conduct as before to get transfer function as:

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \frac{\boldsymbol{C}\ \mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \boldsymbol{D}\ \det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} \tag{2.2.8}$$

2.3 Establishing State Space Model by Differential Equation

Given a single input and single output system,if we have differential equation as:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_n y = b_0 u + b_1 u^{(1)} + b_2 u^{(2)} + ... + b_m u^{(m)} \tag{2.3.1}$$

where $m \le n$.

Condition 1: $m = 0$ let $b_0 = 1$,we have:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_n y = b_0 u + b_1 u^{(1)} + b_2 u^{(2)} + ... + b_m u^{(m)} \tag{2.3.2}$$

We can define state variables as:

$$\begin{cases} x_1 &= y \\ x_2 &= y^{(1)} \\ x_3 &= y^{(2)} \\ &\vdots \\ x_n &= y^{(n-1)} \end{cases} \tag{2.3.3}$$

We can get state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_1 x_n - a_2 x_{n-1} - ... - a_n x_1 + b_0 u \end{cases} \tag{2.3.4}$$

We can rewrite it as vector form:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{2.3.5}$$
$$y = [1 \ \ 0 \ \ \dots \ \ 0]\boldsymbol{x}$$

Condition 2:$m \neq n$

Controllable Canonical Form Method:

Let us note D as $\frac{\mathrm{d}}{\mathrm{d}t}$,we can rewrite Equation (2.3.1) as:

$$y = \frac{b_0 D^m + b_1 D^{m-1} + b_2 D^{m-2} + \dots + b_m}{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n} u \tag{2.3.6}$$

Let us discuss the case when $m < n$

let $\tilde{y}^{(n)} + a_1 \tilde{y}^{(n-1)} + a_2 \tilde{y}^{(n-2)} + \dots + a_{n-1} \tilde{y}^{(1)} + a_n \tilde{y} = u$ also as $\tilde{y} = \frac{1}{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n} u$

we can get:

$$y = b_0 \tilde{y}^{(m)} + b_1 \tilde{y}^{(m-1)} + b_2 \tilde{y}^{(m-2)} + \dots + b_m \tilde{y} \tag{2.3.7}$$

We choose state variables as $x_1 = \tilde{y}, x_2 = \tilde{y}^{(1)}, \dots, x_n = \tilde{y}^{(n-1)}$.We can get state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n + u \end{cases} \tag{2.3.8}$$

and output equation as:

$$y = b_m x_1 + b_{m-1} x_2 + \dots + b_0 x_{m+1} \tag{2.3.9}$$

We can rewrite it as vector form:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \tag{2.3.10}$$
$$y = [b_m, \dots, b_0, 0, \dots, 0]\boldsymbol{x}$$

Let us discuss the case when $m = n$,we can rewrite Equation (2.3.6) as:

$$y = \left[b_0 + \frac{(b_1 - b_0 a_1) D^{n-1} + \dots + (b_n - b_0 a_n)}{D^n + a_1 D^{n-1} + \dots + a_n} \right] u \tag{2.3.11}$$

Also let $\tilde{y}^{(n)} + a_1 \tilde{y}^{(n-1)} + a_2 \tilde{y}^{(n-2)} + \dots + a_{n-1} \tilde{y}^{(1)} + a_n \tilde{y} = u$ We can get:

$$y = (b_n - b_0 a_n) \tilde{y}^{(n-1)} + (b_{n-1} - b_0 a_{n-2}) \tilde{y}^{(n-2)} + \dots + (b_1 - b_0 a_1) \tilde{y} + b_0 u \tag{2.3.12}$$

Thus we can write state equation in vector form in familiar way as:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$
$$y = [b_n - b_0 a_n, b_{n-1} - b_0 a_{n-1}, \dots, b_1 - b_0 a_1] \boldsymbol{x} + b_0 u \tag{2.3.13}$$

Undetermined Canonical Form Method: W.l.o.g,we assume that the equation is in the form of:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + b_2 u^{(n-2)} + \dots + b_n u \tag{2.3.14}$$

We can define state variables as:

$$\begin{cases} x_1 &= y - \beta_0 u \\ x_2 &= \dot{x}_1 - \beta_1 u = \dot{y} - \beta_0 \dot{u} - \beta_1 u \\ x_3 &= \dot{x}_2 - \beta_2 u = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u \\ \vdots \\ x_n &= \dot{x}_{n-1} - \beta_{n-1} u = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \dots - \beta_{n-1} u \end{cases} \tag{2.3.15}$$

Thus we have:

$$\begin{cases} y = x_1 + \beta_0 u \\ \dot{y} = x_2 + \beta_0 \dot{u} + \beta_1 u \\ \ddot{y} = x_3 + \beta_0 \ddot{u} + \beta_1 \dot{u} + \beta_2 u \\ \vdots \\ y^{(n-1)} = x_n + \beta_0 u^{(n-1)} + \beta_1 u^{(n-2)} + \dots + \beta_{n-1} u \end{cases} \tag{2.3.16}$$

Let us introduce a new variables $x_{n+1} = \dot{x}_n - \beta_n u = \dot{x}_{n-1} - \beta_{n-1} u = y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \dots - \beta_n u$.Thus we have:

$$y^{(n)} = x_{n+1} + \beta_0 u^{(n)} + \beta_1 u^{(n-1)} + \dots + \beta_n u \tag{2.3.17}$$

Substitute $y, \dot{y}, \dots, y^{(n)}$ into Equation (2.3.14),we can get:

$$\begin{gathered} (x_{n+1} + a_1 x_n + \dots + a_n x_1) + \beta_0 u^{(n)} + (\beta_1 + a_1 \beta_0) u^{(n-1)} + \\ (\beta_2 + a_1 \beta_1 + a_2 \beta_0) u^{(n-2)} + \dots + (\beta_n + a_1 \beta_{n-1} + a_2 \beta_{n-2} + \dots + a_n \beta_0) u \\ = b_0 u^{(n)} + b_1 u^{(n-1)} + b_2 u^{(n-2)} + \dots + b_n u \end{gathered} \tag{2.3.18}$$

Compare the coefficients of $u^{(n)}, u^{(n-1)}, \dots, u$,we can get:

$$\begin{cases} x_{n+1} + a_1 x_n + \dots + a_n x_1 &= 0 \\ \beta_0 &= b_0 \\ \beta_1 + a_1 \beta_0 &= b_1 \\ \beta_2 + a_1 \beta_1 + a_2 \beta_0 &= b_2 \\ & \vdots \\ \beta_n + a_1 \beta_{n-1} + a_2 \beta_{n-2} + \dots + a_n \beta_0 &= b_n \end{cases} \tag{2.3.19}$$

we can rewrite it as matrix form:

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \dots & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \tag{2.3.20}$$

In summary,we can get state equation as:

$$\begin{cases} \dot{x}_1 = \dot{y} - \beta_0 \dot{u} = x_2 + \beta_1 u \\ \dot{x}_2 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} = x_3 + \beta_2 u \\ \quad \vdots \\ \dot{x}_{n-1} = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \dots - \beta_{n-2} \dot{u} = x_n + \beta_{n-1} u \\ \dot{x}_n = y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \dots - \beta_{n-1} \dot{u} = -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n + \beta_n u \end{cases} \tag{2.3.21}$$

and output equation as:

$$y = x_1 + \beta_0 u \tag{2.3.22}$$

We can rewrite it as vector form:

$$\begin{gathered} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u \\ y = [1, 0, 0, \dots, 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 u \end{gathered} \tag{2.3.23}$$