1. Given a circuit as shown in the figure below,let input be voltage u_1 and output be voltage u_2.Try to select state variables and write down the state space representation of the system.

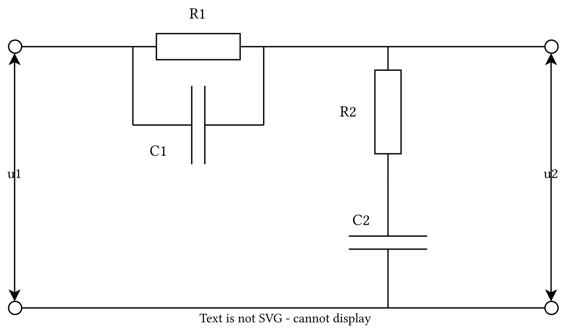

Figure 1: Circuit diagram

Solution:Let we choose u_{c1}, u_{c2} voltage across capacitors C1 and C2 as state variables. We have:

$$\begin{cases} u_{c1} + iR_2 + u_{c2} = u_1 \\ iR_2 + u_{c2} = u_2 \\ C_1 \frac{\mathrm{d}u_{c1}}{\mathrm{d}t} + \frac{u_{c1}}{R_1} = i \\ C_2 \frac{\mathrm{d}u_{c2}}{\mathrm{d}t} = i \end{cases}$$

Substituting $x_1 = u_{c1}, x_2 = u_{c2}$ and reformulating the above equations,we get:

$$\begin{cases} \dot{x_1} = \frac{u_1 - x_1 - x_2}{C_1 R_2} - \frac{x_1}{C_1 R_1} \\ \dot{x_2} = \frac{u_1 - x_1 - x_2}{C_2 R_2} \end{cases}$$

Let input $u_1 = u$ and Rewrite it as vector form:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -\frac{R_1+R_2}{C_1 R_1 R_2} & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} \end{bmatrix} u$$

Let output $u_2 = y$.The output equation is:

$$y = [-1, 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

Therefore,the state space representation of the system is:

$$\begin{cases} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -\frac{R_1+R_2}{C_1 R_1 R_2} & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} \end{bmatrix} u \\ y = [-1, 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \end{cases}$$

2. Given a spring-damper system as shown in the figure below,try to establish the state space representation of the system.

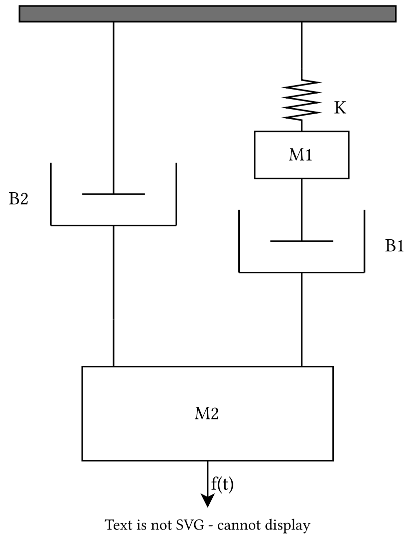

Figure 2: Spring-damper system

Solution:Take downward direction as positive,apply force analysis to the system,we have:

$$\begin{cases} M_1 a_1 = -kx_1 - b_1(v_1 - v_2) \\ M_2 a_2 = f(t) - b_2 v_2 + b_1(v_1 - v_2) \end{cases}$$

where x_1, x_2 stands for the displacement of mass M_1, M_2,v_1, v_2 stands for the velocity of mass M_1, M_2 and a_1, a_2 stands for the acceleration of mass M_1, M_2.

We choose $x_1 = x_1$(displacement of mass)$, x_2 = x_2$(displacement of mass)$, x_3 = v_1, x_4 = v_2$ as state variables and $y_1 = x_1, y_2 = x_2$ as output variables.Then we have:

$$\begin{cases} \dot{x_1} = x_3 \\ \dot{x_2} = x_4 \\ \dot{x_3} = -\frac{k}{M_1}x_1 - \frac{b_1}{M_1}(x_3 - x_4) \\ \dot{x_4} = \frac{f(t)}{M_2} - \frac{b_2}{M_2}x_4 + \frac{b_1}{M_2}(x_3 - x_4) \end{cases}$$

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 \end{cases}$$

Let input $f(t) = u(t)$ and rewrite it as vector form:

$$\begin{cases} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{M_1} & 0 & -\frac{b_1}{M_1} & \frac{b_1}{M_1} \\ 0 & 0 & \frac{b_1}{M_2} & -\frac{b_2+b_1}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \end{bmatrix} u(t) \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{cases}$$

3. Given a state space representation,try to find the transfer function of the system.

$$\begin{cases} \dot{\boldsymbol{x}} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u \\ y = [1, 2]\boldsymbol{x} + 4u \end{cases}$$

Solution:

$$\begin{aligned} g(s) = \frac{Y(s)}{U(s)} &= \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D \\ &= [1, 2] \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 4 \\ &= \frac{12s + 59}{s^2 + 6s + 8} + 4 \end{aligned}$$

4. Given a state space representation,try to find the transfer function of the system.

$$\begin{cases} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

Solution:

$$\begin{aligned} \boldsymbol{G}(s) = \frac{\boldsymbol{Y}(s)}{\boldsymbol{U}(s)} &= \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} s+2 & -1 & 0 \\ 0 & s+3 & 0 \\ 0 & -1 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 \\ 1 & 4 \\ 2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} & 0 \\ 0 & \frac{1}{s+3} & 0 \\ 0 & \frac{1}{(s+3)(s+4)} & \frac{1}{s+4} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 4 \\ 2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2s+7}{(s+3)(s+4)} & \frac{10s+26}{(s+2)(s+3)(s+4)} \\ \frac{1}{s+3} & -\frac{2s+10}{(s+2)(s+3)} \end{bmatrix} \end{aligned}$$