

## 2 Description of State Space

### 2.1 Definition

#### 1. Input variables

We usually use  $\mathbf{u}_t = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n(t)} \end{bmatrix}$  to represent input variables.

#### 2. State variables

We usually use  $\mathbf{x}_t = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n(t)} \end{bmatrix}$  to represent state variables. It is a least set to describe state of system.

#### 3. Output variables

We usually use  $\mathbf{y}_t = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n(t)} \end{bmatrix}$  to represent output variables.

#### 4. State equation

State equation is a first order differential equation that describe relationship between input variables and state variables. We can write it as:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_p, t) \\ \dot{x}_2 = f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_p, t) \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_p, t) \end{cases} \quad (2.1.1)$$

Rewrite it as vector form:

$$\dot{\mathbf{x}}_t = \mathbf{f}(\mathbf{x}_t, \mathbf{u}_t, t) \quad (2.1.2)$$

#### 5. Output equation

Output equation is a equation that describe relationship between state variables and output variables. We can write it as:

$$\begin{cases} y_1 = g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_p, t) \\ y_2 = g_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_p, t) \\ \vdots \\ y_n = g_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_p, t) \end{cases} \quad (2.1.3)$$

Rewrite it as vector form:

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t, \mathbf{u}_t, t) \quad (2.1.4)$$

#### 6. Description of State space of System

We can describe state space of system by equations as:

$$\begin{cases} \dot{\mathbf{x}}_t = \mathbf{f}(\mathbf{x}_t, \mathbf{u}_t, t) \\ \mathbf{y}_t = \mathbf{g}(\mathbf{x}_t, \mathbf{u}_t, t) \end{cases} \quad (2.1.5)$$

When the system is linear, we can write it as:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} \\ \mathbf{y} = \mathbf{C}(t)\mathbf{x} + \mathbf{D}(t)\mathbf{u} \end{cases} \quad (2.1.6)$$

## 2.2 Transfer function

Transfer function is a function that describe relationship between input and output of system. Given a system with different state, the transfer function is still the same which means it is not related to state of system in other words state variables.

### Single input – Single output system

Given a linear single input-single output system, we have state space representation as:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + \mathbf{D}u \end{cases} \quad (2.2.1)$$

To get transfer function, we can use Laplace transform to get:

$$\begin{aligned} s\mathbf{X} - \mathbf{x}(0) &= \mathbf{A}\mathbf{X} + \mathbf{B}U(s) \\ Y(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s) \end{aligned}$$

#### Laplace transfer:

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

$$\mathcal{L}[kf(t)] = kF(s)$$

$$\mathcal{L}[f(t) + g(t)] = F(s) + G(s)$$

$$\mathcal{L}[e^{-at}f(t)] = F(s + a)$$

$$\mathcal{L}[e^{at}f(t)] = F(s - a)$$

$$\mathcal{L}[f(t - T)] = e^{-sT}F(s)$$

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

$$\mathcal{L}\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$$

$$f(\infty) = \lim_{s \rightarrow 0} sF(s)$$

$$f(0) = \lim_{s \rightarrow \infty} sF(s)$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$t$	$\frac{1}{s^2}$
$t^n$	$\frac{n!}{s^{n+1}}$
$e^{-at}$	$s + a$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$u(t)$	$\frac{1}{s}$
$\delta(t)$	1

The equations are organized as follows:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}U(s)]$$

$$Y(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}U(s)] + DU(s)$$

Let initial condition be zero( $\mathbf{x}(0) = 0$ ),we can get:

$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D]U(s)$$

Thus,we can get transfer function as:

$$g(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \quad (2.2.2)$$

Let  $D = 0$ ,we can get:

$$g(s) = \frac{\mathbf{C} \operatorname{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B}}{\det(s\mathbf{I} - \mathbf{A})} \quad (2.2.3)$$

### Multi input – Multi output system

Given a multi input-multi output system,we define transfer function between i-th out  $y_i$  and j-th input  $u_j$  as:

$$g_{ij}(s) = \frac{Y_i(s)}{U_j(s)} \quad (2.2.4)$$

where  $Y_i(s)$  is Laplace transform of  $y_i(t)$  and  $U_j(s)$  is Laplace transform of  $u_j(t)$ . Must mention that if we define transfer function in this way,we assume that all other inputs are zero.Because linear system satisfies the principle of superposition,so when we plus all inputs  $U_1, U_2, \dots, U_p$ ,we can get the i-th output  $Y_i$  as:

$$Y_i = \sum_{j=1}^p g_{ij}U_j \quad (2.2.5)$$

We can write it as matrix form:

$$Y(s) = G(s)U(s) \quad (2.2.6)$$

Thus given a linear multi input-multi output system, we have state space representation as:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (2.2.7)$$

We can conduct as before to get transfer function as:

$$G(s) = C(sI - A)^{-1}B + D = \frac{C \operatorname{adj}(sI - A)B + D \det(sI - A)}{\det(sI - A)} \quad (2.2.8)$$

### Closed-loop System

We have a closed-loop system as figure below:

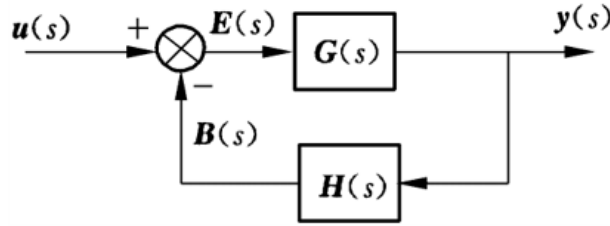


Figure 1: Closed-loop System

We have:

$$\begin{aligned} E(s) &= u(s) - B(s) \\ B(s) &= H(s)y(s) = H(s)G(s)E(s) \\ y(s) &= [I + H(s)G(s)]^{-1}G(s)u(s) \end{aligned}$$

Thus the transfer function of closed-loop system is:

$$G_H(s) = [I + H(s)G(s)]^{-1}G(s) \quad (2.2.9)$$

### Regular

We say a transfer function is regular if and only if when

$$\lim_{s \rightarrow \infty} g(s) = c \quad (2.2.10)$$

where c is a constant. And a transfer function is strictly regular if and only if when

$$\lim_{s \rightarrow \infty} g(s) = 0 \quad (2.2.11)$$

## 2.3 Establishing State Space Model by Differential Equation

Given a single input and single output system, if we have differential equation as:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \dots + a_0y = b_nu^{(n)} + b_{n-1}u^{(n-1)} + \dots + b_0u \quad (2.3.1)$$

where  $m \leq n$ .

**Condition 1:**  $m = 0$

We have:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \dots + a_0y = b_0u \quad (2.3.2)$$

We can define state variables as:

$$\begin{cases} x_1 = y \\ x_2 = y^{(1)} \\ x_3 = y^{(2)} \\ \vdots \\ x_n = y^{(n-1)} \end{cases} \quad (2.3.3)$$

We can get state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0x_1 - a_1x_2 - \dots - a_{n-1}x_n + b_0u \end{cases} \quad (2.3.4)$$

We can rewrite it as vector form:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} \\ y &= [1 \ 0 \ \dots \ 0] \mathbf{x} \end{aligned} \quad (2.3.5)$$

**Condition 2:**  $m \neq n$

**Controllable Canonical Form Method:**

Let us note  $D$  as  $\frac{d}{dt}$ , we can rewrite Equation (2.3.1) as:

$$y = \frac{b_mD^m + b_{m-1}D^{m-1} + b_{m-2}D^{m-2} + \dots + b_0}{D^n + a_{n-1}D^{n-1} + a_{n-2}D^{n-2} + \dots + a_0}u \quad (2.3.6)$$

Let us discuss the case when  $m < n$

Let

$$\tilde{y}^{(n)} + a_{n-1}\tilde{y}^{(n-1)} + a_{n-2}\tilde{y}^{(n-2)} + \dots + a_1\tilde{y}^{(1)} + a_0\tilde{y} = u \quad (2.3.7)$$

Also as

$$\tilde{y} = \frac{1}{D^n + a_{n-1}D^{n-1} + a_{n-2}D^{n-2} + \dots + a_0}u \quad (2.3.8)$$

we can get:

$$y = b_m\tilde{y}^{(m)} + b_{m-1}\tilde{y}^{(m-1)} + b_{m-2}\tilde{y}^{(m-2)} + \dots + b_0\tilde{y} \quad (2.3.9)$$

We choose state variables as  $x_1 = \tilde{y}, x_2 = \tilde{y}^{(1)}, \dots, x_n = \tilde{y}^{(n-1)}$ . We can get state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0x_1 - a_1x_2 - \dots - a_{n-1}x_n + u \end{cases} \quad (2.3.10)$$

and output equation as:

$$y = b_0x_1 + b_1x_2 + \dots + b_mx_{m+1} \quad (2.3.11)$$

We can rewrite it as vector form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [b_0, \dots, b_m, 0, \dots, 0] \mathbf{x} \quad (2.3.12)$$

Let us discuss the case when  $m = n$ , we can rewrite Equation (2.3.6) as:

$$y = \left[ b_n + \frac{(b_{n-1} - b_na_{n-1})D^{n-1} + \dots + (b_0 - b_na_0)}{D^n + a_{n-1}D^{n-1} + \dots + a_0} \right] u \quad (2.3.13)$$

Also let

$$\tilde{y}^{(n)} + a_{n-1}\tilde{y}^{(n-1)} + a_{n-2}\tilde{y}^{(n-2)} + \dots + a_1\tilde{y}^{(1)} + a_0\tilde{y} = u \quad (2.3.14)$$

We can get:

$$y = (b_{n-1} - b_na_{n-1})\tilde{y}^{(n-1)} + (b_{n-2} - b_na_{n-2})\tilde{y}^{(n-2)} + \dots + (b_0 - b_na_0)\tilde{y} + b_nu \quad (2.3.15)$$

Thus we can write state equation in vector form in familiar way as:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [b_0 - b_na_0, b_1 - b_na_1, \dots, b_{n-1} - b_na_{n-1}] \mathbf{x} + b_nu \quad (2.3.16)$$

**Undetermined Canonical Form Method:** W.l.o.g, we assume that the equation is in the form of:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = b_nu^{(n)} + b_{n-1}u^{(n-1)} + \dots + b_0u \quad (2.3.17)$$

We can define state variables as:

$$\begin{cases} x_1 = y - \beta_0u \\ x_2 = \dot{x}_1 - \beta_1u = \dot{y} - \beta_0\dot{u} - \beta_1u \\ x_3 = \dot{x}_2 - \beta_2u = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2u \\ \vdots \\ x_n = \dot{x}_{n-1} - \beta_{n-1}u = y^{(n-1)} - \beta_0u^{(n-1)} - \beta_1u^{(n-2)} - \dots - \beta_{n-1}u \end{cases} \quad (2.3.18)$$

Thus we have:

$$\begin{cases} y = x_1 + \beta_0 u \\ \dot{y} = x_2 + \beta_0 \dot{u} + \beta_1 u \\ \ddot{y} = x_3 + \beta_0 \ddot{u} + \beta_1 \dot{u} + \beta_2 u \\ \vdots \\ y^{(n-1)} = x_n + \beta_0 u^{(n-1)} + \beta_1 u^{(n-2)} + \dots + \beta_{n-1} u \end{cases} \quad (2.3.19)$$

Let us introduce a new variables  $x_{n+1} = \dot{x}_n - \beta_n u = \dot{x}_{n-1} - \beta_{n-1} u = y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \dots - \beta_n u$ . Thus we have:

$$y^{(n)} = x_{n+1} + \beta_0 u^{(n)} + \beta_1 u^{(n-1)} + \dots + \beta_n u \quad (2.3.20)$$

Substitute  $y, \dot{y}, \dots, y^{(n)}$  into Equation (2.3.17), we can get:

$$\begin{aligned} & (x_{n+1} + a_{n-1}x_n + \dots + a_0x_1) + \beta_0 u^{(n)} + (\beta_1 + a_{n-1}\beta_0)u^{(n-1)} + \\ & (\beta_2 + a_{n-1}\beta_1 + a_{n-2}\beta_0)u^{(n-2)} + \dots + (\beta_n + a_{n-1}\beta_{n-1} + a_{n-2}\beta_{n-2} + \dots + a_0\beta_0)u \\ & = b_n u^{(n)} + b_{n-1} u^{(n-1)} + b_{n-2} u^{(n-2)} + \dots + b_0 u \end{aligned} \quad (2.3.21)$$

Compare the coefficients of  $u^{(n)}, u^{(n-1)}, \dots, u$ , we can get:

$$\begin{cases} x_{n+1} + a_{n-1}x_n + \dots + a_0x_1 & = 0 \\ \beta_0 & = b_n \\ \beta_1 + a_{n-1}\beta_0 & = b_{n-1} \\ \beta_2 + a_{n-1}\beta_1 + a_{n-2}\beta_0 & = b_{n-2} \\ \vdots & \\ \beta_n + a_{n-1}\beta_{n-1} + a_{n-2}\beta_{n-2} + \dots + a_0\beta_0 & = b_0 \end{cases} \quad (2.3.22)$$

In summary, we can get state equation as:

$$\begin{cases} \dot{x}_1 = \dot{y} - \beta_0 \dot{u} = x_2 + \beta_1 u \\ \dot{x}_2 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} = x_3 + \beta_2 u \\ \vdots \\ \dot{x}_{n-1} = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \dots - \beta_{n-2} \dot{u} = x_n + \beta_{n-1} u \\ \dot{x}_n = y^{(n)} - \beta_0 u^{(n)} - \beta_1 u^{(n-1)} - \dots - \beta_{n-1} \dot{u} = -a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n + \beta_n u \end{cases} \quad (2.3.23)$$

and output equation as:

$$y = x_1 + \beta_0 u \quad (2.3.24)$$

We can rewrite it as vector form:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u \\ y &= [1, 0, 0, \dots, 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 u \end{aligned} \quad (2.3.25)$$

## 2.4 Establishing State Space Model by Transfer Function

For a actual physical system, the transfer function of the system is always regular.

First, let us discuss the situation where the system is restrict regular in other words order of numerator of the transfer function is less than denominator of the transfer function. If we have a differential equation of system as:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \dots + a_1\dot{y} + a_0y = b_{n-1}u^{(n-1)} + \dots + b_1\dot{u} + b_0u \quad (2.4.1)$$

Then we have transfer function as:

$$g(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad (2.4.2)$$

Introduce a intermediate variables  $Z(s)$  We have:

$$g(s) = \frac{Y(s)}{Z(s)} \frac{Z(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{1} \frac{1}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad (2.4.3)$$

Let us do inverse Laplace transform of  $Z(s)$ , we can get:

$$\begin{cases} y = b_{n-1}z^{(n-1)} + b_{n-2}z^{(n-2)} + \dots + b_1\dot{z} + b_0z \\ z^{(n)} + a_{n-1}z^{(n-1)} + a_{n-2}z^{(n-2)} + \dots + a_1\dot{z} + a_0z = u \end{cases} \quad (2.4.4)$$

We can define state variables as  $x_1 = z, x_2 = \dot{z}, x_3 = \ddot{z}, x_n = z^{(n-1)}$ . We have state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0x_1 - a_1x_2 - \dots - a_{n-1}x_n + u \end{cases} \quad (2.4.5)$$

And output equation as:

$$y = b_0x_1 + b_1x_2 + \dots + b_{n-1}x_n \quad (2.4.6)$$

Let us discuss when the order of numerator of transfer function is as same as order of denominator of transfer function. We have transfer function as:

$$\begin{aligned} g(s) &= \frac{Y(s)}{U(s)} = \frac{b_ns^n + b_{n-1}s^{n-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \\ &= b_n + \frac{(b_{n-1} - b_na_{n-1})s^{n-1} + (b_{n-2} - b_na_{n-2})s^{n-2} + \dots + (b_0 - b_na_0)}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \end{aligned} \quad (2.4.7)$$

Let us note  $h(s)$  as intermediate transfer function:

$$h(s) = \frac{\beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \dots + \beta_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad (2.4.8)$$

where  $\beta_i = b_i - b_na_i$  for  $i = 0, 1, \dots, n-1$ . We do the same thing as before, we can get:

$$\begin{cases} y = \beta_{n-1}z^{(n-1)} + \beta_{n-2}z^{(n-2)} + \dots + \beta_1\dot{z} + \beta_0z \\ z^{(n)} + a_{n-1}z^{(n-1)} + a_{n-2}z^{(n-2)} + \dots + a_1\dot{z} + a_0z = u \end{cases} \quad (2.4.9)$$



And

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0x_1 - a_1x_2 - \dots - a_{n-1}x_n + u \end{cases} \quad (2.4.10)$$

For output equation,all we need is to add a  $b_n u$  term.

$$y = \beta_0x_1 + \beta_1x_2 + \dots + \beta_{n-1}x_n + b_n u \quad (2.4.11)$$

## 2.5 Linear Transformation

Given a state variable vector  $x$ ,the linear combination of the state variable vector is also a state variable vector  $\bar{x}$  if and only if the linear transformation matrix  $P$  is invertible.

$$x = P\bar{x} \quad (2.5.1)$$

In other words:

$$\bar{x} = P^{-1}x \quad (2.5.2)$$

Let us discuss what would happen if we apply liner transformation to a **liner system**.

Given a linear system as:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (2.5.3)$$

Let  $x = P\bar{x}$ ,we have:

$$\begin{cases} \dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu \\ y = CP\bar{x} + Du \end{cases} \quad (2.5.4)$$

We have:

$$\bar{A} = P^{-1}AP \quad (2.5.5)$$

$$\bar{B} = P^{-1}B \quad (2.5.6)$$

$$\bar{C} = CP \quad (2.5.7)$$

$$\bar{D} = D \quad (2.5.8)$$

Let us try to transform state equations to **diagonal canonical form**.

Given a state equation as:

$$\dot{x} = Ax + Bu \quad (2.5.9)$$

The eigenvalues of the system is defined as:

$$\det(\lambda I - A) = 0 \quad (2.5.10)$$

### Diagonal Canonical Form

**If the geometric multiplicity of the system is equal to the order of the system**,we can transform the state equation to diagonal canonical form by linear transformation.

Let  $P = [v_1 \ v_2 \ \dots \ v_n]^{-1}$ . Then the state equation can be transformed to diagonal form as:

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \bar{x} + \bar{B}u \quad (2.5.11)$$

where  $\bar{B} = P^{-1}B$ .

### Trick

If A is a companion matrix, then the state equation can be transformed to diagonal canonical form by transformation matrix  $P$  where  $P$  is an inverse Vandermonde matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, P = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix}^{-1} \quad (2.5.12)$$

### Jordan Canonical Form

**If the geometric multiplicity of the system is less than the order of the system**, we can transform the state equation to Jordan canonical form by linear transformation.

For those eigenvalues with geometric multiplicity less than the order of the system and let  $v_i$  be their corresponding eigenvectors ( $\lambda_i v_i = Av_i$ ), we define generalized eigenvectors as:

$$\begin{cases} (\lambda_i I - A)v_i = 0 \\ (\lambda_i I - A)v'_i = -v_i \\ (\lambda_i I - A)v''_i = -v'_i \\ \vdots \\ (\lambda_i I - A)v_i^{\sigma_i} = -v_i^{\sigma_i-1} \end{cases} \quad (2.5.13)$$

Then let  $P = [v_1 \ v'_1 \ \dots \ v_1^{\sigma_1} \ v_2 \ v'_2 \ \dots \ v_2^{\sigma_2} \ \dots]^{-1}$ . Then the state equation can be transformed to Jordan canonical form as:

$$\dot{\bar{x}} = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_n \end{bmatrix} \bar{x} + \bar{B}u \quad (2.5.14)$$

where  $J_i$  is a Jordan block corresponding to eigenvalue  $\lambda_i$  and  $\bar{B} = P^{-1}B$ .

**Modal Form** If the eigenvalues of the system are complex numbers, we can transform the state equation to modal form.

Let

$$\lambda_1 = \sigma + \omega i, \lambda_2 = \sigma - \omega i \quad (2.5.15)$$

In this situation, the modal form of A is

$$M = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \quad (2.5.16)$$

Let  $v_1$  be the eigenvector of  $\lambda_1$  ( $\lambda_1 v_1 = A v_1$ ).

$$v_1 = \alpha + \beta i \tag{2.5.17}$$

The transformation matrix  $P$  is  $[\alpha \ \beta]^{-1}$ .