2 Description of State Space

2.1 Definition

1. **Input variables**

We usually use $\boldsymbol{u}_t = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n(t)} \end{bmatrix}$ to represent input variables.

2. **State variables**

We usually use $\boldsymbol{x}_t = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n(t)} \end{bmatrix}$ to represent state variables. It is a least set to describe state of system.

3. **Output variables**

We usually use $\boldsymbol{y}_t = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n(t)} \end{bmatrix}$ to represent output variables.

4. **State equation**

State equation is a first order differential equation that describe relationship between input variables and state variables. We can write it as:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ \dot{x}_2 = f_2(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ \quad \vdots \\ \dot{x}_n = f_n(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \end{cases} \tag{2.1.1}$$

Rewrite it as vector form:

$$\dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{2.1.2}$$

5. **Output equation**

Output equation is a equation that describe relationship between state variables and output variables. We can write it as:

$$\begin{cases} y_1 = g_1(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ y_2 = g_2(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \\ \quad \vdots \\ y_n = g_n(x_1, x_2, ..., x_n; u_1, u_2, ..., u_p, t) \end{cases} \tag{2.1.3}$$

Rewrite it as vector form:

$$\boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \tag{2.1.4}$$

6. **Description of State space of System**

We can describe state space of system by equations as:

$$\begin{cases} \dot{\boldsymbol{x}}_t = \boldsymbol{f}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \\ \boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t, \boldsymbol{u}_t, t) \end{cases} \tag{2.1.5}$$

When the system is linear, we can write it as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}(t)\boldsymbol{x} + \boldsymbol{D}(t)\boldsymbol{u} \end{cases} \tag{2.1.6}$$

2.2 Transfer function

Transfer function is a function that describe relationship between input and output of system.Given a system with different state,the transfer function is still the same which means it is not related to state of system in other words state variables.

Single input – Single output system

Given a linear single input-single output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \\ y = \boldsymbol{C}\boldsymbol{x} + Du \end{cases} \tag{2.2.1}$$

To get transfer function,we can use Laplace transform to get:

$$s\boldsymbol{X} - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}U(s)$$
$$Y(s) = \boldsymbol{C}\boldsymbol{X}(s) + DU(s)$$

> **Laplace transfer**:
>
> $$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$
>
> $$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{\{n-1\}}f(0) - s^{\{n-2\}}f^{\{(1)\}}(0) - \dots - f^{\{(n-1)\}}(0)$$
>
> $$\mathcal{L}\left[\int_0^t \mathrm{d}t \int_0^t \mathrm{d}t \dots \int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s^n}\mathcal{L}[f(t)]$$
>
> To be continued...

The equations are organized as follows:

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)]$$
$$Y(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}[\boldsymbol{x}(0) + \boldsymbol{B}U(s)] + DU(s)$$

Let initial condition be zero($\boldsymbol{x}(0) = 0$),we can get:

$$Y(s) = \left[\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D\right]U(s)$$

Thus,we can get transfer function as:

$$g(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D \tag{2.2.2}$$

Let $D = 0$,we can get:

$$g(s) = \frac{\boldsymbol{C}\,\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B}}{\det(s\boldsymbol{I} - \boldsymbol{A})} \tag{2.2.3}$$

Multi input – Multi output system

Given a multi input-multi output system,we define transfer function between i-th out y_i and j-th input u_j as:

$$g_{ij}(s) = \frac{Y_i(s)}{U_j(s)} \tag{2.2.4}$$

where $Y_i(s)$ is Laplace transform of $y_i(t)$ and $U_j(s)$ is Laplace transform of $u_j(t)$. Must mention that if we define transfer function in this way,we assume that all other inputs are zero.Because linear system satisfies the principle of superposition,so when we plus all inputs $U_1, U_2, ..., U_p$,we can get the i-th output Y_i as:

$$Y_i = \sum_{j=1}^{p} g_{ij} U_j \tag{2.2.5}$$

We can write it as matrix form:

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\boldsymbol{U}(s) \tag{2.2.6}$$

Thus given a linear multi input-multi output system,we have state space representation as:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{cases} \tag{2.2.7}$$

We can conduct as before to get transfer function as:

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \frac{\boldsymbol{C} \operatorname{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \boldsymbol{D} \det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} \tag{2.2.8}$$

2.3 Establishing State Space Model by Differential Equation

Given a single input and single output system,if we have differential equation as:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_n y = b_0 u + b_1 u^{(1)} + b_2 u^{(2)} + ... + b_m u^{(m)} \tag{2.3.1}$$

where $m \leq n$.

Condition 1: $m = 0$ let $b_0 = 1$,we have:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_n y = b_0 u + b_1 u^{(1)} + b_2 u^{(2)} + ... + b_m u^{(m)} \tag{2.3.2}$$

We can define state variables as:

$$\begin{cases} x_1 = y \\ x_2 = y^{(1)} \\ x_3 = y^{(2)} \\ \quad \vdots \\ x_n = y^{(n-1)} \end{cases} \tag{2.3.3}$$

We can get state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_1 x_n - a_2 x_{n-1} - ... - a_n x_1 + b_0 u \end{cases} \tag{2.3.4}$$

We can rewrite it as vector form:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{2.3.5}$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \boldsymbol{x}$$

Condition 2: $m \neq n$

Controllable Canonical Form Method:

Let us note D as $\frac{\mathrm{d}}{\mathrm{d}t}$, we can rewrite Equation (2.3.1) as:

$$y = \frac{b_0 D^m + b_1 D^{m-1} + b_2 D^{m-2} + \cdots + b_m}{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n} u \tag{2.3.6}$$

Let us discuss the case when $m < n$

let $\tilde{y}^{(n)} + a_1 \tilde{y}^{(n-1)} + a_2 \tilde{y}^{(n-2)} + \cdots + a_{n-1} \tilde{y}^{(1)} + a_n \tilde{y} = u$ also as $\tilde{y} = \frac{1}{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n} u$

we can get:

$$y = b_0 \tilde{y}^{(m)} + b_1 \tilde{y}^{(m-1)} + b_2 \tilde{y}^{(m-2)} + \cdots + b_m \tilde{y} \tag{2.3.7}$$

We choose state variables as $x_1 = \tilde{y}, x_2 = \tilde{y}^{(1)}, \ldots, x_n = \tilde{y}^{(n-1)}$. We can get state equation as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + u \end{cases} \tag{2.3.8}$$

and output equation as:

$$y = b_m x_1 + b_{m-1} x_2 + \cdots + b_0 x_{m+1} \tag{2.3.9}$$

We can rewrite it as vector form:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \tag{2.3.10}$$

$$y = [b_m, \ldots, b_0, 0, \ldots, 0] \boldsymbol{x}$$

Let us discuss the case when $m = n$, we can rewrite Equation (2.3.6) as:

$$y = \left[b_0 + \frac{(b_1 - b_0 a_1) D^{n-1} + \cdots + (b_n - b_0 a_n)}{D^n + a_1 D^{n-1} + \cdots + a_n} \right] u \tag{2.3.11}$$

Also let $\tilde{y}^{(n)} + a_1 \tilde{y}^{(n-1)} + a_2 \tilde{y}^{(n-2)} + \cdots + a_{n-1} \tilde{y}^{(1)} + a_n \tilde{y} = u$ We can get:

$$y = (b_n - b_0 a_n) \tilde{y}^{(n-1)} + (b_{n-1} - b_0 a_{n-2}) \tilde{y}^{(n-2)} + \cdots + (b_1 - b_0 a_1) \tilde{y} + b_0 u \tag{2.3.12}$$

Thus we can write state equation in vector form in familiar way as:

$$\dot{x} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \qquad (2.3.13)$$

$$y = [b_n - b_0 a_n, \, b_{n-1} - b_0 a_{n-1}, \, ..., \, b_1 - b_0 a_1] x + b_0 u$$