1. Determine the controllability of the following systems:

(1) $\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \boldsymbol{u}$

(2) $\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \boldsymbol{u}$

(3) $\dot{\boldsymbol{x}} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \boldsymbol{u}$

(4) $\dot{\boldsymbol{x}} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \boldsymbol{u}$

(5) $\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \boldsymbol{u}$

Solution:(1)We use Kalman's controllability criterion to determine the controllability of the system.

We have:

$$\begin{aligned} \boldsymbol{Q}_c &= [\boldsymbol{B} \;\; \boldsymbol{AB}] \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ \operatorname{rank}(\boldsymbol{Q}_c) &= 2 = n \end{aligned}$$

Thus the system is controllable.

(2)

$$\begin{aligned} \boldsymbol{Q}_c &= [\boldsymbol{B} \;\; \boldsymbol{AB} \;\; \boldsymbol{A}^2\boldsymbol{B}] \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -5 & -7 \\ -1 & 1 & -5 & -7 & 19 & 15 \end{bmatrix} \\ \operatorname{rank}(\boldsymbol{Q}_c) &= 3 = n \end{aligned}$$

Thus the system is controllable.

(3) We use PBH's controllability criterion to determine the controllability of the system.The eigenvalues of the matrix A are $-3, -3, -1$. We have:

$$\lambda_{1,2} = -3, \lambda_3 = -1$$

$$\operatorname{rank}\big([\lambda_{1,2}I - A \;\; B]\big) = \operatorname{rank}\left(\begin{bmatrix} 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{bmatrix}\right) = 3 = n$$

$$\operatorname{rank}([\lambda_3 I - A \;\; B]) = \operatorname{rank}\left(\begin{bmatrix} 2 & -1 & 0 & 1 & -1 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}\right) = 3 = n$$

Thus the system is controllable.

(4) We use PBH's controllability criterion to determine the controllability of the system. All eigenvalues of the matrix A are λ_1.We have:

$$\operatorname{rank}([\lambda_1 I - A \;\; B]) = \operatorname{rank}\left(\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) = 2 < n$$

Thus the system is not controllable.

(5) We use Kalman's controllability criterion to determine the controllability of the system.

$$\begin{aligned} \boldsymbol{Q}_c &= [\boldsymbol{B} \;\; \boldsymbol{AB} \;\; \boldsymbol{A}^2\boldsymbol{B}] \\ &= \begin{bmatrix} -1 & 12 & 15 \\ 3 & 60 & 480 \\ 0 & -75 & 0 \end{bmatrix} \\ \operatorname{rank}(\boldsymbol{Q}_c) &= 3 = n \end{aligned}$$

Thus the system is controllable.