# Kalman Filter Notes

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Ref1: ON MY COMPUTER: *Guido Gerig - MI37slides-Kalman.pdf*
Ref2: https://www.kalmanfilter.net/alphabeta.html

## 1 Remarks

1. The buidling of the system equaiton is very important.
2. Sample variance = system var + measurement var.

## 2 State Space Equations

### 2.1 Derivative Form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) &= H(t)\mathbf{x}(t) + D(t)\mathbf{u}(t) + \mathbf{v}(t) \end{aligned} \tag{1}$$

Here, $\mathbf{x}(t)$ is the state vector, $\mathbf{y}(t)$ is the observation vector, $\mathbf{u}(t)$ is the input vector, $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are noise vectors.

### 2.2 Recursive Form

$$\begin{aligned} \mathbf{x}(t) &= F(t)\mathbf{x}(t-1) + B(t)\mathbf{u}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) &= H(t)\mathbf{x}(t) + D(t)\mathbf{u}(t) + \mathbf{v}(t) \end{aligned} \tag{2}$$

Where $\mathbf{F}_t = e^{\Delta t \mathbf{A}} = \mathbf{I} + \sum_{i=1}^{\infty} \frac{\Delta t^i \mathbf{A}^i}{i!}$.

## 3 Kalman Filtering

### 3.1 Predict

Based on the state value $\hat{\mathbf{x}}_{t-1|t-1}$ at time $t-1$, we predict $\hat{\mathbf{x}}_{t|t-1}$ at time $t$, using the system equation.

$$\hat{\mathbf{x}}_{t|t-1} = F_t \hat{\mathbf{x}}_{t-1|t-1} + B(t)\mathbf{u}(t) \tag{3}$$

The variance of this prediction is:

$$\hat{\mathbf{P}}_{t|t-1} = F_t \hat{\mathbf{P}}_{t-1|t-1} F_t^T + \mathbf{Q}(t) \tag{4}$$

Where $\mathbf{Q}(t)$ is the process noise variance matrix.

### 3.2 Residual

Calc the residual between observation $y(t)$ and its prediction $\hat{\mathbf{y}}_t = H_t\ \hat{\mathbf{x}}_{t|t-1}$.

$$\tilde{z}_t = \mathbf{y}(t) - H_t\ \hat{\mathbf{x}}_{t|t-1}$$

Calc residual variance matrix $S(t)$, using observation matrix $H_t$.

$$S(t) = H_t\ \hat{\mathbf{P}}_{t|t-1}\ H_t^T + R_t$$

Where $R_t$ is the observation noise variance.

### 3.3 Kalman Filter Gain

$$K_t = P_{t|t-1}\ H_t^T\ S_t^{-1} \tag{5}$$

### 3.4 Update

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t\ \tilde{z}_t \tag{6}$$

$$P_{t|t} = (I - K_t H_t)\ P_{t|t-1} \tag{7}$$

## 4 Prediciton

According to (EQ.3), we can use Kalman Filter to predict the state of the next time with certain uncertainty (EQ.4).

In those equations, the time interval is 1. However, based on current information at time $t$, we can predict further future, such like $t+2$, $t+3$...

$$\hat{\mathbf{x}}_{t+2|t} = F_t^2 \hat{\mathbf{x}}_t + B(t)^2 \mathbf{u}(t) \tag{8}$$

The variance of this prediction is:

$$\hat{\mathbf{P}}_{t+2|t} = (F_t \hat{\mathbf{P}}_{t-1|t-1} F_t^T)^2 + \mathbf{Q}(t) \tag{9}$$

And so on...

The futher the prediction do, the more the uncertainty will be.

Actually, there is a situation in which the filter may lose traction of our object, if the prediction is too far away from now. This means that doing prediction far from that gap may fail sometime.(the filter algorithm will not converge).

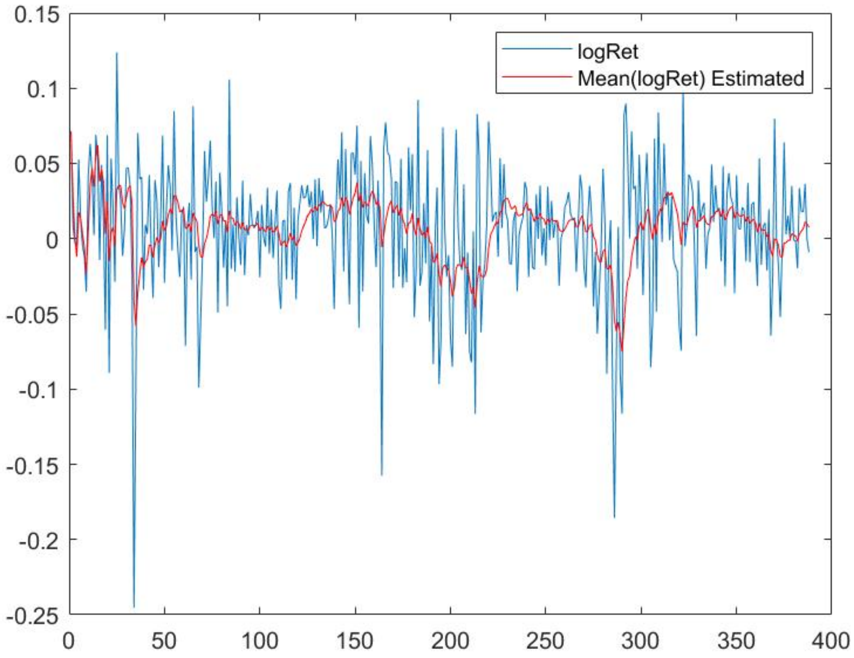



Figure 1: Fig.

# 5 An Example of the $\alpha - \beta - \gamma$ filter

In this example, we are going to track the aircraft that is moving with constant acceleration with the $\alpha - \beta - \gamma$ filter.

Assume a car is been drived with constant acceleration along $a$ straight line.

$$\hat{x}_{n+1,n} = \hat{x}_{n,n} + \hat{\dot{x}}_{n,n}\Delta t + \frac{1}{2}\hat{\ddot{x}}_{n,n}\Delta t^2$$

What we can directly obseave is only its postion $z$. Our measurrment results are noise included:

$$z = x + w$$

Where $w \sim N(0, \sigma^2)$ Gaussian distribution.

## 5.1 State Extrapolation Equation

$$\begin{aligned}
\hat{x}_{n+1,n} &= \hat{x}_{n,n} + \hat{\dot{x}}_{n,n}\Delta t + \frac{1}{2}\hat{\ddot{x}}_{n,n}\Delta t^2 \\
\hat{\dot{x}}_{n+1,n} &= \hat{\dot{x}}_{n,n} + \hat{\ddot{x}}_{n,n}\Delta t \\
\hat{\ddot{x}}_{n+1,n} &= \hat{\ddot{x}}_{n,n}
\end{aligned} \tag{10}$$

Where $\hat{\ddot{x}}_n$ is acceleration (the second order derivative of $x$).

## 5.2 State Update Equation

Representing all state variables using obsearvation variable(s) $z_n$.

$$\begin{aligned}
\hat{x}_{n,n} &= \hat{x}_{n,n-1} + \alpha \left( z_n - \hat{x}_{n,n-1} \right) \\
\hat{\dot{x}}_{n,n} &= \hat{\dot{x}}_{n,n-1} + \beta \left( \frac{z_n - \hat{x}_{n,n-1}}{\Delta t} \right) \\
\hat{\ddot{x}}_{n,n} &= \hat{\ddot{x}}_{n,n-1} + \gamma \left( \frac{z_n - \hat{x}_{n,n-1}}{0.5\Delta t^2} \right)
\end{aligned} \tag{11}$$

## 5.3 Kalman Gain Equation

In a Kalman filter, the $\alpha - \beta - \gamma$ parameters are calculated dynamically for each filter iteration. These parameters are called Kalman Gain and denoted by $K_n$.

$$K_n = [\alpha, \beta, \gamma]$$

$$K_n = \frac{p_{n,n-1}}{p_{n,n-1} + r_n} \tag{12}$$

$$K_n = \frac{Uncertainty\ in\ Estimate}{Uncertainty\ in\ Estimate\ +\ Uncertainty\ in\ Measurement}$$

Where:
$p_{n,n-1}$ is the extrapolated estimate uncertainty
$r_n$ is the measurement uncertainty.

## 5.4 Covariance Extrapolation Equation

$$p_{n+1,n} = p_{n,n} \tag{13}$$

## 5.5 Covariance Update Equation

$$p_{n,n} = \ (1 - K_n)\, p_{n,n-1} \tag{14}$$

Where:
$K_n$ is the Kalman Gain
$p_{n,n-1}$ is the estimate uncertainty that was calculated during the previous filter estimation
$p_{n,n}$ is the estimate uncertainty of the current sate

# 6 Step by Step

## 6.1 Step 0: Initialization

The initialization performed only once, and it provides two parameters:

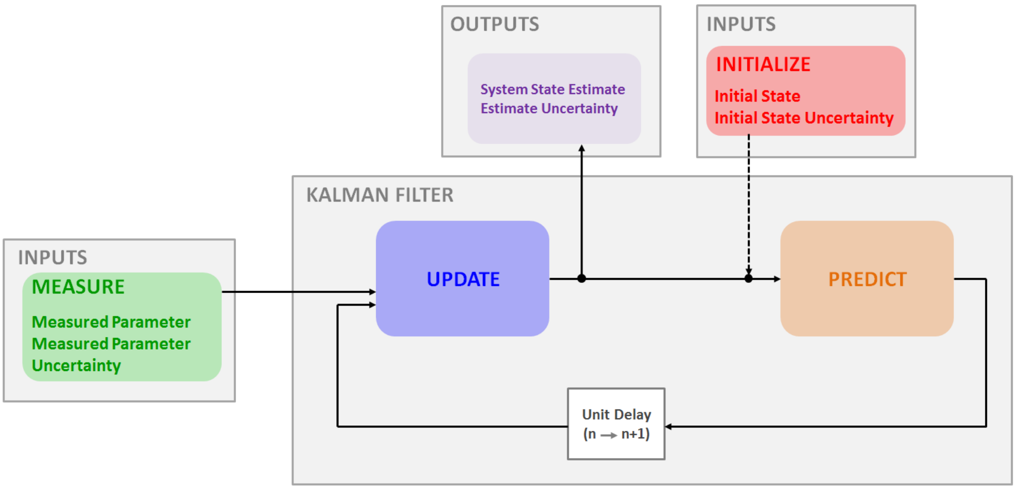



Figure 2: Level-Crossing Rate.

Initial System State $x_{0,0}$
Initial State Uncertainty $p_{0,0}$

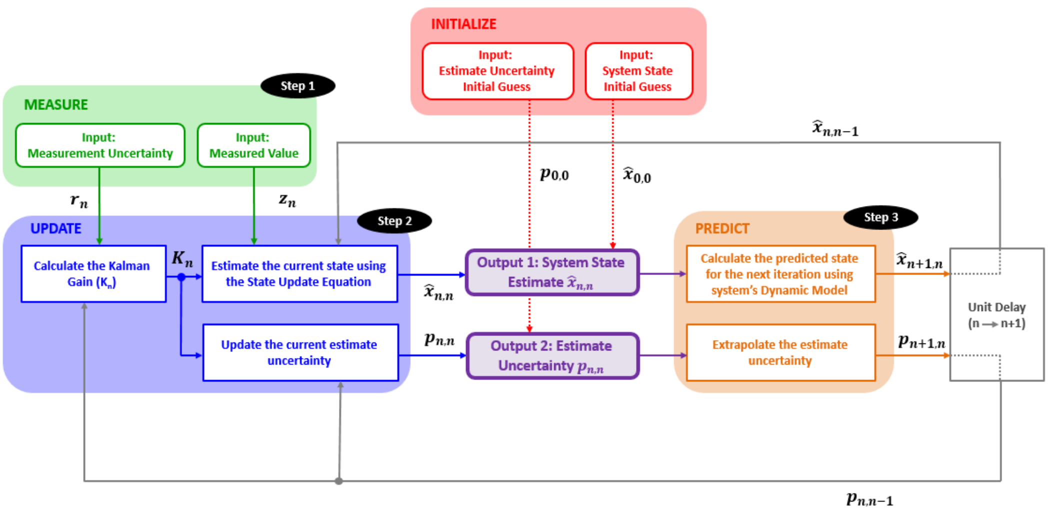
## 6.2 Step 3: Predict

The initialization is followed by prediction, using Eq.10 and Eq.13.

## 6.3 Step 1: Measurement

The measurement process shall provide two parameters:

Measured System State $z_n$
Measurement Uncertainty $r_n$



### 6.4 Step 2: State Update

The state update process is responsible for system's current state estimation. The state update process inputs are:

Measured Value $z_n$
The Measurement Uncertainty $r_n$
Previous System State Estimate $z_{n,n-1}$
Estimate Uncertainty $p_{n,n-1}$

Based on the inputs, the state update process calculates the Kalman Gain (using Eq.12), and provides two outputs (using Eq.11 and Eq.14):

Current System State Estimate $x_{n,n}$
Current State Estimate Uncertainty $p_{n,n}$

These parameters are the Kalman Filter outputs.

### 6.5 Step 3: Prediction

Using Eq.10 and Eq.13, the prediction process extrapolates the current system state and the uncertainty of the current system state estimate to the next system state, based on the system's dynamic model.

At the first filter iteration the initialization outputs are treated as the Previous State Estimate and Uncertainty. On the next filter iterations, the prediction outputs become the Previous State Estimate and Uncertainty.

## 7 An Example

$$logRet(t) = u(t) + w(t)$$

$u(t)$ is smoothy time varing. $w(t) \sim N(0, \sigma(t)^2)$ is additive noise.

$$logRet(t) = u(t) + \sigma(t)z(t)$$

Where $z(t)$ $N(0, 1)$.
I want to use Kalman Filter to track spontanious $u(t)$ and $\sigma(t)$.

### 7.1 System Eq

$$x(t) = u(t), u'(\vec{t}), u''(t)$$

# 8 Extend

## 8.1 Extended KF

1. The system equation is not linear, but we will use Taylor series to approximate it.
2. The noise(err) is still Gaussian distributed.