Stock Price - Notes

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1 Stock Price

Assuming that stock price $X(t)$ follows Geometric Brownian Motion, that is $X(t)$ satisfies:

$$dX(t) = uX(t)dt + \sigma X(t)dW(t) \tag{1}$$

Where μ (the percentage drift) and σ (the percentage volatility) are CONSTANT. The former represents a deterministic trend, and the later is used to simulate the random part. $W(t)$ is the Standard Brownian Motion, or called Winnar Process. Its pdf follows N(0,1) standard normal dist at a fixed time t.

Using Ito Lemma,we can solve Eq.1. At time t, the stock price $X(t)$ is:[1]

$$X(t) = X(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W(t)} \tag{2}$$

Where is $-\frac{1}{2}\sigma^2$ comes from???

Its log:

$$\ln X(t) = \ln X(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \tag{3}$$

From Eq.2 we can see that log-price $\ln X(t)$ is consisted of two items: The first one is $\ln X(0) + (\mu - \frac{1}{2}\sigma^2)t$, The second one is the random part $\sigma W(t)$, which genereate a gaussian fluctration $N(0, \sigma^2)$.

Therefore, $\ln X(t)$ follows Gaussian Didtribution, with mean $\ln X(0) + (\mu - \frac{1}{2}\sigma^2)t$. As time pass by, t grows, so this mean value grows. The variance is σ^2.

2 Log Returns

Define log return $y(t)$ as:

$$y(t + \Delta t) = \ln(X(t + \Delta t)) - \ln(X(t)) \tag{4}$$

[1] when t=0, X(0)!=X(0)

substitute Eq.2 into Eq.4:

$$\begin{aligned} y(t+\Delta t) &= \ln(X(t+\Delta t)) - \ln(X(t)) \\ &= [\ln(X(0)) + (\mu - \frac{1}{2}\sigma^2)(t+\Delta t) + \sigma W(t+\Delta t)] - \\ &\quad [\ln(X(0)) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)] \\ &= (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma[W(t+\Delta t) - W(t)] \end{aligned} \tag{5}$$

Assign:

$$B(t+\Delta t) = W(t+\Delta t) - W(t) \tag{6}$$

We have:

$$y(t+\Delta t) = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma B(t+\Delta t) \tag{7}$$

Because $W(t)$ is a Independent incremental process, $B(t)$ is a Winnar process too. It follows $N(*, *)$ Normal dist at a fixed time.

Comparing Eq.3 with Eq.7, we see that log ret $y(t)$ is also composed of two parts: A deterministec part and a random one.

However, different from log price $\ln X(t)$, the mean value of $y(t)$: $(\mu - \frac{1}{2}\sigma^2)\Delta t$ is time inrelavent but relevant with the time increment Δt.

When the observe time window is fixed, the mean value of $y(t)$ will keep unchanged.

Let $\Delta t = 1$ in (7):

$$y(n) = (\mu - \frac{1}{2}\sigma^2) + \sigma B(n) \tag{8}$$

3 Kalman filtering

If I model $\sigma B(t)$ as the measurement err, then, I won't get the variance of $logRet$, and I need to model the variance sepearately.

Maybe I can model logPrice $logPrice$, instead of $logRet$.

assign $y(t) = logPrice(t)$, then:

$$x(t) = x(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \tag{9}$$

3.1 System Equation

$$\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} 1 & dt \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t-1) \\ x'(t-1) \end{bmatrix} + a(t) \begin{bmatrix} \frac{1}{2}dt^2 \\ dt \end{bmatrix} \tag{10}$$

3.2 Measurement Equation

$$\begin{bmatrix} y(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} + \begin{bmatrix} \sigma_w \\ 0 \end{bmatrix} \tag{11}$$