Exercises

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1 散度 Divergence

Problem 1 用另外一种方式证明 $D(p||q) \geq 0$

$$\begin{aligned} -D(p||q) &= \sum p(x) ln \frac{q(x)}{p(x)} \\ &\leq \sum p(x)(\frac{q(x)}{p(x)} - 1) \\ &= \sum (q(x) - p(x)) \\ &= 1 - 1 \\ &= 0 \end{aligned} \tag{1}$$

hint:$lnx \leq x - 1$

Problem 2 混淆增加熵率

证明：$\vec(p) = (p_1, p_2, ..., p_m)$，$\vec{q} = (p_1, p_2, ..., \frac{p_i+p_j}{2}, ..., \frac{p_j+p_i}{2}, ..., p_m)$，有

$(\vec{q}) \geq H(\vec{p})$

$$
\begin{aligned}
H(\vec{q}) - H(\vec{p}) &= -2 * \frac{p_i + p_j}{2} log \frac{p_i + p_j}{2} + p_i log p_i + p_j log p_j \\
&= -p(log \frac{p_i + p_j}{2} - \frac{p_i}{p} log p_i - \frac{p_j}{p} log p_j)(p = p_i + p_j) \\
&= -p(log \frac{p_i + p_j}{2} - log p - (\frac{p_i}{p} log \frac{p_i}{p}) - (\frac{p_j}{p} log \frac{p_j}{p})) \\
&= -p(log \frac{p_i + p_j}{2} - log p + p_1 log \frac{1}{p_1} + p_2 log \frac{1}{p_2}) \\
&\geq -p(log \frac{p}{2} - log p + log(p_1 * \frac{1}{p_1} + p_2 * \frac{1}{p_2})) \\
&= 0
\end{aligned} \tag{2}
$$

hint:Jensen's Inequality:$\lambda f(x_1) + (1 - \lambda) f(x_2) \leq f(\lambda x_1 + (1 - \lambda) x_2)$, for concave function

2 互信息

Problem 3 设 X,Y,Z,T,满足 $H(T|X) = H(T), H(T|X, Y) = 0, H(T|Y) = H(T), H(Y|Z) = 0, H(T|Z) = 0$,

证明:(1)$H(T|X, Y, Z) = I(Z, T|X, Y) = 0$

$$H(T|X, Y, Z) = H(T|X, Y) - I(T; Z|X, Y) \geq 0$$

Because $H(T|X, Y) = 0$, we can come to the fact that $I(T; Z|X; Y) = 0$, as a consequence we can conclude that $H(T|X, Y, Z) = 0$

(2)$I(X; T|Y, Z) = I(Y; T|X, Z) = 0$

$$I(X; T|Y, Z) = H(T|Y, Z) - H(T|X, Y, Z)$$

From (1), we have the fact that $H(T|X, Y, Z) = 0$. Moreover, we have the fact that $H(T|Z) = 0 = I(T; Y|Z) + H(T|Y, Z)$ and $I(T; Y|Z) \geq 0$, such that $H(T|Y, Z) = 0$. As a consequence, we can conclude that $I(X; T|Y, Z) = 0$. We can draw thE conclusion that $I(Y; T|X, Z) = 0$ with the same method.

(3)$I(X; Y|Z, T) = 0$

$$I(X; Y|Z, T) = H(Y|Z, T) - H(Y, X|Z, T)$$

Because $H(Y|Z) = 0$, we can easily have $H(Y|Z,T) = 0$. Meanwhile, $I(X;Y|Z,T) \geq 0$, such that $H(Y|X,Z,T)$ can only be 0. As a consequence, we can conclude that $I(X;Y|Z,T) = 0$

(4)$I(X;Z) \geq H(T)$