Exercises

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2020 年 9 月 28 日

1　散度 Divergence

Problem 1　用另外一种方式证明 $D(p||q) \geq 0$

$$\begin{aligned}
-D(p||q) &= \sum p(x) ln \frac{q(x)}{p(x)} \\
&\leq \sum p(x)(\frac{q(x)}{p(x)} - 1) \\
&= \sum (q(x) - p(x)) \\
&= 1 - 1 \\
&= 0
\end{aligned} \tag{1}$$

hint:$lnx \leq x - 1$

Problem 2　混淆增加熵率

证明：$\vec(p) = (p_1, p_2, ..., p_m)$，$\vec{q} = (p_1, p_2, ..., \frac{p_i+p_j}{2}, ..., \frac{p_j+p_i}{2}, ..., p_m)$，有

$(\vec{q}) \geq H(\vec{p})$

$$\begin{aligned}
H(\vec{q}) - H(\vec{p}) &= -2 * \frac{p_i + p_j}{2} log \frac{p_i + p_j}{2} + p_i log p_i + p_j log p_j \\
&= -p(log \frac{p_i + p_j}{2} - \frac{p_i}{p} log p_i - \frac{p_j}{p} log p_j)(p = p_i + p_j) \\
&= -p(log \frac{p_i + p_j}{2} - log p - (\frac{p_i}{p} log \frac{p_i}{p}) - (\frac{p_j}{p} log \frac{p_j}{p})) \\
&= -p(log \frac{p_i + p_j}{2} - log p + p_1 log \frac{1}{p_1} + p_2 log \frac{1}{p_2}) \\
&\geq -p(log \frac{p}{2} - log p + log(p_1 * \frac{1}{p_1} + p_2 * \frac{1}{p_2})) \\
&= 0
\end{aligned} \tag{2}$$

hint:Jensen's Inequality:$\lambda f(x_1) + (1-\lambda) f(x_2) \leq f(\lambda x_1 + (1-\lambda) x_2)$, for concave function

2 互信息

Problem 3 设 X,Y,Z,T,满足 $H(T|X) = H(T), H(T|X,Y) = 0, H(T|Y) = H(T), H(Y|Z) = 0, H(T|Z) = 0$，

证明：(1)$H(T|X,Y,Z) = I(Z,T|X,Y) = 0$

$$H(T|X,Y,Z) = H(T|X,Y) - I(T;Z|X,Y) \geq 0$$

Because $H(T|X,Y) = 0$, we can come to the fact that $I(T;Z|X;Y) = 0$, as a consequence we can conclude that $H(T|X,Y,Z) = 0$

(2)$I(X;T|Y,Z) = I(Y;T|X,Z) = 0$

$$I(X;T|Y,Z) = H(T|Y,Z) - H(T|X,Y,Z)$$

From (1), we have the fact that $H(T|X,Y,Z) = 0$. Moreover, we have the fact that $H(T|Z) = 0 = I(T;Y|Z) + H(T|Y,Z)$ and $I(T;Y|Z) \geq 0$, such that $H(T|Y,Z) = 0$. As a consequence, we can conclude that $I(X;T|Y,Z) = 0$. We can draw thE conclusion that $I(Y;T|X,Z) = 0$ with the same method.

(3)$I(X;Y|Z,T) = 0$

$$I(X;Y|Z,T) = H(Y|Z,T) - H(Y,X|Z,T)$$

Because $H(Y|Z) = 0$, we can easily have $H(Y|Z,T) = 0$. Meanwhile, $I(X;Y|Z,T) \geq 0$, such that $H(Y|X,Z,T)$ can only be 0. As a consequence, we can conclude that $I(X;Y|Z,T) = 0$

(4)$I(X;Z) \geq H(T)$

3 信道容量

Problem 4 Binary Erasure Channel(二进制擦拭信道)，问：找到 $p(x)$，使得 $C = Imax(X;Y)$, 图见 exe1_4

$$
\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) = H(Y) - H(Y|X) \\
p(y) &= \sum_x p(x,y) = \sum_x p(y|x)p(x) \\
H(Y) &= -\sum_y p(y)logp(y) \\
&= -\sum_y (\sum_x p(y|x)p(x))log(\sum_x p(y|x)p(x))
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
p(Y=0) &= p_0(1-p-\epsilon) + (1-p_0)p = p_0(1-29-\epsilon) + p \\
p(Y=1) &= p_0 p + (1-p_0)(i-p-\epsilon) = (1-p-\epsilon) + (2p+\epsilon-1)p_0 \\
p(Y=\epsilon) = p_0\epsilon + (1-p_0)\epsilon = \epsilon
\end{aligned} \tag{4}
$$

From equation(4) above, we can get that $p(Y=0) + p(Y=1) = 1-\epsilon$. Apparently, H(Y) reaches its maximum when $p_0 = \frac{1}{2}$.

$$
H(Y|X) = \sum p(x)H(Y|X=x) = p(0)H(Y|X=0) + p(1)H(Y|X=1)
$$

From the transition matrix, we can get that $H(Y|X=0) = H(Y|X=1)$. So that, we have $H(Y|X) = H(Y|X=0) = H(Y|X=1)$, which is only related to ϵ and p, so $I(X;Y)$ reaches its maximum, when $H(Y)$ reaches its maximum, which has nothing to do with $H(Y|X)$.

Problem 5 (转移概率矩阵和及串联信道) 问 X→Y 信道的转移概率，图见 exe1_5 method 1:

$$
\begin{aligned}
p(x=0,y=0) &= p(x=0)p(y=0|y=0) \\
&= p(x=0)(p(y=0,z=0|x=0)+p(y=0,z=1|x=0)) \\
&= p(x=0)(p(y=0|z=0)p(z=0|x=0)+p(y=0|z=1)p(z=1|x=0))
\end{aligned} \tag{5}
$$

So we can get,

$$
\begin{aligned}
p(y=0|x=0) &= (1-p_2)(1-p_1)+p_1p_2 \\
p(y=1|x=0) &= (1-p_1)p_2+(1-p_2)p_1 \\
p(y=0|x=1) &= p_1(1-p_2)+(1-p_1)p_2 \\
p(y=1|x=1) &= (1-p_1)(1-p_2)+p_1p_2
\end{aligned} \tag{6}
$$

method 2: 见图片