The Two Envelope Paradox

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Introduction

In this paper we will look at the Two Envelope Paradox and argue to resolve it.

Problem

First we outline the *Two Envelope Problem.* The "vendor" has two identical (initially empty) envelopes, E_1 and E_2. They then choose a random positive real number k, and slip a k-dollar check in E_1, and a $2k$-dollar check in E_2. Now the "customer" walks up to the vendor and chooses one of the envelopes (arbitrarily). The customer will win the amount of money enclosed in the envelope they choose. Call the customer's choice E, and the other envelope $\overline{E}$.

Then the vendor invites the customer to switch their choice of envelope. To maximize their profit, should the customer switch?

Intuitively we know that switching envelopes (or any other algorithm) should not matter, since the customer has no information about the contents of the envelopes, so they should end up with the more profitable envelope E_2 50% of the time, and the less profitable envelope E_1 50% of the time also.

Expectation

But let's, for a second, calculate the expected value $\mathbb{E}[E]$ of sticking with E and the expected value $\mathbb{E}[\overline{E}]$ of switching to $\overline{E}$. Suppose E has some check for d dollers enclosed. Then $\mathbb{E}[E] = d$. Then we know that $\overline{E}$ has either $2d$ or $\frac{d}{2}$ dollars enclosed inside, with equal probability. Then

$$\mathbb{E}[\overline{E}] = \frac{1}{2} \cdot 2d + \frac{1}{2} \cdot \frac{d}{2} = \frac{5d}{4} > d = \mathbb{E}[E]$$

which means the customer should expect to get more money by switching envelopes.

But we had argued earlier that the customer should be no better off switching envelopes than they are sticking with their choice. What went wrong?

Rayo's Approach

In the course, Professor Rayo argues that there must be an issue with the fact that we are sampling from an infinite set, and that the expected value for playing the game turns out to be infinite[1], which can be problematic.

I am not sure how exactly an infinite expected gain creates an issue within the context of this game. Professor Rayo, too, admits that he does "not feel ready to let the matter rest". I will offer a different explanation for the seeming "paradox" we have on our hands.

Hypothetical Situation

It seems to me that the expected value we calculate in the *Expectation* section describes a different problem, which we will call the *2.0 Envelope Problem.* First consider $\mathbb{E}[E] = d$. This describes a situation in which we know, for certain, that the customer's envelope contains d dollars. So we can imagine that the vendor gives the customer the envelope E with an unknown but specific amount of money d.

What about $\mathbb{E}[\overline{E}] = \frac{1}{2} \cdot 2d + \frac{1}{2} \cdot \frac{d}{2}$? This describes a situation in which the customer can switch to an envelope with either $2d$ or $\frac{d}{2}$, with equal probability. So we can imagine that the vendor holds up the envelope $\overline{E}$ to the customer, with either $2d$ or $\frac{d}{2}$ enclosed.

If you decide to switch envelopes, we calculated earlier that the expected value of switching *another time* should still, symmetrically, be $\frac{5}{4}$ of the expected value of staying at that point. This describes the following situation: after switching envelopes with the vendor let the amount of money that the customer is now holding be d'. The vendor will then again procure a new envelope E' with either $2d'$ or $\frac{d'}{2}$ dollars enclosed, and ask the customer if they want to switch their envelope with E'. This is identical to the situation the customer found themselves in at the beginning, as desired.

Note that in the *2.0 Envelope Problem*, the calculation of the expectations dictates that it is clearly more reasonable to switch, every time. There is no paradox that comes with this conclusion, because if the customer decides to switch, the vendor is able to bring more money to the table in E' depending on how much money the customer is holding at the moment, and the customer's profits can increase without bound. This differs starkly from the *Two Envelope Problem*, where if the customer decides to switch, there is still *the same amount of money* on the table, so switching cannot increase their profits without bound.

Therefore it seems that our calculation in the *Expectation* section and the *Two Envelope Problem* actually represent different situations. But then what should the real expected value calculation be?

[1] The least amount of money you stand to gain is just the contents of E_1, whose value is unbounded.

Expectation, revisited

We can start by modeling the expectation on what actually transpired between the vendor and the customer. First the vendor chooses some positive real number k, and decides that the envelopes E_1 and E_2 must contain k and $2k$ dollars, respectively. Then the customer walks up and chooses one of these envelopes at random. Say their choice is E.

What is the amount of money that E encloses? Well, either the customer has chosen E_1 or E_2, and they choose either envelope with equal probability. Then we can write

$$\mathbb{E}[E] = \frac{1}{2} \cdot k + \frac{1}{2} \cdot 2k = \frac{3k}{2}$$

and symmetrically, $\overline{E}$ is E_1 or E_2 with equal probability, so we can write

$$\mathbb{E}[\overline{E}] = \frac{1}{2} \cdot k + \frac{1}{2} \cdot 2k = \frac{3k}{2}.$$

We can see that we do not expect to do better when we switch rather than stay, and this exactly corresponds with our intuition.

Why were we wrong before?

Let's start from the first assumption we made when calculating the expectation: that the customer's envelope E must contain some number of dollars, d. While this is true, we first have to take into account the distinguishing feature between the *Two Envelope Problem* and the *2.0 Envelope Problem*: the amount of money on the table is *fixed* in the *Two Envelope Problem*. Namely, the total amount of money is $3k$, where k is the random real number that the vendor chose earlier.

Since we have assumed that E contains d dollars, and we know the total amount of money is *fixed* at $3k$ dollars, that means that $\overline{E}$, the other envelope, has no choice but to enclose $3k - d$ dollars. (Note that assuming that $\overline{E}$ has $2d$ or $\frac{d}{2}$ dollars makes the total amount of money change depending on d, which is not allowed in the *Two Envelope Problem*.)

Therefore we can say that $\mathbb{E}[E] = d$ and $\mathbb{E}[E] = 2k - d$. But this isn't very helpful–the outcome for each envelope is fixed once we fix d, and it means nothing to us since we do not know what d is. So instead of assuming that the customer's envelope E contains some d dollars, we can see that it contains either k or $2k$ dollars with equal probability, and proceed as we did in the section *Expectation, revisited*.

Conclusion

In this paper, we have provided an argument that the calculation of expectation for that *Two Envelope Problem* in the course is incorrect, and have given reasons as to why it is incorrect, as well as an alternative calculation of expectation.