1. Cauchy-Schwarz inequality:

$$|u \cdot v| \le \|u\| \|v\| \quad \text{OR} \quad \left(\sum_{i=1}^{n} u_i v_i\right)^2 \le \left(\sum_{i=1}^{n} u_i^2\right) \left(\sum_{i=1}^{n} v_i^2\right)$$

where the equality holds iff $u \cdot v$ are linearly dependent.

proof:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (u_i v_j - u_j v_i)^2 = \sum_{i=1}^{n} u_i^2 \sum_{j=1}^{n} v_j^2 + \sum_{i=1}^{n} v_i^2 \sum_{j=1}^{n} u_j^2 - 2 \sum_{i=1}^{n} u_i v_i \sum_{j=1}^{n} u_j v_j$$

$$= 2\left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{j=1}^{n} v_j^2\right) - 2\left(\sum_{i=1}^{n} u_i v_i\right)^2$$

Since LHS ≥ 0, we have $\left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{j=1}^{n} v_j^2\right) \ge \left(\sum_{i=1}^{n} u_i v_j\right)^2$.

The equality holds iff $u_i v_j = u_j v_i$, i.e. $\frac{u_i}{u_j} = \frac{v_i}{v_j}$ $\forall\, i, j = 1 \dots n$ (linearly dependent).

Q.E.D.

It can be generalized by replacing all finite sums to integrals:

$$\left| \int_{-\infty}^{+\infty} \phi_1(x) \phi_2^*(x) dx \right|^2 \le \int_{-\infty}^{+\infty} |\phi_1(x)|^2 dx \int_{-\infty}^{+\infty} |\phi_2(x)|^2 dx$$

The proof is similar and thus omitted here.

Hölder's inequality generalizes Cauchy-Schwarz inequality.

$$|u \cdot v| \le \|u\|_p \|v\|_q \quad \text{OR} \quad \sum_{i=1}^{n} |u_i v_i| \le \left(\sum_{i=1}^{n} |u_i|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |v_j|^p\right)^{\frac{1}{p}}$$

where $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Note that l_∞ norm reduces to $\|x\|_\infty = \max_i |x_i|$.

2. Detection in AWGN channel with finite input set $\mathcal{A}$ (alphabet)

$$x \in \mathcal{A} \longrightarrow \underset{\substack{\uparrow \\ n \sim (0,\sigma^2)}}{\oplus} \longrightarrow y \longrightarrow \boxed{\text{Detector}} \longrightarrow \hat{x} \in \mathcal{A}$$

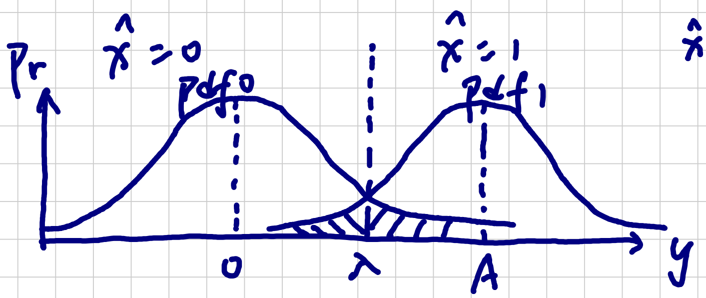
- x is the input letter to transmit
- n is AWGN with zero mean and variance σ^2
- y is the received signal following Gaussian distribution $y \sim (x, \sigma^2)$
- $\hat{x}$ is the detected letter

Success detection: $\hat{x} = x$

For simplicity, let $\mathcal{A} = \{0, A\}$: binary-input $\{0, 1\}$

- If $x = 0$, then $y \sim (0, \sigma^2)$
- If $x = A$, then $y \sim (A, \sigma^2)$
- Decision threshold λ: $\begin{cases} \hat{x} = 0 & \text{if } y < \lambda \\ \hat{x} = 1 & \text{if } y > \lambda \end{cases}$

  $\hat{x} = ?$ if $y = \lambda$ (uncertain)



Error: $\hat{x} \neq x$ $\begin{cases} x = 0, \hat{x} = 1 & ① \\ x = 1, \hat{x} = 0 & ② \end{cases}$

① : $x=0$. $\hat{x}=1$ input probability Gaussian w. pdf 0

$$P_{e0}=P(x=0,\hat{x}=1)=P(x=0)\,P(\hat{x}=1\,|\,x=0)$$

$$P(\hat{x}=1\,|\,x=0)=\int_{\lambda}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{n^2}{2\sigma^2}}dn$$ is a function of λ. σ.

② : $x=1$. $\hat{x}=0$

$$P_{e1}=P(x=1,\hat{x}=0)=P(x=1)\,P(\hat{x}=0\,|\,x=1)$$

$$P(\hat{x}=0\,|\,x=1)=\int_{-\infty}^{\lambda}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(n-A)^2}{2\sigma^2}}dn$$ also depends on A!

$$P_e=P_{e0}+P_{e1}=\underset{P(x=1)}{\underset{\shortparallel}{P_1}}\int_{-\infty}^{\lambda}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(n-A)^2}{2\sigma^2}}dn+\underset{P(x=0)}{\underset{\shortparallel}{(1-P_1)}}\int_{\lambda}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{n^2}{2\sigma^2}}dn$$

$$\frac{dP_e}{d\lambda}=0\Rightarrow\lambda^*=-\frac{\sigma^2}{A}\ln\frac{P_1}{1-P_1}+\frac{A}{2}$$, depends on the input probability.

Consider equiprobable input $P(x=0)=P(x=1)=0.5$.

$\lambda^*=\frac{A}{2}$ ensures $P_e=P(\hat{x}=1|x=0)=P(\hat{x}=0|x=1)$

$$z\triangleq\frac{n}{\sigma}\Rightarrow P_e=\frac{1}{\sqrt{2\pi}}\int_{\frac{A}{2\sigma}}^{\infty}e^{-\frac{z^2}{2}}dz$$

$$\boxed{Q(x)=\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}e^{-\frac{t^2}{2}}dt}$$

$$=Q\left(\frac{A}{2\sigma}\right).$$