PROBLEM SHEET - FT Tables Refresher

EE3-27: Principles of Classical and Modern Radar

Contents

1 Evaluating Functions of δ-lines

Evaluate:

(a) $\int\limits_{-\infty}^{\infty} (t^4 - 3t + 1).\delta(t-2)\ .dt$

$$\int\limits_{-\infty}^{\infty} (t^4 - 3t + 1).\delta(t-2)\ .dt = (t^4 - 3t + 1)|_{t=2} = 2^4 - 3 \times 2 + 1 = 11$$

(b) $\int\limits_{-\infty}^{\infty} \left(\cos(4\pi t) * \delta(t + \frac{1}{4})\right).\delta(t - \frac{1}{8}).dt$

$$\int\limits_{-\infty}^{\infty} \left(\cos(4\pi t) * \delta(t + \tfrac{1}{4})\right).\delta(t - \tfrac{1}{8}).dt = \int\limits_{-\infty}^{\infty} \cos\left(4\pi(t + \tfrac{1}{4})\right).\delta(t - \tfrac{1}{8}).dt =$$
$$= \cos\left(4\pi(t + \tfrac{1}{4})\right)|_{t=\frac{1}{8}} = \cos\left(4\pi(\tfrac{1}{8} + \tfrac{1}{4})\right) = \cos(\tfrac{3}{2}\pi) = 0$$

(c) $\int\limits_{-\infty}^{\infty} (t^3 - 3t^2 - 11).\delta(t-1)\ .dt$ (5%)

$$\int\limits_{-\infty}^{\infty} (t^3 - 3t^2 - 11).\delta(t-1)\ .dt = (t^3 - 3t^2 - 11)|_{t=1} = 1^3 - 3 \times 1^2 - 11 = -13$$

(d) $\int\limits_{-\infty}^{\infty} \left\{(\sin(4\pi t) * \delta(t + \frac{1}{4})\right\}.\delta(t - \frac{1}{4}).dt$ (5%)

$$\int\limits_{-\infty}^{\infty} \left\{(\sin(4\pi t) * \delta(t + \tfrac{1}{4})\right\}.\delta(t - \tfrac{1}{4}).dt = \int\limits_{-\infty}^{\infty} \left(\sin(4\pi\left(t + \tfrac{1}{4}\right)\right).\delta(t - \tfrac{1}{4}).dt =$$
$$\sin\left(4\pi\left(t + \tfrac{1}{4}\right)\right)|_{t=\frac{1}{4}} = \sin\left(4\pi\left(\tfrac{1}{4} + \tfrac{1}{4}\right)\right) = \sin(2\pi) = 0$$

(e) $\int\limits_{-\infty}^{\infty} (t^3 - 2t^2 + 1).\delta(t-2)\ .dt$ (5%)

$$\int\limits_{-\infty}^{\infty} (t^3 - 2t^2 + 1).\delta(t-2)\ .dt = (t^3 - 2t^2 + 1)|_{t=2} = 2^3 - 2 \times 2^2 + 1 = 1$$

(f) $\int\limits_{-\infty}^{\infty} \left\{(\cos(2\pi t) * \delta(t - \frac{1}{4})\right\}.\delta(t - \frac{1}{12}).dt$ (5%)

$$\int\limits_{-\infty}^{\infty} \left(\cos(2\pi t) * \delta(t - \tfrac{1}{4})\right).\delta(t - \tfrac{1}{12}).dt = \int\limits_{-\infty}^{\infty} \cos\left(2\pi(t - \tfrac{1}{4})\right).\delta(t - \tfrac{1}{12}).dt =$$
$$= \cos\left(2\pi(t - \tfrac{1}{4})\right)|_{t=\frac{1}{12}} = \cos\left(2\pi(\tfrac{1}{12} - \tfrac{1}{4})\right) = \cos(-\tfrac{1}{3}\pi) = \tfrac{1}{2}$$

(g) $h(3)$ where $h(t) = \left(t.\text{rect}\left\{\frac{t}{8}\right\}\right) * \delta(t+3)$ (5%)

$h(t) = \left(t.\text{rect}\left\{\frac{t}{8}\right\}\right) * \delta(t+3) = (t+3)\,.\text{rect}\left\{\frac{t+3}{8}\right\}$

$\Rightarrow h(3) = 0$

(h) $h(3)$ where $h(t) = \left(t.\text{rect}\left\{\frac{t}{8T}\right\}\right) * \delta(t-2)$ (10%)

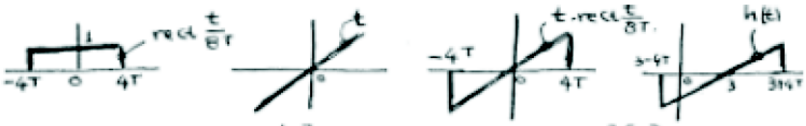
$h(t) = \left(t.\text{rect}\left\{\frac{t}{8T}\right\}\right) * \delta(t-2) = (t-2)\,.\text{rect}\left\{\frac{t-2}{8T}\right\}$

$h(3) = (3-2).\text{rect}\left\{\frac{3-2}{8T}\right\}$

$$\Rightarrow h(t) = \begin{cases} t-2 & \text{if } -0.5 < \frac{t-2}{8T} < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow h(3) = \begin{cases} 1 & \text{if } -0.5 < \frac{t-2}{8T} < 0.5 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } T > \frac{1}{4} \\ 0 & \text{if } T \leq \frac{1}{4} \end{cases}$$

(i) $h(3.5)$ where $h(t) = \left(t.\text{rect}\left\{\frac{t}{8T}\right\}\right) * \delta(t-3)$ (10%)



i.e. $h(t) = (t-3)\,\text{rect}\,\frac{t-3}{8T} \Rightarrow h(3.5) = (3.5-3)\,\text{rect}\,\frac{3.5-3}{8T} \Rightarrow$

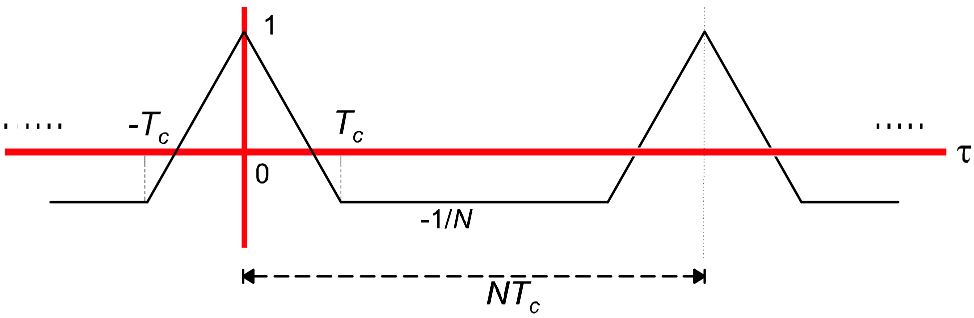
$h(3.5) = 0.5 \times \text{rect}\left(\frac{0.5}{8T}\right)$

$\frac{0.5}{8T} < \frac{1}{2} \Rightarrow T > \frac{1}{8}$

i.e. $h(3.5) = \begin{cases} 0.5 & \text{if } T > \frac{1}{8} \\ 0 & \text{if } T < \frac{1}{8} \end{cases}$

2 Auto-Correlation Function and PSD(f)

The waveform below shows the autocorrelation function $R_{bb}(\tau)$ of what is called in communications a pseudo-random (PN) signal $b(t)$.



(a) Write a mathematical expression, using Woodward's notation, to describe the above autocorrelation function. (15%)

(b) Find the power spectral density $\text{PSD}_b(f)$ of $b(t)$. (20%)

Solution

(a) $R_{bb}(\tau) = \frac{N+1}{N}\mathbf{rep}_{NT_c}\left\{\Lambda\left(\frac{\tau}{T_c}\right)\right\} - \frac{1}{N}$

(b) $\text{PSD}(f) = \text{FT}\{R_{bb}(\tau)\} = \frac{N+1}{N^2}\text{comb}_{\frac{1}{NT_c}}\left\{\text{sinc}^2(fT_c)\right\} - \frac{1}{N}\delta(f)$

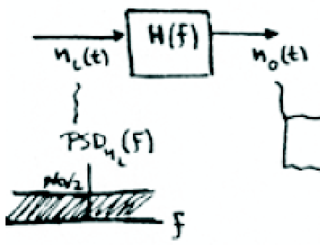
3 Transfer Function and PSD(f)

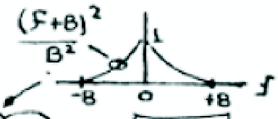
At the input of a filter there is white Gaussian noise of power spectral density $\mathrm{PSD}_{n_i}(f) = \frac{3}{2}10^{-6}$. If the transfer function of the filter is

$$H(f) = \Lambda\left\{\frac{f}{10^6}\right\}\exp\left(-j\phi(f)\right)$$

calculate the power of the signal at the output of the filter. (10%)

Solution





$$\boxed{B = 10^6}$$

$$\frac{N_o}{2} = \frac{3}{2}\times 10^{-6} \Rightarrow \boxed{N_o = 3\times 10^{-6}}$$

$$\mathrm{PSD}_{n_o}(f) = \overbrace{\mathrm{PSD}_{n_i}(f)}^{\frac{N_o}{2}} \cdot \overbrace{|H(f)|^2}^{\frac{(f+B)^2}{B^2}}$$

$$P_{n_o} = \int_{-\infty}^{+\infty} \mathrm{PSD}_{n_i}(f)\cdot|H(f)|^2\, df$$

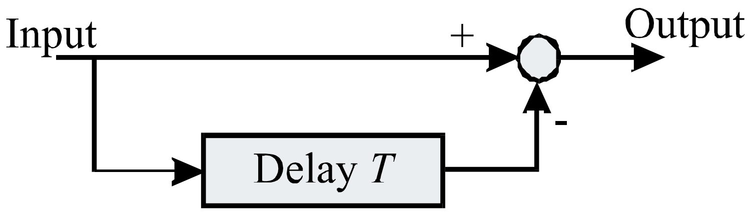
$$= \frac{N_o}{2}\int_{-B}^{+B}|H(f)|^2 df = \frac{N_o}{2}\, 2\int_{-B}^{0}|H(f)|^2 df$$

$$= N_o\int_{-B}^{0}\frac{(f+B)^2}{B^2} df = \frac{N_o}{B^2}\left[\frac{(f+B)^3}{3}\right]_{-B}^{0}$$

$$\Rightarrow \boxed{P_{n_o} = \frac{N_o B}{3}} \boxed{= \frac{3\times10^{-6}\times10^6}{3} = 1}$$

4 Differential Circuit

For the following differential circuit



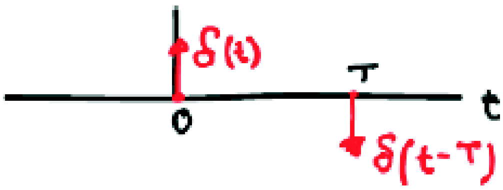
find:

(a) the impulse response and (5%)

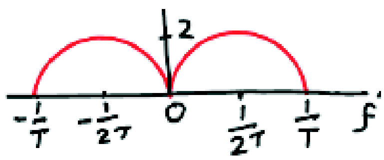
(b) frequency response (5%)

Solution

(a) $h(t) = \delta(t) - \delta(t - T)$



(b) $H(f) = \text{FT}\{h(t)\} = 1 - \exp(-j2\pi fT)$
$= \{\exp(j\pi fT) - \exp(-j\pi fT)\} \exp(-j\pi fT)$
$= 2j \sin(\pi fT) \exp(-j\pi fT)$

$\Rightarrow |H(f)| = 2\,|\sin(\pi fT)| = 2\left|\sin(\frac{\pi f}{1/T})\right| =$ 

5 Impulse Response and PSD(f)

Consider the filter with impulse response

$$h(t) = \text{sinc}^2\left\{10^6(t-3)\right\}$$

and assume that the input signal $n_i(t)$ is white Gaussian noise with double-sided power spectral density $\text{PSD}_{n_i}(f) = 1.5 \times 10^{-6}$ W/Hz.

For the signal $n(t)$ at the output of the filter

(a) find and plot its power spectral density $\text{PSD}_n(f)$; (10%)

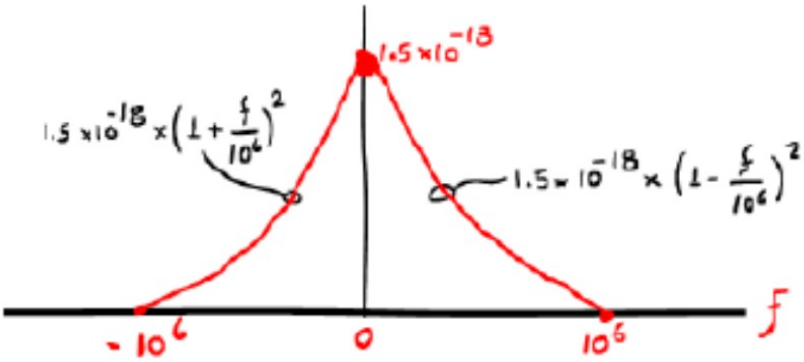
(b) calculate its power P_n (5%)

Solution

(a) $H(f) = \text{FT}\{h(t)\} = \frac{1}{10^6}\Lambda\left(\frac{f}{10^6}\right)\exp\left(-j2\pi f \times 3\right)$

$\Rightarrow \text{PSD}_n(f) = \text{PSD}_{n_i}(f).\left|H(f)\right|^2 =$

$1.5 \times 10^{-6}\left(\frac{1}{10^6}\right)^2\Lambda^2\left(\frac{f}{10^6}\right) = 1.5 \times 10^{-18}\Lambda^2\left(\frac{f}{10^6}\right)$



(b) $P_n = \int\limits_{-10^6}^{10^6}\text{PSD}_n(f).df = 2\int\limits_0^{10^6}\text{PSD}_n(f).df =$

$= 2\int\limits_0^{10^6}\left(1.5 \times 10^{-18} \times \left(1 - \frac{f}{10^6}\right)^2\right).df =$

$= 2 \times 1.5 \times 10^{-18} \times \int\limits_0^{10^6}\left(1 - \frac{f}{10^6}\right)^2.df =$

$= 3 \times 10^{-18} \times \int\limits_0^{10^6}\left(1 - 2\frac{f}{10^6} + \frac{f^2}{10^{12}}\right).df$

$= 3 \times 10^{-18} \times \left(10^6 - 2\frac{10^{12}}{2\times10^6} + \frac{10^{18}}{3\times10^{12}}\right)$

$= 3 \times 10^{-18} \times \left(10^6 - 10^6 + \frac{1}{3}10^6\right)$

$= 10^{-12}$

6 PSD(f) at Bandpass Filter's Output

Consider a bandpass filter with impulse response

$$h(t) = 8 \times 10^3 \mathrm{sinc}\left\{4 \times 10^3 t\right\} . \cos(2\pi 10^4 t)$$

and assume that at the input of this filter there is white Gaussian noise $n_i(t)$ of power spectral density $\mathrm{PSD}_{n_i}(f) = 10^{-6}$.

For the signal $n(t)$ at the output of the filter

(a) find and plot its power spectral density $\mathrm{PSD}_n(f)$; (10%)

(b) calculate its power P_n (5%)

Solution

$$H(f) = FT\{h(t)\} = 8\times10^3 \; FT\{\mathrm{sinc}(4\times10^3 t)\cos(2\pi 10^4 t)\}$$

$$= 8\times10^3 \underbrace{FT\{\mathrm{sinc}(4\times10^3 t)\}}_{\frac{1}{4\times10^3}\mathrm{rect}\frac{f}{4\times10^3}} \circledast \left[\tfrac{1}{2}\delta(f-10^4) + \tfrac{1}{2}\delta(f+10^4)\right]$$

convolution

$$= \mathrm{rect}\frac{f-10^4}{4\times10^3} + \mathrm{rect}\frac{f+10^4}{4\times10^3}$$

$$PSD_n(f) = PSD_{n_i}(f)\cdot|H(f)|^2 = 10^{-6}\,\mathrm{rect}\frac{f-10^4}{4\times10^3} + 10^{-6}\,\mathrm{rect}\frac{f+10^4}{4\times10^3}$$

$$P_n = 4\times10^3 \times 10^{-6} \times 2 = 8\times10^{-3}\,W = \underline{\underline{8mW}}$$

END