Probability and Random Processes

1. If $F_X(x)$ is the distribution function of a random variable X and $x_1 \leq x_2$, show that $F_X(x_1) \leq F_X(x_2)$.

2. Use the definition of cumulative distribution function to write an expression for the probability of a random variable to take values between x_1 and x_2, and take limiting cases to arrive at the definition of the probability density function as the derivative of the distribution function.

3. Show that
$$F_X(x|A) = \frac{P\{A|X \leq x\}F_X(x)}{P\{A\}}$$

4. Show that if two random variables are independent, they are also uncorrelated.

5. Show that the covariance of two random variables $Cov(X,Y) \triangleq E[(X-\mu_X)(Y-\mu_Y)]$ is equal to:
$$Cov(X,Y) = E[XY] - \mu_X\mu_Y,$$
where μ_X and μ_Y are the mean values of X and Y, respectively. Then, show that the covariance of two random variables is zero, the two random variables are uncorrelated.

6. The random variable x is uniform in the interval $(0, 1)$. Find the density of the random variable $y = -\ln x$.
$$xf_X(x) = e^{-y}f_X(e^{-y}) = e^{-y}U(y)$$

7. If $y = \sqrt{x}$ and x is an exponential random variable, show that y represents a Rayleigh random variable.

8. [Tchebycheff Inequality] Let X be a random variable with a finite mean value η and a non-zero variance σ^2. Prove that, for any $\epsilon > 0$
$$P\{|X-\eta| \geq \epsilon\sigma\} \leq \frac{1}{\epsilon^2}.$$

9. For a Poisson random variable x with parameter λ show that (a) $P(0 < x < 2\lambda) > (\lambda-1)/\lambda$; (b) $E[x(x-1)] = \lambda^2$, $E[x(x-1)(x-2)] = \lambda^3$.

10. Show that if the random variable x is $N(\eta; \sigma^2)$, then
$$E\{|x|\} = \sigma\sqrt{\frac{2}{\pi}}e^{-\eta^2/2\sigma^2} + 2\eta G\left(\frac{\eta}{\sigma}\right) - \eta$$

11. X and Y are independent identically distributed normal random variables with zero mean and common variance σ^2, that is, $X \sim \mathcal{N}(0,\sigma^2)$, $Y \sim \mathcal{N}(0,\sigma^2)$ and $f_{XY}(x,y) = f_X(x)f_Y(y)$. Find the p.d.f of (a) $Z = \sqrt{X^2+Y^2}$, (b) $Z = X^2 + Y^2$ (c) $U = X - Y$.

12. The events A, B, C are such that
$$P(A) = P(B) = P(C) = 0.5$$
$$P(AB) = P(AC) = P(BC) = P(ABC) = 0.25$$

Show that the zero-one random variables associated with these events are not independent; they are, however, independent in pairs.

13. Consider the randomly-phased sinusoid

$$n(t) = A\cos(2\pi f_C t + \theta)$$

where A and f_C are constant amplitude and frequency, respectively, and θ is a random phase angle uniformly distributed over the range [0; 2π]. Calculate the mean and mean square of n(t).

14. Let $X(t)$ be a wide-sense stationary random process. $X(t)$ is mixed (i.e., multiplied) by a sinusoidal signal $\cos(2\pi f_C t + \Theta)$, where the phase Θ is a random variable uniformly distributed over the interval (0; 2π). Find the power spectral density of the output process $Y(t)$ defined by

$$Y(t) = X(t)\cos(2\pi f_C t + \Theta).$$

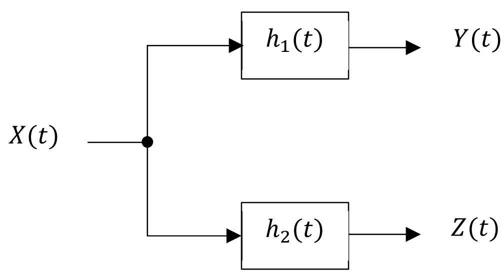
15. The random process $v(t)$ is defined as

$$v(t) = X\cos 2\pi f_c t - Y\sin 2\pi f_c t$$

where X and Y are random variables. Show that $v(t)$ is wide-sense stationary if and only if $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2)$, and $E(XY) = 0$.

16. Let $X(t)$ be a wide-sense stationary process with an autocorrelation function $R_X(\tau)$. Prove that $|R_X(\tau)| \le R_X(0)$ for any τ.

17. A stationary zero-mean Gaussian random process $X(t)$ is passed through two linear filters with impulse responses $h_1(t)$ and $h_2(t)$, yielding processes $Y(t)$ and $Z(t)$, respectively, as shown in the following figure.



Show that $Y(t)$ and $Z(t)$ are statistically independent if the transfer functions $H_1(f)$ and $H_2(f)$do not overlap in the frequency domain (for example, when they are narrowband filters at different frequency bands).