FM and Digital Representation of Signals (Lectures 7 and 8)

1. **An FM modulated signal is given by**

$$\boldsymbol{x(t) = 10\cos(15000\pi t)}$$

for $0 \leq t \leq 1$. Find the message if $k_f = 2000$ and $f_c = 5KHz$.

Solution:

FM modulated signal is given by

$$\begin{aligned} x(t) &= 10\cos\,(15000\pi t) \\ &= A cos(2\pi f_c t + 2\pi f_k \int_0^t m(\tau)\, d\tau) \\ &= A cos(10000\pi t + 4000\pi \int_0^t m(\tau)\, d\tau) \end{aligned}$$

Therefore

$$\int_0^t m(\tau)\, d\tau = 5t/4,$$

and hence,

$$m(t) = 5/4.$$

2. **Given the baseband signal-to-noise ratio $SNR_{Baseband}$, consider an FM detector for singletone modulation, that is, the modulating wave is a sinusoidal wave.**

$$\boldsymbol{m(t) = A_m \cos(2\pi f_m t)}.$$

(a) Compute the output SNR in terms of the modulation index β, where $\beta \triangleq \Delta f/W$.

(b) Comparing with the figure of merit for a full AM system (i.e. $\mu = 1$), at what value of β will FM start to offer improved noise performance?

Solution:

From the lecture, we know

$$SNR_{FM} = 3\beta^2 \frac{P}{m_p^2} SNR_{baseband}.$$

(a) For single-tone modulation, the peak amplitude is $m_p = A_m$, while the message power is $P = \frac{1}{2}A_m^2$. Therefore,

$$SNR_{FM} = \frac{3}{2}\beta^2 SNR_{baseband}$$

(b) The figure of merit for full AM is 1/3. FM will perform better than AM if

$$\frac{3}{2}\beta^2 > \frac{1}{3}$$

That is,

$$\beta > \frac{\sqrt{2}}{3} = 0.471$$

3. **Suppose the modulating signal for FM is modelled as a zero-mean Gaussian**

random process $m(t)$ with standard deviation σ_m. One can make the approximation $m_p = 4\sigma_m$ as the overload probability $|m(t)| > 4\sigma_m$ is very small. Determine the output SNR for the FM receiver in the presence of additive white Gaussian noise, in terms of the deviation ratio β and the baseband SNR.

Solution:

Using the formula for the output SNR for FM

$$SNR_{FM} = 3\beta^2 \frac{P}{m_p^2} SNR_{baseband}$$

And the fact that the message power P=σ_m^2 and $m_p = 4\sigma_m$, one has

$$SNR_{FM} = 3\beta^2 \frac{\sigma_m^2}{16\sigma_m^2} SNR_{baseband}$$

$$= \frac{3}{16}\beta^2 SNR_{baseband}$$

4. **Assume that the bandwidth of a speech signal is between 50 Hz up to 10 KHz. We want to sample this signal at the Nyquist rate, and then quantize using 16 bits per sample. How many megabytes of storage do you need to store one hour of this speech signal?**

Solution:

This signal must be sampled at 20 KHz. Then we will have a bit rate of $16 * 20 = 320$ Kbps. In one hour we will have $60 * 60 * 320$ Kbits = 1152 Mbits = 144 Mbytes.

5. **A PCM output is produced by a uniform quantizer that has 2^n levels. Assume that the input signal is a zero-mean Gaussian process with standard deviation σ.**

(a) **If the quantizer range is required to be $\pm 4\sigma$, show that the quantization signal-to-noise is $6n - 7.3$ dB.**

(b) **Write down an expression for the probability that the input signal will overload the**
quantizer (i.e., when the input signal falls outside of the quantizer range).

Solution:

(a) A uniform quantizer has a mean square error of $P_N = \Delta^2/12$, where Δ is the separation between quantizer levels. Since the input signal has a Gaussian pdf with standard σ, the average power of the source signal is $P_S = \sigma^2$. The quantization step size is chosen such that the range covered by the 2^n quantizing levels is 8σ, giving a quantization step size of

$$\Delta = \frac{8\sigma}{2^n}$$

Substitution yields

$$P_N = \frac{64\sigma^2}{12 \times 2^{2n}} = \frac{16}{3}\frac{\sigma^2}{2^{2n}}$$

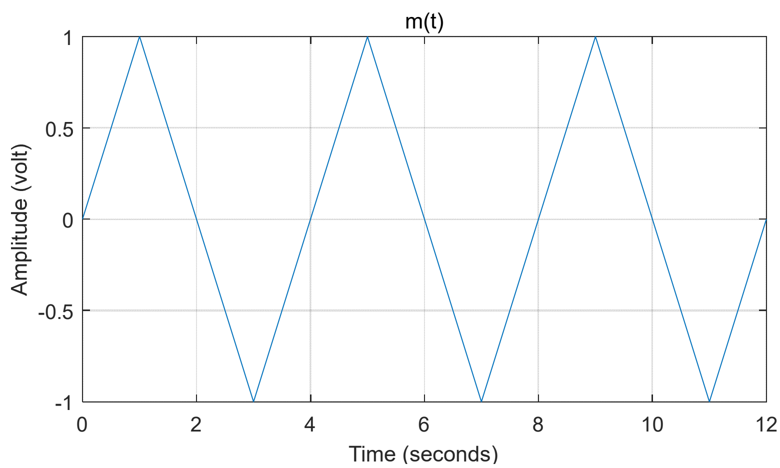
$$SNR = \frac{P_S}{P_N} = \frac{3}{16} \times 2^{2n}$$

$$10 \log_{10} SNR = 10 \log_{10} \left(\frac{3}{16}\right) + 20n \log_{10} 2$$
$$= -7.3 + 6.02n$$

(b) For the probability of overload, denoted by $P_{overload}$, we want to find $P\{X > 4\sigma\} + P\{X < -4\sigma\}$, where X is a Gaussian random variable with mean 0 and variance σ^2, i.e., $X \sim \mathcal{N}(0, \sigma^2)$.Since the pdf is symmetric, we can write

$$P_{overload} = 2P\{X > 4\sigma\}$$
$$= 2 \int_{4\sigma}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx$$
$$= 2Q\left(\frac{4\sigma}{\sigma}\right) = 6.3 \times 10^{-5}.$$

6. The input to a uniform n-bit quantizer is the periodic triangular waveform shown below, which has a period of $T = 4$ seconds, and an amplitude that varies between $+1$ and -1 Volt.

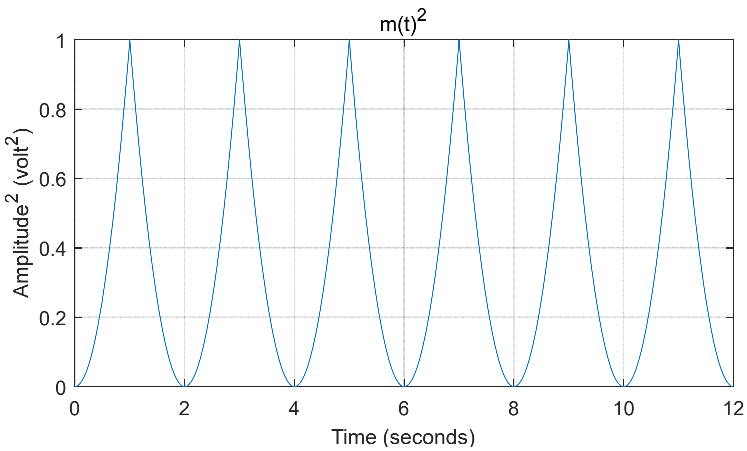


Derive an expression for the signal-to-noise ratio (in decibels) at the output of the quantizer. Assume that the dynamic range of the quantizer matches that of the input signal.

Solution:

With $m^2(t)$ shown below, the signal power is:

$$P_S = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} m^2(t) dt = \frac{4}{T} \int_0^1 m^2(t) dt$$
$$= \int_0^1 t^2 dt = \frac{1}{3}$$



The noise power is

$$P_N = \frac{\Delta^2}{12} = \frac{(\frac{2}{2^n})^2}{12} = \frac{4 \times 2^{-2n}}{12} = \frac{2^{-2n}}{3}$$

The signal to noise ratio is:

$$SNR = \frac{1/3}{2^{-2n}/3} = 2^{2n}$$

or

$$SNR_{dB} = 10log_{10}(2^{2n}) = 20n\log_{10}2 = 6.02n$$