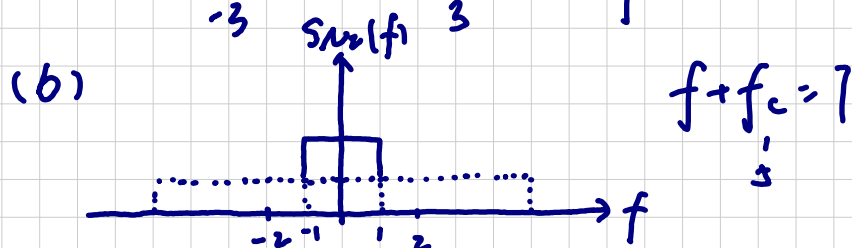
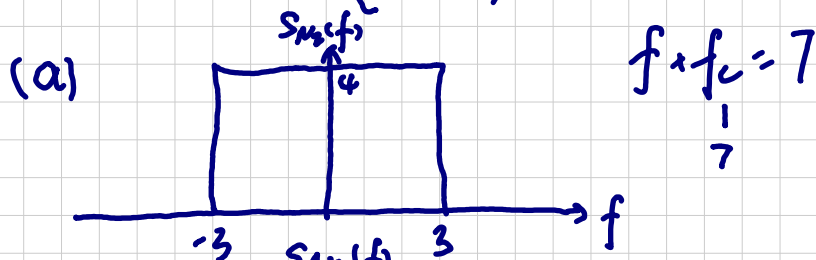


1. Baseband and passband noise PSD satisfy:

$$S_{N_1}(f) = \begin{cases} S_N(f-f_c) + S_N(f+f_c), & |f| < B \\ 0, & \text{otherwise} \end{cases}$$



2. $f_k(t)$ is defined as case function. Alternatively,

$$f_k(t) = \underbrace{e^{-\frac{t}{k}} u(t)}_{f_k^+(t)} - \underbrace{e^{\frac{t}{k}} u(-t)}_{f_k^-(t)}$$

$$F_k^+(f) = \int_0^{+\infty} e^{-\frac{t}{k}} e^{-j2\pi ft} dt = \int_0^{+\infty} e^{-(\frac{1}{k} + j2\pi f)t} dt$$

$$= -\frac{1}{\frac{1}{k} + j2\pi f} e^{-(\frac{1}{k} + j2\pi f)t} \Big|_0^{+\infty} = \frac{1}{\frac{1}{k} + j2\pi f}$$

$$F_k^-(f) = \int_{-\infty}^0 e^{\frac{t}{k}} e^{-j2\pi ft} dt = \frac{1}{\frac{1}{k} - j2\pi f}$$

$$\therefore F_k(f) = F_k^+(f) - F_k^-(f) = -\frac{j4\pi f}{\frac{1}{k^2} + 4\pi^2 f^2}$$

When $k \rightarrow \infty$, $f_k(t) \rightarrow \text{sgn}(t)$ and $F_k(f) \rightarrow \frac{1}{j2\pi f}$

\therefore FT of $\text{sgn}(t)$ is $\frac{1}{j2\pi f}$.

$$\therefore u(t) = \frac{1}{2}(\text{sgn}(t) + 1)$$

$$\therefore u(f) = \frac{1}{2}\left(\frac{1}{j2\pi f} + \delta(f)\right).$$

3. Duality of FT: $x(t) \longleftrightarrow X(f)$
 $X(t) \longleftrightarrow x(-f)$

$$\mathcal{F}(\text{sgn}(t)) = \frac{1}{j\pi f} \Rightarrow \mathcal{F}\left(\frac{1}{j\pi t}\right) = \text{sgn}(-f)$$

$$\therefore \mathcal{F}\left(\frac{1}{\pi t}\right) = j \text{sgn}(-f) = -j \text{sgn}(f)$$

$$\therefore \hat{G}(f) = -j \text{sgn}(f) G(f)$$

4. Hilbert transform: $\begin{cases} \text{shift the angles of all positive frequency components } -90^\circ \\ \text{shift the angles of all negative frequency components } +90^\circ \end{cases}$

- Amplitude spectrum unchanged, phase spectrum changed
- $\hat{g}(t) = g(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau} d\tau = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t-u)}{u} du$
- Linear operation of a "special" filter with impulse response $\frac{1}{\pi t}$
- No change of domains (i.e. FT, LT)

Analytic signal (a.k.a. pre-envelope)

- $u_+(t) = u(t) + j\hat{u}(t)$ removes the negative frequency components
- helpful to compute the instantaneous magnitude/phase of the original signal (e.g. AM, PM, FM)

Let $h(t) = \frac{1}{\pi t}$ be the impulse response of HT.

$$h(f) = \mathcal{F}(h(t)) = \int_{-\infty}^{+\infty} \frac{1}{\pi t} e^{-j2\pi f t} dt = \begin{cases} -j, & f > 0 \\ j, & f < 0 \end{cases}$$

It can be verified that $H(f) = -H(-f)$.

(a) $x(t) = x(-t) \Leftrightarrow X(f) = X(-f)$.

$$\therefore \underbrace{X(f)h(f)}_{\mathcal{F}(\hat{x}(t))} = -\underbrace{X(-f)h(-f)}_{\mathcal{F}(\hat{x}(-t))} \quad \therefore \hat{x}(t) = -\hat{x}(-t)$$

(b) $x(t) = -x(-t) \Leftrightarrow X(f) = -X(-f)$.

$$\therefore X(f)h(f) = X(-f)h(-f) \quad \therefore \hat{x}(t) = \hat{x}(-t)$$

(c) $x(t) = \cos(2\pi f_0 t)$. $X(f) = \frac{1}{2} (\delta(f-f_0) + \delta(f+f_0))$.

$$\therefore X(f)h(f) = \frac{j}{2} (\delta(f-f_0) - \delta(f+f_0)), \text{ which is the FT of } \sin(2\pi f_0 t).$$

(d) $x(t) = \sin(2\pi f_0 t)$. $X(f) = \frac{1}{2j} (\delta(f-f_0) - \delta(f+f_0))$.

$$\therefore X(f)h(f) = \frac{1}{2} (-\delta(f-f_0) - \delta(f+f_0)), \text{ which is the FT of } -\cos(2\pi f_0 t).$$

(e) $h(f) \cdot h(f) = -1$

$$\therefore X(f)h(f)h(f) = -X(f) \quad \therefore \hat{\hat{x}}(t) = x(t)$$

(f) Parseval's theorem: $\int_{-\infty}^{+\infty} f(t)g^*(t)dt = \int_{-\infty}^{+\infty} F(f)G^*(f)df$

Now $f(t) = \hat{x}(t)$. $g(t) = \hat{x}^*(t)$. $F(f) = \hat{X}(f)$. $G(f) = \hat{X}^*(-f)$.

$$\begin{aligned} \therefore \int_{-\infty}^{+\infty} \hat{x}(t)\hat{x}^*(t)dt &= \int_{-\infty}^{+\infty} \hat{X}(f)\hat{X}^*(-f)df = \int_{-\infty}^{+\infty} \hat{X}(f)h(f)\hat{X}(-f)h(-f)df \\ &= \int_{-\infty}^{+\infty} \hat{X}(f)\hat{X}(-f)df = \int_{-\infty}^{+\infty} \hat{x}^*(t)x(t)dt \end{aligned}$$

$$\begin{aligned}
 (g) \int_{-\infty}^{+\infty} \chi(t) \hat{\chi}(t) dt &= \int_{-\infty}^{+\infty} \chi(t) \int_{-\infty}^{+\infty} \frac{\chi(\tau)}{t-\tau} d\tau dt \\
 &= \int_{-\infty}^{+\infty} \chi(\tau) \int_{-\infty}^{+\infty} \frac{\chi(t)}{t-\tau} dt d\tau \\
 &= \int_{-\infty}^{+\infty} \chi(\tau) \cdot (-\hat{\chi}(\tau)) d\tau.
 \end{aligned}$$

$$\therefore \int_{-\infty}^{+\infty} \chi(t) \hat{\chi}(t) dt = 0.$$