1. $F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx$

$\because f_X(x) \geq 0 .\ x_1 \leq x_2 \quad \therefore \int_{x_1}^{x_2} f_X(x)dx \geq 0 .$ Q.E.D.

2. $F_X(x_i) \triangleq \int_{-\infty}^{x_i} f_X(x)dx$

$P(x_1 \leq X < x_2) = \int_{x_1}^{x_2} f_X(x)dx = F_X(x_2) - F_X(x_1)$

$\Delta x \triangleq x_2 - x_1 .$

$\lim_{\Delta x \to 0} \frac{F_X(x_1+\Delta x) - F_X(x_1)}{\Delta x} = f_X(x_1) .$ Q.E.D.

3. $F_X(x|A) = P(X \leq x | A)$

$= \frac{P(A|X \leq x) P(X \leq x)}{P(A)}$

$= \frac{P(A|X \leq x) F_X(x)}{P(A)}$ Q.E.D.

Bayes' theorem

$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

4. X.Y indepent $\Leftrightarrow f_{XY}(x,y) = f_X(x) f_Y(y)$

X.Y uncorrelated $\Leftrightarrow E[XY] = E[X]E[Y]$

$E[XY] = \iint xy f_{XY}(x,y)dxdy$

$= \iint xy f_X(x) f_Y(y) dxdy$

$= \int x f_X(x)dx \int y f_Y(y)dy = E[X]E[Y] .$ Q.E.D.

5. $\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$= E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y]$

$= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X\mu_Y$

$= E[XY] - \mu_X\mu_Y$

$\mathrm{Cov}(X,Y) = 0 \Rightarrow E[XY] = E[X]E[Y]$

$\Rightarrow X, Y$ uncorrelated. Q.E.D.

6. $f_X(x) = 1, \quad x \in (0,1)$.

① $F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y))$

$\underset{g^{-1}(y)=e^{-y}}{\overset{g(x)=-\ln x}{=\!=}} P(X \le e^{-y}) = \int_{-\infty}^{e^{-y}} f_X(x)\,dx = \int_0^{e^{-y}} 1\,dx$.

$e^{-y} \in (0,1)$

$f_Y(y) = \dfrac{dF_Y(y)}{dy} = e^{-y}, \quad y \in (0,\infty)$

② $f_Y(y) = \sum_i \dfrac{1}{|g'(x_i)|} f_X(x_i)$, where x_i's are the solutions to $y = g(x)$.

$y = -\ln x$ has a single solution $x = e^{-y}$. when $y \in (0,\infty)$

$\therefore f_Y(y) = \dfrac{1}{|-\frac{1}{x}|} = e^{-y}, \quad y \in (0,\infty)$.

7. $f_X(x;\frac{1}{\lambda}) = \frac{1}{\lambda}e^{-\frac{x}{\lambda}}, \ x \geq 0$

$y = \sqrt{x}$ has a single solution $x = y^2$ when $y \geq 0$.

$$f_Y(y) = \sum_i \frac{1}{|g'(x_i)|} f_X(x_i) = \frac{1}{|\frac{1}{2\sqrt{x}}|}\frac{1}{\lambda}e^{-\frac{x}{\lambda}}$$

$$= \frac{2y}{\lambda}e^{-\frac{y^2}{\lambda}}, \ y \geq 0.$$

This is a Rayleigh distribution. Q.E.D.

8. $$\sigma^2 \triangleq \int_{-\infty}^{+\infty} (x-\eta)^2 f_X(x)\,dx$$

$$\geq \int_{|x-\eta| \geq \varepsilon\sigma} (x-\eta)^2 f_X(x)\,dx$$

$$\geq \int_{|x-\eta| \geq \varepsilon\sigma} (\varepsilon\sigma)^2 f_X(x)\,dx$$

$$= (\varepsilon\sigma)^2 P(|x-\eta| \geq \varepsilon\sigma)$$

$\therefore \frac{1}{\varepsilon^2} \geq P(|x-\eta| \geq \varepsilon\sigma)$ Q.E.D.

9. $P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$, $E[X]=\lambda$, $\sigma^2=\lambda$.

Chebyshev's inequality
$P(|X-\eta|\geq \varepsilon\sigma)\leq \frac{1}{\varepsilon^2}$

(a) Let $\varepsilon=\sqrt{\lambda}$

$P(|X-\lambda|\geq\lambda)\leq\frac{1}{\lambda}$

$\therefore P(|X-\lambda|<\lambda)>1-\frac{1}{\lambda}$

$\therefore P(0<X<2\lambda)>\frac{\lambda-1}{\lambda}$. Q.E.D.

(b) $E[x(x-1)]=\sum_{k=0}^{\infty}k(k-1)\frac{\lambda^k e^{-\lambda}}{k!}$

$=e^{-\lambda}\sum_{k=2}^{\infty}k(k-1)\frac{\lambda^k e^{-\lambda}}{k!}$

$=e^{-\lambda}\cdot\lambda^2\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!}=e^{\lambda}$ Taylor series

$=\lambda^2$ Q.E.D.

(c) $E[x(x-1)(x-2)]=e^{-\lambda}\cdot\lambda^3\sum_{k=3}^{\infty}\frac{\lambda^{k-3}}{(k-3)!}=\lambda^3$ Q.E.D.

(10) $Y = |X|$ is called "folded normal distribution".

If $\sigma^2 = 0$, then Y is called "half-normal distribution"

$$E[X] = \int_{-\infty}^{0} x f(x) dx + \int_{0}^{+\infty} x f(x) dx$$

$$E[|X|] = \int_{-\infty}^{0} -x f(x) dx + \int_{0}^{+\infty} x f(x) dx$$

$$\therefore \frac{E[X + |X|]}{2} = \int_{0}^{+\infty} x f(x) dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{0}^{\infty} x e^{-\frac{(x-\eta)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{0}^{+\infty} (x-\eta) e^{-\frac{(x-\eta)^2}{2\sigma^2}} dx + \frac{1}{\sqrt{2\pi}\sigma} \int_{0}^{+\infty} \eta e^{-\frac{(x-\eta)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\eta^2}{2\sigma^2}} + \eta G\left(\frac{\eta}{\sigma}\right)$$

$$\therefore E[|X|] = \sqrt{\frac{2}{\pi}} \sigma e^{-\frac{\eta^2}{2\sigma^2}} + 2\eta G\left(\frac{\eta}{\sigma}\right) - \eta. \quad \text{Q.E.D.}$$

11. (a) $F_Z(z) = P(Z \le z) = \iint_{\sqrt{x^2+y^2} \le z} \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} dx\,dy$

$= \int_0^{2\pi} \int_0^{z} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r\,dr\,d\theta = 1 - e^{-\frac{z^2}{2\sigma^2}}$

$\therefore f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}}, \quad z \geq 0.$

(b) $F_Z(z) = \int_0^{2\pi} \int_0^{\sqrt{z}} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r\,dr\,d\theta = 1 - e^{-\frac{z}{2\sigma^2}}$

$\therefore f_Z(z) = \frac{1}{2\sigma^2} e^{-\frac{z}{2\sigma^2}}, \quad z \geq 0.$

(c) Linear combination of independent Gaussian r.v.s are still Gaussian!

$U = X - Y \sim N(0, 2\sigma^2).$

12. $P(A)P(B)P(C)=0.125$. $P(ABC)=0.25$

$\therefore$ A.B.C are not independent.

$P(A)P(B)=0.25=P(AB)$

$P(B)P(C)=0.25=P(BC)$

$P(C)P(A)=0.25=P(CA)$

$\therefore$ A.B.C are independent in pairs.

13. $f_\Theta(\theta)=\frac{1}{2\pi}$. $\theta\in[0,2\pi)$

$$E[n(t)]=\int_{-\infty}^{+\infty}A\cos(2\pi f_c t+\theta)f_\Theta(\theta)d\theta$$

$$=\frac{A}{2\pi}\int_0^{2\pi}\cos(2\pi f_c t+\theta)d\theta=0.$$

$$E[n^2(t)]=\frac{A^2}{2\pi}\int_0^{2\pi}\cos^2(2\pi f_c t+\theta)d\theta$$

$\cos^2\theta=\frac{1}{2}(1+\cos2\theta)$

$$=\frac{A^2}{2\pi}\int_0^{2\pi}\frac{1}{2}(1+\cos(4\pi f_c t+2\theta))d\theta$$

$$=\frac{A^2}{2}$$

$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

14. WSS: $\begin{cases} \mu_X(t) = \mu_X \\ R_X(t, t+\tau) = R_X(\tau) \end{cases}$

n-th order SSS: $\forall c$,

$$f_X(x_1 \dots x_n; t_1 \dots t_n) = f_X(x_1 \dots x_n; t_1 + c \dots t_n + c)$$

WSS input × LTI response ⇒ WSS output.

$$R_Y(\tau) = E[Y(t+\tau)Y(t)]$$

$$= E[X(t+\tau)\cos(2\pi f_c(t+\tau)+\theta)X(t)\cos(2\pi f_c t+\theta)]$$

$$\underset{\text{independent}}{\overset{X(t).\theta}{=\!=}} E[X(t+\tau)X(t)]\, E[\cos(2\pi f_c(t+\tau)+\theta)\cos(2\pi f_c t+\theta)]$$

$$= R_X(\tau)\frac{1}{2}E[\cos(2\pi f_c\tau) + \cos(2\pi f_c(2t+\tau)+2\theta)]$$

$$= \frac{1}{2}R_X(\tau)\cos(2\pi f_c \tau)$$

$$S_Y(f) = \int_{-\infty}^{+\infty} R_Y(\tau) e^{-j2\pi f\tau} d\tau$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} R_X(\tau)\frac{e^{j2\pi f_c\tau} + e^{-j2\pi f_c\tau}}{2} e^{-j2\pi f\tau} d\tau$$

$$= \frac{1}{4}[S_X(f-f_c) + S_X(f+f_c)].$$

15. $\Leftarrow$: 1st order:

$$E[v(t)] = E[X]\cos 2\pi f_c t - E[Y]\sin 2\pi f_c t = 0$$

irrelevant to t.

2nd order: let $E[X^2] = E[Y^2] \triangleq \sigma^2$.

$$E[v(t+\tau)v(t)] = E[X^2]\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t)$$
$$-E[XY]\{\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)$$
$$+\cos(2\pi f_c t)\sin(2\pi f_c(t+\tau))\}$$
$$+E[Y^2]\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t))$$

$$= \sigma^2 \cdot \frac{1}{2}[\cos(2\pi f_c \tau) + \cos(2\pi f_c(2t+\tau))$$
$$+\cos(2\pi f_c \tau) - \cos(2\pi f_c(2t+\tau))]$$

$$= \sigma^2 \cos(2\pi f_c \tau)$$ only depends on τ. not t.

$\Rightarrow$: 1st order:

$$E[v(t)] = E[X]\cos(2\pi f_c t) - E[Y]\sin(2\pi f_c t)$$

has to be constant $\Rightarrow E[X] = E[Y] = 0$

2nd order:

$$E[v(t+\tau)v(t)] = \frac{1}{2}\{[E[X^2]+E[Y^2]]\cos(2\pi f_c \tau)$$
$$+[E[X^2]-E[Y^2]]\cos(2\pi f_c(2t+\tau))$$
$$-2E[XY]\sin(2\pi f_c(2t+\tau))\}$$ has to be irrelevant of t.

$\Rightarrow E[X^2] = E[Y^2]$. $E[XY] = 0$. Q.E.D.

16. $E[(X(t+\tau) \pm X(t))^2] \geq 0$

$\therefore E[X^2(t+\tau)] \pm 2E[X(t+\tau)X(t)] + E[X^2(t)] \geq 0$

$\because X(t)$ is WSS

$\therefore R_X(0) \pm R_X(\tau) \geq 0.$ Q.E.D.

17. Gaussian input $\times$ LTI response $\Rightarrow$ Gaussian output

To prove $Y(t)$. $Z(t)$ uncorrelated is sufficient.

$$R_{YZ}(t,u) = E[Y(t)Z(u)]$$

$$= E\left[\int_{-\infty}^{+\infty} h_1(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{+\infty} h_2(\tau_2)X(u-\tau_2)d\tau_2\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_1(\tau_1)h_2(\tau_2)E[X(t-\tau_1)X(u-\tau_2)]d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_1(\tau_1)h_2(\tau_2)R_X(t-\tau_1, u-\tau_2)d\tau_1 d\tau_2$$

$$\underset{X\ SSS}{\overset{\tau=t-u}{=\!=}} \int_{-\infty}^{+\infty} h_2(\tau_2)\int_{-\infty}^{+\infty} h_1(\tau_1)R_X(\tau-\tau_1+\tau_2)d\tau_1 d\tau_2$$

$$\overset{\Delta}{=} \int_{-\infty}^{+\infty} h_2(\tau_2)g(\tau+\tau_2)d\tau_2 = h_2(-\tau) * g(\tau)$$

$$= h_2(-\tau) * h_1(\tau) * R_X(\tau)$$

$$\therefore S_{YZ}(f) = H_1(f)H_2^*(f)S_X(f)$$

If $H_1(f)$ and $H_2(f)$ do not overlap, then $R_{YZ}(t,u) = 0$,

and $Y(t)$ and $Z(t)$ are uncorrelated.