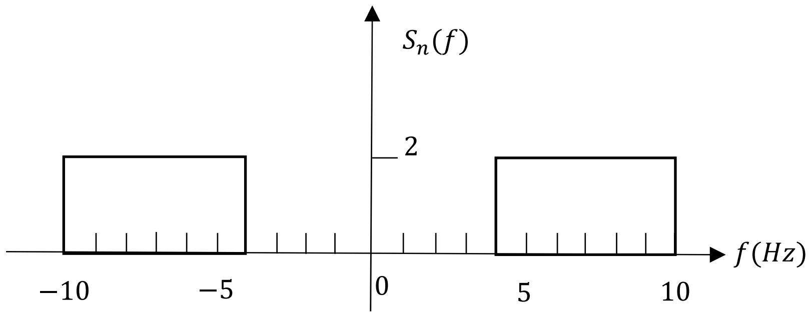
Baseband and Passband Signal and Noise Effects (Lectures 3 & 4)

1. **Consider a bandpass noise signal having the power spectral density shown below. Draw the power spectral density (PSD) of $n_I(t)$ if the center frequency is chosen as:**
   (a) $f_c = 7\ Hz$
   (b) $f_c = 5\ Hz$

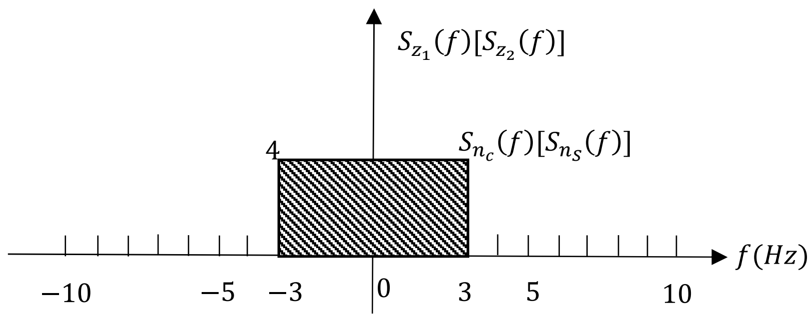


Solution:
Recall that

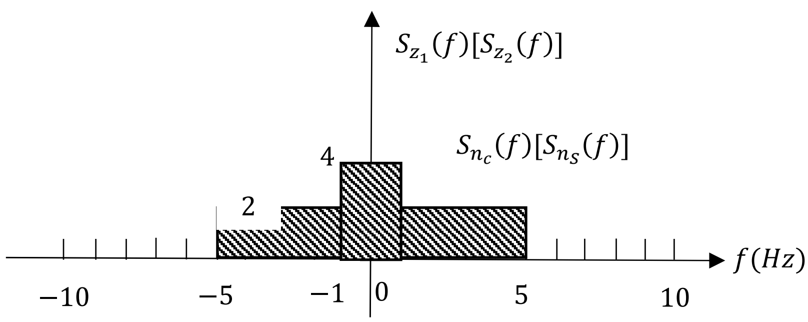
$$S_{N_I}(f) = \begin{cases} S_N(f - f_C) + S_N(f + f_C), & |f| < B \\ 0, & otherwise \end{cases}$$

We add the PSDs of the negative and positive frequency bands shifted to the origin.
In these figures, the shaded region is the PSD of $n_I(t)$.

(a)



(b)



2. Let

$$\boldsymbol{f_k(t) \triangleq \begin{cases} e^{-\frac{t}{k}}, & if\ t > 0, \\ 0, & if\ t = 0, \\ -e^{-\frac{t}{k}}, & if\ t < 0, \end{cases}}$$

Find $F_k(f)$, the Fourier transform of $f_k(t)$. Letting $k \to \infty$, find the Fourier transform of function $sgn(t)$, defined as

$$\boldsymbol{sgn(t) \triangleq \begin{cases} 1, & if\ t > 0, \\ 0, & if\ t = 0, \\ -1, & if\ t < 0. \end{cases}}$$

Using this, find the Fourier transform of unit step function

$$\boldsymbol{u(t) \triangleq \begin{cases} 1, & if\ t > 0, \\ 1/2, & if\ t = 0, \\ 0, & if\ t < 0. \end{cases}}$$

Solution:

We have $f_k(t) = f_k^1(t) + f_k^2(t)$, where

$$f_k^1(t) = e^{-\frac{t}{k}}u(t),$$

$$f_k^2(t) = -e^{\frac{t}{k}}u(-t).$$

Let $\mathcal{F}\left(f_k^i(t)\right) = F_k^i(f)$ for $i = 1{,}2$ and $\forall k$. We have

$$F_k^1(f) = \int_0^\infty e^{-\frac{t}{k}}e^{-j2\pi ft}dt,$$
$$= \frac{1}{\frac{1}{k} + j2\pi f}$$

and

$$F_k^2(f) = \int_{-\infty}^0 -e^{-\frac{t}{k}}e^{-j2\pi ft}dt,$$
$$= \int_0^\infty e^{t(\frac{1}{k}-j2\pi ft)}dt,$$
$$= -\frac{1}{\frac{1}{k} - j2\pi f}$$

Combining the two, we can get

$$F_k(f) = F_k^1(f) + F_k^2(f)$$
$$= \frac{1}{\frac{1}{k} + j2\pi f} - \frac{1}{\frac{1}{k} - j2\pi f}$$
$$= -\frac{j4\pi f}{\frac{1}{k^2} + 4\pi^2 f^2}$$

In the limit, we have

$$f_k(t) \xrightarrow{k\to\infty} sgn(t)$$

and

$$F_k(f) \xrightarrow{k\to\infty} \frac{1}{j\pi f},$$

From which, it follows that

$$\mathcal{F}\big(sgn(t)\big) = \frac{1}{j\pi f}.$$

We have $u(t) = \frac{1}{2}(sgn(t) + 1)$. It directly follows that $U(f) \triangleq \mathcal{F}\big(u(t)\big) = \frac{1}{2}\left(\frac{1}{j\pi f} + \delta(f)\right)$.

3. Hilbert transform of a signal $g(t)$ is defined as

$$\hat{g}(t) = g(t) * \frac{1}{\pi t}$$

Using the result of the previous exercise, find $\hat{G}(f)$, the Fourier transform of $\hat{g}(t)$.

Solution:

From the duality relation of the Fourier transform, since $\mathcal{F}\big(sgn(t)\big) = \frac{1}{j\pi f}$, we have

$$\mathcal{F}\left(\frac{1}{j\pi t}\right) = sgn(-f)$$

It follows that

$$\mathcal{F}\left(\frac{1}{\pi t}\right) = jsgn(-f) = -jsgn(f)$$

We have

$$\hat{G}(f) = -jsgn(f)G(f)$$

4. Prove the following properties of Hilbert transforms:

a) **If $x(t) = x(-t)$, then $\hat{x}(t) = -\hat{x}(-t)$.**
b) **If $x(t) = -x(-t)$, then $\hat{x}(t) = \hat{x}(-t)$.**
c) **If $x(t) = \cos(2\pi f_0 t)$, then $\hat{x}(t) = \sin(2\pi f_0 t)$.**
d) **If $x(t) = \sin(2\pi f_0 t)$, then $\hat{x}(t) = -\cos(2\pi f_0 t)$.**
e) **$\hat{\hat{x}}(t) = -x(t)$.**
f) **$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} \hat{x}^2(t)dt$.**
g) **$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$.**

Solution:

Denote $h(t) = \frac{1}{\pi t}$ as the impulse response of the Hilbert transform and

$$H(f) = \mathcal{F}[h(t)] = \begin{cases} -j, & f \geq 0 \\ +j, & f < 0 \end{cases}.$$

Note that $H(f) = -H(-f)$.

a) It can be directly proved using the property of the Fourier transform that $x(t) = x(-t)X(f)$ is equivalent to $X(f) = X(-f)$. Therefore,

$$X(f)H(f) = X(-f)H(f) \Rightarrow X(f)H(f) = -X(-f)H(-f)$$

As $X(f)H(f) = \mathcal{F}[\hat{x}(t)]$ and $X(-f)H(-f) = \mathcal{F}[\hat{x}(-t)]$,

$$\hat{x}(t) = -\hat{x}(-t)$$

b) Similarly, using the property of the Fourier transform, $x(t) = -x(-t)$ is equivalent to $X(f) = -X(-f)$. Therefore,

$$X(f)H(f) = -X(-f)H(f) \Rightarrow X(f)H(f) = X(-f)H(-f)$$

As $X(f)H(f) = \mathcal{F}[\hat{x}(t)]$ and $X(-f)H(-f) = \mathcal{F}[\hat{x}(-t)]$,

$$\hat{x}(t) = \hat{x}(-t)$$

c) If $x(t) = \cos{(2\pi f_0 t)}$, then $X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$

$$\mathcal{F}[\hat{x}(t)] = X(f)H(f) = [\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)]H(f)$$

$$= -j\frac{1}{2}\delta(f - f_0) + j\frac{1}{2}\delta(f + f_0) = \frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$$

which is the Fourier transformer of $\sin(2\pi f_0 t)$.

d) If $x(t) = \sin{(2\pi f_0 t)}$, then $X(f) = \frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$

$$\mathcal{F}[\hat{x}(t)] = X(f)H(f) = \left[\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)\right]H(f)$$

$$= -j\frac{1}{2j}\delta(f - f_0) - j\frac{1}{2j}\delta(f + f_0) = -\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

which is the Fourier transformer of $-\cos(2\pi f_0 t)$.

e)

$$\mathcal{F}\big[\hat{\hat{x}}(t)\big] = [X(f)H(f)]H(f)$$

As $H(f)H(f) = -1$,

$$\mathcal{F}\big[\hat{\hat{x}}(t)\big] = -X(f) \Rightarrow \hat{\hat{x}}(t) = -x(t)$$

f) According to Parseval's theorem,

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(f)G^*(f)df.$$

If $f(t) = \hat{x}(t)$ and $g(t) = \hat{x}^*(t)$, then $F(f) = \hat{X}(f)$, $G(f) = \hat{X}^*(-f)$ and

$$\int_{-\infty}^{\infty} \hat{x}^2(t)dt = \int_{-\infty}^{\infty} \hat{X}(f)\hat{X}(-f)df = \int_{-\infty}^{\infty} X(f)H(f)X(-f)H(-f)df$$

As $H(f)H(-f) = 1$, we have

$$\int_{-\infty}^{\infty} \hat{x}^2(t)dt = \int_{-\infty}^{\infty} X(f)X(-f)df = \int_{-\infty}^{\infty} x^2(t)dt.$$

g) There are two ways to solve it.

1) Let $I = \int_{-\infty}^{\infty} x(t)\hat{x}(t)dt$

As

$$\hat{x}(t) = \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

Thus

$$\begin{aligned} I &= \int_{-\infty}^{\infty} x(t) \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau dt \\ &= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} \frac{x(t)}{t-\tau} dt \right] d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)(-x(\tau)) d\tau = -I \end{aligned}$$

Hence

$$I = 0$$

2) According to Parseval's theorem, let $f(t) = x(t)$ and $g(t) = \hat{x}^*(t)$. Then $F(f) = X(f)$, $G(f) = \hat{X}^*(-f)$, then

$$\begin{aligned} \int_{-\infty}^{\infty} x(t)\,\hat{x}(t) dt &= \int_{-\infty}^{\infty} X(f)\hat{X}(-f) df \\ &= \int_{-\infty}^{\infty} X(f)X(-f)H(-f) df \end{aligned}$$

However,

$$Z(f) = X(f)X(-f)H(-f) = -X(f)X(-f)H(f) = -Z(-f)$$

Thus, $Z(f)$ is an odd function, so the integral is zero.