1. $I(s_k) = \log_2 \frac{1}{p_k}$ bits

a) $I(x_1) = \log_2 2 = 1$ bit

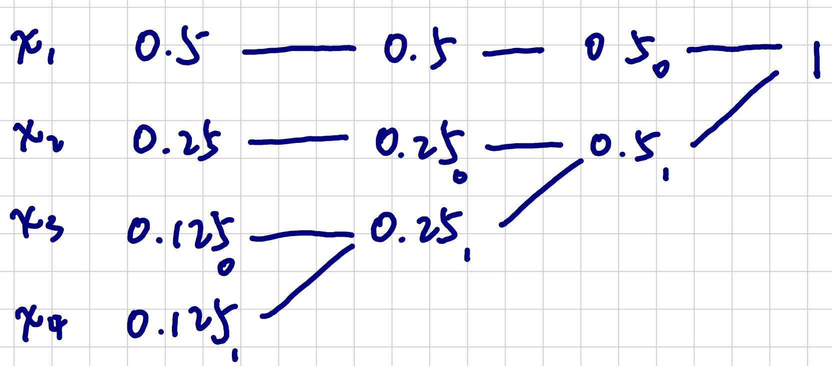
$I(x_2) = \log_2 4 = 2$ bits

$I(x_3) = I(x_4) = \log_2 8 = 3$ bits

b) $H(S) = -\sum_k p_k \log_2 p_k$ bits/symbol

$= \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{8}\times 3 = 1.75$ bits/symbol

2.



x_1: 0

x_2: 10

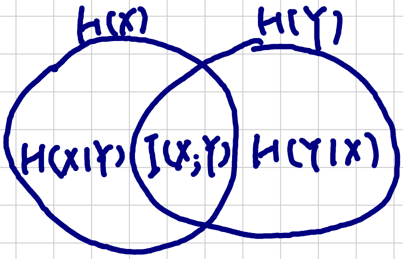
x_3: 110

x_4: 111

4. $C = B\log_2\left(1+\frac{P}{N_0 B}\right)$ bps

$= 10^6 \log_2\left(1+\frac{10}{2\times 10^{-9}\times 10^6}\right) = 12.2$ Mbps

3.



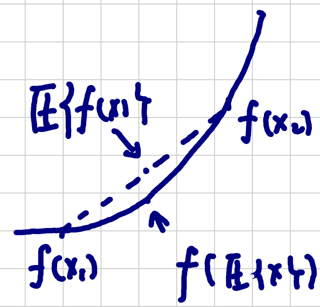
$$I(X;Y) = \sum_x \sum_y P(x,y) \log \frac{P(x,y)}{P(x)P(y)}$$

$$= \mathbb{E}_{X,Y}\left\{ \log \frac{P(x,y)}{P(x)P(y)} \right\} = \mathbb{E}_{X,Y}\left\{ -\log \frac{P(x)P(y)}{P(x,y)} \right\}$$

Jensen's inequality.

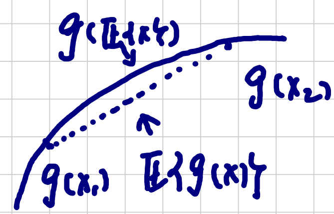
For a convex function $f(\cdot)$,

$$f(\mathbb{E}\{x\}) \le \mathbb{E}(f(x))$$



For a concave function $g(\cdot)$,

$$g(\mathbb{E}\{x\}) \ge \mathbb{E}\{g(x)\}$$



$\log(\cdot)$ is concave $\rightarrow$ $-\log(\cdot)$ is convex

$$\underbrace{\mathbb{E}_{X,Y}\left\{ -\log \frac{P(x)P(y)}{P(x,y)} \right\}}_{I(X;Y)} \ge -\log \mathbb{E}_{X,Y}\left\{ \frac{P(x)P(y)}{P(x,y)} \right\}$$

$$= -\log \sum_x \sum_y P(x,y) \frac{P(x)P(y)}{P(x,y)}$$

$$= -\log \underbrace{\sum_x P(x)}_{1} \underbrace{\sum_y P(y)}_{1}$$

$$= 0.$$

Take equal sign iff X, Y are independent. Q.E.D.

5. $P_{Y|X}(y|x) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & \frac{2}{3} & \frac{1}{3} \\ x_2 & \frac{1}{10} & \frac{9}{10} \end{array}$, $P_X(x) = \begin{array}{c|c} x_1 & \frac{1}{3} \\ x_2 & \frac{2}{3} \end{array}$

a) $H(X) = -\sum_i P_X(x_i) \log_2 P_X(x_i)$

$= \frac{1}{3}\log_2 3 + \frac{2}{3}\log_2 \frac{3}{2} = 0.92$ bits

$P(x_i|y_j) = \frac{P(x_i, y_j)}{P(y_j)} = \frac{P(x_i)}{P(y_j)} P(y_j|x_i)$

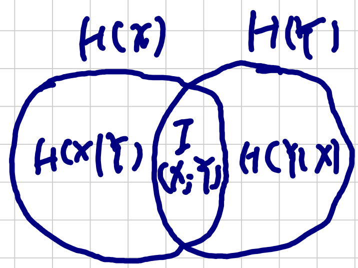
$\Rightarrow P_{X|Y}(x|y) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & \frac{10}{13} & \frac{5}{32} \\ x_2 & \frac{3}{13} & \frac{27}{32} \end{array}$

$P(y_j) = \sum_i P(x_i) P(y_j|x_i)$

$\Rightarrow P_Y(y) = \begin{array}{cc} y_1 & y_2 \\ \hline \frac{13}{45} & \frac{32}{45} \end{array}$

$P(x_i, y_j) = P(x_i) \cdot P(y_j|x_i)$

$\Rightarrow P_{X,Y}(x,y) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & \frac{2}{9} & \frac{1}{9} \\ x_2 & \frac{1}{15} & \frac{3}{5} \end{array}$



$H(X|Y) = -\sum_i \sum_j P(x_i, y_j) \log_2 P(x_i|y_j) = 0.66$ bits

$H(Y) = -\sum_j P_Y(y_j) \log_2 P_Y(y_j) = 0.87$ bits

$H(Y|X) = -\sum_i \sum_j P(x_i, y_j) \log_2 P(y_j|x_i) = 0.62$ bits

$I(X;Y) = H(X) - H(X|Y) = 0.26$ bits

b) $C = \max_{p(x)} I(X;Y)$

Blahut-Arimoto algorithm.

iteratively updates the input distribution until convergence

$$P_X^{(r+1)}(x_i) = P_X^{(r)}(x_i) \frac{c^{(r)}(x_i)}{\sum_{i'} P_X^{(r)}(x_{i'})\, c^{(r)}(x_{i'})}$$

$$c^{(r)}(x_i) = \exp\left(\sum_j P_{Y|X}(y_j|x_i) \log \frac{P_{Y|X}(y_j|x_i)}{\sum_{i'} P_X^{(r)}(x_{i'})\, P_{Y|X}(y_j|x_{i'})} \right)$$

6. $u_1 = [0\,0\,0\,0]$

$c_1 = u_1 G = [0\,0\,0\,0\,0\,0\,0]$

$u_2 = [0\,0\,0\,1]$

$c_2 = u_2 G = [0\,0\,0\,1\,1\,1\,1]$

$u_3 = [0\,0\,1\,0]$

$c_3 = u_3 G = [0\,0\,1\,0\,1\,1\,0]$

$c_4 = [0\,0\,1\,1\,0\,0\,1]$

$c_5 = [0\,1\,0\,0\,1\,0\,1]$

$c_6 = [0\,1\,0\,1\,0\,1\,0]$

$c_7 = [0\,1\,1\,0\,0\,1\,1]$

$c_8 = [0\,1\,1\,1\,1\,0\,0]$

$c_9 = [1\,0\,0\,0\,0\,1\,1]$

$c_{10} = [1\,0\,0\,1\,1\,0\,0]$

$c_{11} = [1\,0\,1\,0\,1\,0\,1]$

$c_{12} = [1\,0\,1\,1\,0\,1\,0]$

$c_{13} = [1\,1\,0\,0\,1\,1\,0]$

$c_{14} = [1\,1\,0\,1\,0\,0\,1]$

$c_{15} = [1\,1\,1\,0\,0\,0\,0]$

$c_{16} = [1\,1\,1\,1\,1\,1\,1]$

Hamming weight = number of non-zero elements

Hamming distance between a, b

$d_H(a,b) = w_H(a+b)$

$d_{min} = d_H(c_1, c_4) = 3$.