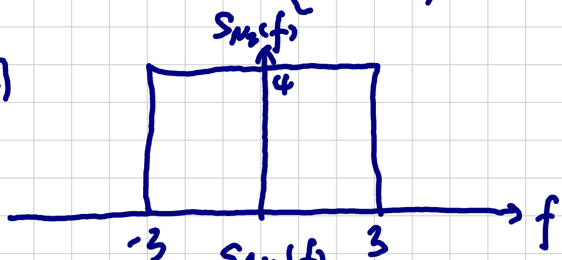
1. Baseband and passband noise PSD satisfy:

$$S_{N_1}(f) = \begin{cases} S_N(f-f_c)+S_N(f+f_c), & |f|<B \\ 0, & \text{otherwise} \end{cases}$$

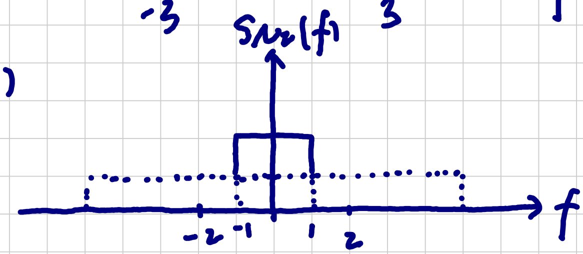
(a)



$f+f_c=7$

7

(b)



$f+f_c=7$

5

2. $f_K(t)$ is defined as case function. Alternatively,

$$f_K(t) = \underbrace{e^{-\frac{t}{K}}u(t)}_{f_K^1(t)} - \underbrace{e^{\frac{t}{K}}u(-t)}_{f_K^2(t)}$$

$$F_K^1(f) = \int_0^{+\infty} e^{-\frac{t}{K}} e^{-j2\pi ft}\,dt = \int_0^{+\infty} e^{-(\frac{1}{K}+j2\pi f)t}\,dt$$

$$= -\frac{1}{\frac{1}{K}+j2\pi f} e^{-(\frac{1}{K}+j2\pi f)t}\Big|_0^{+\infty} = \frac{1}{\frac{1}{K}+j2\pi f}$$

$$F_K^2(f) = \int_{-\infty}^{0} e^{\frac{t}{K}} e^{-j2\pi ft}\,dt = \frac{1}{\frac{1}{K}-j2\pi f}$$

$$\therefore F_K(f) = F_K^1(f) - F_K^2(t) = -\frac{j4\pi f}{\frac{1}{K^2}+4\pi^2 f^2}$$

When $K\to\infty$, $f_K(t)\to \mathrm{sgn}(t)$ and $F_K(t) \to \frac{1}{j\pi f}$

$\therefore$ FT of $\mathrm{sgn}(t)$ is $\frac{1}{j\pi f}$.

$\because u(t) = \frac{1}{2}(\mathrm{sgn}(t)+1)$

$\therefore U(f) = \frac{1}{2}\left(\frac{1}{j\pi f}+\delta(f)\right)$.

3. Duality of FT: $x(t) \longleftrightarrow X(f)$

$$X(t) \longleftrightarrow x(-f)$$

$$\mathcal{F}(\mathrm{sgn}(t)) = \frac{1}{j\pi f} \Rightarrow \mathcal{F}\left(\frac{1}{j\pi t}\right) = \mathrm{sgn}(-f)$$

$$\therefore \mathcal{F}\left(\frac{1}{\pi t}\right) = j\,\mathrm{sgn}(-f) = -j\,\mathrm{sgn}(f)$$

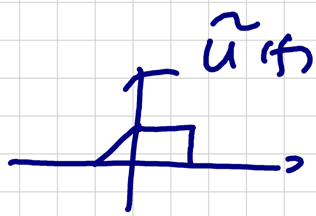
$$\therefore \hat{G}(f) = -j\,\mathrm{sgn}(f)\,G(f)$$

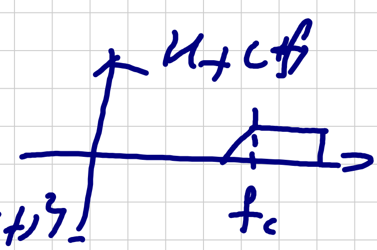
4. Hilbert transform: $\begin{cases} \text{shift the angles of all positive frequency components } -90^\circ \\ \text{shift the angles of all negative frequency components } +90^\circ \end{cases}$

- Amplitude spectrum unchanged, phase spectrum changed
- $\hat{g}(t) = g(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{g(\tau)}{t-\tau}\,d\tau = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{g(t-\tau)}{\tau}\,d\tau$
- Linear operation of a "special" filter with impulse response $\frac{1}{\pi t}$
- No change of domains (c.f. FT. LT)

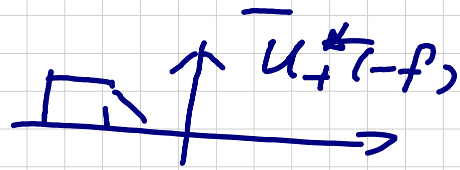
Analytic signal (a.k.a. pre-envelope)

- $u_+(t) = u(t) + j\hat{u}(t)$ removes the negative frequency components
- helpful to compute the instantaneous magnitude/phase of the original signal (eg. AM. PM. FM)





$$\underline{u(t)} = \overbrace{\frac{1}{2}\big(u_+(t) + u^*(t)\big)}^{\mathrm{Re}\{u_+(t)\} =}$$



Let $h(t)=\frac{1}{\pi t}$ be the impulse response of HT.

$$H(f)=\mathcal{F}(h(t))=\int_{-\infty}^{+\infty}\frac{1}{\pi t}e^{-j2\pi ft}dt=\begin{cases}-j, & f\geq 0\\ j, & f<0\end{cases}$$

It can be verified that $H(f)=-H(-f)$.

(a) $x(t)=x(-t)\iff X(f)=X(-f)$.

$\therefore \underbrace{X(f)H(f)}_{\mathcal{F}(\hat{x}(t))}=-\underbrace{X(-f)H(-f)}_{\mathcal{F}(\hat{x}(-t))}$. $\quad\therefore \hat{x}(t)=-\hat{x}(-t)$.

(b) $x(t)=-x(-t)\iff X(f)=-X(-f)$.

$\therefore X(f)H(f)=X(-f)H(-f)$. $\quad\therefore \hat{x}(t)=\hat{x}(-t)$.

(c) $x(t)=\cos(2\pi f_0t)$. $X(f)=\frac{1}{2}\left(\delta(f-f_0)+\delta(f+f_0)\right)$.

$\therefore X(f)H(f)=\frac{1}{2}(-j\delta(f-f_0)+j\delta(f+f_0))$, which is the FT of $\sin(2\pi f_0t)$.

(d) $x(t)=\sin(2\pi f_0t)$. $X(f)=\frac{1}{2j}(\delta(f-f_0)-\delta(f+f_0))$.

$\therefore X(f)H(f)=\frac{1}{2}(-\delta(f-f_0)-\delta(f+f_0))$, which is the FT of $-\cos(2\pi f_0t)$.

(e) $H(f)\cdot H(f)=-1$

$\therefore X(f)H(f)H(f)=-X(f)$ $\quad\therefore \hat{\hat{x}}(t)=-x(t)$

(f) Parseval's theorem: $\int_{-\infty}^{+\infty}f(t)g^*(t)dt=\int_{-\infty}^{+\infty}F(f)G^*(f)df$

Now $f(t)=\hat{x}(t)$. $g(t)=\hat{x}^*(t)$. $F(f)=\hat{X}(f)$. $G(f)=\hat{X}^*(-f)$.

$$\therefore \int_{-\infty}^{+\infty}\hat{x}^2(t)dt=\int_{-\infty}^{+\infty}\hat{X}(f)\hat{X}(-f)df=\int_{-\infty}^{+\infty}X(f)H(f)X(-f)H(-f)df$$

$$=\int_{-\infty}^{+\infty}X(f)X(-f)df=\int_{-\infty}^{+\infty}x^2(t)dt.$$

(g) $\int_{-\infty}^{+\infty} x(t)\,\hat{x}(t)\,dt = \frac{1}{\pi}\int_{-\infty}^{+\infty} x(t)\int_{-\infty}^{+\infty} \frac{x(\tau)}{t-\tau}\,d\tau\,dt$

$= \frac{1}{\pi}\int_{-\infty}^{+\infty} x(\tau)\int_{-\infty}^{+\infty} \frac{x(t)}{t-\tau}\,dt\,d\tau$

$= \int_{-\infty}^{+\infty} x(\tau)\cdot(-\hat{x}(\tau))\,d\tau.$

$\therefore \int_{-\infty}^{+\infty} x(t)\,\hat{x}(t)\,dt = 0.$