Probability and Random Processes (Lectures 1 & 2)

1. **If $\boldsymbol{F_X(x)}$ is the distribution function of a random variable $\boldsymbol{X}$ and $\boldsymbol{x_1 \leq x_2}$, show that $\boldsymbol{F_X(x_1) \leq F_X(x_2)}$.**

Solution:

Let $A \triangleq \{X \leq x_1\}$ and $B \triangleq \{X \leq x_2\}$. We have $P\{A\} = P\{X \leq x_1\} = F_X(x_1)$, and $P\{B\} = P\{X \leq x_2\} = F_X(x_2)$.

Since $x_1 \leq x_2$, we have $A \subseteq B \Rightarrow B = (B - A) \cup A$. Note that events $B - A$ and A do not have common outcomes. Then, according to the axiomatic properties of probabilities,

$$P\{B\} = P\{B - A\} + P\{A\}.$$

Since $P\{B - A\}$ is a non-negative number, $P\{B\} \geq P\{A\}$, that is, $F_X(x_2) \geq F_X(x_1)$.

2. **Use the definition of cumulative distribution function to write an expression for the probability of a random variable to take values between $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, and take limiting cases to arrive at the definition of the probability density function as the derivative of the distribution function.**

Solution:

We have

$$P\{X \leq x_1\} + P\{x_1 < X \leq x_2\} = P\{X \leq x_2\}.$$

Equivalently, we can write

$$P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1).$$

Define $\Delta x \equiv x_2 - x_1$. Then the probability of X to take values between x_1 and x_2 may be written as

$$P\{x_1 < X \leq x_1 + \Delta x\} = F_X(x_1 + \Delta x) - F_X(x_1).$$

To obtain the density, we must divide it by the interval Δx:

$$\frac{F_X(x_1 + \Delta x) - F_X(x_1)}{\Delta x}.$$

Taking the limit of $\Delta x \to 0$, leads to the definition of the probability density function as the derivative of the distribution function.

3. **Show that**

$$\boldsymbol{F_X(x|A) = \frac{P\{A|X \leq x\}F_X(x)}{P\{A\}}}$$

Solution:

We have

$$\begin{aligned} F_X(x|A) &= P\{X \leq x|A\} \\ &= \frac{P\{A|X \leq x\}P\{X \leq x\}}{P\{A\}} \\ &= \frac{P\{A|X \leq x\}F_X(x)}{P\{A\}} \end{aligned}$$

where the second line follows from the Bayes Theorem.

4. **Show that if two random variables are independent, they are also uncorrelated.**

Solution:

Suppose random variables X and Y are independent, then by definition their joint pdf can be factorized as:

$$f_{XY}(x,y) = f_X(x)f_Y(y).$$

Using this property, we derive

$$\begin{aligned} E[XY] &= \iint xyf_{XY}(x,y)dxdy \\ &= \iint xyf_X(x)f_Y(y)dxdy \\ &= \int xf_X(x)dx \int y\,f_Y(y)dy \\ &= E[X]E[Y] \end{aligned}$$

5. **Show that the covariance of two random variables $\boldsymbol{Cov(X,Y) \triangleq E[(X-\mu_X)(Y-\mu_Y)]}$ is equal to:**

$$\boldsymbol{Cov(X,Y) = E[XY] - \mu_X\mu_Y,}$$

where $\boldsymbol{\mu_X}$ and $\boldsymbol{\mu_Y}$ are the mean values of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Then, show that the covariance of two random variables is zero, the two random variables are uncorrelated.

Solution:

By definition

$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\ &= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X\mu_Y \\ &= E[XY] - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y \\ &= E[XY] - \mu_X\mu_Y \end{aligned}$$

The second part goes as follows:

If $Cov(X,Y) = 0$, then $E[XY] - \mu_X\mu_Y = 0$, This means $E[XY] = \mu_X\mu_Y$, which is the definition of uncorrelated random variables.

6. **The random variable $\boldsymbol{x}$ is uniform in the interval $\boldsymbol{(0,1)}$. Find the density of the random variable $\boldsymbol{y = -\ln x}$.**

Solution:

The equation $y = -\ln x$ has a single solution $x = e^{-y}$ for $y > 0$ and no solutions for $y < 0$. Furthermore, $\frac{dy}{dx} = \frac{-1}{x}$. Hence

$$f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x) = xf_X(x) = e^{-y}f_X(e^{-y}) = e^{-y}U(y)$$

Another Solution:

Directly from the definition of the cumulant distribution function, we have

$$F_Y(y) = \Pr\{Y \le y\} = \Pr\{-\ln X \le y\} = \Pr\{-\ln X \le y\}$$
$$= \Pr\{X \ge e^{-y}\} = \begin{cases} 1 - e^{-y}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$

Therefore,

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} e^{-y}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$

7. If $y = \sqrt{x}$ and x is an exponential random variable, show that y represents a Rayleigh random variable.

Solution:

$$y = \sqrt{x} \Rightarrow x = y^2$$

Also,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2y}$$

Thus,

$$f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x) = 2y f_X(y^2)$$

As x is an exponential random variable,

$$f_X(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}$$

Thus,

$$f_Y(y) = \frac{2y}{\lambda} e^{-\frac{y^2}{\lambda}} U(y) = \begin{cases} \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

which represents the Rayleigh density function (with $\lambda = 2\sigma^2$).

8. [Tchebycheff Inequality] Let X be a random variable with a finite mean value η and a non-zero variance σ^2. Prove that, for any $\epsilon > 0$

$$\boldsymbol{P\{|X - \eta| \ge \epsilon\sigma\} \le \frac{1}{\epsilon^2}}.$$

Solution:

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \eta)^2 f_X(x) dx$$

$$\ge \int_{|x-\eta| \ge \varepsilon\sigma} (x - \eta)^2 f_X(x) dx$$

$$\ge \int_{|x-\eta| \ge \varepsilon\sigma} \varepsilon^2 \sigma^2 f_X(x) dx$$

$$= \epsilon^2\sigma^2 P\{|x-\eta| \geq \varepsilon\sigma\}$$

From here, it directly follows that

$$\frac{1}{\epsilon^2} \geq P\{|X-\eta| \geq \epsilon\sigma\}$$

9. For a Poisson random variable x with parameter λ show that (a) $P(0 < x < 2\lambda) > (\lambda - 1)/\lambda$; (b) $E[x(x-1)] = \lambda^2$, $E[x(x-1)(x-2)] = \lambda^3$.

Solution:

$$X \sim P(\lambda) \Rightarrow P(X = k) = e^{\lambda}\frac{\lambda^k}{k!},$$

a) $E(X) = \lambda,\ Var(X) = \sigma_X^2 = \lambda$

From Chebyshev's inequality,

$$P\{|X-\mu| < \lambda\} > 1 - \frac{\sigma_X^2}{\lambda^2} = 1 - \frac{1}{\lambda}$$

But

$$|X-\mu| < \lambda = |X-\lambda| < \lambda \Rightarrow 0 < X < 2\lambda$$

which gives

$$P(0 < X < 2\lambda) > 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}$$

b)

$$E[X(X-1)] = \sum_{k=2}^{\infty} k(k-1)e^{-\lambda}\frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = e^{-\lambda}\lambda^2 e^{\lambda} = \lambda^2.$$

$$E[X(X-1)(X-2)] = \sum_{k=3}^{\infty} k(k-1)(k-2)e^{-\lambda}\frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\lambda^3 \sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!} = \lambda^3.$$

10. Show that if the random variable x is $N(\eta; \sigma^2)$, then

$$\boldsymbol{E\{|x|\} = \sigma\sqrt{\frac{2}{\pi}e^{-\eta^2/2\sigma^2}} + 2\eta G\left(\frac{\eta}{\sigma}\right) - \eta}$$

Solution:

Since $E\{|x|\} = \int_0^\infty x\, f(x)dx - \int_{-\infty}^0 xf(x)dx$, and $\eta = E\{x\} = \int_0^\infty x\, f(x)dx +$

$\int_{-\infty}^0 x\, f(x)dx,$

$$\frac{E\{|x|+\eta\}}{2} = \int_0^\infty x\, f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty x\, e^{-(x-\eta)^2/2\sigma^2}dx.$$

Also,

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\infty}(x-\eta)e^{-(x-\eta)^2/2\sigma^2}\,dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\eta}^{\infty}ye^{-y^2/2\sigma^2}\,dy=\frac{\sigma}{\sqrt{2\pi}}e^{-\eta^2/2\sigma^2}$$

and,

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\infty}e^{-(x-\eta)^2/2\sigma^2}\,dx=G(\frac{\eta}{\sigma})$$

Thus,

$$\begin{aligned}\frac{E\{|x|+\eta\}}{2}&=\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\infty}(x-\eta+\eta)e^{-\frac{(x-\eta)^2}{2\sigma^2}}dx\\&=\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\infty}(x-\eta)e^{-\frac{(x-\eta)^2}{2\sigma^2}}dx+\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\infty}\eta\,e^{-\frac{(x-\eta)^2}{2\sigma^2}}dx\\&=\frac{\sigma}{\sqrt{2\pi}}e^{-\eta^2/2\sigma^2}+\eta G(\frac{\eta}{\sigma}),\end{aligned}$$

from which we have

$$E\{|x|\}=2\left[\frac{\sigma}{\sqrt{2\pi}}e^{-\frac{\eta^2}{2\sigma^2}}+\eta G\left(\frac{\eta}{\sigma}\right)\right]-\eta=\sigma\sqrt{\frac{2}{\pi}}e^{-\eta^2/2\sigma^2}+2\eta G\left(\frac{\eta}{\sigma}\right)-\eta.$$

11. X and Y are independent identically distributed normal random variables with zero mean and common variance σ^2, that is, $X\sim\mathcal{N}(0,\sigma^2)$, $Y\sim\mathcal{N}(0,\sigma^2)$ and $f_{XY}(x,y)=f_X(x)f_Y(y)$. Find the p.d.f of (a) $Z=\sqrt{X^2+Y^2}$, (b) $Z=X^2+Y^2$ (c) $U=X-Y$.

Solution:

(a)

$$F_Z(z)=\Pr\left\{\sqrt{X^2+Y^2}\le z\right\}=\int\int_{\sqrt{x^2+y^2}\le z}f_{xy}(x,y)dxdy$$

$$=\int\int_{\sqrt{x^2+y^2}\le z}\frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}dxdy$$

$$=\int_0^z\int_0^{2\pi}\frac{1}{2\pi\sigma^2}e^{-\frac{r^2}{2\sigma^2}}rd\theta dr=1-e^{-\frac{z^2}{2\sigma^2}},$$

Therefore,

$$f_z(z)=\frac{dF_Z(z)}{dz}=\frac{z}{\sigma^2}e^{-\frac{z^2}{2\sigma^2}}$$

for $z>0$.

(b)

$$F_Z(z)=\Pr\{X^2+Y^2\le z\}=\int\int_{x^2+y^2\le z}f_{xy}(x,y)dxdy$$

$$= \int \int_{x^2+y^2 \le z} \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy$$

$$= \int_0^{\sqrt{z}} \int_0^{2\pi} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r d\theta dr = 1 - e^{-\frac{z}{2\sigma^2}},$$

Therefore,

$$f_z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{2\sigma^2} e^{-\frac{z}{2\sigma^2}}$$

for $z > 0$.

(c)

$$U = X - Y \sim \mathcal{N}(0, 2\sigma^2)$$

Since linear combinations of jointly Gaussian random variables are Gaussian random variables. Here $Var(U) = Var(X) + Var(Y) = 2\sigma^2$.

12. The events $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are such that

$$\boldsymbol{P(A) = P(B) = P(C) = 0.5}$$
$$\boldsymbol{P(AB) = P(AC) = P(BC) = P(ABC) = 0.25}$$

Show that the zero-one random variables associated with these events are not independent; they are, however, independent in pairs.

Solution:

$$P\{x_A = 1, x_B = 1, x_C = 1\} = P\{ABC\} = 1/4$$

And,

$$P\{x_A = 1\} = P(A) = \frac{1}{2} \quad P\{x_B = 1\} = P(B) = \frac{1}{2} \quad P\{x_C = 1\} = P(C) = \frac{1}{2}$$

Hence

$$P\{x_A = 1, x_B = 1, x_C = 1\} \neq P\{x_A = 1\}P\{x_B = 1\}P\{x_C = 1\}$$

Therefore, x_A, x_B, x_C are not independent. But

$$P\{x_A = 1, x_B = 1\} = P(AB) = \frac{1}{4} = P\{x_A = 1\}P\{x_B = 1\}$$

Similarly for any other combination, e.g.,
Since $P(A) = P(AB) + P(A\bar{B})$, we conclude that

$$P(A\bar{B}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$P(\bar{B}) = 1 - P(B) = \frac{1}{2}$$

$$P\{x_A = 1, x_B = 0\} = P(A\bar{B}) = \frac{1}{4}$$

$$P\{x_B = 0\} = P(\bar{B}) = \frac{1}{2}$$

Hence

$$P\{x_A = 1, x_B = 0\} = P\{x_A = 1\}P\{x_B = 0\}$$

13. Consider the randomly-phased sinusoid

$$n(t) = Acos(2\pi f_C t + \theta)$$

where A and f_C are constant amplitude and frequency, respectively, and θ is a random phase angle uniformly distributed over the range [0; 2π]. Calculate the mean and mean square of n(t).

Solution:

The pdf of θ is

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta < 2\pi \\ 0, & otherwise \end{cases}$$

By definition the mean is

$$\begin{aligned} E[n(t)] &= \int_{-\infty}^{\infty} Acos(2\pi f_C t + \theta) f_\Theta(\theta) d\theta \\ &= \frac{A}{2\pi} \int_0^{2\pi} \cos{(2\pi f_C t + \theta)} d \\ &= \frac{A}{2\pi} \times 0 = 0 \end{aligned}$$

By definition the mean square is

$$\begin{aligned} E[n^2(t)] &= \int_{-\infty}^{\infty} A^2 cos^2(2\pi f_C t + \theta) f_\Theta(\theta) d\theta \\ &= \frac{A^2}{2\pi} \int_0^{2\pi} \frac{1}{2}(1 + \cos(4\pi f_C t + 2\theta)) d\theta \\ &= \frac{A^2}{4\pi}([\theta]_0^{2\pi} + [\frac{1}{2}\sin{(4\pi f_C t + 2\theta)}]_0^{2\pi}) \\ &= \frac{A^2}{2} \end{aligned}$$

14. Let $X(t)$ be a wide-sense stationary random process. $X(t)$ is mixed (i.e., multiplied) by a sinusoidal signal $\cos{(2\pi f_C t + \Theta)}$, where the phase Θ is a random variable uniformly distributed over the interval (0; 2π). Find the power spectral density of the output process $Y(t)$ defined by

$$Y(t) = X(t) \cos(2\pi f_C t + \Theta).$$

Solution:

We first find the autocorrelation function of $Y(t)$:

$$\begin{aligned} R_Y(\tau) &= E[Y(t+\tau)Y(t)] \\ &= E[X(t+\tau)\cos{(2\pi f_C(t+\tau) + \Theta)} X(t) \cos{(2\pi f_C t + \Theta)}] \\ &= E[X(t+\tau)X(t)] E[\cos{(2\pi f_C(t+\tau) + \Theta)} \cos{(2\pi f_C t + \Theta)}] \end{aligned}$$

which follows from the independence of Θ and $X(t)$. We then have

$$R_Y(\tau) = \frac{1}{2} R_X(\tau) E[\cos(2\pi f_C \tau) + \cos{(2\pi f_C(2t+\tau) + 2\Theta)}]$$

where we have used the fact that $X(t)$ is a wide-sense stationary process. Therefore, we have

$$R_Y(\tau) = \frac{1}{2} R_X(\tau) \cos(2\pi f_C \tau).$$

Power spectral density is the Fourier transform of the autocorrelation function. We have

$$S_Y(f) = \frac{1}{4}[S_X(f - f_C) + S_X(f + f_C)].$$

15. The random process $v(t)$ is defined as

$$\boldsymbol{v(t) = X \cos 2\pi f_c t - Y \sin 2\pi f_c t}$$

where X and Y are random variables. Show that $v(t)$ is wide-sense stationary if and only if $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2)$, and $E(XY) = 0$.

Solution:

i) Show that if $E(X) = E(Y) = E(XY) = 0$ and $E(X^2) = E(Y^2) = \sigma^2$, then $v(t)$ is wide-sense stationary.

Assume conditions hold, then

$$E\big(v(t)\big) = E(X) \cos 2\pi f_c t - E(Y) \sin 2\pi f_c t = 0 \Rightarrow \text{constant}$$

$$E\big(v(t+\tau)v(t)\big) = E(X^2) \cos\big(2\pi f_c (t+\tau)\big) \cos 2\pi f_c t - E(XY) \cos\big(2\pi f_c (t+\tau)\big) \sin 2\pi f_c t$$
$$-E(XY) \sin\big(2\pi f_c (t+\tau)\big) \cos 2\pi f_c t + E(Y^2) \sin\big(2\pi f_c (t+\tau)\big) \sin 2\pi f_c t$$
$$= \sigma^2\big(\cos\big(2\pi f_c (t+\tau)\big) \cos 2\pi f_c t + \sin\big(2\pi f_c (t+\tau)\big) \sin 2\pi f_c t\big)$$
$$= \frac{\sigma^2}{2}(\cos 2\pi f_c \tau + \cos 2\pi f_c (2t+\tau) + \cos 2\pi f_c \tau - \cos 2\pi f_c (2t+\tau))$$
$$= \sigma^2 \cos 2\pi f_c \tau \Rightarrow \text{not a function of t}$$

Hence, $v(t)$ is wide-sense stationary

ii) Show that if $v(t)$ is wide-sense stationary, then $E(X) = E(Y) = E(XY) = 0$ and $E(X^2) = E(Y^2) = \sigma^2$.

Assume conditions hold, then

- $E\big(v(t)\big)$ is constant. As $E(v(t)) = E(X) \cos 2\pi f_c t - E(Y) \sin 2\pi f_c t$, and the only way $v(t)$ can be constant is that $E(X) = E(Y) = 0$
- $E(v(t+\tau)v(t))$ is not function of t. As

$E\big(v(t+\tau)v(t)\big)$

$$= \frac{1}{2}(\big(E(X^2) + E(Y^2)\big) \cos 2\pi f_c \tau + \big(E(X^2) - E(Y^2)\big) \cos 2\pi f_c (2t+\tau)$$
$$- \big(E(XY) - E(YX)\big) \sin 2\pi f_c \tau - 2\,E(XY) \sin 2\pi f_c (2t+\tau))$$

For this function to be invariant of t, we must have

$$E(X^2) - E(Y^2) = 0 \Longrightarrow E(X^2) = E(Y^2)$$
$$2E(XY) = 0 \Longrightarrow E(XY) = 0$$

16. Let $X(t)$ be a wide-sense stationary process with an autocorrelation function $R_X(\tau)$. Prove that $|R_X(\tau)| \le R_X(0)$ for any τ.

Solution:

We have

$$E[\big(X(t+\tau) \pm X(t)\big)^2] \ge 0,$$

From this we get

$$E[X^2(t+\tau)] \pm 2E[X(t+\tau)X(t)] + E[X^2(t)] \geq 0,$$

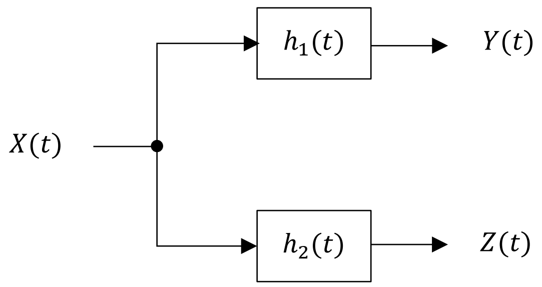
which is equivalent to

$$2R_X(0) \pm 2R_X(\tau) \geq 0,$$

And, finally,

$$-R_X(0) \leq R_X(\tau) \leq R_X(0),$$

17. A stationary zero-mean Gaussian random process $X(t)$ is passed through two linear filters with impulse responses $h_1(t)$ and $h_2(t)$, yielding processes $Y(t)$ and $Z(t)$, respectively, as shown in the following figure.



Show that $Y(t)$ and $Z(t)$ are statistically independent if the transfer functions $H_1(f)$ and $H_2(f)$do not overlap in the frequency domain (for example, when they are narrowband filters at different frequency bands).

Solution:

Note that since both $Y(t)$ and $Z(t)$ are Gaussian, showing their independence is the same as showing that they are uncorrelated. The cross-correlation function of $Y(t)$ and $Z(t)$ is given by

$$\begin{aligned} R_{YZ}(t,u) &= E[Y(t)Z(u)] \\ &= E\left[\int_{-\infty}^{\infty} h_1(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h_2(\tau_2)X(u-\tau_2)d\tau_2\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)E[X(t-\tau_1)X(u-\tau_2)]d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)R_X(t-\tau_1, u-\tau_2)d\tau_1 d\tau_2\,, \end{aligned} \tag{1}$$

where $R_X(t,u)$ is the auto-correlation function of $X(t)$. Since $X(t)$ is stationary, we may set $\tau = t-u$, and rewrite Eq. (1) as follows:

$$R_{YZ}(t,u) = \int_{-\infty}^{\infty} h_2(\tau_2) \underbrace{\int_{-\infty}^{\infty} h_1(\tau_1)R_X(\tau-\tau_1+\tau_2)d\tau_1}_{g(\tau+\tau_2)} d\tau_2$$

$$= \int_{-\infty}^{\infty} h_2(\tau_2) g(\tau + \tau_2) d\tau_2 \tag{2}$$

Where the inner integral is a function $g(\tau + \tau_2)$. Now we recognize Eq. (2) as the convolution

$$R_{YZ}(t,u) = h_2(-\tau) * g(\tau)$$

Moreover,

$$g(\tau) = \int_{-\infty}^{\infty} h_1(\tau_1) R_X(\tau - \tau_1) d\tau_1 = h_1(\tau) * R_X(\tau)$$

Therefore, we have the triple convolution

$$R_{YZ}(t,u) = h_1(\tau) * h_2(-\tau) * R_X(\tau)$$

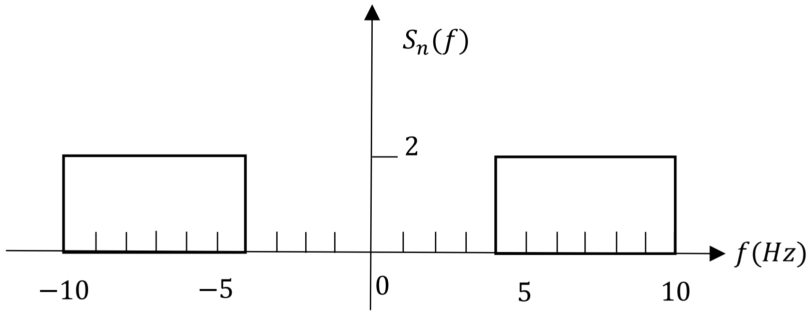
Recall the fact that convolution in time domain corresponds to product in frequency domain, we get

$$S_{YZ}(f) = H_1(f) H_2^*(f) S_X(f)$$

Obviously, $S_{YZ}(f)$ vanishes when $H_1(f)$ and $H_2(f)$ do not overlap in the frequency domain. This implies $R_{YZ}(\tau) \equiv 0$, i.e., $Y(t)$ and $Z(t)$ are uncorrelated.

Baseband and Passband Signal and Noise Effects (Lectures 3 & 4)

1. **Consider a bandpass noise signal having the power spectral density shown below. Draw the power spectral density (PSD) of $\boldsymbol{n_I(t)}$ if the center frequency is chosen as:**
   (a) $\boldsymbol{f_c = 7\ Hz}$
   (b) $\boldsymbol{f_c = 5\ Hz}$

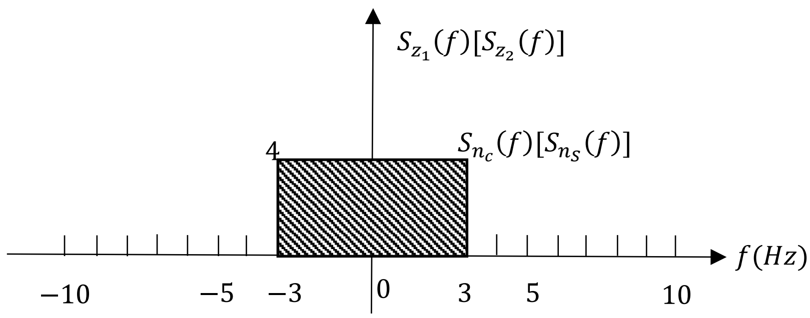


Solution:
Recall that

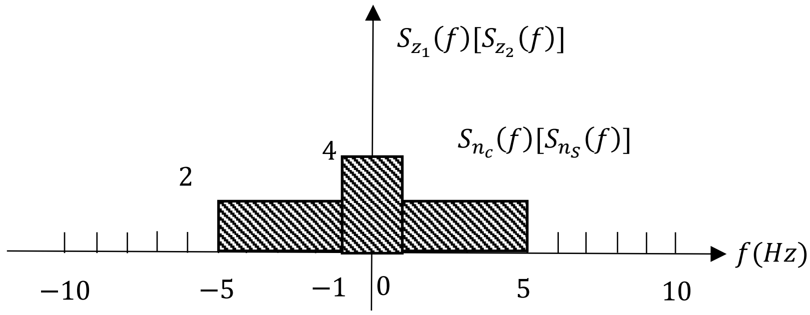
$$S_{N_I}(f) = \begin{cases} S_N(f - f_C) + S_N(f + f_C), & |f| < B \\ 0, & otherwise \end{cases}$$

We add the PSDs of the negative and positive frequency bands shifted to the origin.
In these figures, the shaded region is the PSD of $\boldsymbol{n_I(t)}$.

(a)



(b)



2. Let

$$\boldsymbol{f_k(t)} \triangleq \begin{cases} \boldsymbol{e^{-\frac{t}{k}}}, & \boldsymbol{if\ t > 0}, \\ \boldsymbol{0}, & \boldsymbol{if\ t = 0}, \\ \boldsymbol{-e^{-\frac{t}{k}}}, & \boldsymbol{if\ t < 0}, \end{cases}$$

Find $\boldsymbol{F_k(f)}$, the Fourier transform of $\boldsymbol{f_k(t)}$. Letting $\boldsymbol{k \to \infty}$, find the Fourier transform of function $\boldsymbol{sgn(t)}$, defined as

$$\boldsymbol{sgn(t)} \triangleq \begin{cases} \boldsymbol{1}, & \boldsymbol{if\ t > 0}, \\ \boldsymbol{0}, & \boldsymbol{if\ t = 0}, \\ \boldsymbol{-1}, & \boldsymbol{if\ t < 0}. \end{cases}$$

Using this, find the Fourier transform of unit step function

$$\boldsymbol{u(t)} \triangleq \begin{cases} \boldsymbol{1}, & \boldsymbol{if\ t > 0}, \\ \boldsymbol{1/2}, & \boldsymbol{if\ t = 0}, \\ \boldsymbol{0}, & \boldsymbol{if\ t < 0}. \end{cases}$$

Solution:

We have $f_k(t) = f_k^1(t) + f_k^2(t)$, where

$$f_k^1(t) = e^{-\frac{t}{k}}u(t),$$

$$f_k^2(t) = -e^{\frac{t}{k}}u(-t).$$

Let $\mathcal{F}\left(f_k^i(t)\right) = F_k^i(f)$ for $i = 1{,}2$ and $\forall k$. We have

$$F_k^1(f) = \int_0^\infty e^{-\frac{t}{k}}e^{-j2\pi ft}dt,$$

$$= \frac{1}{\frac{1}{k} + j2\pi f}$$

and

$$F_k^2(f) = \int_{-\infty}^0 -e^{-\frac{t}{k}}e^{-j2\pi ft}dt,$$

$$= \int_0^\infty e^{t(\frac{1}{k} - j2\pi f t)} dt,$$

$$= -\frac{1}{\frac{1}{k} - j2\pi f}$$

Combining the two, we can get

$$F_k(f) = F_k^1(f) + F_k^2(f)$$

$$= \frac{1}{\frac{1}{k} + j2\pi f} - \frac{1}{\frac{1}{k} - j2\pi f}$$

$$= -\frac{j4\pi f}{\frac{1}{k^2} + 4\pi^2 f^2}$$

In the limit, we have

$$f_k(t) \xrightarrow{k\to\infty} sgn(t)$$

and

$$F_k(f) \xrightarrow{k\to\infty} \frac{1}{j\pi f},$$

From which, it follows that

$$\mathcal{F}\big(sgn(t)\big) = \frac{1}{j\pi f}.$$

We have $u(t) = \frac{1}{2}(sgn(t) + 1)$. It directly follows that $U(f) \triangleq \mathcal{F}\big(u(t)\big) = \frac{1}{2}\left(\frac{1}{j\pi f} + \delta(f)\right)$.

3. Hilbert transform of a signal $g(t)$ is defined as

$$\hat{g}(t) = g(t) * \frac{1}{\pi t}$$

Using the result of the previous exercise, find $\hat{G}(f)$, the Fourier transform of $\hat{g}(t)$.

Solution:

From the duality relation of the Fourier transform, since $\mathcal{F}\big(sgn(t)\big) = \frac{1}{j\pi f}$, we have

$$\mathcal{F}\left(\frac{1}{j\pi t}\right) = sgn(-f)$$

It follows that

$$\mathcal{F}\left(\frac{1}{\pi t}\right) = jsgn(-f) = -jsgn(f)$$

We have

$$\hat{G}(f) = -jsgn(f)G(f)$$

4. Prove the following properties of Hilbert transforms:

a) If $x(t) = x(-t)$, then $\hat{x}(t) = -\hat{x}(-t)$.

b) If $x(t) = -x(-t)$, then $\hat{x}(t) = \hat{x}(-t)$.

c) If $x(t) = \cos(2\pi f_0 t)$, then $\hat{x}(t) = \sin(2\,\pi f_0 t)$.

d) If $x(t) = \sin(2\pi f_0 t)$, then $\hat{x}(t) = -\cos(2\,\pi f_0 t)$.

e) $\hat{\hat{x}}(t) = -x(t)$.

f) $\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} \hat{x}^2(t)dt$.

g) $\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$.

Solution:

Denote $h(t) = \frac{1}{\pi t}$ as the impulse response of the Hilbert transform and

$$H(f) = \mathcal{F}[h(t)] = \begin{cases} -j, & f \geq 0 \\ +j, & f < 0 \end{cases}.$$

Note that $H(f) = -H(-f)$.

a) It can be directly proved using the property of the Fourier transform that $x(t) = x(-t)X(f)$ is equivalent to $X(f) = X(-f)$. Therefore,

$$X(f)H(f) = X(-f)H(f) \Rightarrow X(f)H(f) = -X(-f)H(-f)$$

As $X(f)H(f) = \mathcal{F}[\hat{x}(t)]$ and $X(-f)H(-f) = \mathcal{F}[\hat{x}(-t)]$,

$$\hat{x}(t) = -\hat{x}(-t)$$

b) Similarly, using the property of the Fourier transform, $x(t) = -x(-t)$ is equivalent to $X(f) = -X(-f)$. Therefore,

$$X(f)H(f) = -X(-f)H(f) \Rightarrow X(f)H(f) = X(-f)H(-f)$$

As $X(f)H(f) = \mathcal{F}[\hat{x}(t)]$ and $X(-f)H(-f) = \mathcal{F}[\hat{x}(-t)]$,

$$\hat{x}(t) = \hat{x}(-t)$$

c) If $x(t) = \cos(2\,\pi f_0 t)$, then $X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$

$$\mathcal{F}[\hat{x}(t)] = X(f)H(f) = [\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)]H(f)$$

$$= -j\frac{1}{2}\delta(f - f_0) + j\frac{1}{2}\delta(f + f_0) = \frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$$

which is the Fourier transformer of $\sin(2\pi f_0 t)$.

d) If $x(t) = \sin(2\,\pi f_0 t)$, then $X(f) = \frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$

$$\mathcal{F}[\hat{x}(t)] = X(f)H(f) = \left[\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)\right]H(f)$$

$$= -j\frac{1}{2j}\delta(f - f_0) - j\frac{1}{2j}\delta(f + f_0) = -\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

which is the Fourier transformer of $-\cos(2\pi f_0 t)$.

e)

$$\mathcal{F}\left[\hat{\hat{x}}(t)\right] = [X(f)H(f)]H(f)$$

As $H(f)H(f) = -1$,

$$\mathcal{F}\left[\hat{\hat{x}}(t)\right] = -X(f) \Rightarrow \hat{\hat{x}}(t) = -x(t)$$

f) According to Parseval's theorem,

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(f)G^*(f)df\,.$$

If $f(t) = \hat{x}(t)$ and $g(t) = \hat{x}^*(t)$, then $F(f) = \hat{X}(f)$, $G(f) = \hat{X}^*(-f)$ and

$$\int_{-\infty}^{\infty} \hat{x}^2(t)dt = \int_{-\infty}^{\infty} \hat{X}(f)\hat{X}(-f)df = \int_{-\infty}^{\infty} X(f)H(f)X(-f)H(-f)df$$

As $H(f)H(-f) = 1$, we have

$$\int_{-\infty}^{\infty} \hat{x}^2(t)dt = \int_{-\infty}^{\infty} X(f)X(-f)df = \int_{-\infty}^{\infty} x^2(t)dt.$$

g) There are two ways to solve it.

1) Let $I = \int_{-\infty}^{\infty} x(t)\hat{x}(t)dt$

As

$$\hat{x}(t) = \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}d\tau$$

Thus

$$\begin{aligned} I &= \int_{-\infty}^{\infty} x(t)\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}d\tau dt \\ &= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} \frac{x(t)}{t-\tau}dt\right]d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)(-x(\tau))d\tau = -I \end{aligned}$$

Hence

$$I = 0$$

2) According to Parseval's theorem, let $f(t) = x(t)$ and $g(t) = \hat{x}^*(t)$. Then $F(f) = X(f)$, $G(f) = \hat{X}^*(-f)$, then

$$\begin{aligned} \int_{-\infty}^{\infty} x(t)\,\hat{x}(t)dt &= \int_{-\infty}^{\infty} X(f)\hat{X}(-f)df \\ &= \int_{-\infty}^{\infty} X(f)X(-f)H(-f)df \end{aligned}$$

However,

$$Z(f) = X(f)X(-f)H(-f) = -X(f)X(-f)H(f) = -Z(-f)$$

Thus, $Z(f)$ is an odd function, so the integral is zero.

DSB, SSB, and AM (Lectures 5 & 6)

1. **Consider a message signal with a bandwidth of 10 kHz and an average power of P =10 watts. Assume the transmission channel attenuates the transmitted signal by 40 dB, and adds noise with a power spectral density of:**

$$\mathbb{S}(f) = \begin{cases} N_o(1 - \dfrac{|f|}{200 \times 10^3}), & |f| < 200 \times 10^3 \\ 0, & otherwise \end{cases}$$

where $N_o = 10^{-9}$ watts/Hz.

What is the predetection SNR at the receiver if each of the following modulation schemes is used? Assume that a suitable filter is used at the input of the receiver to limit the out-of-band noise.

(a) Baseband

(b) DSB-SC with a carrier frequency of 100kHz and a carrier amplitude of $A_c = 1V$.

(c) DSB-SC with a carrier frequency of 150kHz and a carrier amplitude of $A_c = 1V$.

Solution:

In this question, it is very important to note that the channel noise is not white Gaussian noise. Therefore, we can not directly use the expressions derived in the lecture since these are valid only for white noise. Also note that we are considering the predetection SNR at the receiver (not the output SNR). If the channel attenuates the transmitted signal by 40 dB, then the received signal power will be 10^{-4} times the transmitted signal power.

(a) For baseband, the transmitted signal power is $P_T = 10$ W (i.e., the same as the message power). The received signal power is therefore $P_R = 1$ mW.

The noise power is found by integrating the noise PSD over the transmission bandwidth:

$$P_N = 2\int_0^{10^4} \mathbb{S}(f)df = 2N_o \int_0^{10^4} \left(1 - \frac{f}{200 \times 10^3}\right) df$$

$$= 2N_o \times 10^4 \times 0.975 = 19.5\mu W$$

This gives an SNR at the receiver input of

$$SNR = \frac{1 \times 10^{-3}}{19.5 \times 10^{-6}} = 17.1dB.$$

(b) For DSB-SC, the transmitted signal power is $P_T = \frac{A_c^2 P}{2} = 5$W. The received signal power is therefore $P_R = 0.5$ mW.

The noise power is:

$$P_N = 2\int_{f_c-10^4}^{f_c+10^4} \mathbb{S}(f)df = 2N_o \times 0.5 \times 20 \times 10^3$$

$$= 20\mu W$$

giving a receiver input SNR of

$$SNR = \frac{0.5 \times 10^{-3}}{20 \times 10^{-6}} = 14dB.$$

(c) At this carrier frequency, the noise power becomes:

$$P_N = 2N_o \times 0.25 \times 20 \times 10^3 = 10\mu W,$$

and the receiver input SNR is

$$SNR = \frac{0.5 \times 10^{-3}}{10 \times 10^{-6}} = 17dB.$$

Observe that these results are quite different from what one would obtain if the channel had white Gaussian noise. If the noise PSD were flat, the SNR of DSB-SC would be independent of the carrier frequency.

2. Consider the standard AM modulation, where the transmitted signal is given by

$$\boldsymbol{s(t) = [A + m(t)]\cos(2\pi f_c t),}$$

where $\boldsymbol{m(t)}$ is the message signal. Assume that the modulating wave is a sinusoidal wave, i.e., single-tone modulation,

$$\boldsymbol{m(t) = A_m cos(2\pi f_m t).}$$

Given the baseband signal-to-noise ratio $\boldsymbol{SNR_{Baseband}}$, consider an AM envelope detector when the noise power is small. Compute the output SNR in terms of the modulation index $\boldsymbol{\mu}$, which is defined as $\boldsymbol{\mu \triangleq m_p/A}$, where $\boldsymbol{m_p}$ is the peak value of the message signal. What value of $\boldsymbol{\mu}$ gives the maximum output SNR?

Solution:

In the lecture, the output SNR of an envelope detector (for small noise) was shown to be

$$SNR = \frac{P}{A^2 + P} SNR_{baseband}.$$

For single-tone modulation, we have $P = \frac{1}{2} m_p^2$. Substitution gives

$$SNR = \frac{\frac{1}{2} m_p^2}{A^2 + \frac{1}{2} m_p^2} SNR_{baseband} = \frac{\mu^2}{2 + \mu^2} SNR_{baseband}.$$

We may rewrite it as

$$SNR = \left(1 - \frac{2}{2 + \mu^2}\right) SNR_{baseband}.$$

Now, increasing the value of μ will decrease the second term, thereby increasing the SNR. But, μ cannot be increased arbitrarily, since for an envelope detector to operate the modulation index should satisfy $\mu \leq 1$. Thus, the value of μ that gives the maximum SNR is $\mu = 1$. The resulting SNR expression is

$$SNR = \frac{1}{3} SNR_{baseband}.$$

3. For each of the baseband signals: (i) $\boldsymbol{m(t) = \cos 1000\pi t}$; (ii) $\boldsymbol{m(t) = 2\cos 1000\pi t + \sin 2000\pi t}$; (iii) $\boldsymbol{m(t) = \cos 1000\pi t \cos 3000\pi t}$, do the following.

(a) Sketch the spectrum of $\boldsymbol{m(t)}$.

(b) Sketch the spectrum of the DSB-SC signal $\boldsymbol{m(t)\cos 10000\pi t}$.

(c) Identify the upper sideband (USB) and the lower sideband (LSB) spectra.

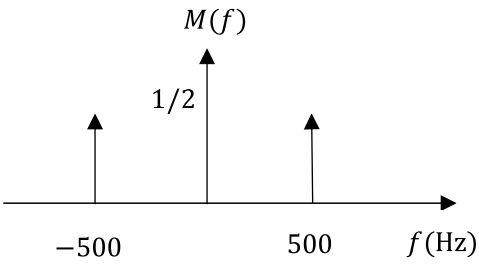
(d) Identify the frequencies in the baseband, and the corresponding frequencies in the

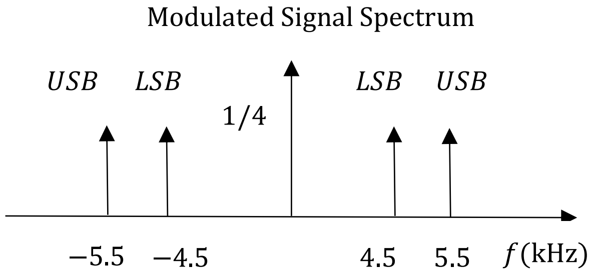
DSB-SC, USB, and LSB spectra. Explain the nature of frequency shifting in each case.

Solution:

(i) $m(t) = \cos w_m t = \cos 2\pi f_m t = \cos 1000\pi t \Rightarrow f_m = 500\text{Hz}.$

$M(f) = \frac{1}{2}\delta(f-500) + \frac{1}{2}\delta(f+500).$



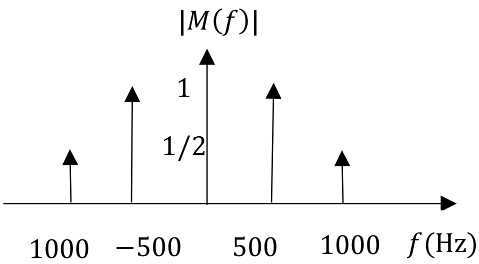


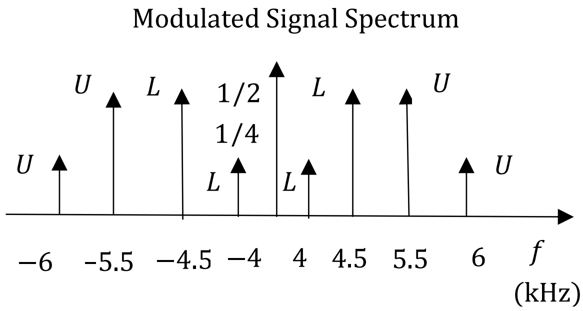
(ii) $m(t) = 2\cos w_{m,1}t + \sin w_{m,2}t = 2\cos 2\pi f_{m,1}t + \sin 2\pi f_{m,2}t = 2\cos 1000\pi t + \sin 2000\pi t$

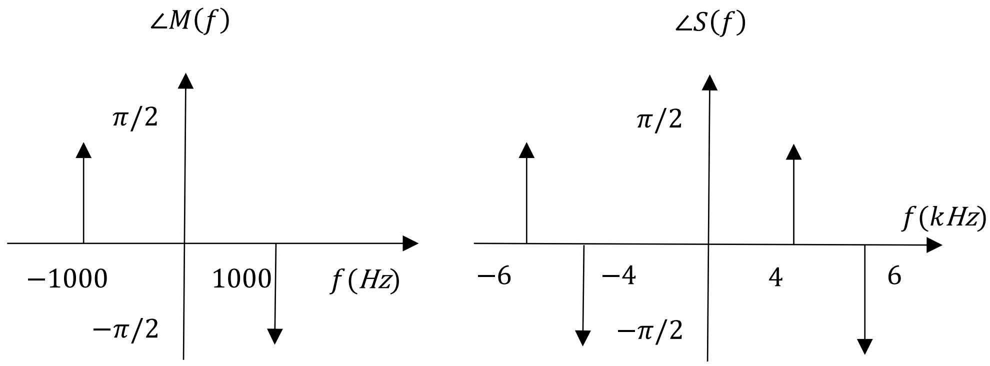
$M(f) = \delta(f-500) + \delta(f+500) + \frac{j}{2}\big(\delta(f+1000) - \delta(f-1000)\big).$

$|M(f)| = \delta(f-500) + \delta(f+500) + \frac{1}{2}\big(\delta(f+1000) + \delta(f-1000)\big).$

$$\angle M(f) = \begin{cases} -\pi/2, & f = 1000 \\ \pi/2, & f = -1000. \\ 0, & else \end{cases}$$

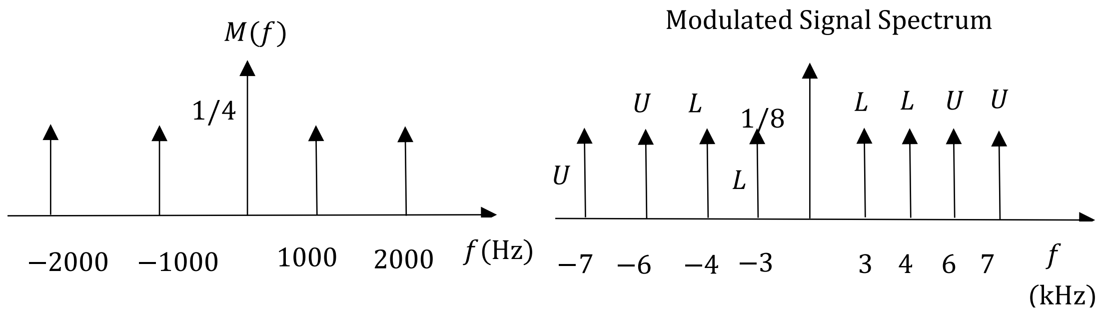






(iii) $m(t) = \cos w_{m,1}t \cos w_{m,2}t = \cos 1000\pi t \cos 3000\pi t = \frac{1}{2}\left(\cos 2\pi f_{m,1}t + \cos 2\pi f_{m,2}t\right) =$

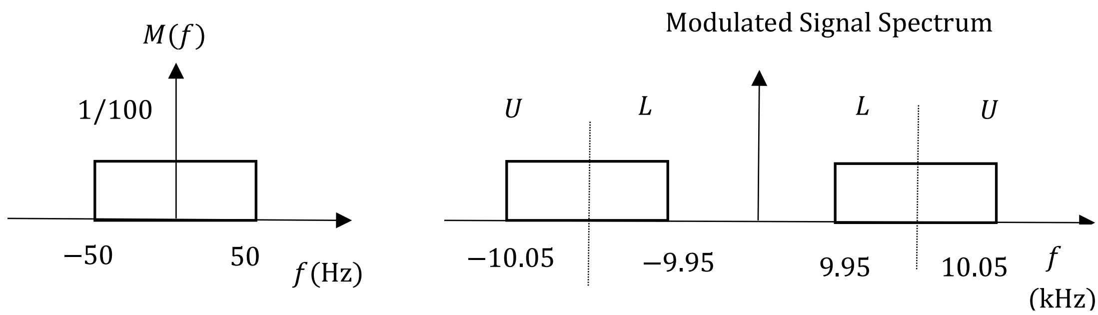
$\frac{1}{2}(\cos 2000\pi t + \cos 4000\pi t) \rightarrow f_{m,1} = 1000Hz\; f_{m,2} = 2000Hz$



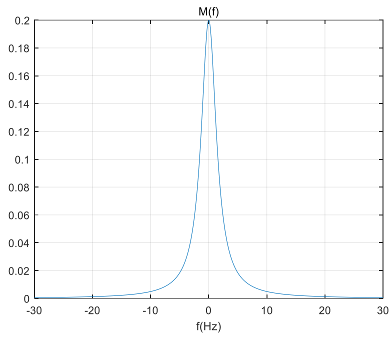
4. **Repeat Prob. 3 [parts (a), (b), and (c) only] if (i) $m(t) = \mathrm{sinc}(100t)$; (ii) $m(t) = e^{-|t|}$; (iii) $m(t) = e^{-|t-1|}$. Observe that $e^{-|t-1|}$ is $e^{-|t|}$ delayed by 1 second. For the last case, you need to consider both the amplitude and the phase spectra.**

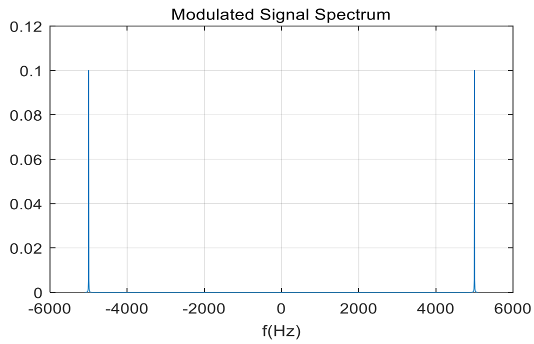
Solution:

(i) $\mathcal{F}\left(m(t)\right) = \frac{1}{100} rect\left(\frac{f}{100}\right) = \begin{cases} 1/100, & |f| < 50 \\ 0, & otherwise \end{cases}$



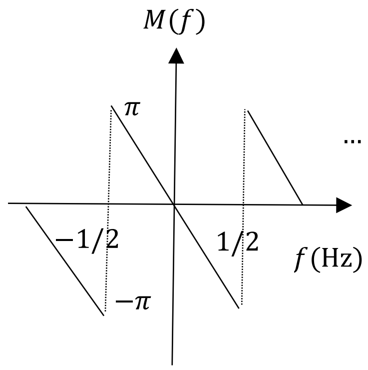
(ii) $\mathcal{F}\big(m(t)\big) = \frac{20}{100+4\pi^2 f^2}$, $\mathcal{F}(m(t)\cos 10000\pi t) = \frac{10}{100+4\pi^2(f-5000)^2} + \frac{10}{100+4\pi^2(f+5000)^2}$

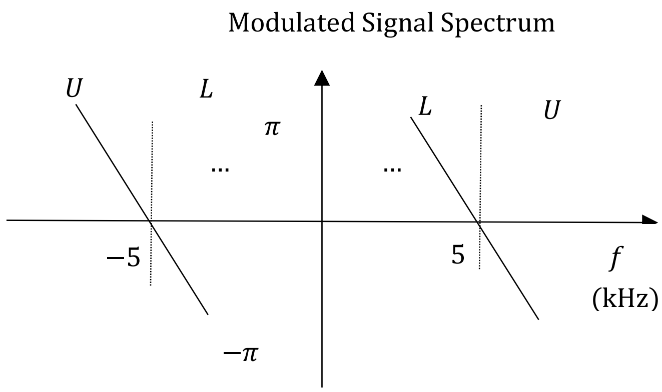




(iii) $\mathcal{F}\big(m(t)\big) = \frac{20}{100+4\pi^2 f^2} e^{-j2\pi f}$, $\mathcal{F}(m(t)\cos 10000\pi t) = \frac{10}{100+4\pi^2(f-5000)^2} e^{-j2\pi(f-5000)} + \frac{10}{100+4\pi^2(f+5000)^2} e^{-j2\pi(f+5000)}$

The amplitude spectra is the same as that in (ii), while the phase spectral is:
$\angle M(f) = -2\pi f$.





5. Sketch the AM signal $[A + m(t)]\cos(2\pi f_c t)$ for the periodic triangle signal $m(t)$ shown in Fig. P4.3-2 corresponding to the modulation indices (a) $\mu = 0.5$; (b) $\mu = 1$; (c) $\mu = 2$; (d) $\mu = \infty$; How do you interpret the case of $\mu = \infty$?

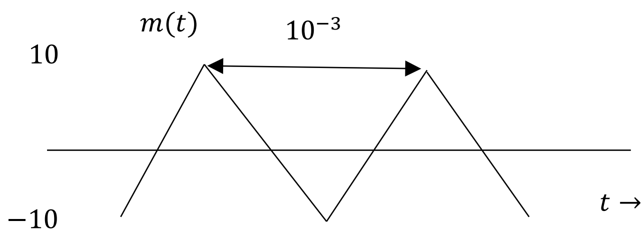
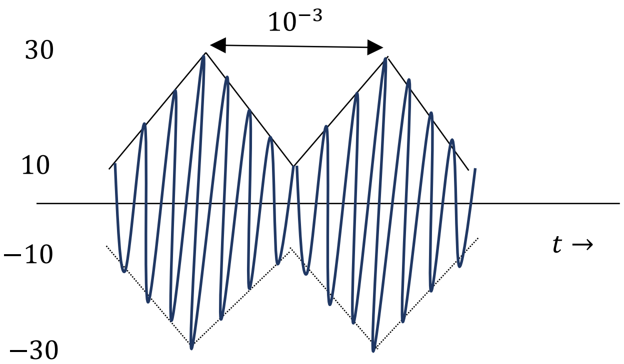


Figure P.4.3-2

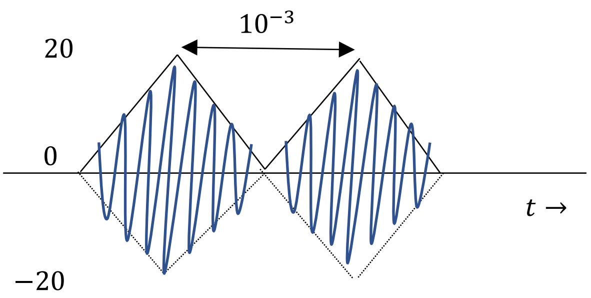
Solution:

As $m_{\min} = -m_{\max} = -10$, we have $\mu = \frac{m_p}{A}$.

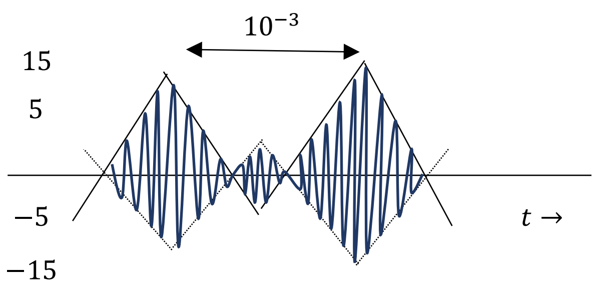
(a) $\mu = \frac{m_p}{A} = 0.5 \rightarrow A = \frac{m_p}{\mu} = \frac{10}{0.5} = 20$



(b) $\mu = \frac{m_p}{A} = 1 \rightarrow A = \frac{m_p}{\mu} = \frac{10}{1} = 10$



(c) $\mu = \frac{m_p}{A} = 2 \rightarrow A = \frac{m_p}{\mu} = \frac{10}{2} = 5$



(d) $\mu = \frac{m_p}{A} = \infty \rightarrow A = \frac{m_p}{\mu} = \frac{10}{\infty} = 0$

This means that $\mu = \infty$ represents the DSB-SC case.

6. For the AM signal with $m(t)$ shown in Fig. P.4.3-2 and $\mu = 0.8$:

(a) Find the amplitude and power of the carrier.

(b) Find the sideband power and the power efficiency η.

Solution:

(a) The carrier amplitude is $A = \frac{m_p}{\mu} = \frac{10}{0.8} = 12.5$. The carrier power is $P_c = \frac{A^2}{2} = \frac{12.5^2}{2} = 78.125$

(b) The sideband power is $\overline{m^2(t)}/2$. Because of symmetry of amplitude values every quarter cycle, the power of $m(t)$ may be computed by averaging the signal energy over a quarter cycle only. Over a quarter cycle $m(t)$ can be represented as $m(t) = 40t/T_0$. Note that $T_0 = 10^{-3}$ Hence,

$$\overline{m^2(t)} = \frac{1}{T_0/4}\int_0^{T_0/4} (\frac{40t}{T_0})^2 dt = 33.34$$

The sideband power is

$$P_s = \frac{\overline{m^2(t)}}{2} = 16.67$$

The efficiency is

$$\eta = \frac{P_s}{P_c + P_s} = \frac{16.67}{78.125 + 16.67} = 17.59\%$$

FM and Digital Representation of Signals (Lectures 7 and 8)

1. **An FM modulated signal is given by**

$$\boldsymbol{x(t) = 10\cos(15000\pi t)}$$

for $0 \le t \le 1$. Find the message if $k_f = 2000$ and $f_c = 5KHz$.

Solution:

FM modulated signal is given by

$$\begin{aligned} x(t) &= 10\cos(15000\pi t) \\ &= A\cos(2\pi f_c t + 2\pi f_k \int_0^t m(\tau)\, d\tau) \\ &= A\cos(10000\pi t + 4000\pi \int_0^t m(\tau)\, d\tau) \end{aligned}$$

Therefore

$$\int_0^t m(\tau)\, d\tau = 5t/4,$$

and hence,

$$m(t) = 5/4.$$

2. **Given the baseband signal-to-noise ratio $SNR_{Baseband}$, consider an FM detector for singletone modulation, that is, the modulating wave is a sinusoidal wave.**

$$\boldsymbol{m(t) = A_m\cos(2\pi f_m t)}.$$

(a) Compute the output SNR in terms of the modulation index β, where $\beta \triangleq \Delta f/W$.

(b) Comparing with the figure of merit for a full AM system (i.e. $\mu = 1$), at what value of β will FM start to offer improved noise performance?

Solution:

From the lecture, we know

$$SNR_{FM} = 3\beta^2 \frac{P}{m_p^2} SNR_{baseband}.$$

(a) For single-tone modulation, the peak amplitude is $m_p = A_m$, while the message power is $P = \frac{1}{2}A_m^2$. Therefore,

$$SNR_{FM} = \frac{3}{2}\beta^2 SNR_{baseband}$$

(b) The figure of merit for full AM is 1/3. FM will perform better than AM if

$$\frac{3}{2}\beta^2 > \frac{1}{3}$$

That is,

$$\beta > \frac{\sqrt{2}}{3} = 0.471$$

3. **Suppose the modulating signal for FM is modelled as a zero-mean Gaussian**

random process $m(t)$ with standard deviation σ_m. One can make the approximation $m_p = 4\sigma_m$ as the overload probability $|m(t)| > 4\sigma_m$ is very small. Determine the output SNR for the FM receiver in the presence of additive white Gaussian noise, in terms of the deviation ratio β and the baseband SNR.

Solution:

Using the formula for the output SNR for FM

$$SNR_{FM} = 3\beta^2 \frac{P}{m_p^2} SNR_{baseband}$$

And the fact that the message power P=σ_m^2 and $m_p = 4\sigma_m$, one has

$$SNR_{FM} = 3\beta^2 \frac{\sigma_m^2}{16\sigma_m^2} SNR_{baseband}$$

$$= \frac{3}{16}\beta^2 SNR_{baseband}$$

4. **Assume that the bandwidth of a speech signal is between 50 Hz up to 10 KHz. We want to sample this signal at the Nyquist rate, and then quantize using 16 bits per sample. How many megabytes of storage do you need to store one hour of this speech signal?**

Solution:

This signal must be sampled at 20 KHz. Then we will have a bit rate of $16 * 20 = 320$ Kbps. In one hour we will have $60 * 60 * 320$ Kbits = 1152 Mbits = 144 Mbytes.

5. **A PCM output is produced by a uniform quantizer that has 2^n levels. Assume that the input signal is a zero-mean Gaussian process with standard deviation σ.**

(a) **If the quantizer range is required to be $\pm 4\sigma$, show that the quantization signal-to-noise is $6n - 7.3$ dB.**

(b) **Write down an expression for the probability that the input signal will overload the**
quantizer (i.e., when the input signal falls outside of the quantizer range).

Solution:

(a) A uniform quantizer has a mean square error of $P_N = \Delta^2/12$, where Δ is the separation between quantizer levels. Since the input signal has a Gaussian pdf with standard σ, the average power of the source signal is $P_S = \sigma^2$. The quantization step size is chosen such that the range covered by the 2^n quantizing levels is 8σ, giving a quantization step size of

$$\Delta = \frac{8\sigma}{2^n}$$

Substitution yields

$$P_N = \frac{64\sigma^2}{12 \times 2^{2n}} = \frac{16}{3}\frac{\sigma^2}{2^{2n}}$$

$$SNR = \frac{P_S}{P_N} = \frac{3}{16} \times 2^{2n}$$

$$10\log_{10} SNR = 10\log_{10}\left(\frac{3}{16}\right) + 20n\log_{10} 2$$

$$= -7.3 + 6.02n$$

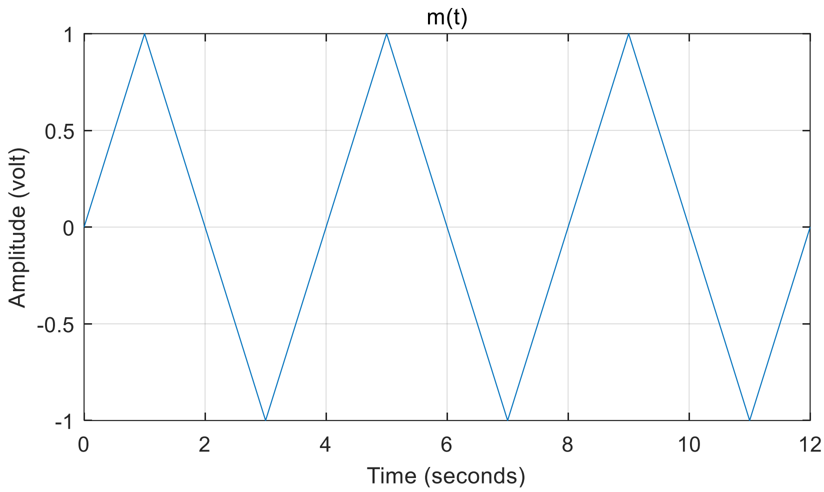
(b) For the probability of overload, denoted by $P_{overload}$, we want to find $P\{X > 4\sigma\} + P\{X < -4\sigma\}$, where X is a Gaussian random variable with mean 0 and variance σ^2, i.e., $X \sim \mathcal{N}(0, \sigma^2)$.Since the pdf is symmetric, we can write

$$P_{overload} = 2P\{X > 4\sigma\}$$

$$= 2\int_{4\sigma}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx$$

$$= 2Q\left(\frac{4\sigma}{\sigma}\right) = 6.3 \times 10^{-5}.$$

6. **The input to a uniform n-bit quantizer is the periodic triangular waveform shown below, which has a period of $T = 4$ seconds, and an amplitude that varies between $+1$ and -1 Volt.**



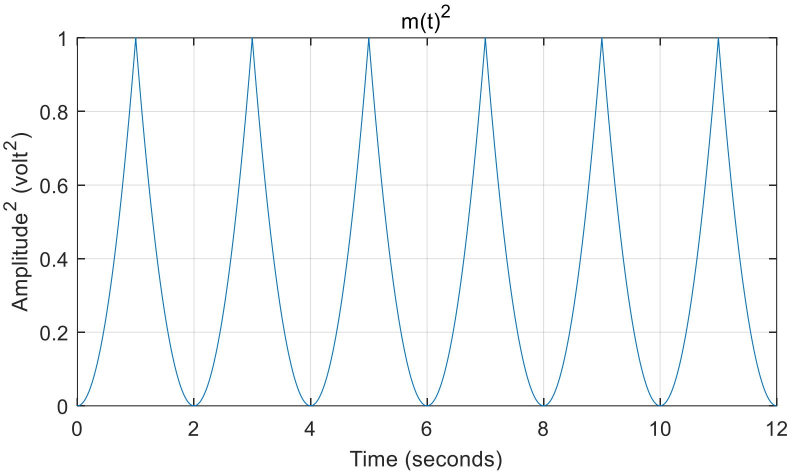
Derive an expression for the signal-to-noise ratio (in decibels) at the output of the quantizer. Assume that the dynamic range of the quantizer matches that of the input signal.

Solution:

With $m^2(t)$ shown below, the signal power is:

$$P_S = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} m^2(t)dt = \frac{4}{T}\int_0^1 m^2(t)dt$$

$$= \int_0^1 t^2 dt = \frac{1}{3}$$



The noise power is

$$P_N = \frac{\Delta^2}{12} = \frac{(\frac{2}{2^n})^2}{12} = \frac{4 \times 2^{-2n}}{12} = \frac{2^{-2n}}{3}$$

The signal to noise ratio is:

$$SNR = \frac{1/3}{2^{-2n}/3} = 2^{2n}$$

or

$$SNR_{dB} = 10log_{10}(2^{2n}) = 20n \log_{10} 2 = 6.02n$$

Baseband Digital Transmission (Lecture 9)

1. **A source transmits 70% of the time 0s and 30% of the time 1s. What is the optimal threshold of detection so that the error rate is minimum? Assume that the noise is zero-mean white Gaussian with variance $\sigma^2 = 9$, and that the amplitude of the transmitted signal when 1 is indicated is A = 5, and 0 if 0 is indicated.**

Solution:

The optimal threshold is given by:

$$T = -\frac{\sigma^2}{A}\ln\frac{p_1}{1-p_1} + \frac{A}{2}$$

(NOTE: You are not expected to remember this formula for the exams. If it is needed, it will either be given, or the question will be phrased in such a way that you will be led through its derivation.)

We apply it with $p_1 = 0.3$, $\sigma^2 = 9$ and $A = 5$:

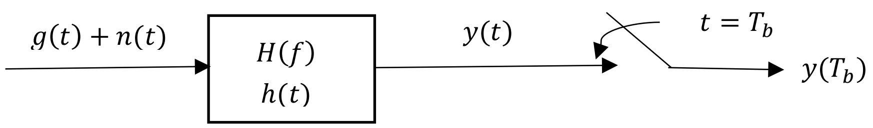
$$T = -\frac{9}{5}\ln\frac{0.3}{0.7} + \frac{5}{2} = -1.8(-0.847) + 2.5 = 4.025.$$

2. **Find the matched filter if the channel noise is zero-mean Gaussian noise and non-white in frequency domain with the following power spectral density,**

$$\boldsymbol{S_N(f) = \frac{1}{\alpha^2 + (2\pi f)^2}}$$

and the transmitted pulse is $\boldsymbol{g(t) = e^{-\alpha t}, t > 0, \alpha > 0}$.

Solution



Direct calculation (also in the lecture slides) yields that

$$y(t) = \int \big(g(\tau) + n(\tau)\big)h(t-\tau)d\tau$$

Denote $g_o(t) = \int g(\tau)h(t-\tau)d\tau$ and $n_o(t) = \int n(\tau)h(t-\tau)d\tau$. As stated in the class, the signal-to-noise ratio at $t = T_b$ can be expressed as

$$\eta = \frac{|g_o(T_b)|^2}{E[n_o^2(T_b)]}.$$

Direct calculation (also in the lecture slides) yields that

$$|g_o(T_b)|^2 = \left|\int H(f)G(f)e^{j2\pi f T_b}df\right|^2.$$

On the other hand, the power spectrum density of noise can be expressed as

$$S_{N_o}(f) = S_N(f)|H(f)|^2,$$

and

$$E[n_o^2(T_b)] = \int S_N(f)|H(f)|^2 df$$

Therefore,

$$\eta = \frac{\left|\int H(f)G(f)e^{j2\pi f T_b} df\right|^2}{\int S_N(f)|H(f)|^2 df}$$

We can denote

$$X(f) = H(f)\sqrt{S_N(f)},$$

and

$$Y(f) = \frac{G(f)e^{j2\pi f T_b}}{\sqrt{S_N(f)}}.$$

Then by applying the Cauchy-Schwarz inequality, we have

$$\eta = \frac{\left|\int_{-\infty}^{\infty} X(f)Y(f)df\right|^2}{\int_{-\infty}^{\infty}|X(f)|^2 df}$$

$$\leq \frac{\int_{-\infty}^{\infty}|X(f)|^2 df \int_{-\infty}^{\infty}|Y(f)|^2 df}{\int_{-\infty}^{\infty}|X(f)|^2 df}$$

$$= \int_{-\infty}^{\infty}|Y(f)|^2 df$$

$$= \int_{-\infty}^{\infty}\frac{|G(f)|^2}{S_N(f)} df$$

with equality if and only if $X(f) = k[Y(f)]^*$ or

$$H(f)\sqrt{S_N(f)} = k\left[\frac{G(f)e^{j2\pi f T_b}}{\sqrt{S_N(f)}}\right]^* = \frac{kG(-f)e^{-j2\pi f T_b}}{\sqrt{S_N(f)}}.$$

Hence, the SNR is maximized if and only if

$$H(f) = k\frac{G(-f)e^{-j2\pi f T_b}}{S_N(f)},$$

where k is an arbitrary constant.

If $g(t) = e^{-\alpha t}u(t)$, its Fourier transform $G(f) = \frac{1}{\alpha + j2\pi f}$ and

$$H(f) = k\frac{\frac{1}{\alpha - j2\pi f}e^{-j2\pi f T_b}}{\frac{1}{\alpha^2 + (2\pi f)^2}} = k(\alpha + j2\pi f)e^{-j2\pi f T_b}$$

Digital modulation and demodulation (Lectures 10 & 11)

1. **(a) Consider a binary ASK modulated-carrier system, which employs coherent demodulation. Let the carrier amplitude at the detector input be 0.7 volts. Assume an additive white Gaussian noise channel with a standard deviation of 0.125 volts. If the binary source stream has equal probabilities of occurrence of a symbol 0 and a symbol 1, estimate the probability of detection error.**
   (b) If PSK was used instead, what is the probability of error?

Solution:

(a) The error probability for ASK is

$$P_e = Q(\frac{A}{2\sigma})$$

where $A = 0.7$ is carrier amplitude and $\sigma = 0.125$ is noise standard deviation. Using the approximation $Q(x) \lesssim \frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}$ yields $P_e = 2.8 \times 10^{-3}$.

Alternatively, it can be approximated from the graph of the Q-function using $\frac{A}{\sigma} = \frac{0.7}{0.125} =$ 15dB.

(b) For PSK, the probability of error is

$$P_e = Q\left(\frac{A}{\sigma}\right) = 1.1 \times 10^{-8}$$

Again, the probability of error can also be approximated from the plot of the Q-function.

2. **Consider the FSK system where symbol 0 and 1 are transmitted by frequency f_0 and f_1, respectively. The unmodulated carrier frequency is $f_c = (f_0 + f_1)/2$. In practice, $f_c T \gg 1$ where T is the symbol period. Define the frequency separation $\Delta f \triangleq |f_1 - f_0|$. Larger Δf means larger bandwidth. FSK using $\Delta f = 1/2T$ is known as minimum-shift keying (MSK). Show that $\Delta f = 1/2T$ is the minimum separation so that the two sinusoids are orthogonal over one symbol period.**

Solution:

The correlation of the two sinusoids over one period is

$$\begin{aligned}
\int_0^T \cos 2\pi f_0 t \cos 2\pi f_1 t \, dt &= \frac{1}{2}\int_0^T \cos 2\pi(f_0 + f_1)t + \cos 2\pi(f_1 - f_0)t \, dt \\
&= \frac{1}{2}\int_0^T \cos 2\pi(2f_c)t + \cos 2\pi\Delta f t \, dt \\
&= \frac{1}{2}T[\frac{\sin 2\pi(2f_c)T}{2\pi(2f_c)T} + \frac{\sin 2\pi\Delta f T}{2\pi\Delta f T}] \\
&= \frac{1}{2}T \operatorname{sinc}(2\pi\Delta f T)
\end{aligned}$$

where the first term is ignored because $f_c T \gg 1$. The correlation is zero when

$$\Delta f = \frac{n}{2T}, \quad n = 1,2,\cdots$$

Therefore, the minimum value of Δf for orthogonal FSK is $1/2T$.

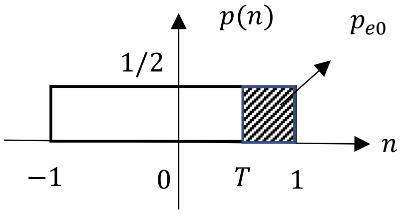
3. **Consider a binary source alphabet where a symbol 0 is represented by 0 volts, and a symbol 1 is represented by 1 volt. Assume these symbols are transmitted over a baseband channel having uniformly distributed noise with a probability density function:**

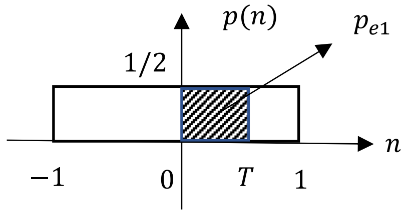
$$\boldsymbol{p(n)} = \begin{cases} \boldsymbol{\frac{1}{2}}, & \boldsymbol{|n| < 1} \\ \boldsymbol{0}, & \textbf{otherwise}. \end{cases}$$

Assume that the decision threshold $\boldsymbol{T}$ is within the range of $\boldsymbol{0}$ to $\boldsymbol{1}$ volt. If the symbols are equally likely, derive an expression for the probability of error.

Solution:

The pdf of the received signals are as shown below, with $P_{e0} = \frac{1}{2}(1-T)$ and $P_{e0} = \frac{1}{2}T$.





P_{e0}: the probability of error when symbol 0 is transmitted.
P_{e1}: the probability of error when symbol 1 is transmitted
For equally-likely symbols,

$$P_e = p_0 P_{e0} + p_1 P_{e1} = \frac{1}{2}(P_{e0} + P_{e1})$$
$$= \frac{1}{2}\left(\frac{1}{2}(1-T) + \frac{1}{2}T\right) = \frac{1}{4}(1 - T + T) = 0.25$$

4. **Show that 16 QAM can be represented as a superposition of two four-phase constant envelope signals where each component is amplified separately before summing, i.e.**

$$\boldsymbol{s(t) = G(A_n \cos 2\pi f_c t - B_n \sin 2\pi f_c t) + (C_n \cos 2\pi f_c t - D_n \sin 2\pi f_c t)}$$

where $\boldsymbol{\{A_n\}}$, $\boldsymbol{\{B_n\}}$, $\boldsymbol{\{C_n\}}$, $\boldsymbol{\{D_n\}}$ are statistically independent binary sequences with elements from the $\boldsymbol{\{+1, -1\}}$ and $\boldsymbol{G}$ is the amplifier gain. Thus, show that the resulting signal is equivalent to

$$\boldsymbol{s(t) = I_n \cos 2\pi f_c t - Q_n \sin 2\pi f_c t}$$

and determine $\boldsymbol{I_n}$ and $\boldsymbol{Q_n}$ in terms of $\boldsymbol{A_n}$, $\boldsymbol{B_n}$, $\boldsymbol{C_n}$ and $\boldsymbol{D_n}$.

Solution:

If we group the terms corresponding to $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$, respectively, in

$$s(t) = G(A_n \cos 2\pi f_c t - B_n \sin 2\pi f_c t) + C_n \cos 2\pi f_c t - D_n \sin 2\pi f_c t,$$

then, we will have $s(t) = (GA_n + C_n)\cos 2\pi f_c t - (GB_n + D_n)\sin 2\pi f_c t$.

For 16-QAM, $Q_n, I_n \in \{\pm 1, \pm 3\}$. If $G = 2$, then $GA_n + C_n$, $GB_n + D_n \in \{\pm 1, \pm 3\}$ since A_n, C_n, $B_n, D_n \in \{\pm 1\}$.

Elementary Information Theory (Lectures 12 & 13)

1. **Consider a source having an M=4 symbol alphabet where $P(x_1) = 1/2;\ P(x_2) = 1/4\ \ P(x_3) = P(x_4) = 1/8$ and symbols are statistically independent.**
   a) **Calculate the information conveyed by the receipt of the symbol x_1, x_2 , x_3. and x_4.**
   b) **What is the source entropy?**

   Solution:

   a.

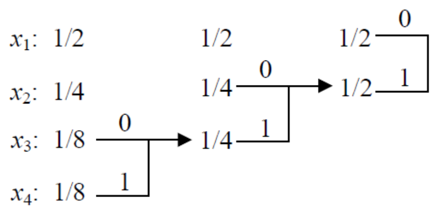
   $I(x_1) = 1$ bit; $I(x_1) = 2$ bits; $I(x_3) = 3$ bits; and $I(x_4) = 3$ bits

   b.

   H=1*1/2+2*1/4+3*1/8+3*1/8=1.75 bits

2. **For the source in Problem 1, find the corresponding codewords for different symbols if Huffman coding is used.**

   Solution: Here is the coding procedure**,**



   Therefore, $x_1 \rightarrow 0,\ x_2 \rightarrow 10,\ x_3 \rightarrow 110,\ x_4 \rightarrow 111$.

3. **Find the capacity of an AWGN channel with a bandwidth B = 1 MHz, signal power of 10W and noise power-spectral density of N0/2 = 10^{-9} W/Hz.**

   Solution:

   C=10^6log(1+10/10^6*2*10^{-9})=12.2Mbps

4. **A binary channel matrix is given by**

$$\textbf{outputs}$$

$$\textbf{inputs}\quad \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\[2ex] x_2 \end{array} & \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\[1ex] \frac{1}{10} & \frac{9}{10} \end{pmatrix} \end{array}$$

This means $P_{y|x}(y_1|x_1) = 2/3$, $P_{y|x}(y_2|x_1) = 1/3$, etc. You are also given that $P_x(x_1) = 1/3$ and $P_x(x_2)=2/3$.

a. Determine H(x), H(x|y), H(y), H(y|x), and I(x;y).

b. How to determine the capacity of the channel?

Solution:

a.

$P(x_1, y_1)= P(x_1) *P(y_1|x_1)=1/3*2/3=2/9,$
$P(x_1, y_2)= P(x_1) *P(y_2|x_1)=1/3*1/3=1/9$

$P(x_2, y_1)= P(x_2) *P(y_1|x_2)=2/3*1/10=1/15$

$P(x_2, y_2)= P(x_2) *P(y_2|x_2)=2/3*9/10=3/5$

$P_y(y_1) = 13/45$

$P_y(y_2) = 32/45$

$P(x_1|y_1)= P(x_1, y_1)/ P_y(y_1)= 10/13$

$P(x_2|y_1)= P(x_2, y_1)/ P_y(y_1)= 3/13$

$P(x_1|y_2)= P(x_1, y_2)/ P_y(y_2)= 5/32$

$P(x_2|y_2)= P(x_2, y_2)/ P_y(y_2) =27/32$

$H(x) = \sum_i P(x_i) \log\frac{1}{P(x_i)} = P(x_1)\log\frac{1}{P(x_1)} + P(x_2)\log\frac{1}{P(x_2)} = 1/3*1.58$
$+2/3*0.58=0.53+0.39=0.92$ bits

$H(x|y) = \sum_i \sum_j P(x_i, y_j) \log\frac{1}{P(x_i|y_j)} =$
$0.22*0.38+0.11*2.68+0.07*2.11+0.6+*0.25=0.08+0.29+0.14+0.15=0.66$ bits

$H(y) = \sum_i P(y_i) \log\frac{1}{P(y_i)} = P(y_1)\log\frac{1}{P(y_1)} + P(y_2)\log\frac{1}{P(y_2)} = 13/45*1.79$
$+32/45*0.49=0.52+0.35=0.87$ bits

$H(y|x) = \sum_i \sum_j P(y_i, x_j) \log\frac{1}{P(y_i|x_j)} =$
$0.22*0.58+0.11*1.58+0.07*3.32+0.6*0.15=0.13+0.18+0.22+0.09=0.62$ bits

$I(x;y) = H(x) - H(x|y) = 0.92-0.66=0.26$ bits

b.

To find the channel capacity, we need to find the input distribution to maximize the mutual information, $I(X;Y)$. Denote $P_x(x_1) = p$, then $P_x(x_2)=1-p$. Then we can express $I(X;Y)$ in terms of p, that is, $f(p)= I(X;Y)$. We can find the p to maximize $f(p)$ using numerical method since analytical method is usually unavailable.

5. Obtain the codewords of the (7; 4) Hamming code using the following generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and show that the minimum distance of this code is dmin = 3.

Solution:

Use C=IG to find all 16 codewords.

Find the weights of all codewords, dmin is the minimum weight of all non-zero codewords, which is 3.