Introduction to Sampling & Hypothesis Testing

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Outline

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Random Variables

In statistics, a random variable is any variable whose value depends on some random phenomenon.

Examples
coin toss, dice roll, choosing a card from a deck,
time of a radioactive decay, ...

The set of all possible outcomes of a random variable is called the sample space, Ω, for that variable.

A probability distribution describes the probability of each possible outcome in the sample space.

A discrete random variable can take only a finite set of values:

Example
rolling 2 dice: $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

A continuous random variable can take an infinite number of values within a given interval (or set of intervals).

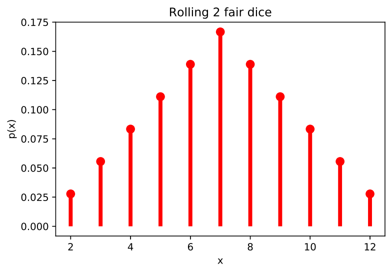
Example
angle of a spinner, in degrees: $\Omega = [0, 360)$

Random Variables | pmf

The probability distribution, $p(x)$, for a discrete random variable X is called a probability mass function, pmf.

Example

rolling 2 fair dice



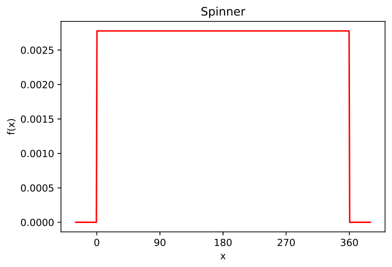
The pmf gives the probability of a particular outcome:

$$\mathbb{P}(X = x) = p(x)$$

The probability distribution, $f(x)$, for a continuous random variable X is called a probability density function, pdf.

Example

angle of a spinner



Integrating the pdf between two values gives the probability of an outcome within that interval:

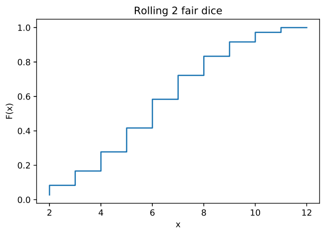
$$\mathbb{P}(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)\mathrm{d}x$$

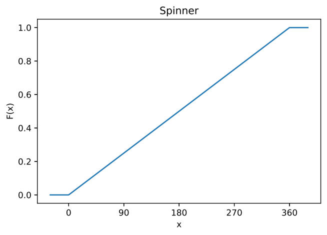
Random Variables | cdf

For convenience in calculating probabilities over intervals, we define the cumulative distribution function, cdf, as

$$F(x) = \mathbb{P}(X \leq x)$$

Examples





The expected value, $\mathbb{E}X$, of a random variable X is the probability-weighted average of all its possible values, also known as its mean, μ.

$$\mu = \mathbb{E}X = \begin{cases} \sum_x xp(x), & X \text{ discrete.} \\ \int xf(x)\mathrm{d}x, & X \text{ continuous.} \end{cases}$$

Examples

rolling 2 fair dice: $\mathbb{E}(X) = 7$

spinner: $\mathbb{E}(X) = 180$

We can also describe the degree to which the values taken by X are spread out from the mean.

The variance, Var$X$ of a random variable X is defined as

$$\text{Var}X = \mathbb{E}(X - \mu)^2$$

The standard deviation, σ is the square root of the variance.

Together, the mean and standard deviation of a random variable give us a simple summary of its distribution.

See the python notebook
`random_variables.ipynb`
for some examples of commonly encountered discrete and continuous random variables.

Sampling

When we take a finite sample of size n from a random variable, the distribution of the sample is *not* the same as the underlying theoretical distribution.

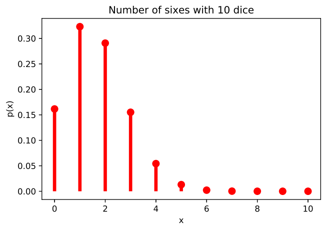
We use $\bar{x}$ and s^2 to represent the sample mean and variance:

$$\bar{x} = \frac{\sum x}{n}$$

$$s^2 = \frac{\sum (x - \bar{x})^2}{n}$$

Example
Number of sixes with 10 dice

theoretical distribution



$\mu = 1.6667$

$\sigma^2 = 1.3889$

sampling $n = 10$ trials

$\bar{x} = 1.200$

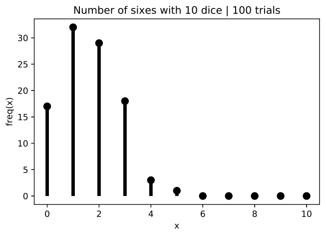
$s^2 = 0.960$

The law of large numbers states that as we take larger and larger samples of a random variable, the sample mean gets closer to the the theoretical (or *population*) mean, μ.

This also implies that the sample variance s^2 approaches the population variance σ^2 as n increases.

Example
Number of sixes with 10 dice

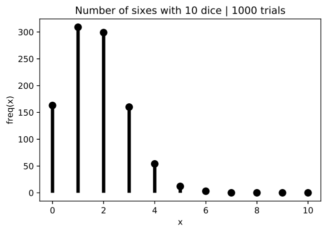
sampling $n = 100$ trials



$\bar{x} = 1.610$

$s^2 = 1.238$

sampling $n = 1000$ trials



$\bar{x} = 1.681$

$s^2 = 1.391$

When we only have access to a finite sample of size n, it is helpful to know how precise our estimate of the population mean will be.

The observed sample mean, $\bar{x}$ behaves as if it is drawn from a continuous random variable $\bar{X}$ with mean μ and a variance that decreases as n increases.

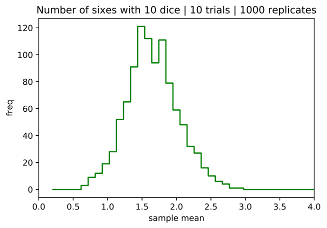
$\bar{X}$ is called the sampling distribution of the mean.

$\bar{x}$ becomes a more precise estimate of μ as we gather more data.

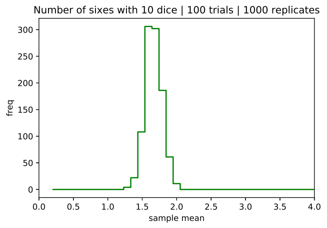
We can see this by repeating the sampling process many times and plotting histograms of $\bar{x}$.

Example
Number of sixes with 10 dice | 1000 replicates

sample mean: $n = 10$ trials



sample mean: $n = 100$ trials



For a sample of size n, the central limit theorem states that $\bar{X}$ *converges* to a normal distribution:

$$\bar{X} \approx N(\mu, \frac{\sigma^2}{n}) \text{ for large } n$$

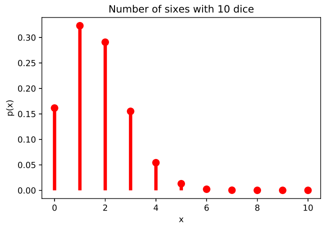
Note that this is true *regardless* of the distribution of X itself.

The central limit theorem is the theoretical justification for many statistical procedures.

Example

Number of sixes with 10 dice | $n = 1000$ trials

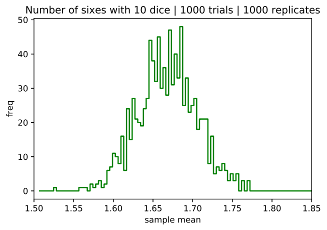
theoretical distribution, X



$\mu = 1.6667$

$\sigma^2 = 1.3889$

sample mean, $\bar{X}$



mean $= 1.6680 \approx \mu$

variance $= 0.0014 \approx \frac{\sigma^2}{n}$

The population standard deviation: σ (unknown)

The sample standard deviation: s (calculated from observed data)

The standard error of the mean: $\frac{\sigma}{\sqrt{n}} \approx \frac{s}{\sqrt{n}}$ for large n

Sampling | Unbiased estimator for population variance

When n is small (say $n < 75$), the sample variance s^2 is not a good approximation for the population variance.

In fact, it is a *biased estimator*, which tends to consistently under-predict the value of σ^2.

We can improve our estimate by using the unbiased sample variance:

$$s^2 = \frac{1}{n-1} \sum (x - \bar{x})^2$$

In many practical applications, the population of interest is not infinite, just very large (e.g. the population of the UK).

There are a variety of ways to try to obtain a representative sample of a finite population, so that the conclusions from the sample are generalisable to the population as a whole.

Simple random: Each individual is chosen randomly and entirely by chance.

Systematic: Every kth individual is sampled from an ordered list.

Stratified: Partition population into heterogenous subpopulations and draw a sample from each one.

Cluster: Total population is split into homogenous clusters, and a subset of clusters is sampled.

Quota: Interviewers told to sample a certain number of a targeted population.

Convenience: The sample is drawn from the most accessible part of the population.

Snowball: Existing study subjects recruit future subjects from their acquaintances.

Voluntary: Study subjects are self-selected.

Parameter Estimation

We have seen how to derive an estimated mean and variance for a population, based on a sample.

$$\hat{\mu} = \bar{x} = \frac{\sum x}{n}$$

$$\hat{\sigma}^2 = s^2 = \frac{1}{n-1} \sum (x - \bar{x})^2$$

These are examples of point estimates, where we quote a single value for a population parameter without an associated uncertainty.

However, it is often more helpful to be able to give a plausible range of values for a parameter, based on the data collected. This is known as a confidence interval.

The python notebook
`confidence_intervals.ipynb`
shows how the central limit theorem can be used to derive confidence intervals for the mean of a population.

Hypothesis Testing

The null hypothesis H_0 and alternative hypothesis H_1 are always two *rival* hypothesis, e.g.

H_0: $\mu = 0$;

H_1: $\mu \neq 0$

Test statistic: A quantity derived from the sample, used in hypothesis testing.

P-value: The probability of obtaining an observation as extreme or more extreme than the test statistic, assuming that the null hypothesis is true.

e.g. $p = 0.03$

The smaller the p-value is, the more unlikely the observation would be to occur if H_0 were true.

The significance level α is how we assess the p-value, and it must be selected in advance of the hypothesis test. We will reject H_0 when $p < \alpha$. From the definition of the p-value, α is the probability of incorrectly rejecting H_0 if it is true. By choosing a smaller α, we can specify a more conservative test.

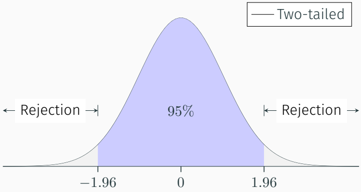
e.g. If $\alpha = 0.05$, $p = 0.03 < \alpha$, reject H_0

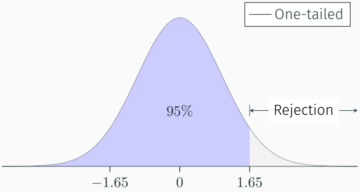
Hypothesis Testing | Procedure

1. Propose a research question
2. Formulate the null hypothesis H_0 and alternative hypothesis H_1
3. Choose an appropriate statistical test
4. Choose an appropriate significance level, α
5. Calculate the test statistic
6. Calculate the p-value
7. Reject H_0 if $p < \alpha$

Hypothesis Testing | One-tailed vs two-tailed

- Two-tailed test: H_0: $\mu = 0$, H_1: $\mu \neq 0$
  - For z-test with a given significance level $\alpha = 0.05$, H_0 is rejected when $P(|X| > z_{\alpha/2}) < \alpha/2$, where $z_{\alpha/2} = 1.96$
- One-tailed test: H_0: $\mu = 0$, H_1: $\mu > 0$
  - For z-test with a given significance level $\alpha = 0.05$, H_0 is rejected when $P(X > z_{\alpha}) < \alpha$, where $z_{\alpha} = 1.65$





Parametric tests rely on a probability distribution of known form as a model for the null hypothesis.

The python notebook
`hypothesis_testing.ipynb`
contains some worked examples of commonly encountered parametric tests.

Rejection Table		
	H_0 True	H_0 False
Reject H_0	*Type I Error* (α)	*Correct Decision* ($1 - \beta$)
Fail to Reject H_0	*Correct Decision*	*Type II Error* (β)

Two possible errors can be made when using p-values to make a decision

- **Type I error**: reject the null hypothesis when it is true
- **Type II error**: not reject the null hypothesis when it is false

Probability of Type I and Type II errors:

- α: the significance level α is the probability of Type I error.
- β: the probability of Type II error relative to the H_1 is called β

The statistical power of a test is given by $1-\beta$, i.e. the probability that H_0 is rejected when H_1 is true.

Methods to reduce errors:

- $\alpha \downarrow \longrightarrow \beta \uparrow$
- $\beta \downarrow \longrightarrow \alpha \uparrow$
- Increase the sample size, n

References

William Mendenhall, Terry Sincich. *Statistics for Engineering and the Sciences*. Pearson/Prentice Hall, Upper Saddle River, New Jersey, 2007.

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Acknowledgments

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