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A General Inner Approximation Algorithm for Nonconvex Mathematical Programs

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Inner approximation algorithms have had two major roles in the mathematical programming literature. Their first role was in the construction of algorithms for the decomposition of large-scale mathematical programs, such as in the Dantzig-Wolfe decomposition principle. However, recently they have been used in the creation of algorithms that locate Kuhn-Tucker solutions to nonconvex programs. Avriel and Williams' [1] complementary geometric programming algorithm, Duffin and Peterson's [4] reversed geometric programming algorithms, Reklaitis and Wilde's [6] primal reversed geometric programming algorithm, and Bitran and Novaes' [2] linear fractional programming algorithm are all examples of this class of inner approximation algorithms. A sequence of approximating convex programs are solved in each of these algorithms. Rosen's [7] inner approximation algorithm is a special case of the general inner approximation algorithm presented in this note.

THE GENERAL inner approximation algorithm locates Kuhn-Tucker solutions to nonconvex mathematical programs. Hence, the general inner approximation algorithm solves the mathematical program

$$(\mathrm{MP})\qquad \min g_0(x)$$
$$g_i(x)\leqq 0,\qquad i=1,\cdots,m$$

where x is an n-dimensional vector, g_i, $i=0,\cdots,l$, are differentiable convex functions, g_i, $i=l+1,\cdots,m$ are differentiable functions, and the feasible region $F=\{x|g_i(x)\leqq 0,\text{ for } i=1,\cdots,m\}$ is a compact set.

The general inner approximation algorithm optimizes a sequence of approximating convex programs. In the kth approximate program AP^k each constraint $g_i(x)\leqq 0$, $i=l+1,\cdots,m$, is replaced by an approximating convex constraint. Duffin and Peterson [4], for instance, construct their approximating convex constraints through the use of the classical geometric-harmonic mean inequality. If the feasible region of AP^k satisfies Slater's constraint qualification condition for convex programs, then the Kuhn-Tucker conditions are both necessary and sufficient conditions for the global optimality of AP^k. The general inner approximation algorithm stops when the Kuhn-Tucker conditions associated with an approximating convex program correspond to the Kuhn-Tucker conditions of MP.

The general inner approximation algorithm for MP is given by the following steps:

Step 0. Choose a starting point $x^0 \in F$ and set $h^0 = g_0(x^0)$. Let $A^0 = \{x | h^0 = g_0(x) \text{ and } x \in F\}$.

Step 1. In the kth iteration replace each constraint $g_i(x) \leqq 0$, $i = l+1, \cdots, m$, by the constraint: $\bar{g}_i(x, x^k) \leqq 0$, where $\bar{g}_i(x, x^k)$ is a differentiable convex function and $x^k \in A^{k-1}$. Each function $\bar{g}_i(x, x^k)$ must have the following properties:

(i) $g_i(x) \leqq \bar{g}_i(x, x^k)$ for all $x \in F^k$
(ii) $g_i(x^k) = \bar{g}_i(x^k, x^k)$
(iii) $\partial g_i(x^k)/\partial x_j = \partial \bar{g}_i(x^k, x^k)/\partial x_j$, $j = 1, \cdots, n$.

The feasible region $F^k = \{x | g_i(x) \leqq 0 \text{ for } i = 1, \cdots, l \text{ and } \bar{g}_i(x, x^k) \leqq 0 \text{ for } i = l+1, \cdots, m\}$ must satisfy Slater's constraint qualification condition for convex programs.

Step 2. Solve the approximate convex program AP^k.

$$(AP^k) \qquad \min g_0(x)$$
$$g_i(x) \leqq 0, \qquad i = 1, \cdots, l$$
$$\bar{g}_i(x, x^k) \leqq 0, \qquad i = l+1, \cdots, m.$$

Let $h^k = \min\{g_0(x) | x \in F^k\}$.

Step 3. If $h^k = h^{k-1}$, then x^k is a Kuhn-Tucker solution to MP. Otherwise, let $A^k = \{x | h^k = g_0(x) \text{ and } x \in F^k\}$ and return to Step 1.

THEOREM 1. *The general inner approximation algorithm stops at a Kuhn-Tucker point, or the limit of any convergent sequence is a Kuhn-Tucker point.*

Proof. To prove this theorem we will show that the algorithm satisfies each of the conditions in Zangwill's [8] Convergence Theorem A. All the points generated by the algorithm are within the compact set F. If x^k is not a solution, then $h^k < h^{k-1}$ because x^k, which is a global solution to AP^{k-1}, is a feasible solution to AP^k. If $h^k = h^{k-1}$, then x^k is a solution to the algorithm. Since AP^k is a convex program that satisfied Slater's constraint qualification condition, the global minimum of AP^k is the Kuhn-Tucker point x^k. Therefore,

$$\partial g_0(x^k)/\partial x_j + \textstyle\sum_{i=1}^{l} \lambda_i \partial g_i(x^k)/\partial x_j + \sum_{i=l+1}^{m} \lambda_i \partial \bar{g}_i(x^k, x^k)/\partial x_j = 0, \qquad j = 1, \cdots, n$$

$$\lambda_i g_i(x^k) = 0, \quad i = 1, \cdots, l; \qquad \lambda_i \bar{g}_i(x^k, x^k) = 0, \quad i = l+1, \cdots, m$$

where λ_i is the dual variable associated with the ith constraint. According to properties (ii) and (iii) in Step 1, $\bar{g}_i(x^k, x^k)$ and $\partial \bar{g}_i(x^k, x^k)/\partial x_j$ can be

replaced by $g(x^k)$ and $\partial g_i(x^k)/\partial x_j$ respectively.
Thus,

$$\partial g_0(x^k)/\partial x_j + \sum_{i=1}^{m} \lambda_i \partial g_i(x^k)/\partial x_j = 0, \qquad j = 1, \cdots, n$$
$$\lambda_i g_i(x^k) = 0, \qquad i = 1, \cdots, m.$$

The point x^k is therefore a Kuhn-Tucker solution to MP. Finally, Dantzig et al. [3] have proven that the point-to-set mapping used in the algorithm is a closed map.

COROLLARY 1 [5]. *If x^k is the solution to the general inner approximation algorithm and x^k is in the interior of F^k, then x^k is a local minimum to MP.*

The general inner approximation algorithm has several distinctive characteristics. First, the algorithm can be used to optimize nonlinear programs even when $g_0(x)$ is not a convex function. The objective function $g_0(x)$ is replaced by the new variable x_{n+1}, and $g_0(x) - x_{n+1} \leqq 0$ is added to the constraint set. The starting point for the algorithm does not necessarily have to be an initial feasible solution to MP, but it must generate an approximate program that has all the properties stated in Step 1. The most difficult property to verify is that the feasible region F^k satisfies Slater's constraint qualification condition. However, a judicial choice of a convex minimization algorithm could assist in the verification. The solution from this algorithm is not only a Kuhn-Tucker point for MP but is also the global minimum for some approximating convex program that is interior to the feasible region F.

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