UNIVERSITEIT VAN AMSTERDAM

INFORMATIE EN COMMUNICATIE

HW-1

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1. ✓

2. (a) Have 52 choices..draw a card..have 51 choices..draw a card....have 1 choice.

   52! = 80658175170943878571660636856403766975289505440883277824000000
   000000
   = 8.0658175170943878571660636856403766975289505440883277824...10^{67}

   (b) Have 52 choices out of 104 cards..draw a card..have 52 choices out of a 103 cards..draw a card..have either 51 choices out of a 102 cards or 52 choices out of a 102 cards..draw a card....have 1 choice.

   $104!/2^{52}$
   = 22868411042921427398174802602703979477313850254825027668749130642
   40686762501672463119886085466145431873911265472729321167650816000000
   000000000000000000
   = 2.2868411042921427398174802602703979477313850254825027668...10^{150}

3. Prove the following inequality for real numbers $p_1, p_2, \ldots, p_n \in [0, 1]$
   (i.e. $0 \leq p_i \leq 1$ $i = 1, 2 \ldots, n$):
   (*)

$$(1 - p_1)(1 - p_2) \ldots (1 - p_n) \geq 1 - p_1 - p_2 - \cdots - p_n$$

Left side

$$\begin{aligned}
(1 - p_1)(1 - p_2) \ldots (1 - p_n) &= (1 - \mathbb{P}(E_1))(1 - \mathbb{P}(E_2)) \ldots (1 - \mathbb{P}(E_n)) && (1)\\
&= (\mathbb{P}(\overline{E_1}))(\mathbb{P}(\overline{E_2})) \ldots (\mathbb{P}(\overline{E_n})) && (2)\\
&= \prod_{i}^{n} \mathbb{P}(\overline{E_i}) && (3)
\end{aligned}$$

Multiplication of independent events is same as the intersection of these independent events.

$$= \mathbb{P}\left(\bigcap_{i}^{n} (\overline{E_i})\right) \qquad (4)$$

Right side

$$\begin{aligned}
1 - p_1 - p_2 - \cdots - p_n &= 1 - \sum_{i}^{n} p_i && (5)\\
&= 1 - \sum_{i}^{n} \mathbb{P}(E_i) && (6)\\
&= 1 - \sum_{i}^{n} (1 - \mathbb{P}(\overline{E_i})) && (7)\\
&= 1 - n - \sum_{i}^{n} (\mathbb{P}(\overline{E_i})) && (8)
\end{aligned}$$

Combined

$$\mathbb{P}\left(\bigcap_{i}^{n}(\overline{E_i})\right) \geq 1 - n - \sum_{i}^{n}(\mathbb{P}(\overline{E_i})) \tag{9}$$

$$n \geq 1 - \sum_{i}^{n}(\mathbb{P}(\overline{E_i})) - \mathbb{P}\left(\bigcap_{i}^{n}(\overline{E_i})\right) \tag{10}$$

$$n \geq 1 - \mathbb{P}\left(\bigcup_{i}^{n}(\overline{E_i})\right) \tag{11}$$

(*new)

$$\mathbb{P}\left(\bigcup_{i}^{n}(\overline{E_i})\right) + n \geq 1$$

(n=1)

$$\mathbb{P}(\overline{E_1}) + 1 \geq 0$$
$$\mathbb{P}(\forall\chi) \in [0,1]$$
$$so \checkmark$$

(n)

$$\mathbb{P}\left(\bigcup_{i}^{n}(\overline{E_i})\right) + n \geq 1$$
$$Since$$
$$\mathbb{P}(\forall\chi) \in [0,1]$$
$$n \in N$$
$$so \checkmark$$

(*new) holds and so does (*)

4. (a)

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[(X-\mu)^2] && (12)\\
&= \mathbb{E}[(X-\mathbb{E}[X])^2] && (13)\\
&= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] && (14)\\
&= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2 && (15)\\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 && (16)
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{Var}(aX) &= \mathrm{Var}(Y) && (17)\\
&= \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 && (18)\\
&= \mathbb{E}[aX^2] - (\mathbb{E}[aX])^2 && (19)\\
&= a^2\mathbb{E}[X^2] - (a\mathbb{E}[X])^2 && (20)\\
&= a^2\mathbb{E}[X^2] - a^2(\mathbb{E}[X])^2 && (21)\\
&= a^2(\mathbb{E}[X^2] - (\mathbb{E}[X])^2) && (22)\\
&= a^2\,\mathrm{Var}(aX) && (23)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X+a) &= \mathrm{Var}(Y) && (24)\\
&= \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 && (25)\\
&= \mathbb{E}[(X+a)^2] - (\mathbb{E}[(X+a)])^2 && (26)\\
&= \mathbb{E}[X^2 + a^2 + 2aX] - (\mathbb{E}[X]+a)^2 && (27)\\
&= \mathbb{E}[X^2] + a^2 + 2a\mathbb{E} = [X] - (\mathbb{E}[X]^2 + a^2 + 2a\mathbb{E}[X]) && (28)\\
&= \mathbb{E}[X^2] + a^2 + 2a\mathbb{E} = [X] - (\mathbb{E}[X])^2 - a^2 - 2a\mathbb{E}[X] && (29)\\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 && (30)
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{Var}(X+Y) &= \mathrm{Var}(Z) && (31)\\
&= \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 && (32)\\
&= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[(X+Y)])^2 && (33)\\
&= \mathbb{E}[X^2 + Y^2 + 2XY] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 && (34)\\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] - (\mathbb{E}[X]^2 + \mathbb{E}[Y]^2 + 2\mathbb{E}[X]\mathbb{E}[Y]) && (35)\\
&= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)) && (36)\\
&= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y) && (37)
\end{aligned}$$

Covariance of two independent variables is 0

$$= \mathrm{Var}(X) + \mathrm{Var}(Y) \quad (38)$$

(d) Let X be a random variable with *Bernoulli distribution*

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x\in X} x\mathbb{P}_x(x) && (39)\\
&= 1 * \mathbb{P}_x(1) + 0 * \mathbb{P}_x(0) && (40)\\
&= 1 * p + 0 * (1-p) && (41)\\
&= p && (42)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 && (43)\\
&= \left(\sum_{x\in X} x^2 P_x(x)\right) - p^2 && (44)\\
&= 1^2 * \mathbb{P}_x(1) + 0^2 * \mathbb{P}_x(0) - p^2 && (45)\\
&= 1 * p + 0 * (1-p) - p^2 && (46)\\
&= p - p^2 && (47)\\
&= p(1-p) && (48)
\end{aligned}$$

(e) Let Y be a random variable with *binomial distribution* $P_Y(y) = \binom{n}{y} p^y (1-p)^{n-y}$

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{n}^{i} X_i\right] \tag{49}$$

Expected value is the expected value of the sum of the independent *Bernoulli distributions.*

$$= \sum_{n}^{i} \mathbb{E}[X_i] \tag{50}$$

$$= \sum_{n}^{i} p \tag{51}$$

$$= np \tag{52}$$

$$\mathrm{Var}[Y] = \mathrm{Var}\left[\sum_{n}^{i} X_i\right] \tag{53}$$

$$= \sum_{n}^{i} \mathrm{Var}[X_i] \tag{54}$$

$$= \sum_{n}^{i} p(1-p) \tag{55}$$

$$= np(1-p) \tag{56}$$

5. An urn contains K balls, of which B are black and $W = K - B$ balls are replaced when drawn.
Classic *Bernoulli distribution*

   (a) $f(n_b; N, B/K) = f(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$

   (b) A Bernoulli distribution has $\mathbb{E}[X] = np$, this one has $n = N$ and $p = B/K$ so the expected value $= N * (B/K)$
   N=5 and N=400 for B=2 and K=10

      i. $5 * (2/10) = 1$

      ii. $400 * (2/10) = 80$

   A Bernoulli distribution has $\mathrm{Var}[X] = np(1-p)$
   so the variance $= N * (B/K)(1 - (B/K))$
   N=5 and N=400 for B=2 and K=10

      i. $5 * 0.2(1 - 0.2) = 0.8$

      ii. $400 * 0.2(1 - 0.2) = 64$