Universiteit van Amsterdam

Informatie en Communicatie

HW-3

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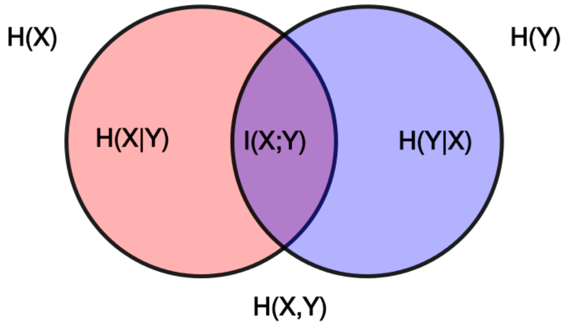
1. (a) $H(f(Y))?H(Y)$

$$H(Y) = H(Y) + H[f(Y)|Y] \tag{1}$$
$$= H(Y, f(Y)) \tag{2}$$
$$= H(f(Y)) + H(Y|f(Y)) \tag{3}$$
$$\geq H(f(Y)) \tag{4}$$
$$H(f(Y)) \leq H(Y) \tag{5}$$

(b) $H(X|f(Y))?H(X|Y)$ The entropy of $f(Y)$ only decreases or is equal to the entropy of Y



If H(f(Y)) is smaller than H(Y) this means:
$H(X|Y) \geq H(X|f(Y))$
$H(X|f(Y)) \leq H(X|Y)$

(c) $I(X,Z|Y) = 0 \implies I(X;Z)?I(X;Y)\&I(X;Z)?I(Y;Z)$

$$I(X;Z|Y) = I(X;Y,Z) - I(X;Y) = 0 \tag{6}$$
$$= I(X;Y|Z) + I(X;Z) - I(X;Y) = 0 \tag{7}$$
$$= I(X;Y|Z) + I(X;Z) = I(X;Y) \implies I(X;Z) \leq I(X;Y) \tag{8}$$

$$I(X;Z|Y) = I(X;Y,Z) - I(X;Y) = 0 \tag{9}$$
$$= I(X;Y|Z) + I(X;Z) - I(X;Y) = 0 \tag{10}$$
$$= I(X;Z) = I(X;Y) - I(X;Y|Z) \tag{11}$$

Chain rule for mutual information

$$= I(X;Z) = I(X;Y) - (I(X;Y) + H(Z|X) + H(Z|Y) - H(Z|X,Y) - H(Z)) \tag{12}$$
$$= I(X;Z) = -H(Z|X) - H(Z|Y) + H(Z|X,Y) + H(Z) \tag{13}$$

$$I(X;Y) = H(X,Y) - H(X|Y) - H(Y|X)$$

$$= I(X;Z) + H(Z,Y) - H(Y|Z) = I(Z;Y) - H(Z|X) + H(Z|X,Y) + H(Z) \tag{14}$$

$$= I(X;Z) + H(Z,Y) - H(Y|Z) + H(Z|X) - H(Z|X,Y) + H(Z) = I(Z;Y) \tag{15}$$

$$= I(X;Z) + H(Z,Y) - (H(Y|Z) - H(Z)) + H(Z|X) - H(Z|X,Y) = I(Z;Y) \tag{16}$$

$$= I(X;Z) + H(Z,Y) - H(Z,Y) + H(Z|X) - H(Z|X,Y) = I(Z;Y) \tag{17}$$

$$= I(X;Z) + H(Z|X) - H(Z|X,Y) = I(Z;Y) \tag{18}$$

$$= I(X;Z) - (-H(Z|X) + H(Z|X,Y)) = I(Z;Y) \tag{19}$$

$$= I(X;Z) - (H(Z|X,Y) - H(Z|X)) = I(Z;Y) \tag{20}$$

$$= I(X;Z) - H(Y|Z|X) = I(Z;Y) \tag{21}$$

$$= I(X;Z) = I(Z;Y) + H(Y|Z|X) \tag{22}$$

$$= I(X;Z) = I(Z;Y) + H(Y|Z|X) \implies I(X;Z) \geq I(Z;Y) \tag{23}$$

Too long but correct.

2. For each statement below, specify a (different) joint distribution P_{XYZ} of random variables X, Y and Z such that the inequalities hold.

   (a) There exists a y, such that $H(X|Y = y) > H(X)$
   Let (X,Y) take values on $(0,0), (0,1)(1,0)$ with equal probability ($p = 1/3$).
   Then $H(X) = h(1/3) \approx 0.92 < 1$.
   But $H(X|Y = 0) = h(1/2) = 1$

   (b) $I(X;Y) > I(X;Y|Z)$ Let Z make X and Y independant. Let (X,Y) take values on $(0,0), (0,1)(1,0)$ with equal probability ($p = 1/3$).
   But (X,Y—Z) take on values $(0,1)(1,0)$ with equal probability ($p = 1/2$).
   Then $I(X;Y|Z) = 0$ and $I(X;Y) \approx 0.6$

   (c) $I(X;Y) < I(X;Y|Z)$ Let Z make X and Y dependant. Let (X,Y) take values on $(0,1)(1,0)$ with equal probability ($p = 1/2$).
   But (X,Y—Z) take on values $(0,1)(1,0)(1,0)$ with equal probability ($p = 1/3$).
   Then $I(X;Y|Z) \approx 0.6$ and $I(X;Y) = 0$

3. **Bottleneck:** Suppose a Markov chain starts in one of n states, necks down to $k < n$ states, and then fans back to $m > k$ states. Thus $X_1 \to X_2 \to X_3$.

   (a) Show that the (unconditional) dependence of X_1 and X_3 is limited by the bottleneck by proving that $I(X_1; X_3) \leq \log k$.

   (b) Evaluate $I(X_1; X_3)$ for $k = 1$, and explain why no dependence can survive such a bottleneck.

4. Which of the three relations $\leq, \geq, =$ holds between the quantities H(A) and H(C)? Prove your answer.

$$
\begin{aligned}
I(A;C|B) &= I(A;B|C) && (24)\\
I(A;C,B) - I(A;B) &= I(A;C,B) - I(A;C) && (25)\\
I(A;B) &= I(A;C) && (26)\\
I(A;B) &= I(A;C) = 0 && (27)\\
& && (28)
\end{aligned}
$$

Both B and C are independent of A, so H(A) != H(C) H(Y—X)=0 if and only if the value of Y is completely determined by the value of X

$$
\begin{aligned}
H(A|B,C) = 0 && (29)\\
&& (30)
\end{aligned}
$$

A can only be fully decided by B,C if $B \leq A$