MATH 507 Homework 3

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Problem 1

1.1

We have that

$$\begin{aligned}
\mathbb{E}[\mathcal{C}(x)] &= \mathbb{E}\int_0^T \tilde{S}_t x_t' \mathrm{d}t \\
&= \mathbb{E}\int_0^T S_t x_t' \mathrm{d}t + \eta\mathbb{E}\int_0^T (x_t')^2 \mathrm{d}t \\
&= \mathbb{E}\left[(S_T x_T - S_0 x_0) - \int_0^T \sigma X_t \mathrm{d}W_t\right] + \eta\mathbb{E}\int_0^T (x_t')^2 \mathrm{d}t \\
&= \mathbb{E}[S_T x_T] - S_0 X + \eta\int_0^T \mathbb{E}[(x_t')^2]\mathrm{d}t
\end{aligned}$$

Note that Fubini's Theorem is needed for the last step.

1.2

Define

$$J(x) := \mathbb{E}[\mathcal{C}(x) + c x_T^2]$$

and accordingly,

$$V(t,x) := \inf_x J(x) \quad \text{with} \quad V(T,x) = c x_T^2$$

which gives us (assuming $\tau > t$)

$$\begin{aligned}
V(t,x_t) &= \inf \mathbb{E}\left[S_T x_T - S_t X_t + \eta\int_t^T (x_s')^2 \mathrm{d}s\right] \\
V(\tau,x_\tau) &= \inf \mathbb{E}\left[S_T x_T - S_\tau X_\tau + \eta\int_\tau^T (x_s')^2 \mathrm{d}s\right]
\end{aligned}$$

Therefore,

$$V(t,x_t) = V(\tau,x_\tau) + \inf \mathbb{E}\left[S_\tau x_\tau - S_t X_t + \eta\int_t^\tau (x_s')^2 \mathrm{d}s\right]$$

The HJB equation is given by

$$V_t + \inf[V_x x' + s x' + \eta(x')^2] = 0$$

Where V_t, V_x are the time and the space derivative, respectively. Note that the $V_x x$ term is gone since x is not a function of W_t.

1.3

Based on the ansatz, the partial derivatives are given by

$$\begin{aligned}
V_t &= \alpha_t' s^2 + \beta_t' x s + \gamma_t' x^2 + \delta_t' \\
V_x &= \beta_t s + 2\gamma_t x
\end{aligned}$$

Plugging the derivatives into the HJB equation gives that

$$\alpha_t' s^2 + \beta_t' xs + \gamma_t' x^2 + \delta_t' + \inf \left[(\beta_t s + 2\gamma_t x)x' + sx' + \eta (x')^2 \right] = 0$$

Note that the infimum is quadratic *w.r.t.* x'. Therefore,

$$x'_{opt} = -\frac{1}{2\eta}(\beta s + s + 2\gamma x) \tag{1}$$

Substitute x'_{opt} into the HJB equation:

$$\alpha_t' s^2 + \beta_t' xs + \gamma_t' x^2 + \delta_t' - \frac{1}{4\eta^2}(\beta s + s + 2\gamma x) = 0$$

Thus

$$\begin{cases} \alpha' s^2 - \frac{1}{4\eta}(\beta+1)^2 + \delta' = 0 \\ \beta' s - \frac{1}{\eta}[\gamma s(\beta+1)] = 0 \\ \gamma' - \frac{\gamma^2}{\eta} = 0 \end{cases}$$

Since we have the terminal condition $V(T, x) = cx_T^2$, we can deduce that $\alpha_T = \beta_T = \delta_T = 0$, $\gamma_T = c$, which which we can solved the ODE's above. It turns out that

$$\begin{cases} \alpha_t & = 0 \\ \beta_t & = \frac{T-t}{t-\frac{\eta}{c}-T} \\ \gamma_t & = \frac{\eta}{T+\frac{\eta}{c}-t} \\ \delta_t & = -\frac{1}{4\eta}\frac{T-t}{t-\frac{\eta}{c}-T} \end{cases}$$

Now substitute $\alpha, \beta, \gamma, \delta$ into Equation **??**:

$$(t - \frac{\eta}{c} - T)x' - x = \frac{s}{2c}$$

Fix $x_t = x$, and the solution is given by $(\tau > t)$

$$x_\tau = \frac{s}{2c}\frac{\tau - t}{t - \frac{\eta}{c} - T} + x\frac{\tau - \frac{\eta}{c} - T}{t - \frac{\eta}{c} - T}$$

1.4

Send $c \to \infty$. We have that $x_\tau = x\frac{T-\tau}{T}$, which apparently satisfies the boundary conditions.