Math 507, Fall 2024.

Homework 2

Due: Tue, Oct 15, 2024, NO LATER than 11:30am.

For the ticker $\widehat{\ }$GSPC, download its closing price for each business day between Jan 1, 2014, and Jan 1, 2022. This is your sample of size M. All questions listed below must be answered using this sample. Assume the existence of a riskless return 0.01 (annualized).

In this homework, you back-test dynamic trading strategies that maximize the expected power utility $\mathbb{E}\left((W_T(\alpha))^\zeta/\zeta\right)$ where $\zeta = -3$ according to the procedure described here. (Do not forget that the riskless return is a part of the universe of available assets.) The number $d = 2$ of basic assets (including the riskless asset).

We index the first day in the overall sample by $t = 1, \ldots, M$. We start by estimating all model parameters based on the sample values in the estimation window $t = 1, \ldots, N$ for $N = 1000$. The model parameters are described at (a). Using the estimated parameters, we compute the optimal portfolio weights α_t^* for every day in the trading window $t = N, \ldots, N+T-1$ with $T = 100$. If we use DPP, the optimal weights will often come in a feedback form: $\alpha_t^* = \alpha^*(t, X_t)$ (recall that X_t is a vector that contains the potential values of all relevant factors at time t).

Using the optimal strategy, we compute the wealth generated over the period $[N, N+t]$ (in the trading window) with the initial investment of \$1:

$$W_0 = 1, \quad W_t = \prod_{s=N+1}^{N+t} (1 + \alpha_{s-1}^* \cdot R_s),$$

for $t = 1, \ldots, T$.

Next, we shift our estimation and trading windows by T forward and estimate the model parameters based on the sample $t = T+1, \ldots, T+N$. Using the new estimated parameters, we find the optimal portfolio weights for the new trading window $t = T+N+1, \ldots, T+N+T$ and extend the wealth process

$$W_{T+t} = W_T \prod_{s=T+N+1}^{T+N+t} (1 + \alpha_{s-1}^* \cdot R_s),$$

for $t = 1, \ldots, T$.

We repeat this procedure until we run out of sample: i.e. until $N + Tk > M$. The size of the very last trading window may need to be adjusted (to be smaller than T, to match the size of the remaining sample).

As a result of this procedure, we obtain a realization of the wealth process $\{W_t\}_{t=1}^{M-N}$. Subtracting one from this process, we obtain the Profit and Loss (PnL) process $\{W_t - 1\}_{t=1}^{M-N}$. To express the performance of the strategy in just one number we compute the Sharpe ratio of the wealth process (or, equivalently, of the PnL process). Namely, we view the wealth pricess as the adjusted price of an asset, compute its returns, compute the sample mean $\hat{\mu}$ and sample standard deviation $\hat{\sigma}$, and output the Sharpe ratio:

$$(\hat{\mu} - R)/\hat{\sigma},$$

where R is the riskless return and

$$\hat{\mu} = \frac{1}{M-N} \sum_{t=N+1}^{M} \frac{W_{t-N} - W_{t-N-1}}{W_{t-N-1}} = \frac{1}{M-N} \sum_{t=N+1}^{M} \alpha_{t-1}^* \cdot R_t,$$

$$\hat{\sigma}^2 = \frac{1}{M-N-1}\sum_{t=N+1}^{M}\left(\frac{W_{t-N}-W_{t-N-1}}{W_{t-N-1}} - \hat{\mu}\right)^2 = \frac{1}{M-N-1}\sum_{t=N+1}^{M}(\alpha^*_{t-1}\cdot R_t - \hat{\mu})^2.$$

The higher is the Sharpe ratio the better is the strategy. Note also that the Sharpe ratio is usually annualized: e.g., if the time is measured in days, then the quantity $(\hat{\mu} - R)/\hat{\sigma}$ needs to be multiplied by $\sqrt{250}$.

A strategy with annualized Sharpe ratio below 1 is not deemed valuable.

1. (a) (8 pts) Construct and back-test the dynamic strategy that maximizes $\mathbb{E}\left((W_T(\alpha))^\zeta/\zeta\right)$ over all strategies $\alpha = (\alpha_0, \ldots, \alpha_{T-1})$. The model for the risky return is

$$R_t = \mu + \varepsilon_t,$$

with i.i.d. $\{\varepsilon_t\}_{t=1}^T$, s.t.

$$\varepsilon_t = \begin{cases} \sigma, & \text{prob. } 1/2, \\ -\sigma, & \text{prob. } 1/2, \end{cases}$$

where μ and σ should be approximated via the sample mean and the sample standard deviation of the returns of the risky asset (over each estimation window).

Plot the PnL process of this strategy. Print the annualized mean, standard deviation and the Sharpe ratio of the returns of the generated wealth process.

In parts b and c, it makes a difference what initial values of the "weights before rebalancing" you use at the beginning of each trading window. While other choices are possible, herein, you need to assume that the initial "weights before rebalancing" at the beginning of a trading window are given by the weights obtained at the end of the previous window.

(b) (4 pts) Using the exact strategy computed in part (a), construct its PnL process in the presence of proportional transaction costs of size $\lambda = 0.02$. This means that at each time step if before trading, the current wealth is ω, fractions of wealth in the risky asset is a and the target fraction of wealth in the risky asset is α. Then, the amount $\lambda\omega|a-\alpha|$ is paid as a transaction cost. Plot the resulting PnL process. Print the annualized mean, standard deviation and the Sharpe ratio of the returns of the generated wealth process.

(c) (16 pts) Repeat part (a) with proportional transaction costs of size $\lambda = 0.02$. Note that, unlike part (b), here you need to find the optimal strategy in the presence of transaction costs, as opposed to re-using the strategy computed in part (a).

To compute the value function and the feedback optimal strategy via DPP, use an equidistant grid on $[-1, 2.5]$, consisting of 100 points, for the possible values of the "weights before rebalancing". To compute the value function and the optimal strategy outside of the grid points, use linear interpolation between the grid points and constant extrapolation outside of $[-1, 2.5]$.

On the very first trading day in your sample, i.e. on day N, your capital is fully invested in the riskless asset before rebalancing (i.e., right before you decide on the optimal portfolio weights to be used at that time).

Plot the PnL process of this strategy. Print the annualized mean, standard deviation and Sharpe ratio of the returns of the generated wealth process.