- Notes on QFT

§ 5.1

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i jk/N}$$

$$U_{QFT} : \sum_{j=0}^{N-1} x_j |j\rangle \longmapsto \sum_{k=0}^{N-1} y_k |k\rangle$$

- Product rep.

$$\underbrace{|j_1 \cdots j_n\rangle}_{= \sum_{k=0}^{n-1} j_{n-k} 2^k} \xrightarrow{U} \sum_{k=0}^{N-1} y_k |k\rangle \quad w/ \quad y_k = \frac{1}{\sqrt{N}} e^{2\pi i jk/N} x_j, \quad j = \sum_{l=0}^{n-1} j_{n-l} 2^l$$

$$(|0\rangle + e^{2\pi i \underbrace{0.j_n}_{= \frac{j_n}{2} \,\equiv\, y_1}} |1\rangle)(|0\rangle + e^{2\pi i \underbrace{0.j_{n-1} j_n}_{= \frac{j_{n-1}}{2} + \frac{j_n}{4} \,\equiv\, y_2}} |1\rangle)(\cdot) \cdots (|0\rangle + e^{2\pi i \underbrace{0.j_1 \cdots j_n}_{= \sum_{l=1}^{n} \frac{1}{2^l} j_l}} |1\rangle)$$

$$y_l \equiv \sum_{l'=1}^{l} \frac{1}{2^{l'}} j_{l'+(n-1)-l}$$

$\rightsquigarrow$ for $|k\rangle = |k_1 \cdots k_n\rangle \quad (k = \sum_{l=1}^{n} k_l 2^{n-l})$

$$\left.\begin{aligned} \tilde{y}_k &= \exp\left[2\pi i \sum_{l=1}^{n} k_l y_l\right] \\ &= \exp\left[2\pi i \sum_{l=1}^{n} k_l \sum_{l'=1}^{l} \frac{1}{2^{l'}} j_{l'+n-l}\right] \\ &= \exp\left[2\pi i \underbrace{\sum_{l=1}^{n} k_l 2^{-l}}_{= 2^{-n} k} \underbrace{\sum_{l''=0}^{l-1} 2^{l''} j_{n-l''}}_{= j}\right] \quad (l'' = l - l') \\ &= \exp\left[2\pi i k j\right] \end{aligned}\right\} \times \sqrt{N} = 2^{n/2}$$

§ 5.2

▷ Order-finding prob.

$\forall y \in \{0,1\}^L$, $\quad U|y\rangle \equiv |xy \pmod N\rangle$, $\quad N = 2^L$ for simplicity

unitary if $x < N$ prime w/ each other

Coprime / relatively prime

∘ $|u_s\rangle \equiv \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \exp\left[-2\pi i \frac{sk}{r}\right] |x^k \bmod N\rangle$

w/ $x^r \bmod N = 1$

$\rightsquigarrow U|u_s\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \exp\left[-2\pi i \frac{sk}{r}\right] |x^{k+1} \bmod N\rangle$ $\quad \exists\ 0 \le s \le r-1$, $\in \mathbb{R}$

$\because x^r \bmod N = 1 = x^0 \bmod N$

$= \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \exp\left[-2\pi i \frac{s(k-1)}{r}\right] |x^k \bmod N\rangle$ $\quad \Uparrow$

$\exp\left[-2\pi i \frac{sr}{r}\right] = 1 = \exp(0)$

$= \exp\left[2\pi i \frac{s}{r}\right] |u_s\rangle$

$\rightsquigarrow$ phase estimation determines $\frac{s}{r}$ $\Rightarrow$ r determined ?

∗ Continued fractions algorithm , $\varphi \mapsto \frac{s}{r}$

ex) $L = 2$. φ $(2L+1)$-bit fraction , r, s : L-bit

$$\varphi = 0.10101 = \frac{2^4+2^2+2^0}{2^5} = \frac{1}{1+\frac{1011}{10101}}$$

$$= \frac{1}{1+\frac{1}{1+\frac{1010}{1011}}} = \frac{1}{\underset{a_1}{1}+\frac{1}{\underset{a_2}{1}+\frac{1}{\underset{a_3}{1}+\frac{1}{\underset{a_4}{1010}}}}}$$

$$\begin{array}{r} 10101 \\ -\ 1011 \\ \hline 1010 \end{array}$$

2nd convergent : $[a_0, a_1, a_2] = \frac{1}{1+\frac{1}{1}} = \frac{1}{2} = 0.10000$

ex. 3rd convergent: $\frac{2}{3}$

$$\begin{array}{r} 0.1010101010\cdots \\ 11 \overline{)\,10} \\ 11 \\ \hline 10 \\ 11 \\ \hline \end{array} \simeq 0.10101 \text{ ?}$$

§5.4.1 ◦ Period finding

$$U|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle \quad \text{w/} \quad f(x+r) = f(x) \in \{0,1\}$$

w/ $0 \le r < 2^L$

◦ $\overset{t\text{-bit}}{|0\rangle}|0\rangle$

$$\longmapsto \frac{1}{\sqrt{2^t}} \sum_{x=0}^{2^t-1} |x\rangle|0\rangle$$

$$\overset{U}{\longmapsto} \frac{1}{\sqrt{2^t}} \sum_{x=0}^{2^t-1} |x\rangle|f(x)\rangle \simeq \frac{1}{\sqrt{r2^t}} \sum_{l=0}^{r-1} \sum_{x=0}^{2^t-1} e^{2\pi i l x/r} |x\rangle|\hat{f}(l)\rangle$$

w/ $|f(x)\rangle = \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} e^{2\pi i l x/r} |\hat{f}(l)\rangle$

$$\underset{\uparrow}{\overset{FT^\dagger}{\longmapsto}} \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} |\widetilde{l/r}\rangle|\hat{f}(l)\rangle$$

(2.22) $$\frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} e^{2\pi i \varphi j}|j\rangle|u\rangle \overset{FT^\dagger}{\longmapsto} |\tilde{\varphi}\rangle|u\rangle$$

$$\underset{\text{1st reg.}}{\overset{\text{meas.}}{\longmapsto}} \widetilde{l/r}$$

$$\overset{\text{continued fraction}}{\longmapsto} r$$

⇒ Look elsewhere in the Sourav's setup

$$\begin{cases} B(t) = B\cos(\omega t + \phi) \\ H(t) = \underbrace{\cancel{\hbar g}}_{1} B(t) \sum_{i=1}^{n_s} Z_i \end{cases}$$

- Partial DD ... $B(t) \mapsto x(t) B(t)$ w/ $|x(t)| \leq 1$

  → X-pulse でゆっくり回転させる

  from $|0\rangle \to |+\rangle \to |1\rangle$ .. $x(t) = 0 \to 1 \to 0$

- Control Hamiltonian $H_c = \sum_k \omega_k |k\rangle\langle k|_Q \otimes \frac{X_S}{2}$

  e bin frequency e sensing unit (ESU)

- Evolution under $H_{tot} = x(t) H(t) + H_c$

  ↑

  effectively couples w/ $n_s g$ to $\{|0\rangle^{\otimes n_s}, |1\rangle^{\otimes n_s}\}$

- RWA w/ $H_{rot} = \frac{1}{2}\omega X_S$

$$e^{iH_{rot}t} H_c e^{-iH_{rot}t} = H_c = H_{rot} + \sum_k \frac{\delta_k}{\cancel{2}} |k\rangle\langle k|_Q \otimes \frac{X_S}{2}$$

w/ $\delta_k \equiv \omega_k - \omega$

$$e^{iH_{rot}t} H(t) e^{-iH_{rot}t} = \{\cos(\tfrac{1}{2}\omega t) + i\sin(\tfrac{1}{2}\omega t) X_S\}\{B(t) Z_S\}\{\cos - i\sin X_S\}$$

$$= B(t)\{(\cos^2 - \sin^2)\overset{Z_S}{\cancel{X_S}} + 2\sin\cos \overset{Y_S}{\cancel{X_S}}\}$$

$$= B(t)\{\cos(\omega t)\overset{Z_S}{\cancel{X_S}} + \sin(\omega t)\overset{Y_S}{\cancel{X_S}}\}$$

$$= \frac{\cancel{B(t)}}{2}\{\sin(\cancel{2\omega t + \phi}) - \sin\phi\}\overset{Y_S}{\cancel{X_S}} + \frac{B(t)}{2}\{\cos(\cancel{2\omega t + \phi}) + \cos\phi\}\overset{Z_S}{\cancel{X_S}}$$

Interaction picture : $H = H_{n.r} + H_{eff}$

$$H_{eff} = \sum_k |k\rangle\langle k|_Q \otimes H_{eff}^{(k)}$$

w/ $H_{eff}^{(k)} = \frac{\delta_k}{2} X_S + \frac{B}{2} x(t) \left(\cos\phi \cdot Z_S - \sin\phi \cdot Y_S \right)$

- Eigenvalues @ time t ... $\pm \frac{1}{2} \sqrt{B^2 x(t)^2 + \delta_k^2}$

@ $t=0$, $H_{eff} = \delta_k \Leftarrow H = \omega_k$ eigenstate

if $x(t)$ is adiabatic, we follow the $+\frac{1}{2}\sqrt{B^2 x(t)^2 + \delta_k^2}$ eigenstate

$$\longrightarrow \theta_k = \frac{1}{2} \int_0^T \sqrt{B^2 x(t)^2 + \delta_k^2} \underbrace{- \frac{\delta_k T}{2}}_{\xi_k} \sim \sum_k |k\rangle\langle k|_Q \otimes e^{-i\theta_k X_S}$$

$\theta_k = 0$ if $B = 0$

$\theta_k \simeq O\left(\frac{B^2 T}{\delta_k}\right)$ if $B \ll \delta_k$ $\boxed{\text{Caused by } e^{iH_{eff}T} \text{ after the measurement}}$

- Simple setup

we may assume $\delta_{k^*} = 0$ $\quad \delta_{k^* \pm 1} \gg B$

w/ equal width freq. bins

then, $\theta_k \simeq \begin{cases} \frac{x_1}{2} B T & (k = k^*) \\ \frac{x_2}{2} B^2 T / \delta_k & (k \neq k^*) \end{cases}$

where $\underset{\underset{x_1}{|||}}{\frac{1}{T}\int_0^T x(t)\,dt}$ & $\underset{\underset{x_2}{|||}}{\frac{1}{T}\int_0^T x(t)^2\,dt}$ $\sim O(1)$ ~~neglected~~

$1 \geq x_1 \geq x_2 \geq 0$

$\Leftarrow \frac{B}{\delta_k} \ll 1$ prevents us to use side bins for global analysis ?

- Take a closer look for t^2 terms

$$H = H_{mot} + \sum_k |k\rangle\langle k|_Q \otimes H_{eff}^{(k)}$$

w/ $$H_{eff}^{(k)} = \frac{\delta_k}{2} X_S + \frac{B}{2} x(t) \left(\cos\phi \cdot Z_S - \sin\phi \cdot Y_S \right)$$

- Initial cond. $$|\Psi\rangle = \frac{1}{\sqrt{N_Q}} \sum_k |k\rangle_Q \otimes \underline{|0\rangle}$$

← abbrev. of $|0\rangle^{\otimes N_S}$

$$\rightarrow \quad |\Psi_I(t)\rangle = \frac{1}{\sqrt{N_Q}} \sum_k |k\rangle_Q \otimes |\varphi_k(t)\rangle$$

w/ $$|\varphi_k(t)\rangle = T \exp\left[-i \int_0^t H_{eff}^{(k)}(t)\, dt \right] |0\rangle$$

$$|\varphi_k(t)\rangle = \sum_{m=\pm} C_m(t) \cancel{e^{-im\frac{\omega}{2}t}} |m\rangle \qquad |\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

← int. picture

w/ $C_\pm(0) = \frac{1}{\sqrt{2}}$

$$i|\dot\varphi_k\rangle = H_{eff}^{(k)} |\varphi_k\rangle$$
$$= C_+(t) \left[\frac{\delta_k}{2} |+\rangle + \frac{B}{2} x(t) \left(\cos\phi \cdot |-\rangle + i \sin\phi |-\rangle \right) \right]$$
$$+ C_-(t) \left[-\frac{\delta_k}{2} |-\rangle + \frac{B}{2} x(t) \left(\cos\phi |+\rangle - i \sin\phi |+\rangle \right) \right]$$

$$\therefore \begin{cases} i\dot C_+ = \frac{\delta_k}{2} C_+ + \frac{B}{2} x(t) e^{-i\phi} C_- \\ i\dot C_- = -\frac{\delta_k}{2} C_- + \frac{B}{2} x(t) e^{i\phi} C_+ \end{cases}$$

$$\Rightarrow \begin{cases} i\dot d_+ = \frac{B}{2} x(t) e^{i(\frac{\delta_k}{2}t - \phi)} d_- \\ i\dot d_- = \frac{B}{2} x(t) e^{-i(\frac{\delta_k}{2}t - \phi)} d_+ \end{cases} \quad \text{w/} \quad d_\pm = e^{\pm i \frac{\delta_k}{2} t} C_\pm$$

$$\rightarrow \quad \ddot d_+ = -\left(\frac{B}{2} x(t) \right)^2 d_+ \quad , \quad d_+(t=0) = \frac{1}{\sqrt{2}}$$

$$d_+ = \frac{1}{\sqrt{2}} \cos\left[\int_0^t \frac{B}{2} x(t')\, dt' \right]$$

→ Maybe no use. Use of control makes sensitivity to $\sqrt{B^2 x^2(t) + \delta_k^2}$ instead of B, which is suppressed if $B \ll \delta_k$ regardless of T

▷ Consider the control Hamiltonian $Z \mapsto X$

$$\begin{cases} B(t) = B\cos(\omega t + \phi) \\ H(t) = B(t) \sum_{i=1}^{us} X_i \equiv B(t) X_s \end{cases}$$

$$H_c = \sum_k \omega_k |k\rangle\langle k|_Q \otimes \frac{Z_s}{2}$$

assume already include qubit freq.

Evolution under $H_{tot} = \alpha(t) H(t) + H_c$

∘ RWA under $H_0 = \mathbb{1}_Q \otimes \omega \frac{Z_s}{2}$

$$e^{iH_0 t} H_c e^{-iH_0 t} = H_c = H_0 + \sum_k \delta_k |k\rangle\langle k|_Q \otimes \frac{Z_s}{2}$$

w/ $\delta_k \equiv \omega_k - \omega$

$$e^{iH_0 t} \alpha(t) H(t) e^{-iH_0 t}$$

$$= \alpha(t) \underbrace{B\cos(\omega t + \phi)}_{\frac{1}{2}\left(e^{i(\omega t + \phi)} + e^{-i(\omega t + \phi)}\right)} \mathbb{1} \otimes \begin{pmatrix} & e^{i\omega t} \\ e^{-i\omega t} & \end{pmatrix}$$

$$\underset{\uparrow}{\simeq} \alpha(t) \frac{B}{2} \mathbb{1} \otimes \begin{pmatrix} & e^{-i\phi} \\ e^{i\phi} & \end{pmatrix}$$

neglect fast-oscillating terms

$$\therefore \quad H_I = \sum_k \delta_k |k\rangle\langle k|_Q \otimes \frac{Z_s}{2} + \alpha(t) \frac{B}{2} \mathbb{1} \otimes \begin{pmatrix} & e^{-i\phi} \\ e^{i\phi} & \end{pmatrix}$$

$$\Updownarrow$$

$$(\cos\phi) X_s + (\sin\phi) Y_s$$

- Starting from $|\Upsilon\rangle = |g\rangle$ @ $t=0$, evolve it ?
  (for simplicity, $\chi(t) = 1$)

- Axis rotation

$$H_I = \underbrace{e^{-\frac{i}{2}\phi Z_B}}_{\cancel{}}\Big[\underbrace{\textstyle\sum_k \delta_k |k\rangle\langle k|_Q \otimes \frac{Z_B}{2} + \cancel{\chi(t)}\frac{\beta}{2} 1_Q \otimes X_B}_{H_I^{(0)}}\Big]\underbrace{e^{\frac{i}{2}\phi Z_B}}_{U}$$

$$|\tilde{\Upsilon}(t)\rangle = U^{\dagger}|\Upsilon_I(t)\rangle$$

$$\rightsquigarrow\ i\frac{d}{dt}|\tilde{\Upsilon}(t)\rangle = \overset{U^{\dagger}}{\vee}\, i\frac{d}{dt}|\Upsilon_I(t)\rangle$$

$$= U^{\dagger} H_I U |\tilde{\Upsilon}(t)\rangle = \underbrace{H_I^{(0)}}_{\text{time-indep}}|\tilde{\Upsilon}(t)\rangle$$

time-indep ?

$$\exp\left[-iH_I^{(0)}t\right] \sum_k \left[\cos\left(\frac{\Omega_k}{2}t\right) - i\sin\left(\frac{\Omega}{2}t\right)\frac{1}{\Omega_k}(\beta X_B + \delta_k Z_B)\right]\otimes |k\rangle\langle k|_Q$$

w/ $\Omega_k \equiv \sqrt{\beta^2 + \delta_k^2}$

$\therefore$ $\tilde{\Upsilon}(0) = \frac{1}{\sqrt{M}}\sum_k |k\rangle_Q \otimes |g\rangle$ から start して,

$M = 2^{n_Q}$

$$\tilde{\Upsilon}(t) = \frac{1}{\sqrt{M}}\sum_k |k\rangle_Q \otimes \Big[\Big\{\cos\left(\frac{\Omega_k}{2}t\right) + i\sin\left(\frac{\Omega_k}{2}t\right)\frac{\delta}{\Omega_k}\Big\}|g\rangle$$

$$- i\sin\left(\frac{\Omega_k t}{2}\right)\frac{\beta}{\Omega_k}|e\rangle\Big]$$

- $|\psi(t)\rangle = e^{-\frac{i}{2}\Phi\sigma_z}\,|\tilde{\psi}(t)\rangle$

$$= \frac{1}{\sqrt{M}}\sum_k |k\rangle \otimes \left[e^{i\frac{\Phi}{2}} \left\{ \cos\left(\frac{\Omega_k}{2}t\right) + i\sin\left(\frac{\Omega_k}{2}t\right)\frac{\delta}{\Omega_k} \right\} |g\rangle - i e^{-i\frac{\Phi}{2}} \sin\left(\frac{\Omega_k t}{2}\right)\frac{\beta}{\Omega_k}|e\rangle \right]$$

- what happenes by QFT on Qubit system

$$U_{QFT}^{\dagger} \otimes \mathbb{1}_{\sigma} = U_{QFT}^{\dagger} \otimes (P_g + P_e)$$

P_e 場に [illegible]

$$|\psi(t)\rangle\Big|_e = \sum_k \underbrace{\left\{ -\frac{i}{\sqrt{M}} e^{-i\frac{\Phi}{2}} \sin\left(\frac{\Omega_k t}{2}\right)\frac{\beta}{\Omega_k} \right\}}_{\equiv\, a_k} |k\rangle \otimes |e\rangle$$

$$\longmapsto \sum_{x_j} \tilde{a}_{x_j} |k\rangle \otimes |e\rangle$$

w/ $$\tilde{a}_{x_j} = \frac{1}{\sqrt{M}} \sum_{a_k} a_k\, e^{2\pi i jk/M}$$