Section 1

Structure of the loop functions

In this section, we examine the structure of the loop functions defined in Sec. **??** and discuss their properties. Recall the form of the loop functions

$$f(x) = \begin{cases} \dfrac{1}{16\pi^2}\displaystyle\int_0^1 dy\,(1-2y)^2 \ln(1-y(1-y)x - i0) & \text{(Scalar)}, \\ \dfrac{1}{16\pi^2}\displaystyle\int_0^1 dy\, y(1-y) \ln(1-y(1-y)x - i0) & \text{(Fermion)}, \end{cases} \tag{1.1}$$

which correspond to the vacuum polarization effects from scalar and fermionic WIMPs, respectively. Note that $f(x=0) = 0$ for both scalar and fermionic cases.

First, we consider the imaginary part of Eq. (1.1). The imaginary part of the integrand is non-zero if and only if $1 - y(1-y)x < 0$, which can be realized when $x > 4$. Under this condition, we can evaluate the integral and obtain

$$\Im f(x) = \begin{cases} -\dfrac{1}{48\pi}\beta^3 & \text{(Scalar, } x > 4), \\ -\dfrac{1}{192\pi}\beta(3-\beta^2) & \text{(Fermion, } x > 4), \end{cases} \tag{1.2}$$

with $\beta \equiv \sqrt{1 - 4/x}$. According to the optical theorem, $\Im f(x)$ is proportional to the parton-level WIMP pair production cross section discussed in Sec. **??** with a negative coefficient, and thus $\Im f(x) < 0$ for any $x > 4$. Note that, if we substitute $x = s'/m_\chi^2$, the β dependence of Eq. (1.2) is consistent with the total pair production cross section that is obtained by integrating Eqs. (**??**) and (**??**) over the solid angle. In the pair production process, β denotes the velocity of the produced WIMPs in the CM frame.

Next, we consider the analytic structure of the loop function $f(z)$ now defined for $z \in \mathbb{C}$. As shown in Fig. 1, $f(z)$ has a branch cut on the real axis with $\Re z > 4$. Note that, in the following discussion, we take account of the Feynman prescription $-i0$ in the loop function by choosing physically interesting points as $z = x + i0$ with $x > 0$ as shown by the black circle, while keeping the position of the branch cut just on the real axis. From the Cauchy's theorem, we have an identity

$$f(z) = \frac{z}{2\pi i} \oint \frac{f(z')}{z'(z'-z)} dz', \tag{1.3}$$

where a contour that surrounds the point z is chosen. The factor z/z' is introduced to realize the required property $f(z=0) = 0$. We can take a contour shown in Fig. 1, which

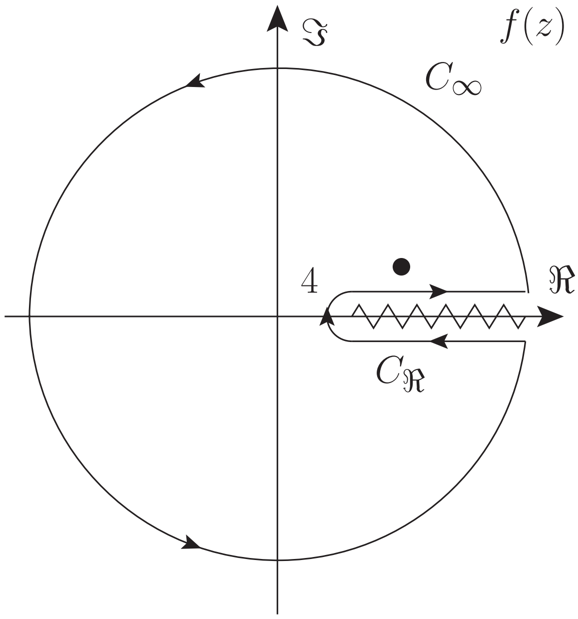

Figure 1: Analytic structure of the loop function $f(z)$ and the contour of the integration.

is decomposed into two parts; C_∞, which exists at $|z'| \to \infty$, and $C_\Re$, which exists just below and above the real axis with $\Re z' > 4$. Since the integrand decreases as $\ln z'/z'^2$ as $|z'|$ increases, the integration along the contour C_∞ gives zero. On the other hand, the integration along $C_\Re$ is equivalent to the integration of $f(z) - f(z)^* = 2i\Im f(z)$ for an analytic function $f(z)$ and we obtain

$$f(z) = \frac{z}{\pi}\int_4^\infty dz' \frac{1}{z'(z'-z)}\Im f(z). \tag{1.4}$$

From Eq. (1.4), we determine the property of $\Re f$ that is important for the discussion in Sec. **??**. For this purpose, we return to the function $f(x)$ of real values and rewrite Eq. (1.4) as

$$f(x) = \begin{cases} \dfrac{x}{\pi}\displaystyle\int_4^\infty dx' \frac{1}{x'(x'-x)}\Im f(x), & (x < 4) \\ \dfrac{x}{\pi}\,\mathrm{P}\displaystyle\int_4^\infty dx' \frac{1}{x'(x'-x)}\Im f(x) + i\Im f(x), & (x > 4) \end{cases} \tag{1.5}$$

where the symbol P deontes the Cauchy principal value. Note that the first term of the second line shows the real part of $f(x)$, while the second term is consistently derived by calculating the residue at $x' \to x$. For $x < 4$, we can see that $\Re f(x) = f(x) < 0$ since the integrand is always negative. For $x > 4$, the integration contains both the negative and positive contributions but we expect $\Re f(x) < 0$ for $x \sim 4$ from the continuity. This explains the behavior of the correction to the cross section from WIMPs shown in Fig. **??**.