Category Theory and Abstract Algebra

Li, Yunsheng

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Introduction

This notes will give an introduction to basic category theory, and show how category theory might be related or applied to abstract algebra by several examples and propositions. Also, we would mention some notions in topology theory.

We would admit facts in set theory and assume the axiom of choice. We would make definitions and notations as detailed as possible, but omit proofs for simple facts, give references or proofs with details omitted for complicated theorems like the Yoneda Lemma. The main references of this thesis are Riehl, Emily. *Category Theory in Context*, Aluffi, Paolo. *Algebra: Chapter 0* and Munkres, James. *Topology*.

1 Category, Functors, Natural Transformations

1.1 Category

In this section we will introduce some elementary notions in category theory and give several examples of categories.

Definition 1.1 (Category). A **category** $\mathcal{C}$ consits of:

- a collection of **objects**. We would denote this collection by $\mathrm{Obj}(\mathcal{C})$. If something X is an object of C, then we say X is in $\mathrm{Obj}(\mathcal{C})$, which is denoted by $X \in \mathrm{Obj}(\mathcal{C})$.

- a collection of **morphisms**. We would denote this collection by $\mathrm{Mor}(\mathcal{C})$. Similarly, if something f is a morphism of $\mathcal{C}$, then we say f is in $\mathrm{Mor}(\mathcal{C})$, denoted by $f \in \mathrm{Mor}(\mathcal{C})$.[1]

where the morphisms satisfy the following properties (or, say, have the following information):

- Each morphism of $\mathcal{C}$ specifies two objects of $\mathcal{C}$ called its **domain** and **codomain**, respectively. The notation $f : X \to Y$, read "f is a morphism from X to Y", signifies that f is a morphism with domain X and codomain Y. With this notation, a morphism is also called an **arrow** pointing from its domain to its codomain. Given a morphism f, we denote its domain by $\operatorname{dom} f$ and its codomain by $\operatorname{cod} f$. Two morphisms f, g with $\operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$ are said to be **parallel**, and the notation $f, g : X \rightrightarrows Y$ signifies that f and g are parallel morphisms with domain X and codomain Y. Given two objects X and Y of $\mathcal{C}$, we denote the collection of all morphisms of $\mathcal{C}$ with X as its domain and Y as its codomain by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.[2] A morphism f with $\operatorname{dom} f = \operatorname{cod} f$ is called an **endomorphism**, and we denote the collection $\mathrm{Hom}_{\mathcal{C}}(X, X)$ by $\mathrm{End}_{\mathcal{C}}(X)$. The subscript $\mathcal{C}$ could be omited, if there comes no confusion about that which category we are focused on.

- For any two morphisms f, g of $\mathcal{C}$ with $\operatorname{cod} f = \operatorname{dom} g$, there exists a morphism h of $\mathcal{C}$ with $\operatorname{dom} h = \operatorname{dom} f$ and $\operatorname{cod} h = \operatorname{cod} g$ specified by f and g according to some rules called the **composition law** in $\mathcal{C}$. We say that h is the **composite morphism** of g composed with f (according to the composition law in $\mathcal{C}$), usually denoting it by gf instead of h.[3] For morphisms f and g satisfying the condition $\operatorname{cod} f = \operatorname{dom} g$, we say that they are **composable**. We may express composition by notations below:

$$f : X \to Y, \quad g : Y \to Z \quad \rightsquigarrow \quad gf : X \to Z.$$

- For any composable triple of morphisms f, g, h of $\mathcal{C}$, we have

$$h(gf) = (hg)f.$$

  That is, the composition law is associative. Therefore, we can omit the parameter and write hgf instead. Namely,

$$f : X \to Y, \quad g : Y \to Z, \quad h : Z \to W \quad \rightsquigarrow \quad hgf : X \to W.$$

- For any object X of $\mathcal{C}$, there exists an **identity morphism** $1_X : X \to X$, which satisfies that

$$1_X f = f, \quad g 1_X = g$$

  for any $f, g \in \mathrm{Mor}(\mathcal{C})$ s.t. $\operatorname{cod} f = \operatorname{dom} g = X$. It is immediate that the identity morphism of an object in a certain category is unique. We would always denote the identity morphism of an object X by 1_X, if there comes no confusion about that of which category the identity morphism is. A category with only identity morphisms is said to be **discrete**.

We state several basic notions about a category before we give examples of categories.

Definition 1.2 (Subcategory). Let $\mathcal{C}, \mathcal{D}$ be two categories, then $\mathcal{C}$ is said to be the **subcategory** of $\mathcal{D}$, if the composition law in $\mathcal{C}$ and $\mathcal{D}$ coincidents and every element in $\mathrm{Obj}(\mathcal{C})$ is in $\mathrm{Obj}(\mathcal{D})$, every element in $\mathrm{Mor}(\mathcal{C})$ is in $\mathrm{Mor}(\mathcal{D})$. We might denote the last two conditions by $\mathrm{Obj}(\mathcal{C}) \subset \mathrm{Obj}(\mathcal{D})$ and $\mathrm{Mor}(\mathcal{C}) \subset \mathrm{Mor}(\mathcal{D})$ for conveniont. A subcategory $\mathcal{C}$ is said to be **full** if $\mathrm{Hom}_{\mathcal{C}}(c, c') = \mathrm{Hom}_{\mathcal{D}}(c, c')$ for all $c, c' \in \mathrm{Obj}\, \mathcal{C}$.

Definition 1.3 (Small, Locally Small). A category $\mathcal{C}$ is **small** if $\mathrm{Mor}(\mathcal{C})$ is a set, or equivalently, there is a bijection from $\mathrm{Mor}(\mathcal{C})$ to some set. $\mathcal{C}$ is **locally small** if for any $X, Y \in \mathrm{Obj}(C)$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set.

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[1] Although we used the notation $\in$ as is in set theory, neither $\mathrm{Obj}(\mathcal{C})$ nor $\mathrm{Mor}(\mathcal{C})$ was asked to be as samll as a set. As we shall see, for the category Set, neither Obj(Set) nor Mor(Set) is a set.

[2] Again, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ might not be a set, though we would let the notation $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ signify that f is a morphism of $\mathcal{C}$ with $\operatorname{dom} f = X$ and $\operatorname{cod} f = Y$.

[3] Sometimes we might want to specify that according to composition law in which category is the composite morphism determined, in which case we would put some notation between g and f. For example, $g \circ_{\mathcal{C}} f$ might denote that $g \circ_{\mathcal{C}} f$ is the composition morphism of g composed with f according to the composition law in the category $\mathcal{C}$.

There are categories that are neither small nor locally small. This thesis would mainly focus on locally small categories. For a small category $\mathcal{C}$, $\mathrm{Obj}(\mathcal{C})$ is also a set since there is an injection[4] $\mathrm{Obj}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C}) : X \mapsto 1_X$.

Definition 1.4 (Initial, Terminal). Given a category $\mathcal{C}$. An object A of $\mathcal{C}$ is an **initial object** of $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(A, X)$ contains only one element for any $X \in \mathrm{Obj}(\mathcal{C})$. We might say such an object is **initial** instead of saying it is an initial object of some category, if there comes no confusion about that which category we are focused on.
Dually, an object A of $\mathcal{C}$ is a **terminal object** of $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(X, A)$ contains only one element for any $X \in \mathrm{Obj}(\mathcal{C})$. Similarly, we might say an object is **terminal** instead of saying it is a terminal object of some category.

In particular, for an object A which is either initial or terminal, $\mathrm{End}(A)$ contains only one element, the identity map 1_A. It's possible for an object to both initial and terminal, as we shall see in the category Grp.

Definition 1.5 (Isomorphism). Given a category $\mathcal{C}$. A morphism $f : X \to Y$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ s.t.

$$gf = 1_X, \quad fg = 1_Y.$$

Such a g is called the **inverse** of f. If there is an isomorphism between X and Y, then X and Y are said to be **isomorphic**, denoted by $X \cong Y$.

It's clear that isomorphic is an equivalence relation, hence we used the word "between" here. Also, the inverse of an isomorphism is unique, hence the morphism g mentioned above might be denoted as f^{-1}.
An immediate fact is that, given a category $\mathcal{C}$, then all initial objects of $\mathcal{C}$ are isomorphic to each other, so do all terminal objects of $\mathcal{C}$.

Definition 1.6 (Groupoid). A **groupoid** is a category whose morphisms are all isomorphisms.

Now we can redefine what we called a "group" in abstract algebra in a categorical view:

Definition 1.7 (Group). A **group** is a locally small groupoid who has only one object.

In fact, the groupoid mentioned above is small. In this point of view, given a group under our traditional notion, it induces a category, namely a locally small groupoid who has only one object.

Example 1.1. We can now state several examples of categories below:

(i) Given a group G, then it induces a category, denoted by $\mathrm{B}G$, in which:

- Objects: an abstract-nonsense point $\bullet$.
- Morphisms: Elements in G, with $\bullet$ being their domains and codomains.
- Composition law: The composition in the group G. That is, if we let $\circ$ denotes the group operation, then for any $f, g \in G(= \mathrm{Mor}(\mathrm{B}G))$,

$$gf := g \circ f.$$

The identity morphism $1_\bullet$ is exactly the identity in G, and every morphism in $\mathrm{B}G$ is an isomorphism, with its inverse in G being its inverse as a morphism.

(ii) The category of sets, denoted by Set, in which:

- Objects: All sets.
- Morphism: All set-functions, with their domain and codomain the same as the domain and codomain when seen as set-functions.[5]
- Composition law: Composition of set-functions.

The identity morphism for each object (set) is the identity map from the set to itself. A morphism of Set is an isomorphism if and only if it is a bijection. The initial object in Set is the empty set $\varnothing$, and the terminal object is the singleton $\{*\}$. Set is locally small but not small.

(iii) The category of groups, denoted by Grp, in which:

- Objects: All groups.
- Morphisms: All group homomorphisms.
- Composition law: Composition of set-functions.

The identity morphism is the identity group homomorphism. A morphisms of Grp is an isomorphism if and only if it is a group isomorphism. The initial object and the terminal object in Grp are both the trivial group $\{*\}$, namely the trivial group is both initial and terminal in Grp. Again, Grp is locally small but not small, for there is an injection from Obj(Set) to Obj(Grp), that sends each set to the free group on it; we shall define what a free group is after we introduce the notion of universal property.

[4] Although what we said as a "collection" might be too large to be a set, we can still define notions like maps, inclusion, injective, surjective, ..., just as how we defined them in set theory.

[5] When we define a category, if the morphism is chosen to be something with pre-defined domains and codomains, we shall omit the words "with their domain and codomain the same as the domain and codomain when seen as set-functions" and take this as default.

(iv) The category of abelian groups, denoted by Ab, in which:

- Objects: All abelian groups.
- Morphisms: All group homomorphisms between abelian groups.
- Composition law: Composition of set-functions.

Ab is a subcategory of Grp. Ab is also locally small but not small, for there is an injection from Obj(Set) to Obj(Ab) that sends each set to the free abelian group on it; we shall define what a free abelian group is after the notion of universal property is introduced. We will see that Ab is better than Grp, one aspect of this is that the product and coproduct for a certain set of objects in Ab are the same, which is not true in Grp, after the notion of limits and colimits are introduced.[6]

(v) The category of rings, denoted by Ring, in which:

- Objects: All rings.
- Morphisms: All ring homomorphisms.
- Composition law: Composition of set-functions.

The identity morphism is the identity ring homomorphism, and a morphism of Ring is an isomorphism if and only if it is a ring isomorphism. The ring $\mathbb{Z}$ of integers is the initial object in Ring, and the trivial ring $\{0, 1\}$ is the terminal object.

(vi) The category of finite-dimensional vector spaces over a given field $\mathbb{F}$, denoted by $\mathrm{Vect}_{\mathbb{F}}$, in which:

- Objects: All finite-dimensional vector spaces over $\mathbb{F}$.
- Morphisms: All linear maps between these vector spaces.
- Composition law: Composition of set-functions.

The zero-dimentional space 0 is the initial and terminal object of $\mathrm{Vect}_{\mathbb{F}}$. Again, $\mathrm{Vect}_{\mathbb{F}}$ is locally small but not small, for its collection of objects contains the collection of all one-dimensional vector spaces, and there is an injection from the collection of all one-point sets that sends every one-point set to the one-dimensional vector space generated by taking that set as its basis. The collection of all one-point sets is not a set, for there is an injection from the collection of all sets that sends every set A to the one-point set $\{A\}$ whose only element is A.

(vii) *Given a topological space X, then it induces a category, denoted by $\mathrm{T}X$, in which:

- Objects: All points in X.
- Morphisms: Path-homotopy classes of pathes in X, with their domain and codomain being the initial point and final point of the pathes, respectively.
- Composition law: For any $[f]$ and $[g]$ morphisms of $\mathrm{T}X$ with $\mathrm{cod}[f] = \mathrm{dom}[g]$, their composition $[gf] = [g][f]$ is the path-homotopy class of

$$gf = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

The category $\mathrm{T}X$ is well-defined, see Munkres, J. *Topology* §51.

(viii) *The category of topological spaces, denoted by Top, in which:

- Objects: All topological spaces.
- Morphisms: All continuous maps between topological spaces.
- Composition law: Composition of set-functions.

A morphism of Top is an isomorphism if and only if it is a homeomorphism.

(ix) The category of non-negative integers no more than n, denoted by [n], in which:

- Objects: Integers $0, 1, \cdots, n$.
- Morphisms: For each pair of integers (m_1, m_2) with $m_1 \leq m_2$, an arrow pointing from m_1 to m_2. That is, the relation “$\leq$”. $\mathrm{Hom}_{[\mathrm{n}]}(m_1, m_2)$ is a singleton if $m_1 \leq m_2$ and is empty if $m_1 > m_2$.
- Composition law: The composition follows from the transitivity of “$\leq$”.

Note that the relation “$\leq$” can't be replaced by “¡”, for the latter does not give identity morphisms. The category [n] is also called the **ordinal category** $\mathrm{n} + 1$.

[6] “Life is simpler in Ab than in Grp.” – Paolo Aluffi.

(x) The category of non-negative integers, denoted by ω, in which:

- Objects: All non-negative integers, namely all elements in $\mathbb{N}$.
- Morphisms: The relation "$\leq$". That is, for two integers a and b there is an arrow $a \to b$ if and only if $a \leq b$.
- Composition law: The transitivity of "$\leq$".

This example, along with (ix), are very simple categories but will appear from time to time in our further study of category theory. In fact, a basic object for us to establish the notion of ∞-category is the category of all categories of non-negative integers, see Example 1.3.(iii).

We are mainly interested in Set, Grp and Ab, and we will take a fancy glance at Top if possible. Note that isomorphic in these categories coincident with our traditional equivalence relations between their objects, namely equinumerous between sets, isomorphic between groups, homeomorphic between topological spaces.

We state a few more notions about morphism before we end this section.

Definition 1.8 (Automorphism). Given a category $\mathcal{C}$. An **automorphism** of $\mathcal{C}$ is an endomorphism which is also an isomorphism. Given $X \in \mathrm{Obj}(\mathcal{C})$, we denote the collection of all automorphisms of $\mathcal{C}$ with domain X by $\mathrm{Aut}_{\mathcal{C}}(X)$. The subscript could be omited, like always.

It's clear that $\mathrm{Aut}_{\mathcal{C}}(X) \subset \mathrm{End}_{\mathcal{C}}(X)$. When $\mathrm{Aut}_{\mathcal{C}}(X)$ is a set, it has the structure of a group, seen as the subcategory of $\mathcal{C}$ where the only object is X and the morphisms are all elements in $\mathrm{Aut}_{\mathcal{C}}(X)$.

Definition 1.9 (Monomorphism, Epimorphism). A morphism f of a category $\mathcal{C}$ is
(i) a **monomorphism** if for any parallel morphisms h, k of $\mathcal{C}$ with $\mathrm{cod}\, h = \mathrm{cod}\, k = \mathrm{dom}\, f$,

$$fh = fk \quad \Rightarrow \quad h = k;$$

(ii) an **epimorphism** if for any parallel morphisms h, k of $\mathcal{C}$ with $\mathrm{dom}\, h = \mathrm{dom}\, k = \mathrm{cod}\, f$,

$$hf = kf \quad \Rightarrow \quad h = k.$$

In the category Set, a monomorphism is exactly an injection and an epimorphism is exactly a surjection. This is also true in the category Grp. However, this needs not hold for all categories having set-functions as its morphisms. For example, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is not surjective, but it is an epimorphism in Ring.

1.2 Functor

Before we state the definition of a functor, we state an important category first, the opposite category.

Definition 1.10 (Opposite Category). Given a category $\mathcal{C}$. The **opposite category** of $\mathcal{C}$, denoted by $\mathcal{C}^{op}$, is a category in which:

- Objects: All objects of $\mathcal{C}$.
- Morphisms: All morphisms of $\mathcal{C}$ with their domain and codomain reversed. That is, given a morphism f of $\mathcal{C}$, we denote its corresponding morphism in $\mathrm{Mor}(\mathcal{C}^{op})$ by f^{op}, then there is $\mathrm{dom}\, f^{op} = \mathrm{cod}\, f$ and $\mathrm{cod}\, f^{op} = \mathrm{dom}\, f$.
- Composition law: The induced composition law in $\mathcal{C}$. That is, for any f^{op}, g^{op} morphisms of $\mathcal{C}^{op}$ with $\mathrm{cod}\, f^{op} = \mathrm{dom}\, g^{op}$, their composition is given by
$$g^{op} f^{op} = (fg)^{op},$$
where fg is the composition of f and g in $\mathcal{C}$.

The opposite category gives us a categorical way to define the notion of opposite group:

Definition 1.11 (Opposite Group). Given a group G, seen as a category $\mathrm{B}G$. The **opposite group** of G is the group G^{op} whose induced category is the opposite category of $\mathrm{B}G$, namely

$$\mathrm{B}(G^{op}) = (\mathrm{B}G)^{op}.$$

We will see soon that the opposite group is a special case of "left action", and that every group G is "naturally isomorphic" to its opposite group G^{op} by the group homomorphism $\varphi : G \to G^{op} : g \mapsto g^{-1}$.

We now state what a functor is:

Definition 1.12 (Covariant Functor). Given two categories $\mathcal{C}$ and $\mathcal{D}$. A **covariant functor** F from $\mathcal{C}$ to $\mathcal{D}$, denoted by $F : \mathcal{C} \to \mathcal{D}$, consists of:

- For each object $c \in \mathrm{Obj}(\mathcal{C})$, an object $Fc \in \mathrm{Obj}(\mathcal{D})$.

- For each morphism $f \in \mathrm{Mor}(\mathcal{C})$, a morphism $Ff \in \mathrm{Obj}(\mathcal{D})$ with $\mathrm{dom}\, Ff = F \,\mathrm{dom}\, f$ and $\mathrm{cod}\, Ff = F\,\mathrm{cod}\, f$, i.e.
$$f : \mathrm{dom}\, f \to \mathrm{cod}\, f \quad \mapsto \quad Ff : F\,\mathrm{dom}\, f \to F\,\mathrm{cod}\, f.$$

s.t.

- For any composable morphisms f, g in $\mathcal{C}$, $(Fg)(Ff) = F(gf)$.
- For each object c in $\mathcal{C}$, $F(1_c) = 1_{Fc}$.

The last two conditions for a functor are called **functoriality axioms**. The category $\mathcal{C}$ is called the **domain** of F, and $\mathcal{D}$ is called the **codomain** of F.

Definition 1.13 (Contravariant Functor). A **contravariant functor** F from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$. Functors have an evident way[7] of composition: Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, their composition $GF : \mathcal{C} \to \mathcal{E}$ is defined by
$$GFc := G(Fc), \quad \forall c \in \mathrm{Obj}(\mathcal{C})$$
and
$$GFf := G(Ff), \quad \forall f \in \mathrm{Mor}(\mathcal{C}).$$

Usually, we would omit the word "covariant" and say only "functor" in place of "covariant functor". To avoid unnatural arrow-theoretic representations, a morphism in the domain of a contravariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$ will always be depicted as an arrow $f : c \to c'$ in $\mathcal{C}$. Graphically, the mapping on morphisms given by a contravariant functor is depicted as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \\ c \arrow[d, "f"'] & Fc \\ c' & Fc' \arrow[u, "Ff"'] \end{tikzcd} \qquad c \overset{f}{\downarrow} c' \;\mapsto\; Fc \overset{Ff}{\uparrow} Fc'$$

Considering the definition of group under the categorical view, we can now redefine what a group homomorphism is:

Definition 1.14 (Group Homomorphism). Given two groups G and H seen as categories, a **group homomorphism** f from G to H is a functor $f : G \to H$.

We now state a lemma about functors here, whose proof is immediate:

Lemma 1.1. *Functors preserve isomorphisms.*

With this lemma, we can redefine and extend the notion of an action of a group:

Definition 1.15 (Action). Let G be a group, seen as the category $\mathrm{B}G$. Given a category $\mathcal{C}$. An **action** of G on an object $X \in \mathrm{Obj}(\mathcal{C})$ is expressed by a functor $F : \mathrm{B}G \to \mathcal{C}$, under which the image of the only object of $\mathrm{B}G$ is X. To be explicit, each element $g \in G$ gives by F a morphism $Fg \in \mathrm{End}_{\mathcal{C}}(X)$ (In fact, $\mathrm{Aut}_{\mathcal{C}}(X)$; see Corollary 1.1.1). For any two elements $h, g \in G$, there is $(Fh)(Fg) = F(hg)$; for the identity element $e \in G$, $Fe = 1_X$.

When $\mathcal{C} = \mathrm{Set}$, the definition above coincidents with what we have defined to be an action of a group in the course of abstract algebra, and the object X endowed with such an action is called a ***G*-set**. When $\mathcal{C} = \mathrm{Vect}_{\mathbb{F}}$, the object X is called a ***G*-representation**. When $\mathcal{C} = \mathrm{Top}$, the object X is called a ***G*-space**.

The action expressed by a functor $\mathrm{B}G \to \mathcal{C}$ is sometimes called a **left action**. A **right action** is expressed by a functor $\mathrm{B}G^{op} \to \mathcal{C}$. Given a right action of a group, then it induces a left action of this group's opposite group.

Lemma 1.1 gives immediately that

Corollary 1.1.1. *When a group G acts (functorially) on an object X of a category $\mathcal{C}$, its elements g must act by automorphisms; moreover, the inverse of the automorphism given by g is the automorphism given by g^{-1}.*

Example 1.2. Here comes several examples of functors below:

(i) Given a locally small category $\mathcal{C}$ and an object $c \in \mathrm{Obj}(\mathcal{C})$, then c induces a functor $\mathrm{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{Set}$ and a contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \to \mathrm{Set}$. See the diagrams below:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(c,-)"] & \mathrm{Set} \\ x \arrow[d, "f"'] & \mathrm{Hom}_{\mathcal{C}}(c,x) \arrow[d, "f_*"] \\ y & \mathrm{Hom}_{\mathcal{C}}(c,y) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^{op} \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(-,c)"] & \mathrm{Set} \\ x \arrow[d, "f"'] & \mathrm{Hom}_{\mathcal{C}}(x,c) \\ y & \mathrm{Hom}_{\mathcal{C}}(y,c) \arrow[u, "f^*"'] \end{tikzcd}$$

[7] We will soon see that natural transformations have two ways of composition.

The sign f_* stands for "composing f by left". That is, it is a set-function from $\mathrm{Hom}_{\mathcal{C}}(c, y)$ to $\mathrm{Hom}_{\mathcal{C}}(c, x)$ induced by f, which maps each element $g \in \mathrm{Hom}_{\mathcal{C}}(c, y)$ to $fg \in \mathrm{Hom}_{\mathcal{C}}(c, x)$. Dually, f^* stands for "composing f by right".

These two functors are significantly important, called **functors represented by** c. We shall learn them more carefully after the notion of natural transformation is introduced.

(ii) Given two categories $\mathcal{J}$ and $\mathcal{C}$. Given an object $c \in \mathrm{Obj}(\mathcal{C})$, then it induces a **constant functor** $c : \mathcal{J} \to \mathcal{C}$, which sends all objects of $\mathcal{J}$ to $c \in \mathrm{Obj}(\mathcal{C})$ and all morphisms of $\mathcal{J}$ to $1_c \in \mathrm{End}_{\mathcal{C}}(c)$. This functor is trivial but useful. We will use it to define the notion of cones, in order to introduce the notion of limits and colimits.

(iii) Given a functor $F : \mathcal{C} \to \mathcal{D}$, the **opposite functor** of F, $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$, is defined by nothing but taking everything to what F brings it to:

$$F^{op}c := Fc, \quad F^{op}f^{op} := (Ff)^{op}, \quad \forall c \in \mathrm{Obj}(\mathcal{C}), f \in \mathrm{Mor}(\mathcal{C}).$$

(iv) Given a category $\mathcal{C}$, then there is an identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that sents everything in $\mathcal{C}$ to itself.

(v) For categories with objects having underlying sets and morphisms having underlying set-functions with composition the composition of set-functions (such as Grp, Ab, Ring, etc.), there is a **forgetful functor** from these categories to Set. For example, the forgetful functor $U : \mathrm{Grp} \to \mathrm{Set}$ sends a group to its underlying set and a group homomorphism to its underlying set-function.

(vi) There is a functor $(-)^* : \mathrm{Vect}_{\mathbb{F}}^{op} \to \mathrm{Vect}_{\mathbb{F}}$ that takes a vector space V to its **dual vector space** $V^* := \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{F}}}(V, \mathbb{F})$. It is somehow very similar to the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-, c)$ in (i), hence we shall not explain more about it.

(vii) There is a functor $F : \mathrm{Set} \to \mathrm{Grp}$ that sends a set X to the **free group** on X. We shall return to this example after we defined what a free group is.

Example 1.3. Also, with functors, we can now have more examples of categories:

(i) The category of all small categories, denoted by Cat, in which:

- Objects: All small categories.
- Morphisms: All functors between small categories.
- Composition: Composition of functors

(ii) The category of all locally small categories, denoted by CAT, in which:

- Objects: All locally small categories.
- Morphisms: All functors between locally small categories.
- Composition: Composition of functors.

(iii) *The category of all categories of non-negative integers, also called the **simplex category**, denoted by $\triangle$, in which:

- Objects: All categories of non-negative integers. That is, [n] for all $n \in \mathbb{N}$.
- Morphisms: All functors between these categories.
- Composition: Composition of functors.

Given a category $\mathcal{C}$, then a functor $F : \triangle^{op} \to \mathcal{C}$ is called a **simplicial object** in $\mathcal{C}$. When $\mathcal{C} = \mathrm{Set}$, it is called a **simplicial set**; when $\mathcal{C} = \mathrm{Grp}$, it is called a **simplicial group**, etc.

It's easy to see that Cat is a subcategory of CAT. The **empty category** which consists of no object is the initial object of both Cat and CAT, and the ordinal category 1 is the terminal object of both. The Russell's paradox suggests that there should not be a category having itself as an object of it, hence there is no category of all categories; and this implies that Cat is not small and CAT is not locally small.

We now state the notion of product of two categories, after which we can "combine" the two functors in Example 1.2 (i) to one functor.

Definition 1.16 (Product of two categories). Given two categories $\mathcal{C}$ and $\mathcal{D}$, their **product**, denoted by $\mathcal{C} \times \mathcal{D}$, is a category in which

- Objects: All ordered pairs (c, d) where $c \in \mathrm{Obj}(\mathcal{C})$ and $d \in \mathrm{Obj}(\mathcal{D})$.
- Morphisms: All ordered pairs $(f, g) : (c, d) \to (c', d')$, where $f \in \mathrm{Hom}_{\mathcal{C}}(c, c')$ and $g \in \mathrm{Hom}_{\mathcal{D}}(d, d')$.

- Composition: Componentwise composition according to the composition in $\mathcal{C}$ and $\mathcal{D}$ respectively.

Similarly, two functors can be "producted" together:

Definition 1.17 (Product of two functors). Given two functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{X} \to \mathcal{Y}$, then there is a **product functor** of them, denoted by $F \times G : \mathcal{C} \times \mathcal{X} \to \mathcal{D} \times \mathcal{Y}$, with everything defined component-wise, i.e.:

$$F \times G(c, x) \coloneqq (Fc, Gx) \in \mathrm{Obj}(\mathcal{D} \times \mathcal{Y}), \quad \forall (c, x) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{X}),$$

and

$$F \times G(f, g) \coloneqq (Ff, Gg) \in \mathrm{Mor}(\mathcal{D} \times \mathcal{Y}), \quad \forall (f, g) \in \mathrm{Mor}(\mathcal{C} \times \mathcal{X}).$$

We end this section by our "combined" functor. Note that it is not the product functor of $\mathrm{Hom}_{\mathcal{C}}(-, c)$ and $\mathrm{Hom}_{\mathcal{C}}(c, -)$. It is an important exmaple in adjunction, but we wouldn't cover that so far.

Definition 1.18 (Two-sided represented functor). If $\mathcal{C}$ is locally small, then there is a **two-sided represented functor**

$$\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set},$$

that maps an object $(x, y) \in \mathrm{Obj}(\mathcal{C}^{op} \times \mathcal{C})$ to the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$, a morphism $(f, h) \in \mathrm{Mor}(\mathcal{C}^{op} \times \mathcal{C})$ to the function (f^*, h_*) defined by:

$$(f^*, h_*)(g) \coloneqq hgf, \quad \forall g \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{cod}\, f, \mathrm{dom}\, h).$$

Note that here f is seen as a morphism of $\mathcal{C}$ instead of its opposite category $\mathcal{C}^{op}$, in order to avoid unnatural notations.

1.3 Natural Transformation

Natural transformations characterize the notion of naturality. After natural transformation is introduced, we will see an interesting example of category, the functor category. After that, we shall introduce the notion of representable functors, the character of which really shines and will lead us deeper into the category theory.

Definition 1.19 (Natural Transformation). Given two parallel functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$, a **natural transformation** α from F to G, denoted by $\alpha : F \Rightarrow G$, consists of:

- for each object $c \in \mathrm{Obj}(\mathcal{C})$, a morphism $\alpha_c \in \mathrm{Hom}_{\mathcal{D}}(Fc, Gc)$, called the **component** of α at c,

s.t. the following diagram

$$\begin{tikzcd} F\,\mathrm{dom}\, f \arrow[r, "\alpha_{\mathrm{dom}\, f}"] \arrow[d, "Ff"'] & G\,\mathrm{dom}\, f \arrow[d, "Gf"] \\ F\,\mathrm{cod}\, f \arrow[r, "\alpha_{\mathrm{cod}\, f}"'] & G\,\mathrm{cod}\, f \end{tikzcd}$$

commutes, i.e., $(Gf)(\alpha_{\mathrm{dom}\, f}) = (\alpha_{\mathrm{cod}\, f})(Ff)$, for all $f \in \mathrm{Mor}(\mathcal{C})$.

A **natural isomorphism** is a natural transformation whose every component is an isomorphism. A natural isomorphism $\alpha : F \Rightarrow G$ may be denoted as $\alpha : F \cong G$. If there is a natural isomorphism between two functors, then they are called to be **naturally isomorphic**. We will see soon that naturally isomorphic is an equivalence relation.

Example 1.4. Here comes some examples of natural transformations:

(i) For any finite-dimensional vector space V, the evaluation map $\mathrm{ev} : V \to V^{**}$ that sends $v \in V$ to the linear function $\mathrm{ev}(v) : V^* \to \mathbb{F} : \varphi \mapsto \varphi(v)$ forms a natural transformation from the idenity functor $1_{\mathrm{Vect}_{\mathbb{F}}}$ to the double dual functor (the composition of the dual functor $(-)^*$ with the opposite functor of itself). The reader can check this directly by checking the definition of natural transformation. In fact, it is a natural isomorphism, for $\mathrm{ev} : V \to V^{**}$ is an injective linear map and $\dim V = \dim V^{**}$. Therefore, the evaluation map tells us that V and V^{**} are "naturally isomorphic", which tends to be an equivalence relation further stronger than isomorphic.

(ii) The opposite group defines a functor $(-)^{op} : \mathrm{Group} \to \mathrm{Group}$ that brings a group to its opposite group and a group homomorphism $\phi : G \to H$ to $\phi^{op} : G^{op} \to H^{op} : g \mapsto \phi(g)$; the fact that ϕ^{op} is a group homomorphism can be verified easily. Now that the homomorphisms $\eta_G : G \to G^{op} : g \mapsto g^{-1}$ forms a natural isomorphism from the identity functor 1_{Grp} to $(-)^{op}$, i.e., the following diagram

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[d, "\phi"'] & G^{op} \arrow[d, "\phi^{op}"] \\ H \arrow[r, "\eta_H"'] & H^{op} \end{tikzcd}$$

commutes, as one can verify.

(iii) Given two parallel functors $X, Y : \mathrm{B}G \to \mathcal{C}$, each defines an action of group G on $X, Y \in \mathrm{Obj}(\mathcal{C})$ respectively, then a natural transformation $\alpha : X \Rightarrow Y$ consists of only one morphism $\alpha : X \to Y$ of $\mathcal{C}$. This single morphism (or equivalently, the natural transformation this morphism consists) is called **G-equivariant**, meaning that for each $g \in G$, the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "Xg"'] & Y \arrow[d, "Yg"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

commutes.

Recall the ordinal category in Example 1.1.(ix). Consider the ordinal categories 1 and 2, there is $\mathrm{Obj}(1) = \{0\}$ and $\mathrm{Obj}(2) = \{0, 1\}$, and there are two evident functors $i_0, i_1 : 1 \to 2$ defined by $i_0 : 0 \mapsto 0$ and $i_1 : 0 \mapsto 1$. We keep these notations, and here comes a characterizing of natural transformations between two functors:[8]

Lemma 1.2. *Given two parallel functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$. Natural transformations from F to G correspond bijectively to functors $H : \mathcal{C} \times 2 \to D$ s.t. the following diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "1_{\mathcal{C}} \times i_0"] \arrow[dr, "F"'] & \mathcal{C} \times 2 \arrow[d, "H"] & \mathcal{C} \arrow[l, "1_{\mathcal{C}} \times i_1"'] \arrow[dl, "G"] \\ & \mathcal{D} & \end{tikzcd}$$

commutes.

Proof. The proof is straightforward but interesting, thus is omitted. The reader is strongly recommended to work this out by hand.[9] □

Natural transformations can also compose with each other, and there are two ways of compositions, called horizontal composition and vertical composition. We first state the strategy of these two compositions, and see what the explicit results of the compositions are later.

Definition 1.20 (Vertical and Horizontal Compositions). Given three parallel functors $F, G, H : \mathcal{C} \rightrightarrows \mathcal{D}$ and natural transformations $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$, the **vertical composition** of β and α is a natural transformation $\beta \cdot \alpha : F \Rightarrow H$. See the diagrams below:

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=60, "F", ""{name=U, below}] \arrow[r, "G" description, ""{name=M}] \arrow[r, bend right=60, "H"', ""{name=D, above}] & \mathcal{D} \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] \end{tikzcd} \xRightarrow{\text{Vertical composition}} \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, "\Downarrow \beta\cdot\alpha"] & \mathcal{D} \end{tikzcd}$$

Given Functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$, $H, K : \mathcal{D} \rightrightarrows \mathcal{E}$ and natural transformations $\alpha : F \Rightarrow G$, $\beta : H \Rightarrow K$, the **horizontal composition** of β and α is a natural transformation $\beta * \alpha : HF \Rightarrow KG$. See the diagrams below:

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D} \arrow[r, bend left=60, "H"] \arrow[r, bend right=60, "K"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathcal{E} \end{tikzcd} \xRightarrow{\text{Horizontal composition}} \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=60, "HF"] \arrow[r, bend right=60, "KG"'] \arrow[r, phantom, "\Downarrow \beta * \alpha"] & \mathcal{E} \end{tikzcd}$$

Let's begin with the vertical composition, with which we will be able to define the category of functors between two categories.

Lemma 1.3 (Vertical Composition). *Given three parallel functors $F, G, H : \mathcal{C} \rightrightarrows \mathcal{D}$ and natural transformations $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$, then the* ***vertical composition*** *$\beta \cdot \alpha$ of α and β, defined by the equation*

$$(\beta \cdot \alpha)_c \coloneqq \beta_c \alpha_c, \quad \forall c \in \mathrm{Obj}(\mathcal{C})$$

is a natural transformation from F to H.

[8] It's rather weak, though.

[9] "Dear reader: don't shy away from trying this, for it is excellent, indispensable practice. Miss this opportunity and you will forever feel unsure about such manipulations." – Paolo Aluffi.

Proof. It follows immediately from the fact that the diagram

$$\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] \arrow[rr, dashed, bend left, "(\beta\cdot\alpha)_c"] & Gc \arrow[r, "\beta_c"] \arrow[d, "Gf"'] & Hc \arrow[d, "Hf"] \\
Fc' \arrow[r, "\alpha_{c'}"'] \arrow[rr, dashed, bend right, "(\beta\cdot\alpha)_{c'}"'] & Gc' \arrow[r, "\beta_{c'}"'] & Hc'
\end{tikzcd}$$

commutes for any morphism $f : c \to c'$ of $\mathcal{C}$. $\square$

One can learn immediately from the diagram above that if α and β are both natural isomorphisms, so is $\beta \cdot \alpha$. Here comes the category of functors:

Definition 1.21 (Category of Functors). Given two category $\mathcal{C}$ and $\mathcal{D}$, then all functors from $\mathcal{C}$ to $\mathcal{D}$ consist a category, denoted by $\mathcal{D}^{\mathcal{C}}$, in which:

- Objects: All functors from $\mathcal{C}$ to $\mathcal{D}$.
- Morphisms: All natural transformations between these functors.
- Composition law: The vertical composition of natural transformations.

Here are some immediate facts about $\mathcal{D}^{\mathcal{C}}$: The identity morphism for an object $F \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$ is the identity natural transformation $1_F : F \Rightarrow F$ defined by $(1_F)_c \coloneqq 1_{Fc}$. The isomorphisms are exactly natural isomorphisms, and given a natural isomorphism $\alpha : F \Rightarrow G$, its inverse $\alpha^{-1} : G \Rightarrow F$ is defined by $(\alpha^{-1})_c \coloneqq (\alpha_c)^{-1}$. Here we conclude that naturally isomorphic is an equivalence relation. If $\mathcal{D}$ has an initial object, then the constant functor at this initial object is initial in $\mathcal{D}^{\mathcal{C}}$. Dually, if $\mathcal{D}$ has a terminal object, then the constant functor at this terminal object is terminal in $\mathcal{D}^{\mathcal{C}}$.

Remark 1 (sizes of functor categories). Care should be taken when discussing functor categories. If $\mathcal{C}$ and $\mathcal{D}$ are both small, then $\mathcal{D}^{\mathcal{C}}$ is again a small category. However, if both are locally small, then $\mathcal{D}^{\mathcal{C}}$ needs not be locally small. One sufficient condition for $\mathcal{D}^{\mathcal{C}}$ to be locally small is that $\mathcal{C}$ is small and $\mathcal{D}$ is locally small: Given two functors $F, G \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$, then

$$\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G) \subset \prod_{c \in \mathrm{Obj}(\mathcal{C})} \mathrm{Hom}_{\mathcal{D}}(Fc, Gc),$$

since a natural transformation $\alpha : F \Rightarrow G$ may be regarded as an element $(\alpha_c)_{c \in \mathrm{Obj}(\mathcal{C})} \in \prod_{c \in \mathrm{Obj}(\mathcal{C})} \mathrm{Hom}_{\mathcal{D}}(Fc, Gc)$.

Before entering the horizontal composition, we may want to introduce to the reader the concept of equivalences of categories.

Definition 1.22 (Equivalence of Categories). Given categories $\mathcal{C}$ and $\mathcal{D}$. An **equivalence of categories** (between $\mathcal{C}$ and $\mathcal{D}$) consists of two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\eta : 1_{\mathcal{C}} \cong GF$ and $\epsilon : FG \cong 1_{\mathcal{D}}$. If there exists an equivalence between $\mathcal{C}$ and $\mathcal{D}$, then $\mathcal{C}$ and $\mathcal{D}$ are said to be **equivalent**.

Equivalent between categories *is* an equivalence relation. The reflexive and symmetry are immediate, but a direct proof to the transitivity is difficult. We may want to use some properties of functors to avoid that:

Definition 1.23 (Full, Faithful and Essentially Surjective on Objects). A functor $F : \mathcal{C} \to \mathcal{D}$ is

- **full** if for each $x, y \in \mathrm{Obj}(\mathcal{C})$, the map $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy) : f \mapsto Ff$ is surjective;
- **faithful** if for each $x, y \in \mathrm{Obj}(\mathcal{C})$, the map $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy) : f \mapsto Ff$ is injective;
- **essentially surjective on objects** if for every object $d \in \mathrm{Obj}(\mathcal{D})$, there exists some $c \in \mathrm{Obj}(\mathcal{C})$ s.t. $d \cong Fc$.

One can verify easily that the composition of two full (faithful, or essentially surjective on objects) functors is again full (faithful, or essentially surjective on objects, respectively). Therefore, the theorem below yields the transitivity of equivalent between categories:

Theorem 1.4 (Characterizing of Equivalences of Categories). *Functors in an equivalence of categories are full, faithful, and essentially surjective on objects. Conversely, any functor with these properties produces an equivalence of categories.*[10] *In particular, two categories are equivalent if and only if there is a functor between them which is full, faithful, and essentially surjective on objects.*

[10] Here we assumed the axiom of choice.

The proof of this theorem used many techniques of diagram chasing, and is too long to be put in this thesis, for it won't be what we mainly concern about. For a complete proof, see Theorem 1.5.9, Riehl, *Category Theory in Context*.

For instance, this characterization applies to explain why matrices is so closely linked with linear algebra: the category of matrices $\mathrm{Mat}_{\mathbb{F}}$, where objects are positive integers (dimension of vector spaces) and a morphism from n to m is a $m \times n$ matrix with entries in $\mathbb{F}$ (linear maps), composition is the matrix multiplication by left (composition of linear maps), is equivalent to the category of finite-dimensional vector spaces $\mathrm{Vect}_{\mathbb{F}}$. In fact, $\mathrm{Mat}_{\mathbb{F}}$ is the skeleton of $\mathrm{Vect}_{\mathbb{F}}$, see Definition 3.12.

Remark 2. For locally small categories, one may regard that two categories are equivalent if and only if they are isomorphic in the category of locally small categories with morphisms the equivalence classes of functors modulo the equivalence relation of naturally isomorphic, i.e., the category in which:

- Objects: All locally small categories.
- Morphisms: All equivalence classes of functors between these categories modulo naturally isomorphic. i.e., a morphism is a collection $[F]$ consists of all functors naturally isomorphic to F.
- Composition law: Given two morphisms $[F]$ and $[G]$ with F and G composable, then

$$[G][F] := [GF].$$

We shall see immediately that the composition in this category is well-defined (i.e., if $F \cong F'$ and $G \cong G'$, then $GF \cong G'F'$), after the construction of horizontal composition is given.

Here comes the horizontal composition:

Lemma 1.5 (Horizontal Composition). *Given everything in the diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

then the ***horizontal composition*** $\beta * \alpha$ *of* α *and* β*, defined by the equation*

$$(\beta * \alpha)_c := (\beta_{Gc})(H\alpha_c) = (K\alpha_c)(\beta_{Fc}),$$

i.e., the diagonal of the commutative diagram

$$\begin{tikzcd} HFc \arrow[r, "\beta_{Fc}"] \arrow[d, "H\alpha_c"'] \arrow[dr, dashed, "(\beta*\alpha)_c"] & KFc \arrow[d, "K\alpha_c"] \\ HGc \arrow[r, "\beta_{Gc}"'] & KGc \end{tikzcd}$$

is a natural transformation from HF *to* KG*.*

Proof. Again, it follows immediately from the fact that the diagram

$$\begin{tikzcd} HFc \arrow[r, "H\alpha_c"] \arrow[d, "HFf"'] \arrow[rr, dashed, bend left, "(\beta*\alpha)_c"] & HGc \arrow[r, "\beta_{Gc}"] \arrow[d, "HGf"'] & KGc \arrow[d, "KGf"] \\ HFc' \arrow[r, "H\alpha_{c'}"'] \arrow[rr, dashed, bend right, "(\beta*\alpha)_{c'}"'] & HGc' \arrow[r, "\beta_{Gc'}"'] & KGc' \end{tikzcd}$$

commutes for any morphism $f : c \to c'$ of $\mathcal{C}$, as one can verify. $\square$

Finnally, and importantly, vertical and horizontal composition are compatible: the order how the composition is done does not matter. That is, they satisfy the rule of **middle four interchange**:

Lemma 1.6 (Middle Four Interchange). *Given functors and natural transformations*

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha"] \arrow[r, phantom, shift right=4, "\Downarrow\beta"] & \mathcal{D} \arrow[r, bend left=60, "J"] \arrow[r, "L" description] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, shift left=4, "\Downarrow\gamma"] \arrow[r, phantom, shift right=4, "\Downarrow\delta"] & \mathcal{E}
\end{tikzcd}
$$

the two natural transformations $(\delta \cdot \gamma) * (\beta \cdot \alpha), (\gamma * \alpha) \cdot (\delta * \beta) : JF \Rightarrow LH$ *are exactly the same:*

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow \beta\cdot\alpha"] & \mathcal{D} \arrow[r, bend left=50, "J"] \arrow[r, bend right=50, "L"'] \arrow[r, phantom, "\Downarrow \delta\cdot\gamma"] & \mathcal{E}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "JF"] \arrow[r, "KG" description] \arrow[r, bend right=60, "LH"'] \arrow[r, phantom, shift left=4, "\Downarrow\gamma*\alpha"] \arrow[r, phantom, shift right=4, "\Downarrow\delta*\beta"] & \mathcal{E}
\end{tikzcd}
$$

Proof. It is nothing but the fact that the diagram

$$
\begin{tikzcd}
JFc \arrow[d, "J\alpha_c"] \arrow[dr, dashed, "(\gamma*\alpha)_c"] \arrow[dd, dashed, bend right, "J(\beta\cdot\alpha)_c"'] \arrow[ddrr, dashed, bend left=60, "(\delta\cdot\gamma)*(\beta\cdot\alpha)_c"] & & \\
JGc \arrow[r, "\gamma_{Gc}"] \arrow[d, "J\beta_c"] & KGc \arrow[d, "K\beta_c"] \arrow[dr, dashed, "(\delta*\beta)_c"] & \\
JHc \arrow[r, "\gamma_{Hc}"'] \arrow[rr, dashed, bend right, "(\delta\cdot\gamma)_{Hc}"'] & KHc \arrow[r, "\delta_{Hc}"'] & LHc
\end{tikzcd}
$$

commutes for any $c \in \mathrm{Obj}(\mathcal{C})$, as one can verify. $\square$

2 The Yoneda Lemma and Universal Property

2.1 Representable functor and the Yoneda lemma

Remind the functors we gave in Example 1.2.(i): given $c \in \mathrm{Obj}(\mathcal{C})$ where $\mathcal{C}$ is locally small, then there are two functors $\mathrm{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{Set}$ and $\mathrm{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \to \mathrm{Set}$. We called them functors represented by c. However, the concept of representable is able to be generalized, and these two functors are so special and so important that we have much to say even for the generalized concept. In this seciton we give the definition of representable functors and raise a few questions to be solved in the next section. We will give an introduction to their solutions in this section, though: we will introduce the Yoneda lemma.

Definition 2.1 (Representable Functors). A covariant (or contravariant) functor F from a locally small category $\mathcal{C}$ to Set is said to be **representable** if there exists an object $c \in \mathrm{Obj}(\mathcal{C})$ s.t. there is a natural isomorphism between F and the functor $\mathrm{Hom}_{\mathcal{C}}(c, -)$ (or $\mathrm{Hom}_{\mathcal{C}}(-, c)$, respectively), in which case we say that the functor F is **represented by** the object c. A **representation** for a representable covariant (or contravariant) functor F is a choice of $c \in \mathrm{Obj}(\mathcal{C})$ together with a specified natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(c, -) \cong F$ (or $\mathrm{Hom}_{\mathcal{C}}(-, c) \cong F$, respectively).

Note that the domain $\mathcal{C}$ of a representable functor is required to be locally small, so that the hom-functors $\mathrm{Hom}_{\mathcal{C}}(c, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, c)$ do send objects of $\mathcal{C}$ to sets.

Example 2.1. Here comes some examples of representable functors which we have already got familiar with.

(i) The identity functor $1_{\mathrm{Set}} : \mathrm{Set} \to \mathrm{Set}$ is represented by the singleton set 1. The natural isomorphism $\mathrm{Hom}_{\mathrm{Set}}(1, -) \cong 1_{\mathrm{Set}}$ consists of maps $\mathrm{Hom}_{\mathrm{Set}}(1, X) \to X$ that maps each element in $\mathrm{Hom}_{\mathrm{Set}}(1, X)$ to the unique element in its image in X. One can verify easily that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Set}}(1, X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & X \arrow[d, "f"] \\ \mathrm{Hom}_{\mathrm{Set}}(1, Y) \arrow[r, "\cong"'] & Y \end{tikzcd}$$

commutes. For this reason, one may denote an element in $\mathrm{Hom}_{\mathrm{Set}}(1, X)$ by its image, i.e., $x \in X$ may also denote the function $x : 1 \to X : 1 \mapsto x$.

(ii) The forgetful functor $U : \mathrm{Grp} \to \mathrm{Set}$ is represented by the additive group $\mathbb{Z}$. The natural isomorphism $\mathrm{Hom}_{\mathrm{Grp}}(\mathbb{Z}, -) \cong U$ consists of bijections $\mathrm{Hom}_{\mathrm{Grp}}(\mathbb{Z}, G) \to UG : \varphi \mapsto \varphi(1)$. The bijectivity follows from that a group homomorphism from $\mathbb{Z}$ to G is determined by the image of 1 the generator of $\mathbb{Z}$. We will see that $\mathbb{Z}$ is the **free group on a single generator** after we define free groups.

(iii) *For a generalization to (ii): Given an index set J. The functor $U(-)_J : \mathrm{Grp} \to \mathrm{Set}$ that sends a group G to the set of J-tuples of elements of G is represented by the **free group** $F(J)$ **on** J. Similarly, the functor $U(-)_J : \mathrm{Ab} \to \mathrm{Set}$ is represented by the **free abelian group** $\bigoplus_{j \in J} \mathbb{Z}_j$ **on** J.[11]

As the notion of representation is given, here raises a few questions:

- If two objects represent a same functor, are they isomorphic?
- What data is involved in the construction of a natural isomorphism in the representation of a functor F?
- *To reader who has known something about universal properties: we assert that the universal property of an object in a locally samll category can be expressed by representable functors. How do the universal property expressed by functors relate to initial and terminal objects?

The answer to the first question is "yes". One may prove it by hand right now, but we shall not put the proof here; it happens to be an immediate result of Yoneda lemma. The Yoneda lemma also provides insights for the other two questions. In fact, the Yoneda lemma is arguably the most important result in category theory:[12]

Theorem 2.1. *(Yoneda lemma) For any functor $F : \mathcal{C} \to \mathrm{Set}$ whose domain $\mathcal{C}$ is locally small, for any object $c \in \mathrm{Obj}(\mathcal{C})$, the function $\Phi : \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(c, -), F) \to Fc : \alpha \mapsto \alpha_c(1_c)$ is a bijection, concluding that*

$$\mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(c, -), F) \cong Fc,$$

hence $\mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(c, -), F)$ is a set. Moreover, Φ is natural with respect to both c and F, i.e., seen as the component of a natural isomorphism at the object $(c, F) \in \mathrm{Obj}(\mathcal{C} \times \mathrm{Set}^{\mathcal{C}})$, it consists a natural isomorphism between bifunctors:

$$\mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(\circ, -), \diamond) : \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \to \mathrm{Set} : (c, F) \mapsto \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(c, -), F)$$

and

$$ev : \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \to \mathrm{Set} : (c, F) \mapsto Fc.$$

[11] $\bigoplus_{j \in J} \mathbb{Z}_j$ is denoted as $\mathbb{Z}^{\oplus J}$ in *Algebra: Chapter 0*

[12] "... although it takes some time to explore the depths of the consequences of this simple statement." – Emily Riehl.

Proof. The main point is to construct the inverse $\Psi : Fc \to \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(c,-),F)$ of Φ. Elements in the codomain of Ψ are natural transformations from $\mathrm{Hom}_{\mathcal{C}}(c,-)$ to F, hence given $x \in Fc$, we proceed by defining each component of the natural transformation $\Psi(x) : \mathrm{Hom}_{\mathcal{C}}(c,-) \Rightarrow F$. In order to make Ψ the inverse of Φ, there must be $\Psi(x)_c(1_c) = \Phi(\Psi(x)) = x$. For any $d \in \mathrm{Obj}(\mathcal{C})$, if $\mathrm{Hom}_{\mathcal{C}}(c,d) = \varnothing$, then we simply let $\Psi(x)_d$ be the empty function; if $\mathrm{Hom}_{\mathcal{C}}(c,d)$ is non-empty, then for any $f \in \mathrm{Hom}_{\mathcal{C}}(c,d)$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(c,c) \arrow[r, "\Psi(x)_c"] \arrow[d, "f_*"'] & Fc \arrow[d, "Ff"] \\
\mathrm{Hom}_{\mathcal{C}}(c,d) \arrow[r, "\Psi(x)_d"'] & Fd
\end{tikzcd}$$

must commute. In particular, $\Psi(x)_d(f) = \Psi(x)_d(f_*(1_c)) = Ff \circ \Psi(x)_c(1_c) = Ff(x)$. Therefore, we have defined Ψ:

$$\Psi : Fc \to \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(c,-),F) \qquad \Psi(x)_d(f) \coloneqq Ff(x), \quad \forall f \in \mathrm{Hom}_{\mathcal{C}}(c,d), \quad \forall d \in \mathrm{Obj}(\mathcal{C}), \quad \forall x \in Fc.$$

It remains to verify that $\Psi(x)$ is natural, and that $\Psi \circ \Phi(\alpha) = \alpha$. Both are straightforward, hence are left to the reader; note that a natural transformation from $\mathrm{Hom}_{\mathcal{C}}(c,-)$ to F is determined by the image of 1_c of its component at c, as we have shown in the construction of Ψ.

The naturality of Φ is again nothing but to check the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(c,-),F) \arrow[r, "\Phi_{(c,F)}"] \arrow[d, "{(f^{**},\alpha_*)}"'] & Fc \arrow[d, "(\alpha_d)(Ff)=(Gf)(\alpha_c)"] \\
\mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(d,-),G) \arrow[r, "\Phi_{(d,G)}"'] & Gd
\end{tikzcd}$$

commutes, hence is left to the reader.[13] $\square$

There is a dual version of the Yoneda lemma, in which the functor F is a contravariant functor from $\mathcal{C}$ to Set, but there is no need to list it alone: it can be accessed immediately by applying the original Yoneda lemma to the functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ (seen as a covariant functor with domain $\mathcal{C}^{op}$), and using the fact that $\mathrm{Hom}_{\mathcal{C}^{op}}(c,-) = \mathrm{Hom}_{\mathcal{C}}(-,c)$.

To emphasis, the Yoneda lemma tells us that there are only a set's worth of natural transformations between F and $\mathrm{Hom}_{\mathcal{C}}(c,-)$. An immediate application of the Yoneda lemma gives the Yoneda embeddings:

Corollary 2.1.1 (Yoneda embedding). *The functors*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, hook, "y"] & & \mathrm{Set}^{\mathcal{C}^{op}} \\
c \arrow[d, "f"'] & & \mathrm{Hom}_{\mathcal{C}}(-,c) \arrow[d, "f_*"] \\
d & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(-,d)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{op} \arrow[rr, hook, "y"] & & \mathrm{Set}^{\mathcal{C}} \\
c \arrow[d, "f"'] & & \mathrm{Hom}_{\mathcal{C}}(c,-) \\
d & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(d,-) \arrow[u, "f^*"']
\end{tikzcd}$$

are both full and faithful. They are called the covariant and contravariant ***Yoneda embeddings****.*

As a simple application of the Yoneda embeddings, we may reprove the Cayley's theorem:

Corollary 2.1.2 (Cayley's Theorem). *Any group is isomorphic to a subgroup of a permutation group.*

Proof. Given a group G, regard it as the one-object category $\mathrm{B}G$. The covariant Yoneda embedding tells us that the embedding

$$\begin{tikzcd}
\mathrm{B}G \arrow[rr, hook, "y"] & & \mathrm{Set}^{\mathrm{B}G^{op}} \\
\bullet \arrow[d, "g"'] & & \mathrm{Hom}_{\mathrm{B}G}(-,\bullet) \arrow[d, "g_*"] \\
\bullet & \longmapsto & \mathrm{Hom}_{\mathrm{B}G}(-,\bullet)
\end{tikzcd}$$

is full and faithful. Note that the forgetful functor $U : \mathrm{Set}^{\mathrm{B}G^{op}} \to \mathrm{Set} : F \mapsto F\bullet$ is also faithful. Since $\mathrm{Hom}_{\mathrm{B}G}(\bullet,\bullet) = G$, the composition Uy yields that any G-equivariant[14] endomorphism on the G-set G is an automorphism in the category Set, and that the set G_{eq} of G-equivariant endomorphisms on the G-set G endowed with the binary operation of composition of set-functions is a group isomorphic to G. The fact that G_{eq} is a subgroup of $\mathrm{Aut}_{\mathrm{Set}}(G) = \mathrm{Sym}(G)$ finishes our proof. $\square$

[13] For a complete proof of the Yoneda lemma, see Emily Riehl, *Category Theory in Context*, Theorem 2.2.4.
[14] See Example 1.4.(iii)

2.2 Universal property

In this section we introduce the universal property. We shall see that the idea of universal property has already played an important role (which we haven't discovered, though) in our previous study of abstract algebra, and it can not (and should not) be avoid if we want to proceed further. We will focus only on the universal properties of objects in locally small categories, since non-locally-small categories tend to be rare in many other mathematics. Also, for locally small categories, we have a very fancy definition for the universal property. However, to begin with, we will still introduce the general definition of universal property to the reader. It tends to be ambiguous to people who never saw it before, though:

Definition 2.2 (Universal Property). A certain mathematical object (or construction) is said to satisfy (or have) a **universal property**, if it could be seen as (a part of) an initial or terminal object of some other category. In particular, an initial or terminal object automatically satisfies an evident universal property.

Here by a mathematical object we mean things to be operated, such as numbers, sets, groups, etc. An explicit definition for mathematical objects is the job for philosophy, hence we shall not be bothered with that. Also, by "could be seen" we usually leave the object unchanged but attatch some constructions on it and generalize the whole to be objects of a category, with morphisms induced from morphisms in the category where the original object is taken from. When we say something satisfies a universal property, the context should be clear enough for the reader to figure out in what kind of category is the object universal. Now we give some examples for expressing universal properties.

Example 2.2. (i) Given a group homomorphism $\varphi : G \to H$, its kernel is universal with respect to the following property: for any group homomorphism $g : K \to G$ s.t. $\varphi \circ g = 0$, there exists a unique group homomorphism $\tilde{g} : K \to \ker \varphi$ s.t. $i \circ \tilde{g} = g$, where $i : \ker \varphi \to G$ is the inclusion. In other words, $\ker \varphi$ could be seen as the terminal object in the category where

- Objects: Group homomorphisms $g : K \to G$ s.t. $\varphi \circ g = 0$;
- Morphisms: Given two objects $f : W \to G$, $g : K \to G$, a morphism from f to g is a group homomorphism $h : W \to K$ s.t. $g \circ h = f$, i.e., the diagram
$$\begin{tikzcd} W \arrow[r, "f"] \arrow[dr, "h"'] & G \\ & K \arrow[u, "g"'] \end{tikzcd}$$
commutes.

(ii) Given a group G, then its **abelianization** $\tilde{G} := G/[xyx^{-1}y^{-1} : x, y \in G]$, where $[xyx^{-1}y^{-1} : x, y \in G]$ stands for the least normal group containing $\{xyx^{-1}y^{-1} : x, y \in G\}$, is universal with respect to the following property: for any group homomorphism $\varphi : G \to A$ where A is abelian, there exists a unique group homomorphism $\tilde{\varphi} : \tilde{G} \to A$ s.t. $\tilde{\varphi} \circ \pi = \varphi$, where $\pi : G \to \tilde{G}$ is the quotient map. In other words, $\tilde{G}$ could be seen as the initial object in the category where

- Objects: Group homomorphisms $g : G \to A$ where A is abelian;
- Morphisms: Given two objects $f : G \to H$, $g : G \to A$, a morphism from f to g is a group homomorphism $h : H \to A$ s.t. $h \circ f = g$, i.e., the diagram
$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "f"'] & A \\ H \arrow[ur, "h"'] & \end{tikzcd}$$
commutes.

(iii) Given two sets A and B, then their cartesian product $A \times B$ is universal with respect to the following property: for any two set-functions $f : C \to A$, $g : C \to B$, there exists a unique function $h : C \to A \times B$ s.t. $\pi_A \circ h = f$ and $\pi_B \circ h = g$, where $\pi_A : A \times B \to A : (a, b) \mapsto a$, $\pi_B : A \times B \to B : (a, b) \mapsto b$ are called the projection maps. In other words, $A \times B$ could be seen as the terminal object in the category where

- Objects: Triples (f, g, C), where $f : C \to A$, $g : C \to B$ are set functions, C stands for an arbitrary set;
- Morphisms: Given two objects (f_1, g_1, C_1), (f_2, g_2, C_2), a morphism from (f_1, g_1) to (f_2, g_2) is a set function $h : C_1 \to C_2$ s.t. $f_2 \circ h = f_1$, $g_2 \circ h = g_1$, i.e. the diagram
$$\begin{tikzcd} & & A \\ C_1 \arrow[r, "h"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "g_1"'] & C_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & \\ & & B \end{tikzcd}$$
commutes.

The cartesian product of sets is a special case of product. The product is a very important universal property, which we shall explore in the next section.

There are many other familiar examples, such as the kernel of linear maps, the product of groups, the product topology, etc. All of which, along with those have been listed above, belong to a kind of special universal property, called limits and colimits. Hence we pause here, and leave a further discussion to the next section. We have already known that initial (or terminal) objects in a certain category are all isomorphic canonically, hence we can use the universal property of an object to re-define the object itself, which gives a definition up to canonical isomorphic. In fact, that is exactly what we are doing most of the time, even without knowing an object priorily. Note that we need to verify the existence of such object if we define something new using universal property.

Example 2.3. Here comes some examples for defining things using universal property.

(i) Given a group homomorphism $\varphi : G \to H$. The **cokernel** of φ, denoted by $\operatorname{coker}\varphi$, is the codomain of the initial object in the category where

- Objects: All group homomorphisms $f : H \to K$ s.t. $f \circ \varphi = 0$;
- Morphisms: Given two objects $f_1 : H \to K_1$, $f_2 : H \to K_2$, a morphism from f_1 to f_2 is a group homomorphism $h : K_1 \to K_2$ s.t. $h \circ f_1 = f_2$, i.e., the diagram
$$\begin{tikzcd} H \arrow[r, "f_1"] \arrow[dr, "f_2"'] & K_1 \arrow[d, "h"] \\ & K_2 \end{tikzcd}$$
commutes.

The reader may verify that $H/[\operatorname{Im}\varphi]$ is the cokernel of φ, using some abstract algebra.

(ii) Given a set A. The **free group** on A, denoted by $F(A)$, is the codomain of the initial object in the category where

- Objects: All set-functions $f : A \to G$, where G is a group;
- Morphisms: Given two objects $f : A \to G$ and $g : A \to H$, a morphism from f to g is a group homomorphism $h : G \to H$ s.t. $h \circ f = g$, i.e., the diagram
$$\begin{tikzcd} G \arrow[r, "h"] & H \\ A \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$
commutes.

(iii) Given a set A. The **free abelian group** on A, denoted by $F^{ab}(A)$, is the codomain of the initial object in the category where

- Objects: All set-functions $f : A \to G$, where G is an abelian group;
- Morphisms: Given two objects $f : A \to G$ and $g : A \to H$, a morphism from f to g is a group homomorphism $h : G \to H$ s.t. $h \circ f = g$, i.e., the diagram
$$\begin{tikzcd} G \arrow[r, "h"] & H \\ A \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$
commutes.

For an explicit construction for free groups and free abelian groups, see either §5, Chapter II, *Algebra: Chapter 0* or §67 and §69, Chapter 11, *Topology*. One will see that $F^{ab}(A) \cong \oplus_{j\in A}\mathbb{Z}_j$, where the latter stands for the direct sum of $\mathbb{Z}$ (see §67, *Topology*, or see the next section), and is also denoted by $\mathbb{Z}^{\oplus A}$, hence we may refer to the free abelian group on A simply by $\mathbb{Z}^{\oplus A}$.

Remark 3. Free groups provide us a new way to deal with groups: every group can be seen as (up to isomorphic) a free group modulo a normal subgroup. If $G \cong F(A)/R$, then $F(A)/R$ along with the isomorphism is called a **presentation** of group G. A group might have a number of very different presentations, while there must be at least one presentation for a group: the free group on the underlying set of group G, by the first isomorphism theorem, will do. See the diagram below.

$$\begin{tikzcd} F(G) \arrow[r, dashed, two heads, "\exists!"] & G \\ G \arrow[u, "j"] \arrow[ur, "id_G"'] & \end{tikzcd}$$

Now we may be ready for our fancy definition of universal property on locally small categories. Recall the bijective relation $\operatorname{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\operatorname{Hom}_{\mathcal{C}}(c, -), F) \cong Fc$ in the Yoneda lemma:

Definition 2.3 (Universal Property). A **universal property** of an object $c \in \operatorname{Obj}(\mathcal{C})$ is (expressed by) a representation by c of a functor F. More explicitly, a universal property of an object $c \in \mathcal{C}$ is a pair (F, x) where F is a representable functor and $x \in Fc$ gives a natural isomorphism $\operatorname{Hom}_{\mathcal{C}}(c, -) \cong F$ or $\operatorname{Hom}_{\mathcal{C}}(-, c) \cong F$ via the Yoneda lemma.

In particular, $\operatorname{Hom}_{\mathcal{C}}(c, -)$ (or $\operatorname{Hom}_{\mathcal{C}}(-, c)$) is a universal property of $c \in \operatorname{Obj}(\mathcal{C})$ (be aware of the choice of which category $\mathcal{C}$ the object c is in), and the reader should have no difficulty translating the general definition to this fancy one; examples will be given right away. To translate this fancy definition to the general one, that is, to find a category and an initial or terminal object out from a representable functor (along with its representation), we need to establish a special kind of category, called **the category of elements**.

Example 2.4. We first gives some example of translations by pointing out the representable functor and its representation; the details are left to the reader.

(i) Given a group homomorphism $\varphi : G \to H$, the universal property of its kernel is expressed by the functor $F : \mathrm{Grp} \to \mathrm{Set} : K \mapsto \{f \in \mathrm{Hom}_{\mathrm{Grp}}(K, G) : \varphi \circ f = 0\}$, which is represented by $\mathrm{Hom}_{\mathrm{Grp}}(-, \ker \varphi)$. The universal element is the inclusion map $j : \ker \varphi \hookrightarrow G$.

(ii) Given a group G, then the universal property of its commutation $\tilde{G}$ is expressed by the functor $F : \mathrm{Ab} \to \mathrm{Set} : H \mapsto \mathrm{Hom}_{\mathrm{Grp}}(G, H)$, which is represented by $\mathrm{Hom}_{\mathrm{Ab}}(\tilde{G}, -)$. The universal element is the quotient map $\pi : G \to \tilde{G}$.

(iii) Given two sets A and B, the universal property of their product $A \times B$ is expressed by the functor $F : \mathrm{Set} \to \mathrm{Set} : S \mapsto \{(f, g) : f \in \mathrm{Hom}_{\mathrm{Set}}(S, A), g \in \mathrm{Hom}_{\mathrm{Set}}(S, B)\}$, which is represented by the $\mathrm{Hom}_{\mathrm{Set}}(-, A \times B)$. The universal element is the pair of projection maps (π_A, π_B).

Definition 2.4 (Category of Elements). (Covariant) The **category of elements** of a covariant functor $F : \mathcal{C} \to \mathrm{Set}$, denoted by $\int F$, consists of

- Objects: All pairs (c, x) where $c \in \mathrm{Obj}(\mathcal{C})$ and $x \in Fc$;
- Morphisms: Given two objects (c, x) and (c', x'), a morphism $(c, x) \to (c', x')$ is a morphism $f : c \to c'$ of $\mathcal{C}$ s.t. $Ff(x) = x'$.

(Contravariant) The **category of elements** of a contravariant functor $F : \mathcal{C}^{op} \to \mathrm{Set}$, denoted by $\int F$, consists of

- Objects: All pairs (c, x) where $c \in \mathrm{Obj}(\mathcal{C})$ and $x \in Fc$;
- Morphisms: Given two objects (c, x) and (c', x'), a morphism $(c, x) \to (c', x')$ is a morphism $f : c \to c'$ of $\mathcal{C}$ s.t. $Ff(x') = x$.

The proposition below ends our translation question:

Proposition 2.1. *A covariant set-valued (i.e., its codomain is* Set*) functor is representable if and only if its category of elements has an initial object. Dually, a contravariant set-valued functor is representable if and only if its category of elements has a terminal object. Explicitly, the representation of a functor is initial (or terminal) in its category of elements.*

Proof. By duallity, we only prove the case where the functor $F : \mathcal{C} \to \mathrm{Set}$ is covariant. The necessity is easy to see: given representation $\alpha : \mathrm{Hom}_{\mathcal{C}}(c, -) \cong F$, then for any $(d, x) \in \mathrm{Obj}(\int F)$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c, c) & \xrightarrow{\alpha_c} & Fc \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle Ff} \\ \mathrm{Hom}_{\mathcal{C}}(c, d) & \xrightarrow{\alpha_d} & Fd \end{array}$$

commutes. We assert that $(c, \alpha_c(1_c))$ is initial. It suffices to show that there exists a unique $f : c \to d$ s.t. $Ff(\alpha_c(1_c)) = x$. By the commutativity, $Ff(\alpha_c(1_c)) = \alpha_d(f)$, and we are done by the bijectivity of α_d.
Now given an initial object (c, x) of $\int F$, we assert that the natural transformation $\alpha : \mathrm{Hom}_{\mathcal{C}}(c, -) \Rightarrow F$ given by x via the Yoneda lemma is a natural isomorphism. In particular, $\alpha_c(1_c) = x$. For any $c' \in \mathrm{Obj}(\mathcal{C})$, we show that $\alpha_{c'}$ is a bijection. Again, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c, c) & \xrightarrow{\alpha_c} & Fc \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle Ff} \\ \mathrm{Hom}_{\mathcal{C}}(c, c') & \xrightarrow{\alpha_{c'}} & Fc' \end{array}$$

commutes. For any $x' \in Fc'$, since $(c, \alpha_c(1_c))$ is initial, there exists a unique $f : c \to c'$ s.t. $Ff(\alpha_c(1_c)) = x'$. Since $Ff(\alpha_c(1_c)) = \alpha_{c'}(f)$, the existence of such f implies that $\alpha_{c'}$ is surjective, and the uniqueness implies that $\alpha_{c'}$ is injective. $\square$

Before entering the next chapter, the reader is suggested to raise some familiar examples of universal property by hand, verify both the general definition and the fancy definition and write out the category of elements, comparing it with the category in the general definition.

3 Limits and Colimits

3.1 Basic Notions

Limits and colimits are special kinds of universal property, but of significant importance and have much better behaviour. Almost every universal property that occurs in a same category can be translated into a limit or colimit, and they can be even seen as functors if conditions provide. To start with, we need to do some preliminaries.

Definition 3.1 (Diagram). A **diagram** of **shape** $\mathcal{J}$ in a category $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$. A diagram is **small** if its **index category** $\mathcal{J}$ is small.

Recall that given an object $c \in \mathrm{Obj}(\mathcal{C})$, it induces a constant functor $c : \mathcal{J} \to \mathcal{C}$ (Example 1.2 (ii)).

Definition 3.2 (Cone). A **cone over** a diagram $F : \mathcal{J} \to \mathcal{C}$ with **summit** or **apex** $c \in \mathrm{Obj}(\mathcal{C})$ is a natural transformation $\lambda : c \Rightarrow F$. The components $(\lambda_j : c \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ of the natural transformation are called the **legs** of the cone. Alternatively, a cone over $F : \mathcal{J} \to \mathcal{C}$ is a choice, for each $j \in \mathrm{Obj}(\mathcal{J})$, a morphism $\lambda_j : c \to Fj$, such that the diagram

$$\begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ Fj \arrow[rr, "Ff"'] & & Fk \end{tikzcd}$$

commutes for every $j \xrightarrow{f} k \in \mathcal{J}$.

Dually, a **cone under** F with **nadir** c is a natural transformation $\lambda : F \Rightarrow c$, whose **legs** are the components $(\lambda_j : Fj \to c)_{j \in \mathrm{Obj}\,\mathcal{J}}$. Also, it could been seen as a choice of morphisms $\lambda_j : Fj \to c$ for each $j \in \mathrm{Obj}(\mathcal{J})$ such that the diagram

$$\begin{tikzcd} Fj \arrow[rr, "Ff"] \arrow[dr, "\lambda_j"'] & & Fk \arrow[dl, "\lambda_k"] \\ & c & \end{tikzcd}$$

commutes for every $j \xrightarrow{f} k \in \mathcal{J}$.

Cones under a diagram are also called **cocones**. In fact a cone under $F : \mathcal{J} \to \mathcal{C}$ is precisely a cone over its opposite functor $F^{op} : \mathcal{J}^{op} \to \mathcal{C}^{op}$.

Since limits and colimits are special kinds of universal property, there are again two equivalent definitions, one by representable functors and one by the category of cones; both of them are based on the notion of cone. As in the discussion of universal property, the latter applies in all cases while the former applies only when the functor is valued in the category Set. To guarantee that our functor is set-valued, we shall keep the assumption that $\mathcal{J}$ is small and $\mathcal{C}$ is locally small, so that $\mathcal{C}^{\mathcal{J}}$ is locally small, whenever we don't mention that they are not.

Definition 3.3 (Limits and Colimits). For any diagram $F : \mathcal{J} \to \mathcal{C}$, there is a functor

$$\mathrm{Cone}(-, F) : \mathcal{C}^{op} \to \mathrm{Set}$$

that sends $c \in \mathrm{Obj}(\mathcal{C})$ to the set of cones over F with summit c, and a morphism $f : c \to d$ to the set function

$$\mathrm{Cone}(d, F) \to \mathrm{Cone}(c, F) : (\lambda_j : d \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}} \mapsto (\lambda_j f : c \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}.$$

A **limit** of F is a representation for $\mathrm{Cone}(-, F)$. That is, a limit consists of an object denoted as $\lim F \in \mathrm{Obj}(\mathcal{C})$ together with a universal cone (element) $\lambda : \lim F \Rightarrow F$, called the **limit cone**, defining via the Yoneda lemma the natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(-, \lim F) \cong \mathrm{Cone}(-, F)$$

Dually, there is a functor

$$\mathrm{Cone}(F, -) : \mathcal{C} \to \mathrm{Set}$$

that sends $c \in \mathrm{Obj}(\mathcal{C})$ to the set of cones under F with nadir c, and morphisms in a similar way. A **colimit** of F is a representation for $\mathrm{Cone}(F, -)$. That is, a colimit consists of an object denoted as $\mathrm{colim}\, F \in \mathrm{Obj}(C)$ together with a universal cone $\lambda : F \Rightarrow \mathrm{colim}\, F$, called the **colimit cone**, defining via the Yoneda lemma the natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, F, -) \cong \mathrm{Cone}(F, -)$$

Note that since a set-valued functor is not always representable, the limit of a diagram does not always exist.

Proposition 2.1 tells that the category of elements of the functors $\mathrm{Cone}(-, F)$ and $\mathrm{Cone}(F, -)$ also defines the notion of limits and colimits. One can see easily that the categories $\int \mathrm{Cone}(-, F)$ and $\int \mathrm{Cone}(F, -)$ are the same as the category of cones given below, while the latter does not rely on the set-valued functors $\mathrm{Cone}(-, F)$ and $\mathrm{Cone}(F, -)$, that is, it can be applied to the general cases where $\mathcal{J}$ and $\mathcal{C}$ are not assumed to be small and locally small.

Definition 3.4 (Category of Cones). Given a diagram $F : \mathcal{J} \to \mathcal{C}$, the **category of cones over** F consists of

- Objects: All cones over F with any summit.
- Morphisms: Given two cones $(\lambda_j : c \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ and $(\mu_j : d \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$, a morphism from λ to μ is a morphism $f : c \to d$ such that $\mu_j f = \lambda_j$ for all $j \in \mathrm{Obj}(\mathcal{J})$. Alternatively, a morphism from λ to μ is a morphism $f : c \to d$ such that the diagram

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[dr, "\lambda_j"'] & d \arrow[d, "\mu_j"] \\ & Fj \end{tikzcd}$$

  commutes for every $j \in \mathrm{Obj}(\mathcal{J})$
- Composition: Composition in $\mathcal{C}$ in the obvious way.

Dually, the **category of cones under** F consists of

- Objects: All cones under F with any nadir.
- Morphisms: Given two cones $(\lambda_j : Fj \to c)_{j \in \mathrm{Obj}\,\mathcal{J}}$ and $(\mu_j : Fj \to d)_{j \in \mathrm{Obj}\,\mathcal{J}}$, a morphism from λ to μ is a morphism $f : c \to d$ such that $f\lambda_j = \mu_j$ for all $j \in \mathrm{Obj}(\mathcal{J})$. Alternatively, a morphism from λ to μ is a morphism $f : c \to d$ such that the diagram

$$\begin{tikzcd} c \arrow[r, "f"] & d \\ Fj \arrow[u, "\lambda_j"] \arrow[ur, "\mu_j"'] & \end{tikzcd}$$

  commutes for every $j \in \mathrm{Obj}(\mathcal{J})$
- Composition: Composition in $\mathcal{C}$ in the obvious way.

Definition 3.5 (Limits and Colimits). Given a diagram $F : \mathcal{J} \to \mathcal{C}$, a **limit** of F is a terminal object in the category of cones over F. The terminal object $(\lambda_j : \lim F \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ is called the **limit cone** with summit denoted as $\lim F$. The terminal condition says exactly that for any cone $(\mu_j : c \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ over F there exists a unique morphism $f : c \to \lim F$ such that the diagram

$$\begin{tikzcd} c \arrow[r, dashed, "\exists! f"] \arrow[dr, "\mu_j"'] & \lim F \arrow[d, "\lambda_j"] \\ & Fj \end{tikzcd}$$

commutes for all $j \in \mathrm{Obj}(\mathcal{J})$.

Dually, a **colimit** is an initial object in the category of cones under F. The initial object $(\lambda_j : Fj \to \mathrm{colim}\, F)_{j \in \mathrm{Obj}\,\mathcal{J}}$ is called the **colimit cone** with nadir denoted as $\mathrm{colim}\, F$. The initial condition says exactly that for any cone $(\mu_j : Fj \to c)_{j \in \mathrm{Obj}\,\mathcal{J}}$ under F there exists a unique morphism $f : \mathrm{colim}\, F \to c$ such that the diagram

$$\begin{tikzcd} \mathrm{colim}\, F \arrow[r, dashed, "\exists! f"] & c \\ Fj \arrow[u, "\lambda_j"] \arrow[ur, "\mu_j"'] & \end{tikzcd}$$

commutes for every $j \in \mathrm{Obj}(\mathcal{J})$.

$\lim F$ and $\mathrm{colim}\, F$ might be also denoted by $\lim_{\mathcal{J}} F$ and $\mathrm{colim}_{\mathcal{J}} F$ if one wants to emphasis the shape of the diagram. As in the case of universal property, the definition up to terminal/initial objects asserts that limits and colimits are well-defined up to a unique isomorphism. In other words, the choice of a single limit cone rarely matters[15], so one can simply say "the" limit of a diagram without refering the choice, leading to no confusion. To conclude, we have the proposition below:

Proposition 3.1 (Essential Uniqueness of Limits and Colimits). *Given any two limit cones $\lambda : l \Rightarrow F$ and $\lambda' : l' \Rightarrow F$ over a common diagram F, there is a unique isomorphism $l \cong l'$ that commutes with the legs of the limit cones. The dual also holds for the colimit cones.* □

[15] We will see that when talking about the functoriality of limits and colimits, a choice issue does rise, but the effect of different choices tends to be weak. In fact, we will see that different choices give natural isomorphic functors.

A trivial example of limit is, consider the category $(\mathbb{R}, \leq)$ where the objects are real numbers and morphisms $a \leq b \Leftrightarrow a \to b$, a diagram in this category admits a limit if and only if it consists of a set of real numbers with a lower boundary, and its limit is exactly the greatest lower boundary. Similarly for the category $(\mathbb{R}, \geq) = (\mathbb{R}, \leq)^{op}$, the limit of a diagram is the least upper boundary of the set of real numbers in the diagram.

The data of a diagram, together with a limit cone over it, is called a **limit diagram** and the data of a diagram, together with a colimit cone under it, is called a **colimit diagram**. There are special names for limit and colimit diagrams of certain shapes.

Definition 3.6 (Product and Coproduct). A **product** is a limit of a diagram indexed by a discrete category[16]. A diagram in $\mathcal{C}$ indexed by a discrete category $\mathcal{J}$ is simply a collection of objects $Fj \in \mathrm{Obj}(\mathcal{C})$ indexed by $j \in \mathrm{Obj}(\mathcal{J})$. A cone over this diagram is simply a $\mathrm{Obj}(\mathcal{J})$-indexed family of morphisms $(\lambda_j : c \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ without any restriction such as commutating condition. The limit is typically denoted by $\prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj$ [17]and the legs are maps

$$\left(\pi_k : \prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj \to Fk\right)_{k \in \mathrm{Obj}\,\mathcal{J}}$$

called (**product**) **projections**.

Dually, a **coproduct** is a colimit of a diagram indexed by a discrete category. The colimit is typically denoted by $\coprod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj$ and the legs are maps

$$\left(\iota_k : Fk \to \coprod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj\right)_{k \in \mathrm{Obj}\,\mathcal{J}}$$

called (**coproduct**) **injections**.

Example 3.1. (i) The cartesian product gives an explicit construction for products in Set, and the disjoint union gives an explicit construction for coproducts in Set.

(ii) The cartesian product provides a construction for products in Top. For any set of topological spaces $(X_i)_{i \in I}$, the cartesian product $\prod_{i \in I} X_i$ gives a product of $(X_i)_{i \in I}$ as sets, and the universal property introduces a topology on $\prod_{i \in I} X_i$ so that it becomes a product of $(X_i)_{i \in I}$ as topological spaces, that is, the coarsest topology such that each projection π_i is continuous. In other words, the topology on $\prod_{i \in I} X_i$ is generated by subbasis $\bigcup_{i \in I} \{\pi_i^{-1}(U) \mid U \text{ is open in } X_i\}$, i.e., every open set in this topology is a union of finite intersections of the subbasis elements. Any cone over $(X_i)_{i \in I}$ in Top descends to a cone over $(X_i)_{i \in I}$ in Set, hence there exists a unique set-function from its summit to $\prod_{i \in I} X_i$ that is compatible with the projections. With the topology on $\prod_{i \in I} X_i$ given above, one can verify easily that this unique set-function lifts to a continuous function, concluding that the construction above does give a product in Top. This construction is in fact the classical definition of products in ordinarty topology theory.

(iii) The cartesian product also provides a construction for products in Grp, where the action of elements in the product is defined componentwisely. The reader should be very familiar with this construction, so let's just skip the description.

(iv) The product in Grp descends to product in Ab, however the coproduct does not. The coproduct in Grp is known as **free product**, with elements "words" of the original groups, see Munkres, *Topology*, §68. The coproduct in Ab is known as **direct sum**, with elements finite formal sums of the original groups, see Munkres, *Topology*, §67. For finite index category $\mathcal{J}$ (that is, it contains only finitely many morphisms), the product and coproduct coincident up to a canonical isomorphism[18] in Ab, while they differ in the infinite case.

Note that for any $c \in \mathrm{Obj}(\mathcal{C})$, $\prod_{k \in \mathrm{Obj}\,\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(c, Fk) = \mathrm{Cone}(c, F)$ as sets if we choose $\prod_{k \in \mathrm{Obj}\,\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(c, Fk)$ to be the cartesian product, post-composition by the projections then induces an isomorphism $\mathrm{Hom}_{\mathcal{C}}(c, \prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj) \cong \prod_{k \in \mathrm{Obj}\,\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(c, Fk)$ via the universal property of products. Such isomorphisms are natural in c as one can verify by hand, and are in fact components of the natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(-, \prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj) \cong \prod_{k \in \mathrm{Obj}\,\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(-, Fk)$ induced by post-composition of the projections provided that $\mathcal{C}$ is small. We will see in later discussion that the product $\prod_{k \in \mathrm{Obj}\,\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(-, Fk)$ makes sense for any small $\mathcal{C}$.

Definition 3.7 (Equalizer and Coequalizer). An **equalizer** is a limit of a diagram indexed by the **parallel pair**, the category $\bullet \rightrightarrows \bullet$ with two objects and two parallel non-identity morphisms. A diagram of this shape simply consists of a parallel pair of morphisms $f, g : A \rightrightarrows B$ in the target category $\mathcal{C}$. A cone over this diagram with summit C consists of a pair of morphisms $a : C \to A$ and $b : C \to B$ so that $fa = b$ and $ga = b$. This is equivalent to giving $a : C \to A$ such that $fa = ga$ and set $b \coloneqq fa = ga$, hence cones over the parallel pair are bijectively represented by morphisms $a : C \to A$ such that $fa = ga$, so we can ommit the morphism b in discussion of these cones, simply write the cone as

[16] Recall that a discrete category is a category with only identity morphisms.

[17] When it is clear that $\mathcal{J}$ is a discrete category, one may also denote the product by $\prod_{j \in \mathcal{J}} Fj$. When it is clear by the context that j is an object, one may drop the sign Obj and simply write $j \in \mathcal{J}$ to indicate that j is an object in the category $\mathcal{J}$.

[18] By this we mean that their summit and nadir coincident.

$$\begin{tikzcd} C \arrow[r, "a"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

and claim that the diagram commutes. The universal property of the equalizer $h : E \to A$ thus could be expressed by the commutative diagram below:

$$\begin{tikzcd} C \arrow[d, dashed, "\exists!"'] \arrow[dr, "a"] & & \\ E \arrow[r, tail, "h"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

where $a : C \to A$ is any morphism in $\mathcal{C}$ such that $fa = ga$.

Dually, a **coequalizer** is a colimit of a diagram indexed by the parallel pair category $\bullet \rightrightarrows \bullet$. Similarly, cones under the parallel pair are bijectively represented by morphisms $b : B \to C$ such that $bf = bg$, hence we can simply write the cone as

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "b"] & C \end{tikzcd}$$

and claim that the diagram commutes. The universal property of the coequalizer $h : B \twoheadrightarrow E$ can be expressed by

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "h"] \arrow[dr, "b"'] & E \arrow[d, dashed, "\exists!"] \\ & & C \end{tikzcd}$$

Note that for an equalizer the arrow $E \rightarrowtail A$ is always a monomorphism and for a coequalizer the arrow $B \twoheadrightarrow E$ is always an epimorphism, as one can verify.

Example 3.2. (i) The kernel and cokernel of a group homomorphism $\varphi : G \to H$ is the equalizer and coequalizer of the diagram

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & H \end{tikzcd}$$

where $0 : G \to H$ is the trivial map.

(ii) **Quotients** can be expressed by coequalizers. For example in Set, the coequalizer of the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

is the quotient of B modulo the relation $f(a) \sim g(a)$, $a \in A$. In other words, this tells the universal property of quotients.

We will see soon that any limit may be expressed as an equalizer of a pair of maps between two products, provided that the equalizer and products exist, and similarly the dual holds. This fact tells that it suffices in a category to show that all equalizers and products exist to show that all limits exist. However, we still have some other types of limits and colimits that are useful in application.

Definition 3.8 (Pullback and Pushout). A **pullback** is a limit of a diagram indexed by the poset category $\bullet \to \bullet \leftarrow \bullet$. Let $B \xrightarrow{f} A \xleftarrow{g} C$ be the image of the diagram in the target category $\mathcal{C}$, a cone over this diagram with summit D consists of three morphisms $a : D \to A$, $b : D \to b$ and $c : D \to C$ such that $a = fb = gc$. Hence as in discussion of equalizers the morphism a could be ommited and we can simply write the cone as

$$\begin{tikzcd} D \arrow[r, "c"] \arrow[d, "b"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

and claim that the diagram commutes. The universal property of pullback can be expressed by

$$\begin{tikzcd} D \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "c"] \arrow[ddr, bend right, "b"'] & & \\ & P \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ & B \arrow[r, "f"'] & A \end{tikzcd}$$

The symbol “⌟” indicates that the square $gk = fh$ is a pullback diagram, i.e., a limit diagram, instead of a common commutative square. The pullback P is also called the **fiber product** and is frequently denoted by $B \times_A C$.

Dually, a **pushout** is a colimit of a diagram indexed by the poset category $\bullet \leftarrow \bullet \rightarrow \bullet$. Similarly a cone under the diagram $B \xleftarrow{f} A \xrightarrow{g} C$ with nadir D can be written as the commutative square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "c"] \\ B \arrow[r, "b"'] & D \end{tikzcd}$$

and the universal property of pushout can be expressed by

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "h"] \arrow[ddr, bend left, "c"] & \\ B \arrow[r, "k"'] \arrow[drr, bend right, "b"'] \arrow[ur, phantom, "\ulcorner"] & P \arrow[dr, dashed, "\exists!"] & \\ & & D \end{tikzcd}$$

where the symbol “⌜” indicates that the commutative square $kf = hg$ is a pushout.

One can show that the assertion that limits may be expressed as an equalizer of a pair of maps between two products holds for pullbacks and the dual holds for pushouts right now. Consider the product $B \xleftarrow{\pi_B} B \times C \xrightarrow{\pi_C} C$, one can show that cones over $B \xrightarrow{f} A \xleftarrow{g} C$ correspond bijectively with cones over $\begin{tikzcd} B \times C \arrow[r, shift left, "g \circ \pi_C"] \arrow[r, shift right, "f \circ \pi_B"'] & A \end{tikzcd}$ and such bijective correspondence is compatible with morphisms between cones, concluding that the pullback of $B \xrightarrow{f} A \xleftarrow{g} C$ is the equalizer of $\begin{tikzcd} B \times C \arrow[r, shift left, "g \circ \pi_C"] \arrow[r, shift right, "f \circ \pi_B"'] & A \end{tikzcd}$. Similarly the pushout of $B \xleftarrow{f} A \xrightarrow{g} C$ is the coequalizer of $\begin{tikzcd} A \arrow[r, shift left, "\iota_C \circ g"] \arrow[r, shift right, "\iota_B \circ f"'] & B \coprod C \end{tikzcd}$.

Example 3.3. (i) In Grp, the pullback of $B \xrightarrow{f} A \xleftarrow{g} C$ is the equalizer of $\begin{tikzcd} B \times C \arrow[r, shift left, "g \circ \pi_C"] \arrow[r, shift right, "f \circ \pi_B"'] & A \end{tikzcd}$, hence is the subgroup $\{(b, c) \in B \times C \mid f(b) = g(c)\}$ of $B \times C$; the pushout of $B \xleftarrow{f} A \xrightarrow{g} C$ is the coequalizer of $\begin{tikzcd} A \arrow[r, shift left, "\iota_C \circ g"] \arrow[r, shift right, "\iota_B \circ f"'] & B * C \end{tikzcd}$ where $*$ stands for the free product, hence is the quotient $B * C/[\{(\iota_C \circ g(a))\,(\iota_B \circ f(a))^{-1} \mid a \in A\}]$.

(ii) The Seifert-van Kampen theorem for $X = U \cup V$ asserts that the diagram

$$\begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \\ \pi_1(V, x_0) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \pi_1(X, x_0) \end{tikzcd}$$

induced by inclusions is a pushout. See Munkres, *Topology*, §70.

(iii) The torus $T \cong S^1 \times S^1$ could be seen as the pushout

$$\begin{tikzcd} S^1 \arrow[r, "aba^{-1}b^{-1}"] \arrow[d, "i"'] & S^1 \vee S^1 \arrow[d, dashed] \\ D^2 \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner"] & T \end{tikzcd}$$

where i is the inclusion $S^1 \cong \partial D^2 \subset D^2$ and $aba^{-1}b^{-1}$ is wrapping S^1 once around one circle and then another, then once around the first circle and the second reversing the orientation. In fact this pushout tells the standard CW-complex structure of the torus.

Definition 3.9 (Inverse and Direct Limit). The limit of a diagram indexed by the category ω^{op} is called an **inverse limit** of a tower or a sequence of morphisms, where ω is the category of non-negative integers, see Example 1.1.(x). A diagram indexed by ω^{op} consists of a sequence of objects and morphisms

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0$$

together with identities and their composites, which are ommited. A cone over this diagram could be seen as a diagarm of shape $(\omega + \infty)^{op}$, the category of non-negative integers along with the infinity. Explicitly, a cone consists of a new object, the summit, with morphisms “all the way to the left” making every triangle commute:

$$\begin{tikzcd}
c \arrow[d, "\cdots"'] \arrow[dr, "\lambda_3"] \arrow[drr, "\lambda_2"] \arrow[drrr, "\lambda_1"] \arrow[drrrr, "\lambda_0"] & & & & \\
\cdots \arrow[r] & F_3 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0
\end{tikzcd}$$

The inverse limit, frequently denoted as $\varprojlim F_n$, is the terminal cone.

Dually, the limit of a diagram indexed by ω

$$F_0 \to F_1 \to F_2 \to F_3 \to \cdots$$

is called a **sequential colimit** or **direct limit**. A cone under this diagram could be seen as a diagram of shape $\omega + \infty$, and the direct limit, frequently denoted as $\varinjlim F_n$, is the initial cone.

Now we state and prove our promised assertion:

Theorem 3.1. *The limit of any diagram $F : \mathcal{J} \to \mathcal{C}$ may be expressed as a equalizer of a pair of morphisms between products*

$$\lim_{\mathcal{J}} F \rightarrowtail \prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj \underset{c}{\overset{d}{\rightrightarrows}} \prod_{f \in \mathrm{Mor}\,\mathcal{J}} F(\mathrm{cod}\, f)$$

provided that the two products exist.

Dually, the colimit of any diagram may be expressed as a coequalizer of a pair of morphisms between coproducts

$$\coprod_{f \in \mathrm{Mor}\,\mathcal{J}} F(\mathrm{dom}\, f) \underset{c}{\overset{d}{\rightrightarrows}} \coprod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj \twoheadrightarrow \mathrm{colim}_{\mathcal{J}} F$$

provided that the two coproducts exist.

There is an elegant proof for this in Riehl, *Category Theory in Context*, Theorem 3.4.12, using the representable nature of limits and colimits, but requiring $\mathcal{J}$ to be small and $\mathcal{C}$ to be locally small. Here we prove this by diagram chasing using only the universal property of products, which applies to the general cases where no restriction for $\mathcal{J}$ and $\mathcal{C}$ is required.

Proof. We only prove the limit case. The proof can be dualized to prove the colimit case. The idea is to show that the category $\mathcal{A}$ of cones over F is isomorphic to the category $\mathcal{B}$ of cones over $\prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj \underset{c}{\overset{d}{\rightrightarrows}} \prod_{f \in \mathrm{Mor}\,\mathcal{J}} F(\mathrm{cod}\, f)$. Let π be the projections, the morphisms d and c are defined via the universal property of products:

$$\begin{tikzcd}
\prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj \arrow[r, dashed, "\exists! d"] \arrow[dr, "\pi_{\mathrm{cod}\, f}"'] & \prod_{f \in \mathrm{Mor}\,\mathcal{J}} F(\mathrm{cod}\, f) \arrow[d, "\pi_f"] \\
& F(\mathrm{cod}\, f)
\end{tikzcd} \quad \text{commutes, } \forall f \in \mathrm{Mor}\,\mathcal{J}$$

and

$$\begin{tikzcd}
\prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj \arrow[r, dashed, "\exists! c"] \arrow[d, "\pi_{\mathrm{dom}\, f}"'] & \prod_{f \in \mathrm{Mor}\,\mathcal{J}} F(\mathrm{cod}\, f) \arrow[d, "\pi_f"] \\
F(\mathrm{dom}\, f) \arrow[r, "Ff"] & F(\mathrm{cod}\, f)
\end{tikzcd} \quad \text{commutes, } \forall f \in \mathrm{Mor}\,\mathcal{J}$$

For this purpose, we construct a functor $G : \mathcal{A} \to \mathcal{B}$ that maps objects and morphisms in $\mathcal{A}$ bijectively to $\mathcal{B}$. Given a cone $(\lambda_j : c \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ over F, the universal property of products says that there exists a unique $\lambda : c \to \prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj$ that is compatible with legs (λ_j) and the projections (π_j), i.e., $\pi_j \circ \lambda = \lambda_j$. We assert that $c \xrightarrow{\lambda} \prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj \underset{c}{\overset{d}{\rightrightarrows}} \prod_{f \in \mathrm{Mor}\,\mathcal{J}} F(\mathrm{cod}\, f)$ is a cone, which defines the action of G on objects.

We need $d \circ \lambda = c \circ \lambda$. Firstly we note that since $(\lambda_j : c \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ is a cone, we have $\lambda_{\mathrm{cod}\, f} = Ff \circ \lambda_{\mathrm{dom}\, f}$ for all $f \in \mathrm{Mor}\,\mathcal{J}$, hence $(\lambda_{\mathrm{cod}\, f} : c \to F_{\mathrm{cod}\, f})_{f \in \mathrm{Mor}\,\mathcal{J}} = (Ff \circ \lambda_{\mathrm{dom}\, f} : c \to F_{\mathrm{cod}\, f})_{f \in \mathrm{Mor}\,\mathcal{J}}$, hence there exists a unique morphism $c \to \prod_{f \in \mathrm{Mor}\,\mathcal{J}} F(\mathrm{cod}\, f)$ that is compatible with legs $(\lambda_{\mathrm{cod}\, f} = Ff \circ \lambda_{\mathrm{dom}\, f})$ and the projections (π_f) by the universal property of products. By definition of c and d, $c \circ \lambda$ and $d \circ \lambda$ both satisfy the commutativity, concluding by the uniqueness that $c \circ \lambda = d \circ \lambda$.

Similarly, given a morphism $\lambda : c \to \prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj$ such that $c \circ \lambda = d \circ \lambda$, the morphisms $(\pi_j \circ \lambda : c \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ form a cone over F as one can verify, which gives the inverse action of G on objects, concluding that G maps objects bijectively.

For morphisms, given a morphism $f : d \to c$ such that $\lambda_j \circ f = \mu_j$ for all $j \in \mathrm{Obj}\,\mathcal{J}$ for cones $(\lambda_j : c \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ and $(\mu_j : d \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$, the uniqueness in universal property of the product $\prod_{j \in \mathrm{Obj}\,\mathcal{J}} Fj$ tells $\lambda \circ f = \mu$ since $\pi_j \circ \lambda \circ f = \lambda_j \circ f = \mu_j = \pi_j \circ \mu$. This gives the action of G on morphisms, and similarly the inverse action is given, concluding that G is bijective on morphisms. The functoriality follows right away, finishing our proof. $\square$

With similar technique we can see that any category that admits all binary products admits all finite products. This follows from the lemma below along with an induction on the number of objects in products. The proof for the lemma uses only the universal property of products, and is much more simple than Theorem 3.1, hence it is left as an exercise for the reader. Its dual holds, of course.

Lemma 3.2 (Associativity of Products). *For any triple of objects X, Y, Z in a category with binary products, there is a unique natural isomorphism $X \times (Y \times Z) \cong (X \times Y) \times Z$ commuting with the projections to X, Y and Z. In fact, $X \times (Y \times Z) \cong (X \times Y) \times Z$ gives the product of the triple X, Y, Z.* □

However, one should note that even when the summits $X \times (Y \times Z)$ and $(X \times Y) \times Z$ are chosen to be equal, the unique natural isomorphism between them may not be the identity, due to the difference between their projections onto X, Y, Z. An example for this needs some other conclusions, hence we leave it to latter sections, see Example 3.6.

(Co)Limits of small diagrams are called **small (co)limits**. It is always good to know if a category admits every limit, and in application we mostly only consider limits that are small, hence there is a name for categories that admit all small limits.

Definition 3.10 (Complete and Cocomplete). A category is **complete** if it admits all small limits and is **cocomplete** if it admits all small colimits.

As we said before, Theorem 3.1 tells that it suffices to show that a category admits all small equalizers and products to show that it is complete. Conversely, if a category is complete, then it of course admits all small equalizers and products. Therefore we see that a category is complete if and only if it admits all small equalizers and products. The dual holds, that a category is cocomplete if and only if it admits all small coequalizers and coproducts. Many familiar categories are both complete and cocomplete. For convenience we fix the diagram $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in description of (co)equalizers.

Example 3.4. (i) Set is both complete and cocomplete, with products the Cartesian product, coproducts the disjoint union, equalizers the subset $\{a \in A \mid f(a) = g(a)\}$ and coequalizers the quotient $B/\left(f(a) \sim g(a)\right)$.

(ii) Grp is both complete and cocomplete, with products, coproducts, equalizers and coequalizers all described before.

(iii) Ab is both complete and cocomplete, with products and coproducts described before, equalizers and coequalziers the same in Grp.

(iv) Top is both complete and cocomplete, with products described before, coproducts the disjoint union in the obvious sense, equalizers the subspace $\{a \in A \mid f(a) = g(a)\}$ of A and coequalizers the quotient space $B/\left(f(a) \sim g(a)\right)$ on which the topology is given by that $U \subset B/\left(f(a) \sim g(a)\right)$ is open if and only if $\pi^{-1}(U) \subset B$ is open, where $\pi : B \to B/\left(f(a) \sim g(a)\right)$ is the quotient map.

(v) $\mathrm{Vect}_{\mathbb{F}}$ is both complete and cocomplete, with products similar as in Ab with scalar multiplications $\lambda(a, b) = (\lambda a, \lambda b)$, coproducts the direct sum, equalizers the subspace $\{a \in A \mid f(a) = g(a)\}$. For coequalizers, choose a basis $a_1, \cdots, a_n$ for A and extend the list $f(a_1), \cdots, f(a_n), g(a_1), \cdots, g(a_n)$ to a span list of B, then reduce the list to a basis of B and quotient the relation $f(a_i) \sim g(a_i)$, the space spanned by the final list with the obvious quotient map from B to this space is the coequalizer.

Remark 4. Observe in Set that we have isomorphisms $\mathrm{Hom}_{\mathrm{Set}}(1, X) \cong X$ that are natural in X, where 1 is the singleton set. For any small diagram $F : \mathcal{J} \to \mathrm{Set}$, a limit of F is a representation $\mathrm{Hom}_{\mathrm{Set}}(-, \lim F) \cong \mathrm{Cone}(-, F)$. Substitute 1 and we obtain $\lim F \cong \mathrm{Hom}_{\mathrm{Set}}(1, \lim F) \cong \mathrm{Cone}(1, F)$, hence the set of cones over F with summit 1 would give a limit of F provided some appropriate legs. With some effort, one can show that the legs $(\lambda_j : \mathrm{Cone}(1, F) \to Fj)$ defined by $\mu : 1 \Rightarrow F \ \mapsto \ \mu_j(1) \in Fj$ will do. In fact, the completeness of Set can be proved without Theorem 3.1 using this construction. For more details, see Emily Riehl, *Category Theory in Context*, §3.2.

So far we have gone through basic examples and properties of limits and colimits. To proceed further, we will enter into the beautiful but somehow-complicated features of limits and colimits. The reader should be familiar enough to the definition of limits and colimits, in order not to lose him/herself.

3.2 Functoriality of limits and colimits

In this section, we establish the functoriality of the construction of limits or colimits of diagrams of a fixed shape, and explore its consequences. As we said in the foot note 15, a choice issue rises, while different choices give natural isomorphic functors.

Proposition 3.2 (Functoriality of Limits and Colimits). *If category $\mathcal{C}$ has all $\mathcal{J}$-shaped limits, then a choice of a limit for each diagram defines the action on objects of a functor $\lim_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$. More explicitly, such choice defines a functor $\mathrm{Obj}\,\mathcal{C}^{\mathcal{J}} \to \mathcal{C}$, which extends uniquely to the functor $\lim_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$. Any two functors $\lim_{\mathcal{J}}, \lim'_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ defined by two different choices are natural isomorphic. The dual holds for colimits, that a choice of colimits defines uniquely a functor $\mathrm{colim}_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ and different choices give natural isomorphic functors.*

Note that the choice is not unique, there might be a number of different choices, but the functor defined by a certain choice is unique.

Proof. Choose a limit for each diagram $F \in \mathcal{C}^{\mathcal{J}}$, let $\lim_{\mathcal{J}} F$ be the summit of this limit, this defines the action of $\lim_{\mathcal{J}}$ on objects. Note that morphisms in $\mathcal{C}^{\mathcal{J}}$ are natural transformations, for any morphism $\alpha : F \Rightarrow G$, its vertical composition with the limit cone over F defines a cone over G

$$\lim_{\mathcal{J}} F \Longrightarrow F \overset{\alpha}{\Longrightarrow} G$$

hence by the universal property of limits there exists a unique morphism $\lim_{\mathcal{J}} F \to \lim_{\mathcal{J}} G$ that is compatible with the legs of cones; seen as a natural transformation between constant functors, it is the unique morphism such that the diagram

$$\begin{tikzcd} \lim_{\mathcal{J}} F \arrow[r, Rightarrow] \arrow[drr, Rightarrow, "\exists!"'] & F \arrow[r, Rightarrow, "\alpha"] & G \\ & & \lim_{\mathcal{J}} G \arrow[u, Rightarrow] \end{tikzcd}$$

commutes in the category $\mathcal{C}^{\mathcal{J}}$. $\lim_{\mathcal{J}} \alpha$ is thus defined to be this unique morphism. The functoriality of $\lim_{\mathcal{J}}$ follows from the uniqueness. Recall by the universal property that for two limits $\lim_{\mathcal{J}} F, \lim'_{\mathcal{J}} F$ there is a unique isomorphism $\lim_{\mathcal{J}} F \cong \lim'_{\mathcal{J}} F$ that is compatible with the legs of limit cones, one can show that these isomorphisms consist the natural isomorphism $\lim_{\mathcal{J}} \cong \lim'_{\mathcal{J}}$ by the same technique in the proof of Theorem 3.1. The dual can be proved similarly. ☐

More generally, the proof of Proposition 3.2 implies that a natural transformation between diagrams give rise to a morphism between their (co)limit cones whenever these exist. Whether the codomain category has all (co)limits of that shape is irrelevant. Moreover, such rise is functorial, i.e., it is compatible with vertical composition of natural transformations and preserves isomorphisms. One can see this directly, but the proof of the corollary below illustrates a clearer picture.

Corollary 3.2.1. *A natural isomorphism between diagrams induces a naturally-defined isomorphism between their limits or colimits, whenever these exist.*

Proof. Consider the full subcategory of $\mathcal{C}^{\mathcal{J}}$ spanned by the functors admitting (co)limits in $\mathcal{C}$, then the proof of proposition 3.2 defines a (co)limit functor from this subcategory to $\mathcal{C}$. Since functors preserve isomorphisms (Lemma 1.1), this corollary follows. ☐

The functor that takes a topological space to the set of its path components could be seen as a result of post-composing a colim functor. To see this, we need to formulate the notion of path components by the categorical language.

Example 3.5. The functor $\mathrm{Hom}_{\mathrm{Top}}(I, -) : \mathrm{Top} \to \mathrm{Set}$ brings a topological space to the set of paths in it, where I is the standard unit interval $[0, 1]$. Consider the bifunctor $\mathrm{Hom}_{\mathrm{Top}}(-, -) : \mathrm{Top}^{op} \times \mathrm{Top} \to \mathrm{Set}$, substituting the endpoint inclusions $0, 1 : * \rightrightarrows I$ into its left component[19] give rise to a functor $P : \mathrm{Top} \to \mathrm{Set}^{\bullet \rightrightarrows \bullet}$ that sends a topological space X to the diagram $\mathrm{Hom}_{\mathrm{Top}}(I, X) \underset{ev_1}{\overset{ev_0}{\rightrightarrows}} \mathrm{Hom}_{\mathrm{Top}}(*, X)$, where ev stands for "evaluating". The coequalizer $\pi_0 X$ of this diagram

$$\mathrm{Hom}_{\mathrm{Top}}(I, X) \underset{ev_1}{\overset{ev_0}{\rightrightarrows}} \mathrm{Hom}_{\mathrm{Top}}(*, X) \twoheadrightarrow \pi_0 X$$

is the set of **path components** of X, the set of points in X quotient the relation that identifies any path-connected points. Since Set is complete, we can see that the composition

$$\pi_0 := \mathrm{Top} \xrightarrow{P} \mathrm{Set}^{\bullet \rightrightarrows \bullet} \xrightarrow{\mathrm{colim}} \mathrm{Set}$$

defines the **path components functor**.

3.3 Preservation, reflection, and creation of limits and colimits

Before discussing the representable nature of limits and colimits, we introduce language to describe a variety of possible ways in which a functor can mediate between the limits and colimits in its domain and codomain categories.

Definition 3.11. For any class $\mathfrak{K}$ of diagrams $K : \mathcal{J} \to \mathcal{C}$ valued in $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{D}$

- **preserves** those limits if for any $K \in \mathfrak{K}$ and limit cone over K, the image of this cone under F is a limit cone over the composite diagram $FK : \mathcal{J} \to \mathcal{D}$;

[19] Speaking more formally, we are in fact pre-composing the functor $(0, 1 : * \rightrightarrows I) : (\bullet \rightrightarrows \bullet) \to \mathrm{Top}$ by the left component of $\mathrm{Hom}_{\mathrm{Top}}(-, -)$.

- **reflects** those limits if any cone over $K \in \mathfrak{K}$ whose image under F is a limit cone over FK is a limit cone over K;
- **creates** those limits if F reflects those limits and whenever $FK : \mathcal{J} \to \mathcal{D}$ has a limit in $\mathcal{D}$, there is some limit cone over FK that is in the image of F, hence can be lifted via F to a limit cone over K.

Changing every word "limit" to "colimit" and "over" to "under" defines the notion of preserve, reflect and create for colimits. The terms "preserve", "reflect" and "create" can be used more generally, for instance, a fully faithful functor preserves, reflects and creates isomorphisms.

The context in which it is of greatest interest to have a functor $F : \mathcal{C} \to \mathcal{D}$ that creates limits or colimits is when the codomain category $\mathcal{D}$ is already known to have the (co)limits in question. In such cases, a functor that creates (co)limits both preserves and reflects them:

Proposition 3.3. *If $F : \mathcal{C} \to \mathcal{D}$ creates limits for a class $\mathfrak{K}$ of diagrams in $\mathcal{C}$ and $\mathcal{D}$ has limits of the image of those diagrams under F, then $\mathcal{C}$ admits those limits and F preserves them. The dual holds similarly.*

Proof. For any $K \in \mathfrak{K}$, since D admits a limit of FK and F creates the limits, there exists a limit cone $\lambda : c \Rightarrow K$ over K whose image under F is a limit cone over FK. Since K is an arbitrary diagram in $\mathfrak{K}$, this tells that $\mathcal{C}$ admits all limits of diagram in $\mathfrak{K}$. Note that since an object isomorphic to a terminal object is also terminal ($A \cong B$ induces a bijection $\mathrm{Hom}(A, X) \cong \mathrm{Hom}(B, X)$), a cone that is isomorphic to a limit cone is also a limit cone. Since a limit cone over K is isomorphic to λ and F preserves isomorphisms, its image under F is isomorphic to the image of λ under F, hence is a limit cone over FK since $F\lambda$ is, concluding that F preserves those limits. □

With the fact that an object isomorphic to a terminal (or initial) object is also terminal (or initial) in mind, one can prove the following two lemmas easily:

Lemma 3.3. *Any full and faithful functor reflects any limits and colimits that are present in its codomain.* □

Lemma 3.4. *Any equivalence of categories preserves, reflects and creates any limits and colimits that are present in its domain or codomain.* □

Recall that an equivalence of categories is precisely a full, faithful and essentially surjective functor (Theorem 1.4).

Lemma 3.4 gives rise to the example promised after Lemma 3.2, but we need to introduce the notion of skeletal category, in order to give that example.

Definition 3.12 (Skeletal). A category $\mathcal{C}$ is **skeletal** if it contains just one object in each isomorphism class. The **skeleton** $\mathrm{sk}\mathcal{C}$ of a category $\mathcal{C}$ is the unique (up to isomorphism) skeletal category that is equivalent to $\mathcal{C}$.

Given a category $\mathcal{C}$, its skeleton may be constructed by choosing one object in each isomorphism class of its objects and define $\mathrm{sk}\mathcal{C}$ to be the subcategory consists of all the chosen objects and morphisms between them. It is clear that $\mathrm{sk}\mathcal{C}$ is skeletal. Since the inclusion functor $\mathrm{sk}\mathcal{C} \hookrightarrow \mathcal{C}$ is full, faithful and essentially surjective, we see that $\mathrm{sk}\mathcal{C}$ built in this way is indeed the skeleton of $\mathcal{C}$. Lemma 3.4 tells that if $\mathcal{C}$ is complete, then so does its skeleton, and here comes our example:

Example 3.6. Consider sk(Set) the skeleton of Set. Since Set is complete, so does sk Set. Let C be the corresponding object in sk(Set) of the countably infinite set $\mathbb{N}$. Since $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, we have $C \times C = C$ by Lemma 3.4. Since product projections in Set are epimorphisms and equivalences of categories preserve epimorphisms, the product projections $\pi_1, \pi_2 : C \times C \to C$ are both epimorphisms. Given any two morphisms $f, g : C \to C$, their product $f \times g : C \times C \to C \times C$ is defined via the universal property:

$$\begin{tikzcd}
 & C \times C \arrow[dl, "\pi_1"'] \arrow[d, dashed, "\exists! f\times g"] \arrow[dr, "\pi_2"] & \\
C \arrow[d, "f"'] & C \times C \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & C \arrow[d, "g"] \\
C & & C
\end{tikzcd}$$

hence for any triple of morphisms $f, g, h : C \to C$ the natural isomorphism $C \times (C \times C) \cong (C \times C) \times C$ gives the commutative square

$$\begin{tikzcd}
C \times (C \times C) \arrow[r, "\cong"] \arrow[d, "f\times(g\times h)"'] & (C \times C) \times C \arrow[d, "(f\times g)\times h"] \\
C \times (C \times C) \arrow[r, "\cong"] & (C \times C) \times C
\end{tikzcd}$$

Assuming the natural isomorphism is the identity $C \times (C \times C) = (C \times C) \times C$, we have immediately $f \times (g \times h) = (f \times g) \times h$. Since $C \times C = C$, $f \times (g \times h) = (f \times g) \times h$ could be seen as a morphism between the product $C \times C$. By the universal property of products, their equality is equivalent to the equality of their projections onto components, i.e., equivalent to the commutativity of the diagram below:

$$\begin{tikzcd}
C \arrow[r, shift left, "g\times h"] \arrow[r, shift right, "h"'] & C \\
C\times C \arrow[u, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[r, shift left, "f\times(g\times h)"] \arrow[r, shift right, "(f\times g)\times h"'] & C\times C \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \\
C \arrow[r, shift left, "f"] \arrow[r, shift right, "f\times g"'] & C
\end{tikzcd}$$

Since π_1, π_2 are epimorphisms, this implies that $f = f \times g$ and $g \times h$. Since the choice of the triple f, g, h is arbitrary, we obtain $f = f \times g = g$, concluding that any pair of maps $f, g : C \rightrightarrows C$ must be equal, which is absurd since $\mathrm{Hom}_{\mathrm{sk(Set)}}(C, C) \cong \mathrm{Hom}_{\mathrm{Set}}(\mathbb{N}, \mathbb{N})$. Therefore the natural isomorphism $C \times (C \times C) \cong (C \times C) \times C$ must not be the identity.

In spite of the equivalences of categories, functors that one meets in practice that create certain limits and colimits tend to do so *strictly*:

Definition 3.13 (Strictly Creates). A functor $F : \mathcal{C} \to \mathcal{D}$ **strictly creates** limits for a given class $\mathfrak{K}$ of diagrams if for any diagram $\mathfrak{K} \ni K : \mathcal{J} \to \mathcal{C}$ and limit cone over $FK : \mathcal{J} \to \mathcal{D}$, there exists a unique lift of that limit cone, and this unique lift turns out to be a limit cone over K in $\mathcal{C}$.

A category is **connected** if it is non-empty and $\mathrm{Hom}(A, B)$ is non-empty for any pair of its objects A, B. A diagram is **connected** if it is indexed by a connected category. A **connected (co)limit** is a (co)limit over(under) a connected diagram. An example of strictly creating functor is given by the following proposition.

Proposition 3.4. *For any $F : \mathcal{C} \to \mathrm{Set}$, the projection functor $\prod : \int F \to \mathcal{C}$:* $\begin{array}{c} (c,x) \\ \downarrow f \\ (c',x') \end{array} \mapsto \begin{array}{c} c \\ \downarrow f \\ c' \end{array}$ *:*

(i) strictly creates all limits that $\mathcal{C}$ admits and that F preserves, and

(ii) strictly creates all connected colimits that $\mathcal{C}$ admits.

The size of codomain of F here does not matter: one can replace the category Set by the category of classes of a certain cardinal, and the result still holds. Therefore an immediate application tells that the forgetful functor $\prod : c/\mathcal{C} \to \mathcal{C}$, where $c/\mathcal{C}$ is the **slice category** (see Riehl, *Category Theory in Context*, Exercise 1.1.iii.), strictly creates all limits and connected colimits, since $c/\mathcal{C} = \int \mathrm{Hom}_{\mathcal{C}}(c, -)$ and $\mathrm{Hom}_{\mathcal{C}}(c, -)$ preserves all limits.

Proof. A diagram $(K, k) : \mathcal{J} \to \int F$ in $\int F$ consists of a diagram $K : \mathcal{J} \to \mathcal{C}$ in $\mathcal{C}$ together with a cone $k : 1 \Rightarrow FK$, where 1 is the singleton set. Note that $\prod(K, k) = K$ and a cone (φ, Φ) over (K, k) consists of a cone φ over K together with a cone Φ over the cone diagram of $F\varphi$ with summit 1 such that $\Phi_j = k_j$ for all $j \in \mathrm{Obj}(\mathcal{J})$.[20] Suppose K has a limit $\lambda : c \Rightarrow K$ in $\mathcal{C}$ and is preserved by F, then $F\lambda : Fc \Rightarrow FK$ is a limit cone over FK, hence there exists a unique morphism $1 \to Fc$ that is compatible with legs of k and $F\lambda$.

$$\begin{tikzcd}
Fc \arrow[r, Rightarrow, "F\lambda"] & FK \\
1 \arrow[u, dashed, "\exists!"] \arrow[ur, Rightarrow, "k"'] &
\end{tikzcd}$$

This unique arrow, along with legs of k, consists the unique cone Ψ over the cone diagram of $F\lambda$ such that (λ, Ψ) is a cone over (K, k). This gives the unique lift of λ. To show that (λ, Ψ) is a limit cone, given a cone (φ, Φ) where φ is with summit d, then there exists a unique morphism $f : d \to c$ that is compatible with legs of λ and φ. Since F preserves the limit cone λ, $Ff : Fd \to Fc$ is the unique morphism that is compatible with legs of $F\lambda$ and $F\varphi$, telling that (λ, Ψ) is indeed terminal. See the commutative diagram below.

$$\begin{tikzcd}
1 \arrow[r, "\Phi"] \arrow[rr, bend left, "\Psi"] \arrow[dr, Rightarrow, "k"'] & Fd \arrow[r, dashed, "\exists! Ff"] \arrow[d, Rightarrow, "F\varphi"] & Fc \arrow[dl, Rightarrow, "F\lambda"] \\
& FK &
\end{tikzcd}$$

The proof of the second statement is immediate using the fact that $\mathcal{J}$ is connected, hence is left to the reader. $\square$

[20]Given a diagram $K : \mathcal{J} \to \mathcal{C}$ and a cone $\lambda : c \Rightarrow K$, its **cone diagram** is a diagram K' indexed by the category $\mathcal{J}'$ where objects are $\mathrm{Obj}(\mathcal{J}) \sqcup \{c\}$ and morphisms are morphisms in $\mathcal{J}$ along with an arrow pointing from c to each object in $\mathcal{J}$. K' maps the arrows from c to objects in $\mathcal{J}$ to the legs of the cone λ in the obvious way, and restricts to K on $\mathcal{J}$.

Recall the functor category $\mathcal{C}^{\mathcal{A}}$ whose objects are functors from $\mathcal{A}$ to $\mathcal{C}$ and morphisms are natural transformations between them. A final result shows that functor categories inherit limits and colimits in the target category: that is, given a diagram $K : \mathcal{J} \to \mathcal{C}^{\mathcal{A}}$, substituting $a \in \mathrm{Obj}(\mathcal{A})$ gives diagrams $K - a : \mathcal{J} \to \mathcal{C}$ in $\mathcal{C}$. If (co)limits of $K - a$ exists for each $a \in \mathrm{Obj}(A)$, then a choice of (co)limit cones consists a limit of $\mathcal{J} \xrightarrow{K} \mathcal{C}^{\mathcal{A}} \xrightarrow{U} \mathcal{C}^{\mathrm{Obj}\,\mathcal{A}}$, where U is the forgetful functor, and the (co)limit lifts uniquely to a (co)limit of K in $\mathcal{C}^{\mathcal{A}}$. See the proposition below.

Proposition 3.5. *The forgetful functor $U : \mathcal{C}^{\mathcal{A}} \to \mathcal{C}^{\mathrm{Obj}\,\mathcal{A}}$ strictly creates all limits and colimits that exist in $\mathcal{C}$. These limits are defined objectwise, meaning that for each $a \in \mathcal{A}$, the evaluation functor $ev_a : \mathcal{C}^{\mathcal{A}} \to \mathcal{C}$ preserves all limits and colimits existing in $\mathcal{C}$.*

Proof. Seeing $\mathcal{C}^{\mathrm{Obj}\,\mathcal{A}}$ as an $\mathrm{Obj}\,\mathcal{A}$-tuple of copies of $\mathcal{C}$, it is easy to see that a limit cone over a diagram $UK : \mathcal{J} \to \mathcal{C}^{\mathrm{Obj}\,\mathcal{A}}$ consists of limit cones over $ev_a UK : \mathcal{J} \to \mathcal{C}$ for each $a \in \mathrm{Obj}\,\mathcal{A}$, hence $\mathcal{C}^{\mathrm{Obj}\,\mathcal{A}}$ has all limits or colimits that $\mathcal{C}$ does, and these are preserved by the evaluation functors $ev_a : \mathcal{C}^{\mathrm{Obj}\,\mathcal{A}} \to \mathcal{C}$.

With the conclusion above, to show that the forgetful functor $U : \mathcal{C}^{\mathcal{A}} \to \mathcal{C}^{\mathrm{Obj}\,\mathcal{A}}$ strictly creates all limits and colimits is to show that for any $K : \mathcal{J} \to \mathcal{C}^{\mathcal{A}}$, any $\mathrm{Obj}(\mathcal{A})$-indexed family of (co)limits $\lim_{\mathcal{J}} K - a$, seeing as a functor $\lim_{\mathcal{J}} K - \circ : \mathrm{Obj}(\mathcal{A}) \to \mathcal{C}$, extends uniquely to a functor from $\mathcal{A}$ to $\mathcal{C}$ which is a (co)limit of K. The proof for this fact is nothing but diagram chasing using the universal property without anything new, hence is left to the reader. □

3.4 The representable nature of limits and colimits

The representable nature of limits and colimits asserts that all (small) limits and colimits (in locally small categories) are defined representably in terms of limits in the category of sets. To be precise:

Theorem 3.5. *For any diagram $F : \mathcal{J} \to \mathcal{C}$ whose limit exists there are isomorphisms*

$$\mathrm{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{J}} F) \cong \lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X, F-)$$

that are natural in $X \in \mathrm{Obj}(\mathcal{C})$.

The natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(\circ, \lim_{\mathcal{J}} F) \cong \lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(\circ, F-)$ expresses the **representable universal property of the limit**: the limit $\lim_{\mathcal{J}} F$ is defined representably as the limit of $\mathcal{J}$-indexed diagrams $\mathrm{Hom}_{\mathcal{C}}(X, F-)$ valued in Set.

Proof. Fix a small diagram $F : \mathcal{J} \to \mathcal{C}$ and $X \in \mathrm{Obj}(\mathcal{C})$, since Set is complete, the limit of the composite functor

$$\mathrm{Hom}_{\mathcal{C}}(X, F-) : \mathcal{J} \xrightarrow{F} \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X,-)} \mathrm{Set}$$

exists. Remark 4 tells that $\lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X, F-) \cong \mathrm{Cone}(1, \mathrm{Hom}_{\mathcal{C}}(X, F-))$. An element in $\mathrm{Cone}(1, \mathrm{Hom}_{\mathcal{C}}(X, F-))$ consists of morphisms $(\lambda_j : X \to Fj)_{j \in \mathrm{Obj}\,\mathcal{J}}$ such that

$$\begin{tikzcd} & 1 \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ \mathrm{Hom}_{\mathcal{C}}(X, Fj) \arrow[rr, "(Ff)_*"'] & & \mathrm{Hom}_{\mathcal{C}}(X, Fk) \end{tikzcd}$$

commutes for all $j \xrightarrow{f} k \in \mathcal{J}$. This is exactly that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ Fj \arrow[rr, "f"'] & & Fk \end{tikzcd}$$

commutes for all $j \xrightarrow{f} k \in \mathcal{J}$, hence elements in $\mathrm{Cone}(1, \mathrm{Hom}_{\mathcal{C}}(X, F-))$ are exactly cones over F with summit X, thus $\lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X, F-) \cong \mathrm{Cone}(1, \mathrm{Hom}_{\mathcal{C}}(X, F-)) \cong \mathrm{Cone}(X, F)$. The naturality in X of the isomorphism $\lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X, F-) \cong \mathrm{Cone}(X, F)$ is immediate hence is left to the reader. Since the limit of F is defined to be the representation $\mathrm{Cone}(-, F) \cong \mathrm{Hom}_{\mathcal{C}}(-, \lim_{\mathcal{J}} F)$, we conclude that $\lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X, F-) \cong \mathrm{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{J}} F)$ along with the desired naturality. □

Dually, we have

Theorem 3.6. *For any diagram $F : \mathcal{J} \to \mathcal{C}$ whose colimit exists, there are isomorphisms*

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{J}} F, X) \cong \lim_{\mathcal{J}^{op}} \mathrm{Hom}_{\mathcal{C}}(F-, X)$$

that are natural in X.

Example 3.7 (Iterated Products and Coproducts). Iterated products of an object $A \in \mathrm{Obj}(\mathcal{C})$ are called **powers** or **cotensors**. Given a set I, the I-indexed power of A is denoted as $\prod_I A$ or A^I. Theorem 3.5 tells that

$$\mathrm{Hom}_{\mathcal{C}}(X, A^I) \cong \mathrm{Hom}_{\mathcal{C}}(X, A)^I$$

i.e., a map $h : X \to A^I$ corresponds uniquely to an I-indexed family of maps $h_i : X \to A$. In fact one can see that $\pi_i \circ h = h_i$ where $\pi_i : A^I \to A$ is the product projection.

Dually, iterated coproducts of an object $A \in \mathrm{Obj}(\mathcal{C})$ are called **copowers** or **tensors**. Given a set I, the I-indexed copower of A is denoted as $\coprod_I A$ or $I \cdot A$. Theorem 3.6 tells that

$$\mathrm{Hom}_{\mathcal{C}}(\coprod_I A, X) \cong \mathrm{Hom}_{\mathcal{C}}(A, X)^I$$

i.e., a map $h : \coprod_I A \to X$ corresponds uniquely to an I-indexed family of maps $h_i : A \to X$. In fact there is $h \circ \iota_i = h_i$ where $\iota_i : A \to \coprod_I A$ is the coproduct inclusion.

As a consequence of Theorem 3.5, we have

Theorem 3.7. *For any locally small category $\mathcal{C}$:*

(i) The covariant representable functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserve all limits that exist in $\mathcal{C}$.

(ii) The covariant Yoneda embedding $y : \mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{C}^{op}}$ both preserves and reflects limits, i.e., a cone over a diagram in $\mathcal{C}$ is a limit cone if and only if its image under y is a limit cone in $\mathrm{Set}^{\mathcal{C}^{op}}$.

Proof. The first statement follows immediately from Theorem 3.5, since $\mathrm{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{J}} F) \cong \lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X, F-)$ and the functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves the legs of the limit cone by the commutative diagram concluded from the proof of Theorem 3.5:

$$\begin{tikzcd}
\lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X, F-) \arrow[r, "\cong"] \arrow[dr] & \mathrm{Cone}(X, F) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{J}} F) \arrow[dl] \\
& \prod_{j \in \mathrm{Obj}\, \mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(X, Fj) &
\end{tikzcd}$$

The second statment that y preserves limits follows similarly since $\mathrm{Hom}_{\mathcal{C}}(\circ, \lim_{\mathcal{J}} F) \cong \lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(\circ, F-)$. Recall by Corollary 2.1.1 the Yoneda embedding y is full and faithful, hence Lemma 3.3 tells that y reflects limits. □

Dually, as a consequence of Theorem 3.6 we have

Theorem 3.8. *For any locally small category $\mathcal{C}$:*

(i) Contravariant representable functors $\mathrm{Hom}_{\mathcal{C}}(-, X)$ carry colimits in $\mathcal{C}$ to limits in Set.

(ii) The contravariant Yoneda embedding $y : \mathcal{C}^{op} \hookrightarrow \mathrm{Set}^{\mathcal{C}}$ both preserves and reflects limits in $\mathcal{C}^{op}$, i.e., a cone under a diagram in $\mathcal{C}$ is a colimit cone if and only if its image under y is a limit cone in $\mathrm{Set}^{\mathcal{C}}$.

The good behavior of Yoneda embeddings provides an elegant proof of Theorem 3.1, i.e., the proof of Emily Riehl, *Category Theory in Context*, Theorem 3.4.12.

3.5 Interactions between limits and colimits

Given a functor whose codomain is a product of several categories, for example a bifunctor $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$, it is natural to consider its (co)limits in each variable regarded as a functor $F : \mathcal{I} \to \mathcal{C}^{\mathcal{J}}$ or $F : \mathcal{J} \to \mathcal{C}^{\mathcal{I}}$, just like the limit of a multiple-variable function in calculus.

As in calculus, a first question is that whether taking limits in each variable is commutative, that is, whether $\lim_{\mathcal{I}} \lim_{\mathcal{J}} F(i,j) = \lim_{\mathcal{J}} \lim_{\mathcal{I}} F(i,j)$. The answer is yes, up to isomorphism, provided that the limits all exist.

Theorem 3.9. *If the limits $\lim_{\mathcal{I}} \lim_{\mathcal{J}} F(i,j)$ and $\lim_{\mathcal{J}} \lim_{\mathcal{I}} F(i,j)$ associated to a diagram $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ exist in $\mathcal{C}$, they are isomorphic and define the limit $\lim_{\mathcal{I} \times \mathcal{J}} F$.*

This turns out to be a special case of the fact that right adjoint functors preserve limits, since the limit can be seen as the right adjoint functor of the constant diagram-functor $\mathcal{C} \to \mathcal{C}^{\mathcal{J}}$.

Proof. The good behavior of Yoneda embeddings allows us to reduce the general case to the case of $\mathcal{C} = \mathrm{Set}$: To show that $\lim_{\mathcal{I}} \lim_{\mathcal{J}} F(i,j) \cong \lim_{\mathcal{I} \times \mathcal{J}} F \cong \lim_{\mathcal{J}} \lim_{\mathcal{I}} F(i,j)$ is equivalent to show that

$$\mathrm{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}} \lim_{\mathcal{J}} F(i,j)) \cong \mathrm{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I} \times \mathcal{J}} F(i,j)) \cong \mathrm{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{J}} \lim_{\mathcal{I}} F(i,j))$$

where the isomorphisms are natural in $X \in \operatorname{Obj}\mathcal{C}$. Theorem 3.5 tells that

$$\operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}} \lim_{\mathcal{J}} F(i,j)) \cong \lim_{\mathcal{I}} \operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{J}} F(i,j)) \cong \lim_{\mathcal{I}} \lim_{\mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(X, F(i,j))$$

$$\operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{J}} \lim_{\mathcal{I}} F(i,j)) \cong \lim_{\mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}} F(i,j)) \cong \lim_{\mathcal{J}} \lim_{\mathcal{I}} \operatorname{Hom}_{\mathcal{C}}(X, F(i,j))$$

and

$$\operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}\times\mathcal{J}} F(i,j)) \cong \lim_{\mathcal{I}\times\mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(X, F(i,j))$$

where the isomorphisms are all natural in X. By Proposition 3.5, consider

$$\mathcal{I} \times \mathcal{J} \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(-,F(i,j))} \mathrm{Set}^{\mathcal{C}} \xrightarrow{U} \mathrm{Set}^{\operatorname{Obj}\mathcal{C}}$$

if we can show that

$$\lim_{\mathcal{J}} \lim_{\mathcal{I}} \operatorname{Hom}_{\mathcal{C}}(X, F(i,j)) \cong \lim_{\mathcal{I}\times\mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(X, F(i,j))$$

for each X, then in $\mathrm{Set}^{\operatorname{Obj}\mathcal{C}}$ both $\lim_{\mathcal{J}} \lim_{\mathcal{I}} \operatorname{Hom}_{\mathcal{C}}(-, F(i,j))$ and $\lim_{\mathcal{I}\times\mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(-, F(i,j))$ defines $\lim_{\mathcal{I}\times\mathcal{J}} U \operatorname{Hom}_{\mathcal{C}}(-, F(i,j))$, hence they both extends to define the limit $\lim_{\mathcal{I}\times\mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(-, F(i,j))$ in $\mathrm{Set}^{\mathcal{C}}$, therefore they are isomorphic as functors from $\mathcal{C} \to \mathrm{Set}$, which tells that the isomorphisms

$$\lim_{\mathcal{J}} \lim_{\mathcal{I}} \operatorname{Hom}_{\mathcal{C}}(X, F(i,j)) \cong \lim_{\mathcal{I}\times\mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(X, F(i,j))$$

are natural in X,[21] giving our desired natural isomorphisms

$$\operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}} \lim_{\mathcal{J}} F(i,j)) \cong \lim_{\mathcal{I}} \lim_{\mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(X, F(i,j)) \cong \lim_{\mathcal{I}\times\mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(X, F(i,j)) \cong \operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}\times\mathcal{J}} F(i,j))$$

and similarly $\operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}\times\mathcal{J}} F(i,j)) \cong \operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{J}} \lim_{\mathcal{I}} F(i,j))$. Therefore it remains only to show that for any set-valued functor $H : \mathcal{I} \times \mathcal{J} \to \mathrm{Set}$ we have

$$\lim_{\mathcal{I}} \lim_{\mathcal{J}} H(i,j) \cong \lim_{\mathcal{I}\times\mathcal{J}} H(i,j) \cong \lim_{\mathcal{J}} \lim_{\mathcal{I}} H(i,j)$$

On account of the isomorphism of categories $\mathcal{I}\times\mathcal{J} \cong \mathcal{J}\times\mathcal{I}$, it suffices to show the left-hand isomorphism $\lim_{\mathcal{I}} \lim_{\mathcal{J}} H(i,j) \cong \lim_{\mathcal{I}\times\mathcal{J}} H(i,j)$.

Consider the construction of limits in Set given in Remark 4, it is easy to establish a bijection between $\lim_{\mathcal{I}} \lim_{\mathcal{J}} H(i,j)$ and $\lim_{\mathcal{I}\times\mathcal{J}} H(i,j)$, finishing the proof. The reader should be able to do this by hand. For a guidance of establishing this bijection, see Emily Riehl, *Category Theory in Context*, Proof of Theorem 3.8.1. $\square$

Theorem 3.9 dualizes to give the commutativity of colimits:

Theorem 3.10. *If the colimits* $\operatorname{colim}_{\mathcal{I}} \operatorname{colim}_{\mathcal{J}} F(i,j)$ *and* $\operatorname{colim}_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F(i,j)$ *associated to a diagram* $F : \mathcal{I}\times\mathcal{J} \to \mathcal{C}$ *exist in* $\mathcal{C}$*, they are isomorphic and define the colimit* $\operatorname{colim}_{\mathcal{I}\times\mathcal{J}} F$*.*

The uniqueness statements in the universal properties for limits and colimits tell that for any choices of limit and colimit functors in the diagrams (not necessarily commutative) below

$$\begin{tikzcd} \mathcal{C}^{\mathcal{I}\times\mathcal{J}} \arrow[r, "\lim_{\mathcal{J}}"] \arrow[d, "\lim_{\mathcal{I}}"'] & \mathcal{C}^{\mathcal{I}} \arrow[d, "\lim_{\mathcal{I}}"] \\ \mathcal{C}^{\mathcal{J}} \arrow[r, "\lim_{\mathcal{J}}"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^{\mathcal{I}\times\mathcal{J}} \arrow[r, "\operatorname{colim}_{\mathcal{J}}"] \arrow[d, "\operatorname{colim}_{\mathcal{I}}"'] & \mathcal{C}^{\mathcal{I}} \arrow[d, "\operatorname{colim}_{\mathcal{I}}"] \\ \mathcal{C}^{\mathcal{J}} \arrow[r, "\operatorname{colim}_{\mathcal{J}}"'] & \mathcal{C} \end{tikzcd}$$

there exist natural isomorphisms $\lim_{\mathcal{I}} \lim_{\mathcal{J}} \cong \lim_{\mathcal{J}} \lim_{\mathcal{I}}$ and $\operatorname{colim}_{\mathcal{I}} \operatorname{colim}_{\mathcal{J}} \cong \operatorname{colim}_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}}$.

However, there is seldom an isomorphism between $\operatorname{colim}_{\mathcal{I}} \lim_{\mathcal{J}} F(i,j)$ and $\lim_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F(i,j)$. But there is always a canonical comparison morphism from $\operatorname{colim}_{\mathcal{I}} \lim_{\mathcal{J}} F(i,j)$ to $\lim_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F(i,j)$:

Lemma 3.11. *For any bifunctor* $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ *so that the displayed limits and colimits exist, there is a canonical map*

$$\kappa : \operatorname{colim}_{\mathcal{I}} \lim_{\mathcal{J}} F(i,j) \to \lim_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F(i,j)$$

To remember the direction of this canonical map, recall that colimits are defined via a "mapping out" universal property, while limits are defined via a "mapping in" universal property.

[21]This comes from nowhere but the unique way of extending the functors $\operatorname{Obj}\mathcal{C} \to \mathrm{Set}$ to functors $\mathcal{C} \to \mathrm{Set}$.

Proof. By the universal property of the colimit, the map κ may be defined by specifying components

$$\left(\lim_{\mathcal{J}} F(i,j) \xrightarrow{\kappa_i} \lim_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F(i',j)\right)_{i\in \operatorname{Obj}\mathcal{I}}$$

which define a cone under the diagram $\lim_{\mathcal{J}} F(-,j) : \mathcal{I} \to \mathcal{C}$. By the universal property of the limit, each κ_i may be defined by specifying components

$$\left(\lim_{\mathcal{J}} F(i,j') \xrightarrow{\kappa_{i,j}} \operatorname{colim}_{\mathcal{I}} F(i',j)\right)_{j\in \operatorname{Obj}\mathcal{J}}$$

which define a cone over the diagram $\operatorname{colim}_{\mathcal{I}} F(i,-) : \mathcal{J} \to \mathcal{C}$. The legs of the limit and colimit gives the canonical $\kappa_{i,j}$, namely the composition:

$$\kappa_{i,j} : \lim_{\mathcal{J}} F(i,j') \xrightarrow{\pi_{i,j}} F(i,j) \xrightarrow{\iota_{i,j}} \operatorname{colim}_{\mathcal{I}} F(i',j)$$

where $\pi_{i,j}$ is the leg of the limit cone $\lim_{\mathcal{J}} F(i,j') \Rightarrow F(i,-)$ and $\iota_{i,j}$ is the leg of the colimit cone $F(-,j) \Rightarrow \operatorname{colim}_{\mathcal{I}} F(i',j)$. The construction of $\operatorname{colim}_{\mathcal{I}}$ seen as a functor from $(\mathcal{C}^{\mathcal{J}})^{\mathcal{I}}$ to $\mathcal{C}^{\mathcal{J}}$ tells that the collection $(\kappa_{i,j})_{j\in \operatorname{Obj}\mathcal{J}}$ defined in this way does give a cone over $\operatorname{colim}_{\mathcal{I}} F(i,-)$, i.e., the diagram below

$$\begin{tikzcd}
\lim_{\mathcal{J}} F(i,j') \arrow[r, "\pi_{i,j}"] \arrow[dr, "\pi_{i,k}"'] & F(i,j) \arrow[r, "\iota_{i,j}"] \arrow[d, "F(1_i,f)"] & \operatorname{colim}_{\mathcal{I}} F(i',j) \arrow[d, "\operatorname{colim}_{\mathcal{I}} F(1_{i'},f)"] \\
 & F(i,k) \arrow[r, "\iota_{i,k}"'] & \operatorname{colim}_{\mathcal{I}} F(i',k)
\end{tikzcd}$$

commutes for all $j \xrightarrow{f} k \in \mathcal{J}$. Hence $(\kappa_{i,j})_{j\in \operatorname{Obj}\mathcal{J}}$ gives κ_i, and similarly the collection $(\kappa_i)_{i\in \operatorname{Obj}\mathcal{I}}$ does give a cone over $\lim_{\mathcal{J}} F(-,j)$, giving our desired κ. $\square$

Let $\mathcal{C} = (\mathbb{R}, \leq)$, seeing a set as a discrete category, note that the limit of a function $X \to (\mathbb{R}, \leq)$ is its infima (the maximum lower bound) and the colimit is its suprema, Lemma 3.11 tells immediately that

Corollary 3.11.1. *For any pair of sets X and Y and any function $f : X \times Y \to \mathbb{R}$*

$$\sup_{x\in X} \inf_{y\in Y} f(x,y) \leq \inf_{y\in Y} \sup_{x\in X} f(x,y)$$

whenever these infima and suprema exist.

In Set there is a sufficient conditioin for colimit and limit to be commutative.

Definition 3.14 (Filtered). A category $\mathcal{J}$ is **filtered** if there is a cone under every finite diagram in $\mathcal{J}$.

Theorem 3.12. *Filtered colimits commute with finite limits in Set.*

The proof of this theorem is not categorical, relying an explicit description of filtered colimits in the category of sets, and this result is not what we are concerned, so let's just skip it. Curious reader may see Emily Riehl, *Category Theorey in Context*, Proof of Theorem 3.8.9, for the proof of this result.

4 Adjunctions

4.1 Adjoint functors

An adjunction consists of an opposing pair of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ that enjoy a special relationship to one another:

Definition 4.1 (Adjunction I). An **adjunction** consists of a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with an isomorphism (a bijection)

$$\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$$

for each $c \in \mathrm{Obj}\,\mathcal{C}$ and $d \in \mathrm{Obj}\,\mathcal{D}$ that is natural in both variables. Here F is **left adjoint** to G and G is **right adjoint** to F. The morphisms

$$Fc \xrightarrow{f^\sharp} d \qquad \leftrightsquigarrow \qquad c \xrightarrow{f^\flat} Gd$$

corresponding under the bijection are **adjuncts** or are **transposes** of each other.

The "left" and "right" in an adjunction matter. In particular, if $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$, there are natural isomorphisms $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$, but there need not be isomorphisms $\mathrm{Hom}_{\mathcal{D}}(d, Fc) \cong \mathrm{Hom}_{\mathcal{C}}(Gd, c)$. We shall use the notation $F \dashv G$ to express that F is left adjoint to G. For instance, $F \dashv G$, $G \vdash F$ and

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, shift right, "F"'] \arrow[r, phantom, "\top"] & \mathcal{D} \arrow[l, shift right, "G"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "F"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, shift left, "G"] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, shift left, "F"] \end{tikzcd}$$

all tell that F is left adjoint to G.

If $\mathcal{C}$ and $\mathcal{D}$ are locally small, the naturality statement tells that the isomorphisms $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ give a natural isomorphism between functors

$$\begin{tikzcd} \mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(F-,-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G-)"'] \arrow[r, phantom, "\Downarrow\cong"] & \mathrm{Set} \end{tikzcd}$$

The following lemma provides an equivalent expression of the naturality of a collection of isomorphisms $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$, in a form that tends to be convenient to use in practice.

Lemma 4.1. *Consider a pair of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ equipped with isomorphisms $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ for all $c \in \mathrm{Obj}\,\mathcal{C}$ and $d \in \mathrm{Obj}\,\mathcal{D}$, naturality of these isomorphisms in both c and d is equivalent to the assertion that for any morphisms with domains and codomains as displayed below:*

$$\begin{tikzcd} Fc \arrow[r, "f^\sharp"] \arrow[d, "Fh"'] & d \arrow[d, "k"] \\ Fc' \arrow[r, "g^\sharp"'] & d' \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} c \arrow[r, "f^\flat"] \arrow[d, "h"'] & Gd \arrow[d, "Gk"] \\ c' \arrow[r, "g^\flat"'] & Gd' \end{tikzcd}$$

the left-hand square commutes in $\mathcal{D}$ if and only if the right-hand transposed square commutes in $\mathcal{C}$.

Proof. If the isomorphisms $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ are natural in both c and d, then consider the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Fc, d) \arrow[r, phantom, "\cong"] \arrow[d, "(1^*_{Fc}, k_*)"'] & \mathrm{Hom}_{\mathcal{D}}(c, Gd) \arrow[d, "(1^*_c, Gk_*)"] \\ \mathrm{Hom}_{\mathcal{C}}(Fc, d') \arrow[r, phantom, "\cong"] & \mathrm{Hom}_{\mathcal{D}}(c, Gd') \\ \mathrm{Hom}_{\mathcal{C}}(Fc', d') \arrow[r, phantom, "\cong"] \arrow[u, "(Fh^*, 1_{d'*})"] & \mathrm{Hom}_{\mathcal{D}}(c', Gd') \arrow[u, "(h^*, 1_{Gd'*})"'] \end{tikzcd}$$

It gives exactly that

$$k \cdot f^\sharp = g^\sharp \cdot Fh \quad \Longleftrightarrow \quad Gk \cdot f^\flat = g^\flat \cdot h,$$

i.e. the left square commutes if and only if the right does.

Conversely, if we have the equivalence that $k \cdot f^\sharp = g^\sharp \cdot Fh \quad \Longleftrightarrow \quad Gk \cdot f^\flat = g^\flat \cdot h$, observe that to show the naturality in $d \in \mathrm{Obj}\,\mathcal{D}$ is to show that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(Fc, d) \arrow[r, phantom, "\cong"] \arrow[d, "k_*"'] & \mathrm{Hom}_{\mathcal{C}}(c, Gd) \arrow[d, "Gk_*"] \\ \mathrm{Hom}_{\mathcal{D}}(Fc, d') \arrow[r, phantom, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(c, Gd') \end{tikzcd}$$

commutes for all $d \xrightarrow{k} d' \in \mathcal{D}$, which is equivalent to show that for any $f^\sharp : Fc \to d$, tracking along the upper path gets the same result as tracking along the lower one, i.e., $Gk \cdot f^\flat = (k \cdot f^\sharp)^\flat$, or alternatively, the diagram

$$\begin{tikzcd}
c \arrow[r, "f^\flat"] \arrow[dr, "(k\cdot f^\sharp)^\flat"'] & Gd \arrow[d, "Gk"] \\
& Gd'
\end{tikzcd}$$

commutes for all $Fc \xrightarrow{f^\sharp} d \in \mathcal{D}$. Now for any $f^\sharp : Fc \to d$, consider the commutative square

$$\begin{tikzcd}
Fc \arrow[r, "f^\sharp"] \arrow[d, "1_{Fc}"'] & d \arrow[d, "k"] \\
Fc \arrow[r, "k\cdot f^\sharp"'] & d'
\end{tikzcd}$$

i.e., let $h = 1_c$ and $g^\sharp = k \cdot f^\sharp$, the equivalence tells that the transposed square

$$\begin{tikzcd}
c \arrow[r, "f^\flat"] \arrow[d, "1_c"'] & Gd \arrow[d, "Gk"] \\
c \arrow[r, "(k\cdot f^\sharp)^\flat"'] & Gd'
\end{tikzcd}$$

also commutes, hence $Gk \cdot f^\flat = (k \cdot f^\sharp)^\flat$, concluding the naturality in d. Similarly the naturality in c is obtained. $\square$

There are numerous examples of adjunctions[22]:

Example 4.1. (i) The inclusion of posets $\mathbb{Z} \hookrightarrow \mathbb{R}$, with the usual ordering $\leq$, has both left and right adjoints, namely the ceiling and floor functions. For any integer n and real number r, $n \leq r$ if and only if $n \leq \lfloor r \rfloor$, where $\lfloor r \rfloor$ is the floor of r, namely the greatest integer no more than r. The floor function is order-preserving, defining a functor $\lfloor - \rfloor : \mathbb{R} \to \mathbb{Z}$ which is right adjoint to the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. Dually, $r \leq n$ if and only if $\lceil r \rceil \leq n$, where $\lceil r \rceil$ is the ceil of r, namely the least integer no less than r. The ceil function is also order-preserving, thus defines a functor $\lceil - \rceil : \mathbb{R} \to \mathbb{Z}$ that is left adjoint to the inclusion.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"'] & \mathbb{R} \arrow[l, bend right=50, "\lceil - \rceil"'] \arrow[l, bend left=50, "\lfloor - \rfloor"]
\end{tikzcd}$$

(ii) A **propositional function** is a function $P : X \to \Omega = \{0, 1\}$ that assigns for each $x \in X$ a true (1) or false (0) for $P(x)$. The set Ω^X is then the set of propositional functions on X. The set Ω is given the partial order $0 \leq 1$, from which Ω^X inherits a pointwise-defined order: $P \leq Q$ if and only if $P(x) \leq Q(x)$ for all $x \in X$, which is the case if and only if P implies Q ($P(x) = 1 \;\Rightarrow\; Q(x) = 1$).

The partial orders given above turns the sets Ω and Ω^X into categories as usual. The logical operations of universal and existential quantification define functors

$$\forall_X, \exists_X : \Omega^X \rightrightarrows \Omega$$

in the expected way: $\forall_X P = 1$ if and only if $P(x) = 1$ for all $x \in X$ and $\exists_X P = 1$ if and only if there exists $x \in X$ such that $P(x) = 1$. Let $\Delta_X : \Omega \to \Omega^X$ be the functor that sends elements $\omega \in \Omega$ to the constant function $\omega : X \to \Omega : x \mapsto \omega$, then these functors define a triple of adjoints, as one can verify:

$$\begin{tikzcd}
\Omega \arrow[rr, "\Delta_X"] \arrow[rr, phantom, shift left=4, "\perp"] \arrow[rr, phantom, shift right=4, "\perp"'] & & \Omega^X \arrow[ll, bend right=50, "\exists_X"'] \arrow[ll, bend left=50, "\forall_X"]
\end{tikzcd}$$

Example 4.2. There is a large and very important family of "free $\dashv$ forgetfull" adjunctions

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{S} \arrow[l, shift right=1.5, "F"']
\end{tikzcd}$$

[22] "Adjoint functors arise everywhere." – Saunders Mac Lane, *Categories for the Working Mathematician.*

with the forgetful functor U defining the right adjoint and the free functor F defining the left adjoint. Given a forgetful functor $U : \mathcal{A} \to \mathcal{S}$, if one can construct a universal object of $\mathcal{A}$ on an object of $\mathcal{S}$ and this construction defines a left adjoint to the forgetful functor U, then the term "free" is used in place of "universal" to convey this particular relationship. The reason why left adjoints to forgetful functors are more common than right adjoints has to do with the handedness of the universal properties that one meets in practice. "Cofree" constructions, which define right adjoints to some forgetful functor, are somehow less common.

(i) The left adjoint to the forgetful functors $U : \mathcal{C} \to \mathrm{Set}$ where $\mathcal{C} = \mathrm{Grp}, \mathrm{Ab}, \mathrm{Vect}_{\mathbb{F}}$ or Mod_R (the category of R-modules) gives the free (abelian) groups, vector spaces or R-modules on sets, as has already been introduced. The universal property of the pre-introduced "free"

$$\begin{tikzcd} F(A) \arrow[r, dashed, "\exists!"] & d \\ A \arrow[u] \arrow[ur] & \end{tikzcd}$$

gives exactly the natural isomrphisms $\mathrm{Hom}_{\mathcal{C}}(F(A), d) \cong \mathrm{Hom}_{\mathrm{Set}}(A, Ud)$.

(ii) The notion of the pre-introduced "free" can be extended now. For example, the forgetful functor $U : \mathrm{Ring} \to \mathrm{Ab}$ forgetting the multiplicative structure admits a left adjoint, giving the free rings on abelian groups. To be explicit, the free ring on an abelian group A is $\bigoplus_{n\geq 0} A^{\otimes n}$.

(iii) The functor $(-)^{\times} : \mathrm{Ring} \to \mathrm{Grp}$ carrying a ring to its multiplicative group of units admits a left adjoint, giving the free rings on groups. The free ring on a group G is the **group ring** $\mathbb{Z}[G] = \bigoplus_G \mathbb{Z}$, whose elements are finite formal sums of group elements and the multiplication is defined by the group operation.

Example 4.3. None of the functors

$$\mathrm{Field} \xrightarrow{U} \mathrm{Ring}, \quad \mathrm{Field} \xrightarrow{U} \mathrm{Ab}, \quad \mathrm{Field} \xrightarrow{(-)^{\times}} \mathrm{Ab}, \quad \mathrm{Field} \xrightarrow{U} \mathrm{Set}$$

that forget algebraic structure on fields admit left or right adjoints. To see that no left adjoint is possible, note that there exist maps in the target categories from $\mathbb{Z}$ to fields of any characteristic. As there are no field homomorphisms between fields of differing characteristic, it is not possible to define the value of a hypothetical left adjoint on $\mathbb{Z}$. A similar argument can be used to show that no right adjoints to these functors exist, since the categories $\mathrm{Ring}, \mathrm{Ab}, \mathrm{Set}$ all have terminal objects.

There are a number of other examples on Emily Riehl, *Category Theory in Context*, §4.1, which we shall not mention here.

4.2 The unit and counit as universal arrows

Given an adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$, the natural isomorphisms $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ consist a natural isomorphism between functors from $\mathcal{D}$ to Set, $\mathrm{Hom}_{\mathcal{D}}(Fc, -) \cong \mathrm{Hom}_{\mathcal{C}}(c, G-)$. The Yoneda lemma then gives a natural bijection

$$\mathrm{Hom}_{\mathrm{Set}^{\mathcal{D}}}(\mathrm{Hom}_{\mathcal{D}}(Fc, -), \mathrm{Hom}_{\mathcal{C}}(c, G-)) \cong \mathrm{Hom}_{\mathcal{C}}(c, GFc) \cong \mathrm{Hom}_{\mathcal{D}}(Fc, Fc)$$

hence the natural isomorphism $\mathrm{Hom}_{\mathcal{D}}(Fc, -) \cong \mathrm{Hom}_{\mathcal{C}}(c, G-)$ is determined by an element of $\mathrm{Hom}_{\mathcal{C}}(c, GFc)$. Denote the natural isomorphism as $\alpha : \mathrm{Hom}_{\mathcal{D}}(Fc, -) \cong \mathrm{Hom}_{\mathcal{C}}(c, G-)$, we see that the element is $\Phi(\alpha) = \alpha_{Fc}(1_{Fc})$, the transpose of $1_{Fc} \in \mathrm{Hom}_{\mathcal{D}}(Fc, Fc)$. We shall put $\eta_c := \alpha_{Fc}(1_{Fc})$, and the notion of unit follows:

Lemma 4.2. *Given an adjunction $F \dashv G$, there is a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow GF$, called the **unit** of the adjunction, whose component $\eta_c : c \to GFc$ at each $c \in \mathrm{Obj}\, \mathcal{C}$ is defined to be the transpose of the identity morphism 1_{Fc}.*

Proof. We need to verify that η is natural, that is, the square

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f"'] & GFc \arrow[d, "GFf"] \\ c' \arrow[r, "\eta_{c'}"'] & GFc' \end{tikzcd}$$

commutes for all $c \xrightarrow{f} c' \in \mathcal{C}$. Since η_c is the transpose of 1_{Fc}, take $d = Fc$, $h = f$ and $k = Ff$ in Lemma 4.1, the commutativity of the square

$$\begin{tikzcd} Fc \arrow[r, "1_{Fc}"] \arrow[d, "Ff"'] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "1_{Fc'}"'] & Fc' \end{tikzcd}$$

finishes the proof. $\square$

Dually, fix $d \in \operatorname{Obj} \mathcal{D}$ and we have natural isomorphism $\operatorname{Hom}_{\mathcal{D}}(F-, d) \cong \operatorname{Hom}_{\mathcal{C}}(-, Gd) = \operatorname{Hom}_{\mathcal{C}^{op}}(Gd, -) : \mathcal{C}^{op} \to \mathrm{Set}$. The Yoneda lemma gives natural bijection

$$\operatorname{Hom}_{\mathrm{Set}^{\mathcal{C}^{op}}}(\operatorname{Hom}_{\mathcal{C}^{op}}(Gd, -), \operatorname{Hom}_{\mathcal{D}}(F-, d)) \cong \operatorname{Hom}_{\mathcal{D}}(FGd, d) \cong \operatorname{Hom}_{\mathcal{C}}(Gd, Gd)$$

thus the natural isomorphism is represented by an element of $\operatorname{Hom}_{\mathcal{D}}(FGd, d)$, the transpose of 1_{Gd}, denoted as ϵ_d, and the notion of counit follows:

Lemma 4.3. *Given an adjunction $F \dashv G$, there is a natural transformation $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ called the* ***counit*** *of the adjunction, whose component $\epsilon_d : FGd \to d$ at each $d \in \operatorname{Obj} \mathcal{D}$ is defined to be the transpose of the identity morphism 1_{Gd}.*

These two lemmas show that any adjunction has a unit and counit. Conversely, if $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ are opposing functors equipped with natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ satisfying a dual pair of conditions, then these two natural transformations would give a natural isomorphism $\operatorname{Hom}_{\mathcal{D}}(Fc, d) \cong \operatorname{Hom}_{\mathcal{C}}(c, Gd)$, exhibiting F and G as adjoint functors. Thus an equivalent definition of adjunction follows:

Definition 4.2 (Adjunction II). An **adjunction** consists of an opposing pair of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$, together with natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ that satisfy the **triangle identities**:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] & FGF \arrow[d, Rightarrow, "\epsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, Rightarrow, "1_G"'] & GFG \arrow[d, Rightarrow, "G\epsilon"] \\ & G \end{tikzcd}$$

To remember the "left" and "right" in this definition, one can tell that F is left adjoint to G from the position of F in the counit $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$.

In particular, the triangle identities assert that "the counit is a left inverse of the unit modulo translation": $\epsilon_{Fc} : FGFc \to Fc$ is the left inverse of $F\eta_c : Fc \to FGFc$. Dually, the unit is a right inverse of the counit: $\eta_{Gd} : Gd \to GFGd$ is the right inverse of $G\epsilon_d : GFGd \to Gd$.

The proposition below shows that the definition is equivalent to the previous one:

Proposition 4.1. *Given a pair of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$, there exist natural isomorphisms $\operatorname{Hom}_{\mathcal{D}}(Fc, d) \cong \operatorname{Hom}_{\mathcal{C}}(c, Gd)$ if and only if there exists a pair of natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ satisfying the triangle identities.*

Proof. Since we already have Lemma 4.2 and Lemma 4.3, it remains to show that the unit and counit built from the two lemmas satisfy the triangle identities. This follows from applying Lemma 4.1 as below, where the left squares are equivalent to the triangle identities, whose commutatitvity follows from the commutativity of the right squares:

$$\begin{tikzcd} Fc \arrow[r, "1_{Fc}"] \arrow[d, "F\eta_c"'] & Fc \arrow[d, "1_{Fc}"] \\ FGFc \arrow[r, "\epsilon_{Fc}"'] & Fc \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "\eta_c"'] & GFc \arrow[d, "1_{GFc}"] \\ GFc \arrow[r, "1_{GFc}"'] & GFc \end{tikzcd}$$

$$\begin{tikzcd} Gd \arrow[r, "\eta_{Gd}"] \arrow[d, "1_{Gd}"'] & GFGd \arrow[d, "G\epsilon_d"] \\ Gd \arrow[r, "1_{Gd}"'] & Gd \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} FGd \arrow[r, "1_{FGd}"] \arrow[d, "1_{FGd}"'] & FGd \arrow[d, "\epsilon_d"] \\ FGd \arrow[r, "\epsilon_d"'] & d \end{tikzcd}$$

Conversely, given unit and counit satisfying the triangle identities, we can use them to define natural isomorphisms $\operatorname{Hom}_{\mathcal{D}}(Fc, d) \cong \operatorname{Hom}_{\mathcal{C}}(c, Gd)$. Given $f^\sharp : Fc \to d$ and $g^\flat : c \to Gd$, their transposes are defined to be the composites:

$$f^\flat := c \xrightarrow{\eta_c} GFc \xrightarrow{Gf^\sharp} Gd, \qquad g^\sharp := Fc \xrightarrow{Fg^\flat} FGd \xrightarrow{\epsilon_d} d.$$

The triangle identities imply that these operations are inverse. The transpose of the transpose of $f^\sharp : Fc \to d$ is equal to the top composite of diagram

$$\begin{tikzcd}
Fc \arrow[r, "F\eta_c"] \arrow[drr, bend right, "1_{Fc}"'] & FGFc \arrow[r, "FGf^\sharp"] \arrow[dr, "\epsilon_{Fc}"] & FGd \arrow[r, "\epsilon_d"] & d \\
& & Fc \arrow[ur, "f^\sharp"'] &
\end{tikzcd}$$

which is commutative by the triangle identity and naturality of ϵ, hence the transpose of the transpose of $f^\sharp$ is again $f^\sharp$. The dual diagram chase

$$\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[dr, "g^\flat"'] & GFc \arrow[r, "GFg^\flat"] & GFGd \arrow[r, "G\epsilon_d"] & Gd \\
& Gd \arrow[ur, "\eta_{Gd}"] \arrow[urr, bend right, "1_{Gd}"'] & &
\end{tikzcd}$$

tells that the transpose of the transpose of $g^\flat$ is again $g^\flat$, finishing the proof. $\square$

Remark 5. We have shown two equivalent definitions of adjunctions, namely it consists of a pair of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ together with

(i) a natural family of isomorphisms $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ for all $c \in \mathrm{Obj}\,\mathcal{C}$ and $d \in \mathrm{Obj}\,\mathcal{D}$,

or, equivalently,

(ii) natural transformations $\eta : 1_{\Rightarrow}GF$ and $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ so that $G\epsilon \cdot \eta G = 1_G$ and $\epsilon F \cdot F\eta = 1_F$.

In fact, either of the unit and the counit alone satisfying an appropriate universal property gives another equivalent condition:

(iii) a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow GF$ so that the composition

$$\mathrm{Hom}_{\mathcal{D}}(Fc, d) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(GFc, Gd) \xrightarrow{(\eta_c)^*} \mathrm{Hom}_{\mathcal{C}}(c, Gd)$$

defines an isomorphism for all $c \in \mathrm{Obj}\,\mathcal{C}$ and $d \in \mathrm{Obj}\,\mathcal{D}$.

(iv) a natural transformation $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ so that the composition

$$\mathrm{Hom}_{\mathcal{C}}(c, Gd) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(Fc, FGd) \xrightarrow{(\epsilon_d)_*} \mathrm{Hom}_{\mathcal{D}}(Fc, d)$$

defines an isomorphism for all $c \in \mathrm{Obj}\,\mathcal{C}$ and $d \in \mathrm{Obj}\,\mathcal{D}$.

With these in mind, several equivalent definitions of **morphisms between adjunctions** can be introduced: A morphism from $F \dashv G$ to $F' \dashv G'$ is comprised of a pair of functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] \arrow[d, shift right, "F"'] \arrow[d, phantom, "\dashv"] & \mathcal{C}' \arrow[d, shift right, "F"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{D} \arrow[r, "K"'] \arrow[u, shift right, "G"'] & \mathcal{D}' \arrow[u, shift right, "G"']
\end{tikzcd}$$

so that the square with the left adjoints and the square with the right adjoints both commute, i.e., $KF = F'H$ and $HG = G'K$, and satisfying one of the three equivalent conditions:

(i) $H\eta = \eta' H$, where η and η' are the respective units of the adjunctions.

(ii) $K\epsilon = \epsilon' K$, where ϵ and ϵ' are the respective counits of the adjunctions.

(iii) Transposition across the adjunctions commutes with application of the functors H and K, i.e., for every $c \in \mathrm{Obj}\,\mathcal{C}$ and $d \in \mathrm{Obj}\,\mathcal{D}$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(Fc, d) \arrow[d, "K"'] \arrow[r, phantom, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(c, Gd) \arrow[d, "H"] \\
\mathrm{Hom}_{\mathcal{D}'}(KFc, Kd) \arrow[d, equal] & \mathrm{Hom}_{\mathcal{C}'}(Hc, HGd) \arrow[d, equal] \\
\mathrm{Hom}_{\mathcal{D}'}(F'Hc, Kd) \arrow[r, phantom, "\cong"] & \mathrm{Hom}_{\mathcal{C}'}(Hc, G'Kd)
\end{tikzcd}$$

commutes.

All the verifications are left to the reader.

4.3 The calculus of adjunctions

This section introduces the basic calculus of adjoint functors: proving that adjoints are unique up to unique natural isomorphism, that adjunctions can be composed and whiskered, and that any equivalence can be promoted to an adjoint equivalence in which the inverse equivalences are also adjoints, with either choice of handedness. These results follow from nowhere but a pure categorical approach, hence their proofs are also of great importance for us, and there will be two different versions of the proof for each of them, one is pure diagram chasing while the other uses Yoneda-style arguments.

Proposition 4.2. *If F and F' are left adjoint to G, then $F \cong F'$, and moreover there is a unique natural isomorphism $\theta : F \cong F'$ commuting with the units and counits of the adjunctions:*

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta'"'] & GF \arrow[d, Rightarrow, "G\theta"] \\ & GF' \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, Rightarrow, "\epsilon"] \arrow[d, Rightarrow, "\theta G"'] & 1_{\mathcal{D}} \\ F'G \arrow[ur, Rightarrow, "\epsilon'"'] & \end{tikzcd}$$

Proof. Diagram Chasing Version: To define a natural transformation $\theta : F \Rightarrow F'$, it suffices, by Lemma 4.1, to define a transposed natural transformation from $1_{\mathcal{C}}$ to GF', which we take to be the unit $\eta'_c : 1_{\mathcal{C}} \Rightarrow GF'$

$$\begin{tikzcd} Fc \arrow[r, "\theta_c"] \arrow[d, "Ff"'] & F'c \arrow[d, "F'f"] \\ Fc' \arrow[r, "\theta_{c'}"'] & F'c' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} c \arrow[r, "\eta'_c"] \arrow[d, "f"'] & GF'c \arrow[d, "GF'f"] \\ c' \arrow[r, "\eta'_{c'}"'] & GF'c' \end{tikzcd}$$

The proof of Proposition 4.1 (or the statement in Remark 5.(iv)) gives us an explicit formula for θ:

$$\theta := F \xRightarrow{F\eta'} FGF' \xRightarrow{\epsilon F'} F'.$$

Similarly we define a natural transformation $\theta' : F' \Rightarrow F$ to be the transpose of $\eta : 1_{\mathcal{C}} \Rightarrow GF$, given by the formula:

$$\theta' := F' \xRightarrow{F'\eta} F'GF \xRightarrow{\epsilon' F} F.$$

The hope is that θ and θ' defined above are the inverse of each other. In fact, the triangle identities guarantee this. By symmetry we need only show that $\theta' \cdot \theta = 1_F$. It suffices to show that their transposes equal, i.e., to show that $\eta : 1_{\mathcal{C}} \Rightarrow GF$ equals the composite

$$1_{\mathcal{C}} \xRightarrow{\eta} GF \xRightarrow{GF\eta'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF.$$

Naturality of η tells that the square

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta'"] & GF \arrow[d, Rightarrow, "GF\eta'"] \\ GF' \arrow[r, Rightarrow, "\eta GF'"] & GFGF' \end{tikzcd}$$

commutes, hence the composite turns to

$$1_{\mathcal{C}} \xRightarrow{\eta'} GF' \xRightarrow{\eta GF'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

By the triangle identity $G\epsilon \cdot \eta G = 1_G$, this reduces to

$$1_{\mathcal{C}} \xRightarrow{\eta'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

By naturality of η', it turns to

$$1_{\mathcal{C}} \xRightarrow{\eta} GF \xRightarrow{\eta' GF} GF'GF \xRightarrow{G\epsilon' F} GF$$

and by the triangle identity $G\epsilon' \cdot \eta' G = 1_G$ we obtain

$$1_{\mathcal{C}} \xRightarrow{\eta} GF$$

as desired.

The formula for θ gives straightforward the fact that θ commutes with the units and counits by "half" of the diagram chasing displayed above.

The left triangle asserts that the transpose of θ across the adjunction $F \dashv G$ is η' (see the formula of transposes in proof of Proposition 4.1), thereby proving the uniqueness. In fact this proof just takes θ to be the transpose of η', and the result follows. $\square$

Proof. Yoneda-style Version: By the Yoneda lemma, the composite natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F'c, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd) \cong \mathrm{Hom}_{\mathcal{D}}(Fc, d)$$

defines a natural isomorphism $\theta : F \cong F'$, whose component θ_c is defined to be the image of $1_{F'c}$ under the bijection $\mathrm{Hom}_{\mathcal{D}}(F'c, F'c) \cong \mathrm{Hom}_{\mathcal{D}}(Fc, F'c)$, since

$$\mathrm{Hom}_{\mathrm{Set}^{\mathcal{D}}}(\mathrm{Hom}_{\mathcal{D}}(F'c, -), \mathrm{Hom}_{\mathcal{D}}(Fc, -)) \cong \mathrm{Hom}_{\mathcal{D}}(Fc, F'c) \cong \mathrm{Hom}_{\mathcal{D}}(F'c, F'c)$$

Since the isomorphism $\mathrm{Hom}_{\mathcal{D}}(F'c, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ carries $1_{F'c}$ to $\eta'_c : c \to GF'c$, we see that θ_c is the transpose of η'_c along $F \dashv G$, obtaining by the formula in proof of Proposition 4.1 the compatibility of $\theta : F \cong F'$ with the units of the adjunctions, or alternatively by the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F'c, Fc) & \xrightarrow{\theta_{c*}} & \mathrm{Hom}_{\mathcal{D}}(F'c, F'c) \\ \wr\| & & \wr\| \\ \mathrm{Hom}_{\mathcal{C}}(c, GFc) & \xrightarrow{(G\theta_c)_*} & \mathrm{Hom}_{\mathcal{C}}(c, GF'c) \\ \wr\| & & \wr\| \\ \mathrm{Hom}_{\mathcal{D}}(Fc, Fc) & \xrightarrow{\theta_{c*}} & \mathrm{Hom}_{\mathcal{D}}(Fc, F'c) \end{array}$$

The image of $1_{F'c} \in \mathrm{Hom}_{\mathcal{D}}(F'c, F'c)$ gives η'_c and θ_c, and θ_c pulls back to $1_{Fc} \in \mathrm{Hom}_{\mathcal{D}}(Fc, Fc)$. The transpose of 1_{Fc} is $\eta_c \in \mathrm{Hom}_{\mathcal{C}}(c, GFc)$, hence by commutativity we obtain $\eta'_c = G\theta_c \cdot \eta_c$ as desired.

The compatibility of θ with the counits follows from the commutative diagram below:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F'Gd, F'Gd) & \xrightarrow{\epsilon'_{d*}} & \mathrm{Hom}_{\mathcal{D}}(F'Gd, d) \\ \wr\| & & \wr\| \\ \mathrm{Hom}_{\mathcal{C}}(Gd, GF'Gd) & \xrightarrow{(G\epsilon'_d)_*} & \mathrm{Hom}_{\mathcal{C}}(Gd, Gd) \\ \wr\| & & \wr\| \\ \mathrm{Hom}_{\mathcal{D}}(FGd, F'Gd) & \xrightarrow{\epsilon'_{d*}} & \mathrm{Hom}_{\mathcal{D}}(FGd, d) \end{array}$$

The counits are the images of $1_{Gd} \in \mathrm{Hom}_{\mathcal{C}}(Gd, Gd)$. Since $\epsilon'_d \in \mathrm{Hom}_{\mathcal{D}}(F'Gd, d)$ pulls back to $1_{F'Gd} \in \mathrm{Hom}_{\mathcal{D}}(F'Gd, F'Gd)$ whose image in $\mathrm{Hom}_{\mathcal{D}}(FGd, F'Gd)$ is θ_{Gd}, we obtain $\epsilon_d = \epsilon'_d\theta_{Gd}$ as desired. □

Proposition 4.3. *Given adjunctions $F \dashv G$ and $F' \dashv G'$, the composite $F'F$ is left adjoint to the composite GG':*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "G'"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F'F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "GG'"] \end{tikzcd}$$

Proof. Using Definition I: There are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{E}}(F'Fc, e) \cong \mathrm{Hom}_{\mathcal{D}}(Fc, G'e) \cong \mathrm{Hom}_{\mathcal{C}}(c, GG'e)$$

the first using $F' \dashv G'$ and the second using $F \dashv G$. □

Proof. Using Definition II: Define the unit and counit of $F'F \dashv GG'$ by

$$\bar{\eta} := 1_{\mathcal{C}} \xRightarrow{\eta} GF \xRightarrow{G\eta' F} GG'F'F \qquad \bar{\epsilon} := F'FGG' \xRightarrow{F'\epsilon G'} F'G' \xRightarrow{\epsilon'} 1_{\mathcal{E}}$$

In fact they are natural transformations that result from passing the identity $1_{F'F}$ and $1_{GG'}$ across the natural isomorphisms in the previous version of proof. The verification of the triangle identities is an entertaining diagram chase left to the reader. □

Proposition 4.4. *Any equivalence of categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & D \arrow[l, shift left, "G"] \end{tikzcd}, \qquad \eta : 1_{\mathcal{C}} \cong GF, \quad \epsilon : FG \cong 1_{\mathcal{D}}$$

can be promoted to an adjoint equivalence, in which the natural isomrophisms satisfy the triangle identities, by replacing either one of the orginally specified natural isomorphisms by a new unit or counit.

Note that by symmetry the equivalence $F : \mathcal{C} \to \mathcal{D}$ may be regarded as either a left or right adjoint. Indeed, any chosen equivalence inverse $G : \mathcal{D} \to \mathcal{C}$ is both left and right adjoint to F.

Proof. Version I: Since $\eta : 1_{\cong} GF$ and $\epsilon : FG \cong 1_{\mathcal{D}}$ are both isomorphisms, so is the composite

$$\gamma := G \xRightarrow{\eta G} GFG \xRightarrow{G\epsilon} G$$

but it need not be the identity. Redefining either ϵ or η so as to absorb the isomorphism γ^{-1}, we will show that the resulting pair of natural isomorphisms define the unit and the counit of an adjunction $F \dashv G$.

Let $\epsilon' := \epsilon \cdot F\gamma^{-1}$. By naturality of η, the diagram

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, "\gamma^{-1}"'] & GFG \arrow[r, Rightarrow, "GF\gamma^{-1}"] & GFG \arrow[r, Rightarrow, "G\epsilon"] & G \\ & G \arrow[ur, Rightarrow, "\eta G"] \arrow[urr, Rightarrow, "\gamma"'] & & \end{tikzcd}$$

commutes, proving one triangle identity $G\epsilon' \cdot \eta G = 1_G$. By naturality of η and ϵ', and by this first triangle identity, the diagram

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] & FGF \arrow[r, Rightarrow, "\epsilon' F"] \arrow[d, Rightarrow, "FGF\eta"] & F \arrow[d, Rightarrow, "F\eta"] \\
FGF \arrow[r, Rightarrow, "F\eta GF"] \arrow[dr, equal] & FGFGF \arrow[r, Rightarrow, "\epsilon' FGF"] \arrow[d, Rightarrow, "FG\epsilon' F"] & FGF \arrow[d, Rightarrow, "\epsilon' F"] \\
& FGF \arrow[r, Rightarrow, "\epsilon' F"'] & F
\end{tikzcd}$$

commutes, proving that $\epsilon' F \cdot F\eta$ is an idempotent, i.e., $(\epsilon' F \cdot F\eta)^2 = \epsilon' F \cdot F\eta$. Since $\epsilon' F \cdot F\eta$ is an isomorphism, cancellation tells that $\epsilon' F \cdot F\eta = 1_F$, finishing the proof. $\square$

Proof. Version II: If $\eta : 1_{\mathcal{C}} \cong GF$ is one of the natural isomorphisms defining an equivalence of categories $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$, then the function

$$\mathrm{Hom}_{\mathcal{D}}(Fc, d) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(GFc, Gd) \xrightarrow{(\eta_c)^*} \mathrm{Hom}_{\mathcal{C}}(c, Gd)$$

defines a natural isomorphism for all $c \in \mathrm{Obj}\,\mathcal{C}$ and $d \in \mathrm{Obj}\,\mathcal{D}$: the first map is an isomorphism because G is fully faithful and the second map is an isomorphism because η_c is an isomorphism. By Remark 5 (iii), it follows that F and G define an adjunction with unit η. $\square$

Proposition 4.5. *Given an adjunction*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"]
\end{tikzcd}$$

post-composition with F and G defines a pair of adjoint functors

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{J}} \arrow[r, shift left, "F_*"] \arrow[r, phantom, "\perp"] & \mathcal{D}^{\mathcal{J}} \arrow[l, shift left, "G_*"]
\end{tikzcd}$$

for any small category $\mathcal{J}$, and pre-composition with F and G also defines an adjunction

$$\begin{tikzcd}
\mathcal{E}^{\mathcal{C}} \arrow[r, shift left, "G^*"] \arrow[r, phantom, "\perp"] & \mathcal{E}^{\mathcal{D}} \arrow[l, shift left, "F^*"]
\end{tikzcd}$$

for any locally small category $\mathcal{E}$. [23]

Proof. For the adjoint

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{J}} \arrow[r, shift left, "F_*"] \arrow[r, phantom, "\perp"] & \mathcal{D}^{\mathcal{J}} \arrow[l, shift left, "G_*"]
\end{tikzcd}$$

we use natural the isomorphisms $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ to establish natural bijections

$$\mathrm{Hom}_{\mathcal{D}^{\mathcal{J}}}(FK, L) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(K, GL)$$

for all $K \in \mathrm{Obj}\,\mathcal{C}^{\mathcal{J}}$ and $L \in \mathrm{Obj}\,\mathcal{D}^{\mathcal{J}}$. An element in $\mathrm{Hom}_{\mathcal{D}^{\mathcal{J}}}(FK, L)$ is a natural transformation $\alpha : FK \Rightarrow L$, which consists of a family of morphisms $(\alpha_j)_{j \in \mathrm{Obj}\,\mathcal{J}}$ satisfying the naturality condition, i.e., the square

$$\begin{tikzcd}
FKi \arrow[r, "\alpha_i"] \arrow[d, "FKf"'] & Li \arrow[d, "Lf"] \\
FKj \arrow[r, "\alpha_j"'] & Lj
\end{tikzcd}$$

commutes for all $i \xrightarrow{f} j \in \mathcal{J}$. By Lemma 4.1, the transposes of α_j's consist a natural transformation from K to GL, since the square

$$\begin{tikzcd}
Ki \arrow[r, "\alpha_i^\flat"] \arrow[d, "Kf"'] & GLi \arrow[d, "GLf"] \\
Kj \arrow[r, "\alpha_j^\flat"'] & GLj
\end{tikzcd}$$

[23] Note that the functor categories $\mathcal{E}^{\mathcal{C}}$ and $\mathcal{E}^{\mathcal{D}}$ will not be locally small unless $\mathcal{C}$ and $\mathcal{D}$ are small. If local smallness of the functor categories is not a concern, then the hypothesis that $\mathcal{J}$ is small can be dropped.

commutes for all $i \xrightarrow{f} j \in \mathcal{J}$. Hence we just map $\alpha = (\alpha_j)$ to $(\alpha_j^\flat)$. The inverse of this map can be defined in the same way, hence we know that it is a bijection. It remains only to verify the naturality, which is straightforward by the naturality of $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd)$.

For the adjoint

$$\begin{tikzcd} \mathcal{E}^{\mathcal{C}} \arrow[r, shift left, "G^*"] \arrow[r, phantom, "\perp"] & \mathcal{E}^{\mathcal{D}} \arrow[l, shift left, "F^*"] \end{tikzcd}$$

we use the unit $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and counit $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ to establish natural bijections

$$\mathrm{Hom}_{\mathcal{E}^{\mathcal{D}}}(KG, L) \cong \mathrm{Hom}_{\mathcal{E}^{\mathcal{C}}}(K, LF)$$

for each $K \in \mathrm{Obj}\,\mathcal{E}^{\mathcal{C}}$ and $L \in \mathrm{Obj}\,\mathcal{E}^{\mathcal{D}}$. An element in $\mathrm{Hom}_{\mathcal{E}^{\mathcal{D}}}(KG, L)$ is a natural transformation $\alpha : KG \Rightarrow L$, we maps it to the natural morphism $\beta : K \Rightarrow LF$ whose component at $c \in \mathrm{Obj}\,\mathcal{C}$ is the composite

$$\beta_c \coloneqq Kc \xrightarrow{K\eta_c} KGFc \xrightarrow{\alpha_{Fc}} LFc$$

The naturality of β follows from these of η and α. The opposite map can be defined in a same way, which is indeed the inverse of this map by the triangle identities. The naturality of $\mathrm{Hom}_{\mathcal{E}^{\mathcal{D}}}(KG, L) \cong \mathrm{Hom}_{\mathcal{E}^{\mathcal{C}}}(K, LF)$ is again straightforward. $\square$

4.4 Adjunctions, limits, and colimits

In this section, we explore the interaction between adjoint functors and limits and colimits, giving our promised result that Theorem 3.9 is a special case of right adjoint functors preserving limits. First we see that how limits and colimits are special cases of adjoint functors:

Proposition 4.6. *A category $\mathcal{C}$ admits all limits of diagrams indexed by a small category $\mathcal{J}$ if and only if the constant diagram functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ admits a right adjoint, and admits all colimits of $\mathcal{J}$-indexed diagrams if and only if Δ admits a left adjoint:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\Delta"] & & \mathcal{C}^{\mathcal{J}} \arrow[ll, bend right=50, "\mathrm{colim}"', "\perp"] \arrow[ll, bend left=50, "\mathrm{lim}", "\perp"'] \end{tikzcd}$$

When these adjoints exist, their value at each $F \in \mathrm{Obj}\,\mathcal{C}^{\mathcal{J}}$ define exactly the (co)limit of F, in other words, they are the (co)limit functors introduced in Proposition 3.2.

Proof. These dual statements follow immediately from the defining universal properties of a limit and colimit. We give only the proof of the statement about limit; the proof for the statement about colimit follows similarly.

For $c \in \mathrm{Obj}\,\mathcal{C}$ and $F \in \mathrm{Obj}\,\mathcal{C}^{\mathcal{J}}$, the collection $\mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta c, F)$ is exactly the collection of cones over F with summit c, hence there is an object $\lim F \in \mathcal{C}$ together with an isomorphism

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta c, F) \cong \mathrm{Hom}_{\mathcal{C}}(c, \lim F)$$

that is natural in $c \in \mathcal{C}$ if and only if this limit exists (see Definition 3.3; note that $\mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta -, F)$ is exactly $\mathrm{Cone}(-, F)$). It follows that if Δ admits a right adjoint, then all these limits exist. Conversely, if all these limits exist, then all such natural isomorphisms exist and by Proposition 3.2 we have the functor $\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$, which is a right adjoint of Δ by those natural isomorphisms. Note that the naturality in $F \in \mathrm{Obj}\,\mathcal{C}^{\mathcal{J}}$ follows from the construction of the functor lim. $\square$

Here comes our promised fact:

Theorem 4.4 (RAPL). *Right adjoints preserve limits.*

Proof. Given a right adjoint $G : \mathcal{D} \to \mathcal{C}$. Given a diagram $K : \mathcal{J} \to \mathcal{D}$ with a limit cone $\lambda : \lim K \Rightarrow K$ in $\mathcal{D}$, which could be illustrated by commutative diagrams

$$\begin{tikzcd} \lim K \arrow[d, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ Ki \arrow[r, "Kf"'] & Kj \end{tikzcd}$$

for all $i \xrightarrow{f} j \in \mathcal{J}$. Apply G to λ and we obtain a cone $G\lambda : G \lim K \Rightarrow GK$ in $\mathcal{C}$:

$$
\begin{tikzcd}
G\lim K \arrow[d, "G\lambda_i"'] \arrow[dr, "G\lambda_j"] & \\
GKi \arrow[r, "GKf"'] & GKj
\end{tikzcd}
$$

For any cone $\mu : c \Rightarrow GK$ over GK, we need to show that there exists a unique morphism $c \to G\lim K$ that is compatible with the legs of the two cones. Transpose each leg of μ via $\mathrm{Hom}_{\mathcal{C}}(c, GKj) \cong \mathrm{Hom}_{\mathcal{D}}(Fc, Kj)$ and by the naturality we obtain a cone $(\mu_j^\sharp) : Fc \Rightarrow K$ over K. Now there exists a unique morphism $\tau : Fc \to \lim K$ such that

$$
\begin{tikzcd}
Fc \arrow[d, dashed, "\exists!\tau"] \arrow[dd, bend right=50, "\mu_i^\sharp"'] \arrow[ddr, bend left=40, "\mu_j^\sharp"] & \\
\lim K \arrow[d, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
Ki \arrow[r, "Kf"'] & Kj
\end{tikzcd}
$$

commutes for all $i \xrightarrow{f} j \in \mathcal{J}$. The map τ transposes to $\tau^\flat : c \to G\lim K$ and by Lemma 4.1 $\tau^\flat$ satisfies that

$$
\begin{tikzcd}
c \arrow[d, "\tau^\flat"] \arrow[dd, bend right=50, "\mu_i"'] \arrow[ddr, bend left=40, "\mu_j"] & \\
G\lim K \arrow[d, "G\lambda_i"'] \arrow[dr, "G\lambda_j"] & \\
GKi \arrow[r, "GKf"'] & Kj
\end{tikzcd}
$$

commutes for all $i \xrightarrow{f} j \in \mathcal{J}$ (or one can simply use the naturality). Such $\tau^\flat$ is unique by the bijectivity of the transposition, hence we are done. $\square$

More concisely, the argument just presented describes a series of natural isomorphisms:

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta c, GK) \cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{J}}}(F\Delta c, K) \cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{J}}}(\Delta Fc, K) \cong \mathrm{Hom}_{\mathcal{D}}(Fc, \lim_{\mathcal{J}} K) \cong \mathrm{Hom}_{\mathcal{C}}(c, G\lim_{\mathcal{J}} K)$$

which says that $G\lim_{\mathcal{J}} K$ represents the functor $\mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta -, GK) \cong \mathrm{Cone}(-, F)$, telling us that it defines a limit for the diagram GK.

Dually, of course:

Theorem 4.5 (LAPC). *Left adjoints preserve colimits.*

For a certain important class of adjoint functors, Theorem 4.4 and Theorem 4.5 take on a stronger form.

Definition 4.3. A **reflective subcategory** of a category $\mathcal{C}$ is a full subcategory $\mathcal{D}$ so that the inclusion admits a left adjoint, called the **reflector** or **localization**:

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "L"']
\end{tikzcd}
$$

Where possible, we identify the full subcategory $\mathcal{D}$ with its image in $\mathcal{C}$, declining in particular to introduce notation for the inclusion functor. With this convention in mind, the components of the unit have the form $c \to Lc$. The following lemma implies that the components of the counit $Ld \to d$ are isomorphisms. Via the counit, any object in a reflective subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ is naturally isomorphic to its reflection back into that subcategory.

A monomorphism $s : x \to y$ is a **split monomorphism** if it has a left inverse, i.e., a morphism $r : y \to x$ such that $rs = 1_x$. Dually, a epimorphism is a **split epimorphism** if it has a right inverse. Note that the left (right) inverse would imply the mono (epi) condition.

Lemma 4.6. *Consider an adjunction*

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"']
\end{tikzcd}
$$

with counit $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$. Then:

(i) G is faithful if and only if each component of ϵ is an epimorphism.

(ii) G is full if and only if each component of ϵ is a split monomorphism.

(iii) G is full and faithful if and only if ϵ is an isomorphism.

Dually, writing $\eta : 1_{\mathcal{C}} \Rightarrow GF$ for the unit:

(i) F is faithful if and only if each component of η is a monomorphism.

(ii) F is full if and only if each component of η is a split epimorphism.

(iii) F is full and faithful if and only if η is an isomorphism.

Proof. By duality we prove only the counit statements.
(i): For any $d \in \operatorname{Obj} \mathcal{D}$, we have the component $\epsilon_d : FGd \to d$. ϵ_d is an epimorphism if and only if for any $f, g : d \to d'$ such that $f\epsilon_d = g\epsilon_d$ we have $f = g$. By the formula in the proof of Proposition 4.1, the transpose of $f\epsilon_d = g\epsilon_d$ gives $Gf = Gg$, hence each component of ϵ is an epimorphism if and only if $Gf = Gg$ implies $f = g$, which is exactly that G is faithful.
(ii) As is observed above, for any $d \in \operatorname{Obj} \mathcal{D}$ and any morphism $f : d \to d'$, the transpose of $f\epsilon_d$ is $Gf : Gd \to Gd'$. If each component of ϵ is a split monomorphism, then for any $d \in \operatorname{Obj} \mathcal{D}$ there is a morphism $r_d : d \to FGd$ such that $r_d\epsilon_d = 1_{FGd}$. For any $h : Gd \to Gd'$, its transpose is $h^\sharp = h^\sharp r_d \epsilon_d : FGd \to d'$, telling that $Gh^\sharp r_d = h$, hence G is full. If G is full, then for each $d \in \operatorname{Obj} \mathcal{D}$, consider the component of unit $\eta_{Gd} : Gd \to GFGd$, there exists $r_d : d \to FGd$ such that $Gr_d = \eta_{Gd}$. Since the transpose of η_{Gd} is 1_{FGd} by Lemma 4.2, we have $r_d\epsilon_d = 1_{FGd}$, telling that each component of ϵ is a split monomorphism.
(iii) It suffices to show that a morphism is an isomorphism if and only if it is both an epimorphism and a split monomorphism. It is obvious for an isomorphism to be both an epimorphism and a split monomorphism. For the converse, let $f : x \to y$ be our morphism and $r : y \to x$ be a left inverse of f, then $rf = 1_x$. Compose f by left and we have $frf = f = 1_y f$. Since f is epic, this gives $fr = 1_y$, finishing the proof. $\square$

There is a simple example of reflective category:

Example 4.4. Abelian groups define a reflective subcategroy $\mathrm{Ab} \hookrightarrow \mathrm{Grp}$. The reflector carries a group G to its abelianization, see Example 2.2 (ii). The universal property of abelianization tells rightaway the action of reflector on morphisms and the fact that the reflector is left adjoint to the inclusion.

It is good to know the following conclusions, which strengthen Theorem 4.4 and Theorem 4.5. We shall not prove them here though.[24]

Proposition 4.7. *If $\mathcal{D} \hookrightarrow \mathcal{C}$ is a reflective subcategory, then:*

(i) The inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ creates all limits that $\mathcal{C}$ admits.

(ii) $\mathcal{D}$ has all colimits that $\mathcal{C}$ admits, formed by applying the reflector to the colimit in $\mathcal{C}$.

Corollary 4.6.1. Cat *is complete and cocomplete.*

4.5 Existence of adjoint functors

Given a functor $F : \mathcal{C} \to \mathcal{D}$ and an object $d \in \operatorname{Obj} \mathcal{D}$, the **comma category** $d \downarrow F$ consists of pairs $(c \in \operatorname{Obj} \mathcal{C}, f : d \to Fc \in \operatorname{Mor} \mathcal{D})$ as objects and a morphism from $f : d \to Fc$ to $f' : d \to Fc'$ is a morphism $h : c \to c' \in \operatorname{Mor} \mathcal{C}$ such that the triangle

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[dr, "f'"'] & Fc \arrow[d, "Fh"] \\ & Fc' \end{tikzcd}$$

commutes. If $\mathcal{D}$ is locally small, then $d \downarrow F$ is isomorphic to the category of elements for the functor $\operatorname{Hom}_{\mathcal{D}}(s, F-) : \mathcal{C} \to \mathrm{Set}$, as one can verify.

Lemma 4.7. *A functor $G : \mathcal{D} \to \mathcal{C}$ admits a left adjoint if and only if for each $c \in \operatorname{Obj} \mathcal{C}$ the comma category $c \downarrow G$ has an initial object.*

Proof. If a left joint $F \dashv G$ exists, then the component of the unit at c defines an initial object $\eta_c : c \to GFc$ in the comma category $c \downarrow G$: for any $f : c \to Gd \in \operatorname{Mor} \mathcal{D}$, the transpose formula in proof of Proposition 4.1 tells that $f = (G(\epsilon_d \cdot Ff))\, \eta_c$, and the bijectivity gives us the uniqueness. Moreover, if $\mathcal{C}$ is locally small, then the left adjoint F gives a representation $\operatorname{Hom}_{\mathcal{D}}(Fc, -) \cong \operatorname{Hom}_{\mathcal{C}}(c, G-)$, hence Proposition 2.1 tells that there is an initial object in the category $\int \operatorname{Hom}_{\mathcal{C}}(c, G-) \cong c \downarrow G$.

Conversely, forget the left adjoint $F \dashv G$, suppose each $c \downarrow G$ admits an initial object, which we denote by $\eta_c : c \to GFc$, where Fc is an object in $\mathcal{D}$ given by the chosen initial object in $c \downarrow G$, namely we constructed a function $F : \operatorname{Obj} \mathcal{C} \to \operatorname{Obj} \mathcal{D}$ in this way. We now extend F to a functor which is supposed to be a left adjoint to G. For each morphism $f : c \to c' \in \operatorname{Mor} \mathcal{C}$, we take $Ff : Fc \to Fc'$ to be the unique morphism in $\mathcal{D}$, by the initiality of η_c, such that the square

[24] For interested reader who wants to know the proof, they are Proposition 4.5.15 and Corollary 4.5.16 in Emily Riehl, *Category Theory in Context.*

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f"'] & GFc \arrow[d, dashed, "GFf"] \\ c' \arrow[r, "\eta_{c'}"'] & GFc' \end{tikzcd}$$

commutes. The functoriality of F follows from the uniqueness of these choices, hence this construction defines a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow GF$.

Now we have an opposite pair of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ along with a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow GF$ which is very likely to be a unit. Enlightened by the formula of transpose in proof of Proposition 4.1, we consider the natural transformation $\phi : \mathrm{Hom}_{\mathcal{D}}(F-, -) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-, G-)$ whose components

$$\phi_{c,d} : \mathrm{Hom}_{\mathcal{D}}(Fc, d) \to \mathrm{Hom}_{\mathcal{C}}(c, Gd)$$

are defined by

$$\phi_{c,d}(g) \coloneqq c \xrightarrow{\eta_c} GFc \xrightarrow{Gg} Gd$$

for any $g : Fc \to d \in \mathrm{Mor}\,\mathcal{D}$. For each $c \in \mathrm{Obj}\,\mathcal{C}$ and $d \in \mathrm{Obj}\,\mathcal{D}$, the initiality of η_c in $c \downarrow G$ gives an inverse of $\phi_{c,d}$, hence ϕ turns to be a natural isomorphism, concluding that F is left adjoint to G. □

A functor that preserves all limits is called **continuous**; a functor is called **cocontinuous** if it preserves all colimits. Lemma 4.7 reduces the problem of finding a left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ to the problem of finding an initial object in the comma category $c \downarrow G$ for each $c \in \mathrm{Obj}\,\mathcal{C}$. Since $c \downarrow G$ is isomorphic to the category of elements for $\mathrm{Hom}_{\mathcal{C}}(c, G-) : \mathcal{D} \to \mathrm{Set}$,[25] Proposition 3.4 applies and we see that

Lemma 4.8. *For any functor $G : \mathcal{D} \to \mathcal{C}$ and object $c \in \mathrm{Obj}\,\mathcal{C}$, the associated forgetful functor $\prod : c \downarrow G \to \mathcal{D}$ strictly creates the limit of any diagram whose limit exists in $\mathcal{D}$ and is preserved by G. In particular, if $\mathcal{D}$ is complete and G is continuous, then $c \downarrow G$ is complete.*

Observe that an initial object is equally the limit of the identity functor: Given a limit cone $(\lambda_c : l \to c)_{c \in \mathrm{Obj}\,\mathcal{C}}$ over the identity $1_{\mathcal{C}}$, for any morphism $f : l \to c$, we show that $f = \lambda_c$. By the cone condition we have $f\lambda_l = \lambda_c$. Considering λ_c as a morphism in the diagram $1_{\mathcal{C}}$, the cone condition tells us that $\lambda_c \lambda_l = \lambda_c$, hence λ_l defines a factorization of the cone $\lambda : l \Rightarrow 1_{\mathcal{C}}$ through iteself. By the uniqueness of such factorizations, we see that $\lambda_l = 1_l$, concluding that $f = \lambda_c$.

Applying Lemma 4.8, for continuous G, a limit of the identity functor on $c \downarrow G$ exists if the limit of the forgetful functor $\prod : c \downarrow G \to \mathcal{D}$ exists in $\mathcal{D}$. This argument seems to say that all continuous functors whose domains are complete should admit left adjoints, which is not the case since $c \downarrow G$ needs not to be small, hence $\mathcal{D}$, though complete, may not admit a limit of the diagram $\prod : c \downarrow G \to \mathcal{D}$. The *adjoint functor theorems*, the two most common of which are discussed here, supply conditions under which this large limit can be reduced to a small one that $\mathcal{D}$ possesses.

Theorem 4.9 (General Adjoint Functor Theorem). *Let $G : \mathcal{D} \to \mathcal{C}$ be a continuous functor whose domain is locally small and complete. Suppose that G satisfies the following **solution set condition**:*

- *For every $c \in \mathcal{C}$ there exists a set of morphisms $\Phi_c = \{f_i : c \to Gd_i\}$ so that any $f \to c \to Gd$ factors through some $f_i \in \Phi_c$ along a morphism $d_i \to d$ in $\mathcal{D}$*

Then G admits a left adjoint.

The solution set condition says exactly that $\{f_i : c \to Gd_i\}$ is a set of jointly weakly initial objects in the category $c \downarrow G$.

Definition 4.4. (i) An object c in a category $\mathcal{C}$ is **weakly initial** if for every $c' \in \mathrm{Obj}\,\mathcal{C}$ there exists a morphism $c \to c'$ (possibly not unique).

(ii) A collection of objects $\Phi = \{c_i\}$ in a category $\mathcal{C}$ is **jointly weakly initial** if for every $c \in \mathrm{Obj}\,\mathcal{C}$ there exists some $c_i \in \Phi$ for which there exists a morphism $c_i \to c$.

In the presence of certain limits, an initial object can be built from a jointly weakly initial set of objects, which proves Theorem 4.9 along with the previous arguments:

Lemma 4.10. *If $\mathcal{C}$ is complete, locally small and has a jointly weakly initial set of objects Φ, then $\mathcal{C}$ has an initial object.*

Proof. Form the limit l of the inclusion into $\mathcal{C}$ of the full subcategory spanned by the objects of Φ; because $\mathcal{C}$ is locally small, this diagram is small and because $\mathcal{C}$ is complete, this limit exists. Using the limit cone $(\kappa_k : l \to k)_{k \in \Phi}$, we may define a morphism $\lambda_c : l \to c$ for each $c \in \mathrm{Obj}\,\mathcal{C}$:

[25] Here we assumed $\mathcal{C}$ to be locally small. This assumption can be weakened: If there exists an upper boundary for the size of $\mathrm{Hom}_{\mathcal{C}}(c, c')$, then replace Set by the category of all classes no larger than that upper boundary, all arguments apply and the result still holds.

- For objects $k \in \Phi$, define $\lambda_k := \kappa_k$
- For each remaining object $c \in \mathrm{Obj}\,\mathcal{C} - \Phi$, choose a morphism $h_c : k \to c$ with $k \in \Phi$, and define $\lambda_c := l \xrightarrow{\kappa_k} k \xrightarrow{h_c} c$.

The maps λ_c show that l is weakly initial. To show that l is initial, we show that the morphisms λ_c define a cone $\lambda : l \Rightarrow 1_{\mathcal{C}}$ with summit l over the identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ with the property that the component λ_l is the identity morphism, which gives that λ is a limit cone, concluding that l is initial.

To show that the maps λ_c define a cone, consider any morphism $f : c \to c' \in \mathcal{C}$. Since $\mathcal{C}$ is complete, the pullback over $k \xrightarrow{h_c} c \xrightarrow{f} c' \xleftarrow{h_{c'}} k'$ exists. In particular, there is a cone over that diagram; let p be the summit of the cone:

$$\begin{tikzcd}
& p \arrow[r] \arrow[dl] & k' \arrow[d, "h_{c'}"] \\
k \arrow[r, "h_c"'] & c \arrow[r, "f"'] & c'
\end{tikzcd}$$

Since Φ is jointly weakly initial, there is some $k'' \to p$ with $k'' \in \Phi$. Now consider the whole picture:

$$\begin{tikzcd}
& l \arrow[d, "\kappa_{k''}"] \arrow[ddl, bend right, "\kappa_k"'] \arrow[dr, bend left, "\kappa_{k'}"] & \\
& k'' \arrow[d] \arrow[ddl, dashed] \arrow[dr, dashed] & \\
& p \arrow[r] \arrow[dl] & k' \arrow[d, "h_{c'}"] \\
k \arrow[r, "h_c"'] & c \arrow[r, "f"'] & c'
\end{tikzcd}$$

The dashed composite morphisms live in the full subcategory spanned by the objects in Φ, hence the top triangles, living in the cone diagram of the limit cone κ, commutes. It follows that $f\lambda_c = fh_c\kappa_k = h_{c'}\kappa_{k'} = \lambda_{c'}$, telling us that λ defines a cone.

Finally, the cone condition for λ implies that the triangle

$$\begin{tikzcd}
l \arrow[d, "\lambda_l"'] \arrow[dr, "\lambda_k = \kappa_k"] & \\
l \arrow[r, "\kappa_k"'] & k
\end{tikzcd}$$

commutes for all $k \in \Phi$. This tells us that λ_l defines a factorization of the limit cone κ through itself; whence $\lambda_l = 1_l$.$\square$

Our next adjoint functor theorem requires a few preliminary definitions.

Definition 4.5. A **separating set** or **separator** for a category $\mathcal{C}$ is a set Φ of objects that can distinguish between distinct parallel morphisms in the following sense: given $f, g : x \rightrightarrows y$, if $f \neq g$ then there exists some $h : c \to x$ with $c \in \Phi$ so that $fh \neq gh$. A **coseparating set** in $\mathcal{C}$ is a separating set in $\mathcal{C}^{op}$ (just reverse the arrows).

Definition 4.6. A **subobject** of an object $c \in \mathrm{Obj}\,\mathcal{C}$ is a monomorphism $c' \rightarrowtail c$ with codomain c. Isomorphic subobjects, that is, subobjects $c' \rightarrowtail c \leftarrowtail c''$ with a commuting isomorphism $c' \cong c''$, are typically identified (to be a same subobject).

Definition 4.7. The **intersection** of a family of subobjects of c is the limit of the diagram of monomorphisms with codomain c.

The induced map from the limit to c is again a monomorphism as one can verify. It follows that the leges of the limit cone are also monomorphisms, since if gf is monic, then f is monic. Thus the intersection is the "maximal" subobject that is contained in, i.e., is a subobject of, every member of the family of subobjects.

In particular, for two subobjects $c' \rightarrowtail c \leftarrowtail c''$, their intersection is the pullback

$$\begin{tikzcd}
p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & c'' \arrow[d, tail] \\
c' \arrow[r, tail] & c
\end{tikzcd}$$

If these two subojects are isomorphic, then their intersection is again isomorphic two each of them.

Theorem 4.11 (Special Adjoint Functor Theorem). *Let $G : \mathcal{D} \to \mathcal{C}$ be a continuous functor whose domain is complete and whose domain and codomain are locally small. If $\mathcal{D}$ has a small coseparating set and every collection of subobjects of a fixed object in $\mathcal{D}$ admits an intersection, then G admits a left adjoint.*

In many categories, each object admits only a set's worth of subobjects up to isomorphis, in which case completeness implies that every collection of subobjects admits an intersection. The purpose for these hypotheses is that they are used to construct the initial objects sought for in Lemma 4.7.

Lemma 4.12. *Suppose $\mathcal{C}$ is locally small, complete, has a small coseparating set Φ, and has the property that every collection of subobjects has an intersection, then $\mathcal{C}$ has an initial object.*

Proof. Form the product $p = \prod_{k\in\Phi} k$ of the objects in the coseparating set and form the intersection $i \hookrightarrow p$ of all of the subobjects of p. We claim that i is initial.

To simplify the notation, recall the notation of iterated products introduced in Example 3.7, given $c \in \operatorname{Obj}\mathcal{C}$, we have $\prod_{k\in\Phi} k^{\operatorname{Hom}_{\mathcal{C}}(c,k)} = \prod_{f\in\bigcup_{k\in\Phi}\operatorname{Hom}_{\mathcal{C}}(c,k)} \operatorname{cod} f$. To say that Φ is coseparating is to say that for every $c \in \operatorname{Obj}\mathcal{C}$ the canonical map

$$c \rightarrowtail \prod_{k\in\Phi} k^{\operatorname{Hom}_{\mathcal{C}}(c,k)}$$

whose component at an element $f \in \operatorname{Hom}_{\mathcal{C}}(c,k)$ is that morphism $f : c \to k$, is a monomorphism. Since $\mathcal{C}$ is complete, there is a map $\prod_{k\in\Phi} k \to \prod_{k\in\Phi} k^{\operatorname{Hom}_{\mathcal{C}}(c,k)}$, which is the product over $k \in \Phi$ of maps $\Delta_k : k \to k^{\operatorname{Hom}_{\mathcal{C}}(c,k)}$ defined to be the identity 1_k on each component of the power $k^{\operatorname{Hom}_{\mathcal{C}}(c,k)}$. The pullback

$$\begin{tikzcd}
p_c \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & c \arrow[d, tail] \\
p = \prod_{k\in\Phi} k \arrow[r, "\prod_{k\in\Phi}\Delta_k"'] & \prod_{k\in\Phi} k^{\operatorname{Hom}_{\mathcal{C}}(c,k)}
\end{tikzcd}$$

defines a subobject[26] p_c of p and thus a map $i \to p_c \to c$ from the intersection to c. There cannot be more than one arrow from i to c because for any pair of parallel arrows $i \rightrightarrows c$, the equalizer $e \rightarrowtail i \rightrightarrows c$ gives a subobject of i, hence gives a subobject of p, while i is the intersection of all subjects of p, so the diagram below, which is a part of the limit diagram of i with legs f and g, commutes:

$$\begin{tikzcd}
i \arrow[r, tail, "f"] \arrow[d, tail, "g"'] & e \arrow[drrr] & & & \\
e \arrow[r, tail] & i \arrow[r, tail, "f"'] & e \arrow[r, tail] & p
\end{tikzcd}$$

The monomorphism $i \xrightarrow{f} e \to p$ tells that the composition $i \xrightarrow{g} e \to i$ is the identity on i, hence the monomorphism $e \to i$ tells that the composition $e \to i \xrightarrow{g} e$ is the identity on e, concluding that $e \cong i$; whence the parallel arrows $i \rightrightarrows c$ are equal to each other. Therefore i is initial. $\square$

Now we prove Theorem 4.11.

Proof. By Lemma 4.8, the comma category $c \downarrow G$ for each $c \in \operatorname{Obj}\mathcal{C}$ is complete, and since monomorphisms in $c \downarrow G$ are preserved and reflected by the forgetful functor $\prod : c \downarrow G \to \mathcal{D}$, $c \downarrow G$ has intersections of subobjects created by $\prod$. $c \downarrow G$ is locally small since $\mathcal{D}$ is. The small coseparating set Φ of $\mathcal{D}$ gives a small coseparating set

$$\Phi' := \{c \to Gd \mid d \in \Phi\}$$

of $c \downarrow G$ since $\mathcal{C}$ is locally small. Apply Lemma 4.12 and we know that $c \downarrow G$ has an initial object, hence G admits a left adjoint by Lemma 4.7. $\square$

The adjoint functor theorems have a number of corollaries.

Corollary 4.12.1. *Suppose $\mathcal{C}$ is locally small, complete, has a small coseparating set, and has the property that every collection of subobjects of a fixed object has an intersection. Then $\mathcal{C}$ is cocomplete.*

Proof. For any small category $\mathcal{J}$, if $\mathcal{C}$ is locally small, then $\mathcal{C}^{\mathcal{J}}$ is again locally small. The constant diagram functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ preserves limits as one can verify, hence is continuous. Applying Theorem 4.11, Δ has a left adjoint, which by Proposition 4.6 tells that $\mathcal{C}$ has all $\mathcal{J}$-shaped colimits. $\square$

The argument used to prove the adjoint functor theorems also specialize to give conditions under which a set-valued limit-preserving functor is representable.

Corollary 4.12.2. *Suppose $\mathcal{C}$ is locally small, complete, has a small coseparating set, and has the property that every collection of subobjects of a fixed object has an intersection. Then any continuous functor $F : \mathcal{C} \to \mathrm{Set}$ is representable.*

[26]Verify that for the pullback $\begin{tikzcd} p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & c \arrow[d, tail] \\ b \arrow[r] & a \end{tikzcd}$ where $c \rightarrowtail a$ is a monomorphism, its leg $p \to b$ is a monomorphism.

Proof. By Theorem 4.11, F has a left adjoint $L : \mathrm{Set} \to \mathcal{C}$. In particular, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(L(*), c) \cong \mathrm{Hom}_{\mathrm{Set}}(*, Fc) \cong Fc$$

where $* \in \mathrm{Obj}\,\mathrm{Set}$ denotes the singleton set. By naturality in c, the object $L(*) \in \mathrm{Obj}\,\mathcal{C}$ represents F. □

Theorem 4.13 (Freyd's Representability Theorem). *Let $F : \mathcal{C} \to \mathrm{Set}$ be a continuous functor and suppose that $\mathcal{C}$ is complete and locally small. If F satisfies the solution set condition:*

- *There exists a set Φ of objects of $\mathcal{C}$ so that for any $c \in \mathrm{Obj}\,\mathcal{C}$ and any element $x \in Fc$, there exists an $s \in \Phi$, an element $y \in Fs$, and a morphism $f : s \to c$ so that $Ff(y) = x$.*

then F is representable.

Proof. The solution set condition defines a jointly weakly initial set of objects in the comma category $* \downarrow F \cong \int F$, where $* \in \mathrm{Obj}\,\mathrm{Set}$ is a singleton set. By Lemma 4.8, since Set is complete, $\int F$ is complete, hence Lemma 4.10 tells that it has an initial object. Proposition 2.1 thus tells that F is representable. □