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Multiattribute decision making based on interval-valued intuitionistic fuzzy values

Shyi-Ming Chen [a,b,*], Li-Wei Lee [c], Hsiang-Chuan Liu [d], Szu-Wei Yang [e]

[a] Department of Computer Science and Information Engineering, National Taiwan University of Science and Technology, Taipei, Taiwan, ROC
[b] Graduate Institute of Educational Measurement and Statistics, National Taichung University of Education, Taichung, Taiwan, ROC
[c] Department of Computer and Communication Engineering, De Lin Institute of Technology, New Taipei City, Taiwan, ROC
[d] Department of Biomedical Informatics, Asia University, Taichung, Taiwan, ROC
[e] Department of Education, National Taichung University of Education, Taichung, Taiwan, ROC

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ABSTRACT

In this paper, we present a new multiattribute decision making method based on the proposed interval-valued intuitionistic fuzzy weighted average operator and the proposed fuzzy ranking method for intuitionistic fuzzy values. First, we briefly review the concepts of interval-valued intuitionistic fuzzy sets and the Karnik–Mendel algorithms. Then, we propose the intuitionistic fuzzy weighted average operator and interval-valued intuitionistic fuzzy weighted average operator, based on the traditional weighted average method and the Karnik–Mendel algorithms. Then, we propose a fuzzy ranking method for intuitionistic fuzzy values based on likelihood-based comparison relations between intervals. Finally, we present a new multiattribute decision making method based on the proposed interval-valued intuitionistic fuzzy weighted average operator and the proposed fuzzy ranking method for intuitionistic fuzzy values. The proposed method provides us with a useful way for multiattribute decision making based on interval-valued intuitionistic fuzzy values.

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1. Introduction

Atanassov (1986) proposed the concepts of intuitionistic fuzzy sets, where each element in an intuitionistic fuzzy set has a membership degree and a non-membership degree between zero and one, respectively. Intuitionistic fuzzy sets are the extension of fuzzy sets (Zadeh, 1965), where each element in a fuzzy set has a membership degree between zero and one. Moreover, vague sets (Gau & Buehrer, 1993) and interval-valued fuzzy sets (Zadeh, 1975) are the extensions of fuzzy sets. Burillo and Bustince (1996) pointed out that vague sets are intuitionistic fuzzy sets. An interval-valued fuzzy set is characterized by an interval-valued membership function, where the membership degree of an element in an interval-valued fuzzy set is represented by an interval (Zadeh, 1975). Li (2010b) pointed out that it was proven that the interval-valued fuzzy set is isomorphic to the intuitionistic fuzzy set in the sense of Atanassov (Atanassov & Gargov, 1989; Deschrijver & Kerre, 2003, 2007; Dubois, Gottwald, Hajek, Kacprzyk, & Prade, 2005). Dubois et al. (2005) pointed out that Atanassov's intuitionistic fuzzy sets have similar notions and are isomorphic to interval-valued fuzzy sets, even if their interpretive settings and motivation are quite different. They also pointed out that the concepts of interval-valued fuzzy sets capture the idea of ill-known membership degree, whereas the concepts of intuitionistic fuzzy sets start from the idea of evaluating the membership degree and the non-membership degree of an element in intuitionistic fuzzy sets independently. In recent years, some researchers focused on the research topics of the applications of intuitionistic fuzzy sets (Atanassov, Pasi, & Yager, 2005; De, Biswas, & Roy, 2001; Li, 2005; Li & Cheng, 2002; Szmidt & Kacprzyk, 2002; Tan & Chen, 2010; Xu, 2007; Ye, 2009).

Atanassov and Gargov (1989) proposed the concepts of interval-valued intuitionistic fuzzy sets which are extensions of intuitionistic fuzzy sets, where the membership degree and the non-membership degree of an element in an interval-valued intuitionistic fuzzy set are represented by intervals in [0,1] rather than crisp values between zero and one, respectively. In recent years, some researchers focused on the research topics of the applications of interval-valued intuitionistic fuzzy sets (Chen & Lee, 2011; Li, 2010a, 2010b; Nayagam, Muralikrishnan, & Sivaraman, 2011; Wu & Chen, 2007).

One of the well-known multiattribute decision making methods is the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method (Hwang & Yoon, 1981). Li (2010b) presented a TOPSIS-based nonlinear-programming method for multiattribute decision making in which both the ratings of alternatives with respect to the attributes and the weights of attributes are represented by interval-valued intuitionistic fuzzy values. He also pointed out that it is very difficult to solve the nonlinear-programming models with many unknown variables and that it will

* Corresponding author. Tel.: +886 2 27376417; fax: +886 2 27301081.
E-mail address: smchen@mail.ntust.edu.tw (S.-M. Chen).

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cost a large amount of computation time. Moreover, he also pointed out that the choice of the interval-valued intuitionistic fuzzy positive ideal solution and the interval-valued intuitionistic fuzzy negative ideal solution is a sensitive problem and is not easy to determine.

Karnik and Mendel (2001) presented the Karnik–Mendel algorithms, which are iterative procedures and have successfully been used to deal with the fuzzy weighted average (FWA) (Liu & Mendel, 2008) and the linguistic weighted average (LWA) (Wu & Mendel, 2007). Mendel and Liu (2007) pointed out that the Karnik–Mendel algorithms (Karnik & Mendel, 2001) are known to converge monotonically and super-exponentially fast.

In this paper, we present a new multiattribute decision making method based on the proposed interval-valued intuitionistic fuzzy weighted average operator and the proposed fuzzy ranking method for intuitionistic fuzzy values. First, we briefly review the concepts of interval-valued intuitionistic fuzzy sets (Atanassov & Gargov, 1989) and the Karnik–Mendel algorithms (Karnik & Mendel, 2001). Then, we propose the intuitionistic fuzzy weighted average operator and the interval-valued intuitionistic fuzzy weighted average operator, based on the traditional weighted average method and the Karnik–Mendel algorithms. Then, we propose a fuzzy ranking method for intuitionistic fuzzy values based on likelihood-based comparison relations between intervals. Finally, we present a new multiattribute decision making method based on the proposed interval-valued intuitionistic fuzzy weighted average operator and the proposed fuzzy ranking method for intuitionistic fuzzy values. The proposed method provides us with a useful way for multiattribute decision making based on interval-valued intuitionistic fuzzy values.

The rest of this paper is organized as follows. In Section 2, we briefly review the concepts of interval-valued intuitionistic fuzzy sets (Atanassov & Gargov, 1989) and the Karnik–Mendel algorithms (Karnik & Mendel, 2001). In Section 3, we briefly review the TOPSIS-based nonlinear-programming method for multiattribute decision making from (Li, 2010b). In Section 4, we propose the intuitionistic fuzzy weighted average operator and the interval-valued intuitionistic fuzzy weighted average operator. In Section 5, we propose a new fuzzy ranking method for intuitionistic fuzzy values based on likelihood-based comparison relations between intervals. In Section 6, we propose a new multiattribute decision making method based on the proposed interval-valued intuitionistic fuzzy weighted average operator and the proposed fuzzy ranking method for intuitionistic fuzzy values. The conclusions are discussed in Section 7.

2. Preliminaries

2.1. Intuitionistic fuzzy sets (Atanassov, 1986)

Atanassov (1986) presented the concepts of intuitionistic fuzzy sets. An intuitionistic fuzzy set A in a finite set of the universe of discourse $X = \{x_1, x_2, \ldots, x_m\}$ can be represented as follows:

$$A = \{\langle x_i, \mu_A(x_i), \upsilon_A(x_i)\rangle | x_i \in X\}, \tag{1}$$

where $\mu_A(x_i)$ denotes the membership degree of element x_i belonging to the intuitionistic fuzzy set A, $\upsilon_A(x_i)$ denotes the non-membership degree of element x_i belonging to the intuitionistic fuzzy set A, $0 \leqslant \mu_A(x_i) \leqslant 1$, $0 \leqslant \upsilon_A(x_i) \leqslant 1$ and $0 \leqslant \mu_A(x_i) + \upsilon_A(x_i) \leqslant 1$. For convenience, an intuitionistic fuzzy value $\langle x_i, \mu_A(x_i), \upsilon_A(x_i)\rangle$ can be abbreviated into $\langle \mu_A(x_i), \upsilon_A(x_i)\rangle$, where the hesitation degree $\pi_A(x_i)$ of element x_i belonging to the intuitionistic fuzzy set A is shown as follows:

$$\pi_A(x_i) = 1 - \mu_A(x_i) - \upsilon_A(x_i). \tag{2}$$

2.2. Interval-valued intuitionistic fuzzy sets (Atanassov & Gargov, 1989)

Atanassov and Gargov (1989) presented the concepts of interval-valued intuitionistic fuzzy sets. An interval-valued intuitionistic fuzzy set A in a finite set in the universe of discourse $X = \{x_1, x_2, \ldots, x_m\}$ can be represented as follows:

$$A = \{\langle x_i, [\underline{\mu}_A(x_i), \bar{\mu}_A(x_i)], [\underline{\upsilon}_A(x_i), \bar{\upsilon}_A(x_i)]\rangle | x_i \in X\}, \tag{3}$$

where $[\underline{\mu}_A(x_i), \bar{\mu}_A(x_i)]$ denotes the interval membership degree of element x_i belonging to the interval-valued intuitionistic fuzzy set A, $[\underline{\upsilon}_A(x_i), \bar{\upsilon}_A(x_i)]$ denotes the interval non-membership degree of element x_i belonging to the interval-valued intuitionistic fuzzy set A, $0 \leqslant \underline{\mu}_A(x_i) \leqslant \bar{\mu}_A(x_i) \leqslant 1$, $0 \leqslant \underline{\upsilon}_A(x_i) \leqslant \bar{\upsilon}_A(x_i) \leqslant 1$ and $0 \leqslant \bar{\mu}_A(x_i) + \bar{\upsilon}_A(x_i) \leqslant 1$. For convenience, an interval-valued intuitionistic fuzzy value $\langle x_i, [\underline{\mu}_A(x_i), \bar{\mu}_A(x_i)], [\underline{\upsilon}_A(x_i), \bar{\upsilon}_A(x_i)]\rangle$ can be abbreviated into $\langle [\underline{\mu}_A(x_i), \bar{\mu}_A(x_i)], [\underline{\upsilon}_A(x_i), \bar{\upsilon}_A(x_i)]\rangle$, where the interval hesitation degree $\pi_A(x_i)$ of element x_i belonging to the interval-valued intuitionistic fuzzy set A is shown as follows:

$$\pi_A(x_i) = [1 - \bar{\mu}_A(x_i) - \bar{\upsilon}_A(x_i), 1 - \underline{\mu}_A(x_i) - \underline{\upsilon}_A(x_i)]. \tag{4}$$

If $\underline{\mu}_A(x_i) = \bar{\mu}_A(x_i)$ and $\underline{\upsilon}_A(x_i) = \bar{\upsilon}_A(x_i)$ for every element x_i in the interval-valued intuitionistic fuzzy set A, then the interval-valued intuitionistic fuzzy set A shown in Eq. (3) is reduced into an intuitionistic fuzzy set A shown in Eq. (1).

Xu (2007) pointed out that the intuitionistic fuzzy value $\langle \mu_A(x_i), \upsilon_A(x_i)\rangle = [\mu_A(x_i), 1 - \upsilon_A(x_i)]$ has a physical interpretation. For example, if $\langle \mu_A(x_i), \upsilon_A(x_i)\rangle = \langle 0.5, 0.2\rangle = [0.5, 0.8]$, then we can see that $\mu_A(x_i) = 0.5$, $\upsilon_A(x_i) = 0.2$, $1 - \upsilon_A(x_i) = 0.8$ and $\pi_A(x_i) = 1 - \mu_A(x_i) - \upsilon_A(x_i) = 0.3$. It can be interpreted as "the vote for resolution is 5 in favor, 2 against and 3 abstentions". In the same way, we can see that the interval-valued intuitionistic fuzzy value $\langle [\underline{\mu}_A(x_i), \bar{\mu}_A(x_i)], [\underline{\upsilon}_A(x_i), \bar{\upsilon}_A(x_i)]\rangle = [[\underline{\mu}_A(x_i), \bar{\mu}_A(x_i)], [1 - \bar{\upsilon}_A(x_i), 1 - \underline{\upsilon}_A(x_i)]]$ also has a physical interpretation. For example, if $\langle [\underline{\mu}_A(x_i), \bar{\mu}_A(x_i)], [\underline{\upsilon}_A(x_i), \bar{\upsilon}_A(x_i)]\rangle = \langle [0.5, 0.6], [0.2, 0.3]\rangle = [[0.5, 0.6], [0.7, 0.8]]$, then we can see that $\underline{\mu}_A(x_i) = 0.5, \bar{\mu}_A(x_i) = 0.6$, $\underline{\upsilon}_A(x_i) = 0.2, \bar{\upsilon}_A(x_i) = 0.3, 1 - \bar{\upsilon}_A(x_i) = 0.7, 1 - \underline{\upsilon}_A(x_i) = 0.8$ and $\pi_A(x_i) = [1 - \bar{\mu}_A(x_i) - \bar{\upsilon}_A(x_i), 1 - \underline{\mu}_A(x_i) - \underline{\upsilon}_A(x_i)] = [0.1, 0.3]$. It can be interpreted as "the vote for resolution is between 5 and 6 in favor, between 2 and 3 against and between 1 and 3 abstentions".

2.3. Karnik–Mendel algorithms (Karnik & Mendel, 2001)

Karnik and Mendel (2001) presented the Karnik–Mendel algorithms. Let x_i and w_i be two intervals, where $x_i = [\underline{x}_i, \bar{x}_i]$, $w_i = [\underline{w}_i, \bar{w}_i], \underline{x}_i \leqslant \bar{x}_i, \underline{w}_i \leqslant \bar{w}_i$ and $1 \leqslant i \leqslant n$. The weighted average Y of the intervals x_i and w_i, where $1 \leqslant i \leqslant n$, is defined as follows (Karnik & Mendel, 2001):

$$Y = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i} = [\underline{y}, \bar{y}], \tag{5}$$

where

$$\underline{y} = \min_{w_i \in [\underline{w}_i, \bar{w}_i]} \frac{\sum_{i=1}^{n} \underline{x}_i w_i}{\sum_{i=1}^{n} w_i} = \frac{\sum_{i=1}^{L} \underline{x}_i \bar{w}_i + \sum_{i=L+1}^{n} \underline{x}_i \underline{w}_i}{\sum_{i=1}^{L} \bar{w}_i + \sum_{i=L+1}^{n} \underline{w}_i}, \tag{6}$$

$$\bar{y} = \max_{w_i \in [\underline{w}_i, \bar{w}_i]} \frac{\sum_{i=1}^{n} \bar{x}_i w_i}{\sum_{i=1}^{n} w_i} = \frac{\sum_{i=1}^{R} \bar{x}_i \underline{w}_i + \sum_{i=R+1}^{n} \bar{x}_i \bar{w}_i}{\sum_{i=1}^{R} \underline{w}_i + \sum_{i=R+1}^{n} \bar{w}_i}, \tag{7}$$

$\min\{\underline{x}_i\} \leqslant \underline{y} \leqslant \bar{y} \leqslant \max\{\bar{x}_i\}, 1 \leqslant i \leqslant n$, and L and R are the switch points. The Karnik–Mendel algorithms are reviewed as follows (Karnik & Mendel, 2001).

(1) Karnik–Mendel algorithm for computing $\underline{y}$ (Karnik & Mendel, 2001):

Step 1: Sort $\underline{x}_i$ in an increasing order and call the sorted $\underline{x}_i$ by the same name, i.e., $\underline{x}_1 \leqslant \underline{x}_2 \leqslant \cdots \leqslant \underline{x}_n$. Match the weights w_i with their respective $\underline{x}_i$ and renumber them, such that the weight w_i corresponds to the renumbered $\underline{x}_i$, where $1 \leqslant i \leqslant n$.

Step 2: Calculate the value y, shown as follows:

$$y = \frac{\sum_{i=1}^{n} \underline{x}_i w_i}{\sum_{i=1}^{n} w_i},$$

where $w_i = \frac{\underline{w}_i + \bar{w}_i}{2}$ and $1 \leqslant i \leqslant n$.

Step 3: Find the switch point k, such that

$$\underline{x}_k \leqslant y \leqslant \underline{x}_{k+1},$$

where $1 \leqslant k \leqslant n-1$.

Step 4: Calculate the value y', shown as follows:

$$y' = \frac{\sum_{i=1}^{n} \underline{x}_i w_i}{\sum_{i=1}^{n} w_i},$$

where

$$w_i = \begin{cases} \bar{w}_i, & \text{if } i \leqslant k \\ \underline{w}_i, & \text{if } i > k \end{cases}$$

and $1 \leqslant i \leqslant n$.

Step 5: If $y' = y$, then let $\underline{y} = y$, $L = k$ and **Stop**. Otherwise, let $y = y'$ and go to **Step 3**.

(2) Karnik–Mendel algorithm for computing $\bar{y}$ (Karnik & Mendel, 2001).

Step 1: Sort $\bar{x}_i$ in an increasing order and call the sorted $\bar{x}_i$ by the same name, i.e., $\bar{x}_1 \leqslant \bar{x}_2 \leqslant \cdots \leqslant \bar{x}_n$. Match the weights w_i with their respective $\bar{x}_i$ and renumber them, such that the weight w_i corresponds to the renumbered $\bar{x}_i$, where $1 \leqslant i \leqslant n$.

Step 2: Calculate the value y, shown as follows:

$$y = \frac{\sum_{i=1}^{n} \bar{x}_i w_i}{\sum_{i=1}^{n} w_i},$$

where $w_i = \frac{\underline{w}_i + \bar{w}_i}{2}$ and $1 \leqslant i \leqslant n$.

Step 3: Find the switch point k, such that

$$\bar{x}_k \leqslant y \leqslant \bar{x}_{k+1},$$

where $1 \leqslant k \leqslant n-1$.

Step 4: Calculate the value y', shown as follows:

$$y' = \frac{\sum_{i=1}^{n} \bar{x}_i w_i}{\sum_{i=1}^{n} w_i},$$

where

$$w_i = \begin{cases} \underline{w}_i, & \text{if } i \leqslant k, \\ \bar{w}_i, & \text{if } i > k, \end{cases}$$

and $1 \leqslant i \leqslant n$.

Step 5: If $y' = y$, then let $\bar{y} = y, R = k$ and **Stop**. Otherwise, let $y = y'$ and go to **Step 3**.

3. A review of the TOPSIS-based nonlinear-programming method for multiattribute decision making

In this section, we briefly review the TOPSIS-based nonlinear-programming method (Li, 2010b) for multiattribute decision making. Assume that there is a set X of alternatives, a set A of attributes and a set W of the weights of attributes, where $X = \{x_1, x_2, \ldots, x_m\}$, $A = \{a_1, a_2, \ldots, a_n\}$, $W = \{w_1, w_2, \ldots, w_n\}$, $w_j = \langle[\underline{\omega}_j, \bar{\omega}_j], [\underline{\rho}_j, \bar{\rho}_j]\rangle$ denotes the weight of attribute a_j represented by an interval-valued intuitionistic fuzzy value and $1 \leqslant j \leqslant n$. Assume that the decision matrix Y is shown as follows:

$$Y = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \end{array} \\ \begin{bmatrix} \langle[\underline{\mu}_{11}, \bar{\mu}_{11}], [\underline{\upsilon}_{11}, \bar{\upsilon}_{11}]\rangle & \langle[\underline{\mu}_{12}, \bar{\mu}_{12}], [\underline{\upsilon}_{12}, \bar{\upsilon}_{12}]\rangle & \cdots & \langle[\underline{\mu}_{1n}, \bar{\mu}_{1n}], [\underline{\upsilon}_{1n}, \bar{\upsilon}_{1n}]\rangle \\ \langle[\underline{\mu}_{21}, \bar{\mu}_{21}], [\underline{\upsilon}_{21}, \bar{\upsilon}_{21}]\rangle & \langle[\underline{\mu}_{22}, \bar{\mu}_{22}], [\underline{\upsilon}_{22}, \bar{\upsilon}_{22}]\rangle & \cdots & \langle[\underline{\mu}_{2n}, \bar{\mu}_{2n}], [\underline{\upsilon}_{2n}, \bar{\upsilon}_{2n}]\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle[\underline{\mu}_{m1}, \bar{\mu}_{m1}], [\underline{\upsilon}_{m1}, \bar{\upsilon}_{m1}]\rangle & \langle[\underline{\mu}_{m2}, \bar{\mu}_{m2}], [\underline{\upsilon}_{m2}, \bar{\upsilon}_{m2}]\rangle & \cdots & \langle[\underline{\mu}_{mn}, \bar{\mu}_{mn}], [\underline{\upsilon}_{mn}, \bar{\upsilon}_{mn}]\rangle \end{bmatrix}_{m \times n} \end{array}$$

where $\langle[\underline{\mu}_{ij}, \bar{\mu}_{ij}], [\underline{\upsilon}_{ij}, \bar{\upsilon}_{ij}]\rangle$ denotes the interval-valued intuitionistic fuzzy value in which the decision-maker evaluates the alternative x_i with respect to the attribute a_j, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. The TOPSIS-based nonlinear-programming method (Li, 2010b) for multiattribute decision making is reviewed as follows:

Step 1: Calculate the relative-closeness coefficient $C_i((\mu_{ij})_{m\times n}, (\upsilon_{ij})_{m\times n}, (\omega_j)_{n\times 1}, (\rho_j)_{n\times 1})$ of alternative x_i with respect to the interval-valued intuitionistic fuzzy positive ideal solution x^+, shown as follows:

$$\begin{aligned} &C_i((\mu_{ij})_{m\times n}, (\upsilon_{ij})_{m\times n}, (\omega_j)_{n\times 1}, (\rho_j)_{n\times 1}) \\ &= \frac{d(x_i, x^-)}{d(x_i, x^-) + d(x_i, x^+)} = [\underline{C}_i, \overline{C}_i], \end{aligned} \tag{8}$$

where $(\mu_{ij})_{m\times n}$ and $(\upsilon_{ij})_{m\times n}$ are $m \times n$ matrices with elements $\mu_{ij} \in [\underline{\mu}_{ij}, \bar{\mu}_{ij}]$ and $\upsilon_{ij} \in [\underline{\upsilon}_{ij}, \bar{\upsilon}_{ij}]$, respectively, $(\omega_j)_{n\times 1}$ and $(\rho_j)_{n\times 1}$ are column vectors of n-dimensions with elements $\omega_j \in [\underline{\omega}_j, \bar{\omega}_j]$ and $\rho_j \in [\underline{\rho}_j, \bar{\rho}_j]$, respectively, $x^+ = (\langle[1,1],[0,0]\rangle)_{1\times n}$ denotes the interval-valued intuitionistic fuzzy positive ideal solution and $x^- = (\langle[0,0],[1,1]\rangle)_{1\times n}$ denotes the interval-valued intuitionistic fuzzy negative ideal solution,

$$d(x_i, x^-) = \sqrt{\sum_{j=1}^{n} ((\omega_j \mu_{ij})^2 + (\rho_j(1 - \upsilon_{ij}))^2)}, \tag{9}$$

and

$$d(x_i, x^+) = \sqrt{\sum_{j=1}^{n} ((\omega_j(1 - \mu_{ij}))^2 + (\rho_j \upsilon_{ij})^2)}. \tag{10}$$

Construct the auxiliary nonlinear-programming models for alternatives x_i to calculate the values of $\underline{C}_i$ and $\overline{C}_i$, respectively, shown as follows:

$$\begin{aligned} \underline{C}_i &= \min\{C_i((\underline{\mu}_{ij})_{m\times n}, (\bar{\upsilon}_{ij})_{m\times n}, (\omega_j)_{n\times 1}, (\rho_j)_{n\times 1})\} \\ &= \min \frac{\sqrt{\sum_{j=1}^{n}\left((\omega_j \underline{\mu}_{ij})^2 + (\rho_j(1-\bar{\upsilon}_{ij}))^2\right)}}{\sqrt{\sum_{j=1}^{n}((\omega_j \underline{\mu}_{ij})^2 + (\rho_j(1-\bar{\upsilon}_{ij}))^2)} + \sqrt{\sum_{j=1}^{n}((\omega_j(1-\underline{\mu}_{ij}))^2 + (\rho_j \bar{\upsilon}_{ij})^2)}} \\ &\text{s.t.} \quad ((\omega_j)_{n\times 1}, (\rho_j)_{n\times 1}) \in \Omega_\omega \times \Omega_\rho, \end{aligned} \tag{11}$$

where $\Omega_\omega = \{(\omega_j)_{n\times 1} | \underline{\omega}_j \leqslant \omega_j \leqslant \bar{\omega}_j \text{ and } 1 \leqslant j \leqslant n\}$ and $\Omega_\rho = \{(\rho_j)_{n\times 1} | \underline{\rho}_j \leqslant \rho_j \leqslant \bar{\rho}_j \text{ and } 1 \leqslant j \leqslant n\}$;

$$\begin{aligned} \overline{C}_i &= \max\{C_i((\bar{\mu}_{ij})_{m\times n}, (\underline{\upsilon}_{ij})_{m\times n}, (\omega_j)_{n\times 1}, (\rho_j)_{n\times 1})\} \\ &= \max \frac{\sqrt{\sum_{j=1}^{n}((\omega_j \bar{\mu}_{ij})^2 + (\rho_j(1-\underline{\upsilon}_{ij}))^2)}}{\sqrt{\sum_{j=1}^{n}((\omega_j \bar{\mu}_{ij})^2 + (\rho_j(1-\underline{\upsilon}_{ij}))^2)} + \sqrt{\sum_{j=1}^{n}((\omega_j(1-\bar{\mu}_{ij}))^2 + (\rho_j \underline{\upsilon}_{ij})^2)}} \\ &\text{s.t.} \quad ((\omega_j)_{n\times 1}, (\rho_j)_{n\times 1}) \in \Omega_\omega \times \Omega_\rho, \end{aligned} \tag{12}$$

where $\Omega_\omega = \{(\omega_j)_{n\times 1} | \underline{\omega}_j \leqslant \omega_j \leqslant \bar{\omega}_j \text{ and } 1 \leqslant j \leqslant n\}$ and $\Omega_\rho = \{(\rho_j)_{n\times 1} | \underline{\rho}_j \leqslant \rho_j \leqslant \bar{\rho}_j \text{ and } 1 \leqslant j \leqslant n\}$.

Step 2: Calculate the inclusion-comparison probability $p(x_i \geqslant x_k)$, shown as follows:

$$\begin{aligned} p(x_i \geqslant x_k) &= p(C_i \supseteq C_k) \\ &= \max\left(1 - \max\left(\frac{\overline{C}_k - \underline{C}_i}{\pi(C_i) + \pi(C_k)}, 0\right), 0\right), \end{aligned} \tag{13}$$

where $C_i = [\underline{C}_i, \overline{C}_i] = \langle \underline{C}_i, 1-\overline{C}_i \rangle; C_k = [\underline{C}_k, \overline{C}_k] = \langle \underline{C}_k, 1-\overline{C}_k \rangle; \pi(C_i) = 1-\underline{C}_i-(1-\overline{C}_i) = \overline{C}_i-\underline{C}_i$ and $\pi(C_k) = 1-\underline{C}_k-(1-\overline{C}_k) = \overline{C}_k-\underline{C}_k$ are hesitation degrees of intuitionistic fuzzy values C_i and C_k, respectively.

Step 3: Construct the judgment matrix P, shown as follows:

$$P = (p_{ik})_{m\times m} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}_{m\times m}, \tag{14}$$

where $p_{ik} = p(x_i \geqslant x_k) = p(C_i \supseteq C_k)$ and the symbol "$\supseteq$" denotes the inclusion relations.

Step 4: Calculate the ranking value θ_i of alternative x_i, where $1 \leqslant i \leqslant m$, shown as follows:

$$\theta_i = \frac{1}{m(m-1)}\left(\sum_{k=1}^{m} p_{ik} + \frac{m}{2} - 1\right). \tag{15}$$

Rank the alternatives $x_1, x_2, \ldots$, and x_m according to their corresponding ranking values $\theta_1, \theta_2, \ldots$, and θ_m, respectively. The larger the value of θ_i, the better the ranking order of alternative x_i, where $1 \leqslant i \leqslant m$.

Li (2010b) also presented some features of his proposed auxiliary nonlinear-programming models, shown as follows:

(1) If the interval-valued intuitionistic fuzzy positive ideal solution x^+ and the interval-valued intuitionistic fuzzy negative ideal solution x^- are not fixed, where $x^+ = \left(\left\langle \left[g_j^+, g_j^+\right], \left[b_j^+, b_j^+\right]\right\rangle\right)_{1\times n}, x^- = \left(\left\langle \left[g_j^-, g_j^-\right], \left[b_j^-, b_j^-\right]\right\rangle\right)_{1\times n}$, $g_j^+ = \max\{\bar{\mu}_{ij} | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}, b_j^+ = \min\{\underline{\upsilon}_{ij} | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$, $g_j^- = \min\{\underline{\mu}_{ij} | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ and $b_j^- = \max\{\bar{\upsilon}_{ij} | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$, then Eqs. (11) and (12) can be rewritten as Eqs. (16) and (17), respectively.

(2) If the weights $w_1, w_2, \ldots, w_n$ of the attributes $a_1, a_2, \ldots, a_n$, respectively, are crisp values, then Eqs. (11) and (12) can be rewritten as Eqs. (18) and (19), respectively.

$$\begin{aligned} \underline{C}_i &= C_i((\underline{\mu}_{ij})_{m\times n}, (\bar{\upsilon}_{ij})_{m\times n}, (w_j)_{n\times 1}, (w_j)_{n\times 1}) \\ &= \frac{\sqrt{\sum_{j=1}^{n}((w_j\underline{\mu}_{ij})^2 + (w_j(1-\bar{\upsilon}_{ij}))^2)}}{\sqrt{\sum_{j=1}^{n}((w_j\underline{\mu}_{ij})^2 + (w_j(1-\bar{\upsilon}_{ij}))^2)} + \sqrt{\sum_{j=1}^{n}((w_j(1-\underline{\mu}_{ij}))^2 + (w_j\bar{\upsilon}_{ij})^2)}}, \end{aligned} \tag{18}$$

$$\begin{aligned} \overline{C}_i &= C_i((\bar{\mu}_{ij})_{m\times n}, (\underline{\upsilon}_{ij})_{m\times n}, (w_j)_{n\times 1}, (w_j)_{n\times 1}) \\ &= \frac{\sqrt{\sum_{j=1}^{n}((w_j\bar{\mu}_{ij})^2 + (w_j(1-\underline{\upsilon}_{ij}))^2)}}{\sqrt{\sum_{j=1}^{n}((w_j\bar{\mu}_{ij})^2 + (w_j(1-\underline{\upsilon}_{ij}))^2)} + \sqrt{\sum_{j=1}^{n}((w_j(1-\bar{\mu}_{ij}))^2 + (w_j\underline{\upsilon}_{ij})^2)}}. \end{aligned} \tag{19}$$

(3) If Eqs. (11) and (12) based on the weighted-Euclidean distances are replaced by the weighted-Hamming distances, respectively, then Eqs. (11) and (12) can be rewritten as follows:

$$\begin{aligned} \underline{C}_i &= \min\{C_i((\underline{\mu}_{ij})_{m\times n}, (\bar{\upsilon}_{ij})_{m\times n}, (\omega_j)_{n\times 1}, (\rho_j)_{n\times 1})\} \\ &= \min \frac{\sum_{j=1}^{n}(\omega_j\underline{\mu}_{ij} + \rho_j(1-\bar{\upsilon}_{ij}))}{\sum_{j=1}^{n}(\omega_j\underline{\mu}_{ij} + \rho_j(1-\bar{\upsilon}_{ij})) + \sum_{j=1}^{n}(\omega_j(1-\underline{\mu}_{ij}) + \rho_j\bar{\upsilon}_{ij})} \\ &= \min \frac{\sum_{j=1}^{n}(\omega_j\underline{\mu}_{ij} + \rho_j(1-\bar{\upsilon}_{ij}))}{\sum_{j=1}^{n}(\omega_j + \rho_j)} \\ \text{s.t.} \quad & ((\omega_j)_{n\times 1}, (\rho_j)_{n\times 1}) \in \Omega_\omega \times \Omega_\rho, \end{aligned} \tag{20}$$

$$\begin{aligned} \overline{C}_i &= \max\{C_i((\bar{\mu}_{ij})_{m\times n}, (\underline{\upsilon}_{ij})_{m\times n}, (\omega_j)_{n\times 1}, (\rho_j)_{n\times 1})\} \\ &= \max \frac{\sum_{j=1}^{n}(\omega_j\bar{\mu}_{ij} + \rho_j(1-\underline{\upsilon}_{ij}))}{\sum_{j=1}^{n}(\omega_j\bar{\mu}_{ij} + \rho_j(1-\underline{\upsilon}_{ij})) + \sum_{j=1}^{n}(\omega_j(1-\bar{\mu}_{ij}) + \rho_j\underline{\upsilon}_{ij})} \\ &= \max \frac{\sum_{j=1}^{n}(\omega_j\bar{\mu}_{ij} + \rho_j(1-\underline{\upsilon}_{ij}))}{\sum_{j=1}^{n}(\omega_j + \rho_j)}, s.t. ((\omega_j)_{n\times 1}, (\rho_j)_{n\times 1}) \\ &\in \Omega_\omega \times \Omega_\rho. \end{aligned} \tag{21}$$

$$\begin{aligned} \underline{C}_i &= \min\{C_i((\underline{\mu}_{ij})_{m\times n}, (\bar{\upsilon}_{ij})_{m\times n}, (\omega_j)_{n\times 1}, (\rho_j)_{n\times 1})\} \\ &= \min \frac{\sqrt{\sum_{j=1}^{n}\left(\left(\omega_j\left(\underline{\mu}_{ij} - g_j^-\right)\right)^2 + \left(\rho_j\left(b_j^- - \bar{\upsilon}_{ij}\right)\right)^2\right)}}{\sqrt{\sum_{j=1}^{n}\left(\left(\omega_j\left(\underline{\mu}_{ij} - g_j^-\right)\right)^2 + \left(\rho_j\left(b_j^- - \bar{\upsilon}_{ij}\right)\right)^2\right)} + \sqrt{\sum_{j=1}^{n}\left(\left(\omega_j\left(g_j^+ - \underline{\mu}_{ij}\right)\right)^2 + \left(\rho_j\left(\bar{\upsilon}_{ij} - b_j^+\right)\right)^2\right)}}, \\ \text{s.t.} \quad & ((\omega_j)_{n\times 1}, (\rho_j)_{n\times 1}) \in \Omega_\omega \times \Omega_\rho, \end{aligned} \tag{16}$$

$$\begin{aligned} \overline{C}_i &= \max\{C_i((\bar{\mu}_{ij})_{m\times n}, (\underline{\upsilon}_{ij})_{m\times n}, (\omega_j)_{n\times 1}, (\rho_j)_{n\times 1})\} \\ &= \max \frac{\sqrt{\sum_{j=1}^{n}\left(\left(\omega_j\left(\bar{\mu}_{ij} - g_j^-\right)\right)^2 + \left(\rho_j\left(b_j^- - \underline{\upsilon}_{ij}\right)\right)^2\right)}}{\sqrt{\sum_{j=1}^{n}\left(\left(\omega_j\left(\bar{\mu}_{ij} - g_j^-\right)\right)^2 + \left(\rho_j\left(b_j^- - \underline{\upsilon}_{ij}\right)\right)^2\right)} + \sqrt{\sum_{j=1}^{n}\left(\left(\omega_j\left(g_j^+ - \bar{\mu}_{ij}\right)\right)^2 + \left(\rho_j\left(\underline{\upsilon}_{ij} - b_j^+\right)\right)^2\right)}}, \\ \text{s.t.} \quad & ((\omega_j)_{n\times 1}, (\rho_j)_{n\times 1}) \in \Omega_\omega \times \Omega_\rho. \end{aligned} \tag{17}$$

4. The proposed intuitionistic fuzzy weighted average operator and the proposed interval-valued intuitionistic fuzzy weighted average operator

In this section, we propose the intuitionistic fuzzy weighted average operator and the interval-valued intuitionistic fuzzy weighted average operator. Assume that there are n values $x_1, x_2, \ldots, \text{and}\, x_n$, where the weights of $x_1, x_2, \ldots, \text{and}\, x_n$ are $w_1, w_2, \ldots, \text{and}\, w_n$, respectively. Then, their weighted average Y of x_i and w_i, where $1 \leqslant i \leqslant n$, is calculated as follows:

$$Y = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}. \tag{22}$$

If x_i and w_i are crisp values, then Y is called the traditional weighted average (WA) (Wu & Mendel, 2010). Based on Eq. (22), the interval weighted average (IWA) (Wu & Mendel, 2010) is derived, shown as follows:

Case 1: If $x_1, x_2, \ldots, \text{and}\, x_n$ are intervals and $w_1, w_2, \ldots, \text{and}\, w_n$ are crisp values, where $x_i = [\underline{x}_i, \bar{x}_i], \underline{x}_i < \bar{x}_i$ and $1 \leqslant i \leqslant n$, then the interval weighted average Y is calculated as follows:

$$Y = \frac{\sum_{i=1}^{n} [\underline{x}_i, \bar{x}_i] w_i}{\sum_{i=1}^{n} w_i} = \left[\frac{\sum_{i=1}^{n} \underline{x}_i w_i}{\sum_{i=1}^{n} w_i}, \frac{\sum_{i=1}^{n} \bar{x}_i w_i}{\sum_{i=1}^{n} w_i} \right] = [\underline{y}, \bar{y}]. \tag{23}$$

Case 2: If $x_1, x_2, \ldots, \text{and}\, x_n$ are crisp values and $w_1, w_2, \ldots, \text{and}\, w_n$ are intervals, where $w_i = [\underline{w}_i, \bar{w}_i], \underline{w}_i < \bar{w}_i$ and $1 \leqslant i \leqslant n$, then the interval weighted average Y is calculated as follows:

$$Y = \frac{\sum_{i=1}^{n} [x_i, x_i][\underline{w}_i, \bar{w}_i]}{\sum_{i=1}^{n} [\underline{w}_i, \bar{w}_i]} = [\underline{y}, \bar{y}], \tag{24}$$

where $\underline{y}$ and $\bar{y}$ are calculated by the Karnik–Mendel algorithms (Karnik & Mendel, 2001), respectively.

Case 3: If $x_1, x_2, \ldots, \text{and}\, x_n$ are intervals and $w_1, w_2, \ldots, \text{and}\, w_n$ are intervals, where $x_i = [\underline{x}_i, \bar{x}_i], w_i = [\underline{w}_i, \bar{w}_i], \underline{x}_i < \bar{x}_i, \underline{w}_i < \bar{w}_i$ and $1 \leqslant i \leqslant n$, then the interval weighted average Y is calculated as follows:

$$Y = \frac{\sum_{i=1}^{n} [\underline{x}_i, \bar{x}_i][\underline{w}_i, \bar{w}_i]}{\sum_{i=1}^{n} [\underline{w}_i, \bar{w}_i]} = [\underline{y}, \bar{y}], \tag{25}$$

where $\underline{y}$ and $\bar{y}$ are calculated by the Karnik–Mendel algorithms (Karnik & Mendel, 2001), respectively.

In the following, we propose the intuitionistic fuzzy weighted average operator. Based on Eq. (22), the intuitionistic fuzzy weighted average operator is derived, shown as follows:

Case 1: If $x_1, x_2, \ldots, \text{and}\, x_n$ are intuitionistic fuzzy values and $w_1, w_2, \ldots, \text{and}\, w_n$ are crisp values, where $x_i = \langle \mu_i, \upsilon_i \rangle = [\mu_i, 1 - \upsilon_i]$, $0 \leqslant \mu_i + \upsilon_i \leqslant 1$ and $1 \leqslant i \leqslant n$, then the intuitionistic fuzzy weighted average Y is calculated as follows:

$$Y = \frac{\sum_{i=1}^{n} [\mu_i, 1 - \upsilon_i] w_i}{\sum_{i=1}^{n} w_i} = \left[\frac{\sum_{i=1}^{n} \mu_i w_i}{\sum_{i=1}^{n} w_i}, \frac{\sum_{i=1}^{n} (1 - \upsilon_i) w_i}{\sum_{i=1}^{n} w_i} \right] = [\underline{y}, \bar{y}], \tag{26}$$

where $Y = [\underline{y}, \bar{y}] = \langle \underline{y}, 1 - \bar{y} \rangle$ is an intuitionistic fuzzy value.

Case 2: If $x_1, x_2, \ldots, \text{and}\, x_n$ are intuitionistic fuzzy values and $w_1, w_2, \ldots, \text{and}\, w_n$ are intervals, where $x_i = \langle \mu_i, \upsilon_i \rangle = [\mu_i, 1 - \upsilon_i], 0 \leqslant \mu_i \leqslant 1, 0 \leqslant \upsilon_i \leqslant 1, 0 \leqslant \mu_i + \upsilon_i \leqslant 1, w_i = [\underline{w}_i, \bar{w}_i]$ and $1 \leqslant i \leqslant n$, then the intuitionistic fuzzy weighted average Y is calculated as follows:

$$Y = \frac{\sum_{i=1}^{n} [\mu_i, 1 - \upsilon_i][\underline{w}_i, \bar{w}_i]}{\sum_{i=1}^{n} [\underline{w}_i, \bar{w}_i]} = [\underline{y}, \bar{y}], \tag{27}$$

where $Y = [\underline{y}, \bar{y}] = \langle \underline{y}, 1 - \bar{y} \rangle$ is an intuitionistic fuzzy value; $\underline{y}$ and $\bar{y}$ are calculated by the Karnik–Mendel algorithms (Karnik & Mendel, 2001), respectively.

Case 3: If $x_1, x_2, \ldots, \text{and}\, x_n$ are intuitionistic fuzzy values and $w_1, w_2, \ldots, \text{and}\, w_n$ are intuitionistic fuzzy values, where $x_i = \langle \mu_i, \upsilon_i \rangle = [\mu_i, 1 - \upsilon_i]$, $0 \leqslant \mu_i \leqslant 1$, $0 \leqslant \upsilon_i \leqslant 1$, $0 \leqslant \mu_i + \upsilon_i \leqslant 1$, $w_i = \langle \omega_i, \rho_i \rangle = [\omega_i, 1 - \rho_i]$, $0 \leqslant \omega_i \leqslant 1$, $0 \leqslant \rho_i \leqslant 1$, $0 \leqslant \omega_i + \rho_i \leqslant 1$ and $1 \leqslant i \leqslant n$, then the intuitionistic fuzzy weighted average Y is calculated as follows:

$$Y = \frac{\sum_{i=1}^{n} [\mu_i, 1 - \upsilon_i][\omega_i, 1 - \rho_i]}{\sum_{i=1}^{n} [\omega_i, 1 - \rho_i]} = [\underline{y}, \bar{y}], \tag{28}$$

where $Y = [\underline{y}, \bar{y}] = \langle \underline{y}, 1 - \bar{y} \rangle$ is an intuitionistic fuzzy value; $\underline{y}$ and $\bar{y}$ are calculated by the Karnik–Mendel algorithms (Karnik & Mendel, 2001), respectively.

In the following, we propose the interval-valued intuitionistic fuzzy weighted average operator. Based on Eq. (22), the interval-valued intuitionistic fuzzy weighted average is derived, shown as follows:

Case 1: If $x_1, x_2, \ldots, \text{and}\, x_n$ are interval-valued intuitionistic fuzzy values and $w_1, w_2, \ldots, \text{and}\, w_n$ are crisp values, where $x_i = \langle [\underline{\mu}_i, \bar{\mu}_i], [\underline{\upsilon}_i, \bar{\upsilon}_i] \rangle = [[\underline{\mu}_i, \bar{\mu}_i], [1 - \bar{\upsilon}_i, 1 - \underline{\upsilon}_i]], 0 \leqslant \underline{\mu}_i \leqslant \bar{\mu}_i \leqslant 1$, $0 \leqslant \underline{\upsilon}_i \leqslant \bar{\upsilon}_i \leqslant 1$, $0 \leqslant \bar{\mu}_i + \bar{\upsilon}_i \leqslant 1$ and $1 \leqslant i \leqslant n$, then the interval-valued intuitionistic fuzzy weighted average Y is calculated as follows:

$$\begin{aligned} Y &= \frac{\sum_{i=1}^{n} [[\underline{\mu}_i, \bar{\mu}_i], [1 - \bar{\upsilon}_i, 1 - \underline{\upsilon}_i]] \times w_i}{\sum_{i=1}^{n} w_i} \\ &= \left[\frac{\sum_{i=1}^{n} [\underline{\mu}_i, \bar{\mu}_i] w_i}{\sum_{i=1}^{n} w_i}, \frac{\sum_{i=1}^{n} [1 - \bar{\upsilon}_i, 1 - \underline{\upsilon}_i] w_i}{\sum_{i=1}^{n} w_i} \right] \\ &= \left[\left[\frac{\sum_{i=1}^{n} \underline{\mu}_i w_i}{\sum_{i=1}^{n} w_i}, \frac{\sum_{i=1}^{n} \bar{\mu}_i w_i}{\sum_{i=1}^{n} w_i} \right], \left[\frac{\sum_{i=1}^{n} (1 - \bar{\upsilon}_i) w_i}{\sum_{i=1}^{n} w_i}, \frac{\sum_{i=1}^{n} (1 - \underline{\upsilon}_i) w_i}{\sum_{i=1}^{n} w_i} \right] \right] \\ &= [[\underline{y}_1, \bar{y}_1], [\underline{y}_2, \bar{y}_2]], \end{aligned} \tag{29}$$

where $Y = [[\underline{y}_1, \bar{y}_1], [\underline{y}_2, \bar{y}_2]] = \langle [\underline{y}_1, \bar{y}_1], [1 - \bar{y}_2, 1 - \underline{y}_2] \rangle$ is an interval-valued intuitionistic fuzzy value.

Case 2: If $x_1, x_2, \ldots, \text{and}\, x_n$ are interval-valued intuitionistic fuzzy values and $w_1, w_2, \ldots, \text{and}\, w_n$ are intervals, where $x_i = \langle [\underline{\mu}_i, \bar{\mu}_i], [\underline{\upsilon}_i, \bar{\upsilon}_i] \rangle = [[\underline{\mu}_i, \bar{\mu}_i], [1 - \bar{\upsilon}_i, 1 - \underline{\upsilon}_i]], 0 \leqslant \underline{\mu}_i \leqslant \bar{\mu}_i \leqslant 1$, $0 \leqslant \underline{\upsilon}_i \leqslant \bar{\upsilon}_i \leqslant 1, 0 \leqslant \bar{\mu}_i + \bar{\upsilon}_i \leqslant 1, w_i = [\underline{w}_i, \bar{w}_i]$ and $1 \leqslant i \leqslant n$, then the interval-valued intuitionistic fuzzy weighted average Y is calculated as follows:

$$\begin{aligned} Y &= \frac{\sum_{i=1}^{n} [[\underline{\mu}_i, \bar{\mu}_i], [1 - \bar{\upsilon}_i, 1 - \underline{\upsilon}_i]][\underline{w}_i, \bar{w}_i]}{\sum_{i=1}^{n} [\underline{w}_i, \bar{w}_i]} \\ &= \left[\frac{\sum_{i=1}^{n} [\underline{\mu}_i, \bar{\mu}_i][\underline{w}_i, \bar{w}_i]}{\sum_{i=1}^{n} [\underline{w}_i, \bar{w}_i]}, \frac{\sum_{i=1}^{n} [1 - \bar{\upsilon}_i, 1 - \underline{\upsilon}_i][\underline{w}_i, \bar{w}_i]}{\sum_{i=1}^{n} [\underline{w}_i, \bar{w}_i]} \right] \\ &= [[\underline{y}_1, \bar{y}_1], [\underline{y}_2, \bar{y}_2]], \end{aligned} \tag{30}$$

where $Y = [[\underline{y}_1, \bar{y}_1], [\underline{y}_2, \bar{y}_2]] = \langle [\underline{y}_1, \bar{y}_1], [1 - \bar{y}_2, 1 - \underline{y}_2] \rangle$ is an interval-valued intuitionistic fuzzy value; $\underline{y}_1, \bar{y}_1, \underline{y}_2$ and $\bar{y}_2$ are calculated by the Karnik–Mendel algorithms (Karnik & Mendel, 2001), respectively.

Case 3: If $x_1, x_2, \ldots, \text{and}\, x_n$ are interval-valued intuitionistic fuzzy values and $w_1, w_2, \ldots, \text{and}\, w_n$ are intuitionistic fuzzy values, where $x_i = \langle [\underline{\mu}_i, \bar{\mu}_i], [\underline{\upsilon}_i, \bar{\upsilon}_i] \rangle = [[\underline{\mu}_i, \bar{\mu}_i], [1 - \bar{\upsilon}_i, 1 - \underline{\upsilon}_i]]$, $0 \leqslant \underline{\mu}_i \leqslant \bar{\mu}_i \leqslant 1, 0 \leqslant \underline{\upsilon}_i \leqslant \bar{\upsilon}_i \leqslant 1, 0 \leqslant \bar{\mu}_i + \bar{\upsilon}_i \leqslant 1, w_i = \langle \omega_i, \rho_i \rangle = [\omega_i, 1 - \rho_i], 0 \leqslant \omega_i \leqslant 1, 0 \leqslant \rho_i \leqslant 1, 0 \leqslant \omega_i + \rho_i \leqslant 1$ and $1 \leqslant i \leqslant n$, then the interval-valued intuitionistic fuzzy weighted average Y is calculated as follows:

$$Y = \frac{\sum_{i=1}^{n}[[\underline{\mu}_i, \bar{\mu}_i], [1-\bar{\upsilon}_i, 1-\underline{\upsilon}_i]][\omega_i, 1-\rho_i]}{\sum_{i=1}^{n}[\omega_i, 1-\rho_i]} = \left[\frac{\sum_{i=1}^{n}[\underline{\mu}_i, \bar{\mu}_i][\omega_i, 1-\rho_i]}{\sum_{i=1}^{n}[\omega_i, 1-\rho_i]}, \frac{\sum_{i=1}^{n}[1-\bar{\upsilon}_i, 1-\underline{\upsilon}_i][\omega_i, 1-\rho_i]}{\sum_{i=1}^{n}[\omega_i, 1-\rho_i]}\right] = [[\underline{y}_1, \bar{y}_1], [\underline{y}_2, \bar{y}_2]], \quad (31)$$

where $Y = [[\underline{y}_1, \bar{y}_1], [\underline{y}_2, \bar{y}_2]] = \langle[\underline{y}_1, \bar{y}_1], [1-\bar{y}_2, 1-\underline{y}_2]\rangle$ is an interval-valued intuitionistic fuzzy value; $\underline{y}_1, \bar{y}_1, \underline{y}_2$ and $\bar{y}_2$ are calculated by the Karnik–Mendel algorithms (Karnik & Mendel, 2001), respectively.

Case 4: If $x_1, x_2, \ldots,$ and x_n are interval-valued intuitionistic fuzzy values and $w_1, w_2, \ldots,$ and w_n are interval-valued intuitionistic fuzzy values, where $x_i = \langle[\underline{\mu}_i, \bar{\mu}_i], [\underline{\upsilon}_i, \bar{\upsilon}_i]\rangle = [[\underline{\mu}_i, \bar{\mu}_i], [1-\bar{\upsilon}_i, 1-\underline{\upsilon}_i]], 0 \leqslant \underline{\mu}_i \leqslant \bar{\mu}_i \leqslant 1, 0 \leqslant \underline{\upsilon}_i \leqslant \bar{\upsilon}_i \leqslant 1, 0 \leqslant \bar{\mu}_i + \bar{\upsilon}_i \leqslant 1, w_i = \langle[\underline{\omega}_i, \bar{\omega}_i], [\underline{\rho}_i, \bar{\rho}_i]\rangle = [[\underline{\omega}_i, \bar{\omega}_i], [1-\bar{\rho}_i, 1-\underline{\rho}_i]], 0 \leqslant \underline{\omega}_i \leqslant \bar{\omega}_i \leqslant 1, 0 \leqslant \underline{\rho}_i \leqslant \bar{\rho}_i \leqslant 1, \quad 0 \leqslant \bar{\omega}_i + \bar{\rho}_i \leqslant 1$ and $1 \leqslant i \leqslant n$, then the interval-valued intuitionistic fuzzy weighted average Y is calculated as follows:

$$Y = \left[\frac{\sum_{i=1}^{n}[\underline{\mu}_i, \bar{\mu}_i][\underline{\omega}_i, \bar{\omega}_i]}{\sum_{i=1}^{n}[\underline{\omega}_i, \bar{\omega}_i]}, \frac{\sum_{i=1}^{n}[1-\bar{\upsilon}_i, 1-\underline{\upsilon}_i][1-\bar{\rho}_i, 1-\underline{\rho}_i]}{\sum_{i=1}^{n}[1-\bar{\rho}_i, 1-\underline{\rho}_i]}\right] = [[\underline{y}_1, \bar{y}_1], [\underline{y}_2, \bar{y}_2]], \quad (32)$$

where $Y = [[\underline{y}_1, \bar{y}_1], [\underline{y}_2, \bar{y}_2]] = \langle[\underline{y}_1, \bar{y}_1], [1-\bar{y}_2, 1-\underline{y}_2]\rangle$ is an interval-valued intuitionistic fuzzy value; $\underline{y}_1, \bar{y}_1, \underline{y}_2$ and $\bar{y}_2$ are calculated by the Karnik–Mendel algorithms (Karnik & Mendel, 2001), respectively.

5. The proposed fuzzy ranking method for intuitionistic fuzzy values based on likelihood-based comparison relations

In this section, we propose a fuzzy ranking method for intuitionistic fuzzy values based on likelihood-based comparison relations between intervals. Xu and Da (2003) presented a likelihood-based method for ranking the priority between intervals. Assume that there are two intervals $X = [\underline{x}, \bar{x}]$ and $Y = [\underline{y}, \bar{y}]$, where $\underline{x} \leqslant \bar{x}$ and $\underline{y} \leqslant \bar{y}$. The likelihood-based comparison relation $p(X \geqslant Y)$ between the intervals X and Y is defined as follows (Xu & Da, 2003):

$$p(X \geqslant Y) = \max\left(1 - \max\left(\frac{\bar{y} - \underline{x}}{L(X) + L(Y)}, 0\right), 0\right), \quad (33)$$

where $L(X) = \bar{x} - \underline{x}$ and $L(Y) = \bar{y} - \underline{y}$. The likelihood $p(X \geqslant Y)$ of $X \geqslant Y$ has the following properties (Xu & Da, 2003):

(1) $0 \leqslant p(X \geqslant Y) \leqslant 1$.
(2) $p(X \geqslant Y) + p(Y \geqslant X) = 1$.
(3) If $\bar{x} \leqslant \underline{y}$, then $P(X \geqslant Y) = 0$.,
(4) If $\underline{x} \geqslant \bar{y}$, then $P(X \geqslant Y) = 1$.
(5) $p(X \geqslant X) = 0.5$.

If $\underline{x} = \bar{x}$ and $\underline{y} = \bar{y}$, then the likelihood $p(X \geqslant Y)$ of $X \geqslant Y$ is defined as follows (Xu & Da, 2003):

$$p(X \geqslant Y) = \begin{cases} 1, & \text{if } \underline{x} > \underline{y}, \\ \frac{1}{2}, & \text{if } \underline{x} = \underline{y}, \\ 0, & \text{if } \underline{x} < \underline{y}, \end{cases} \quad (34)$$

In the following, we propose the definition of likelihood-based comparison relations between intuitionistic fuzzy values. Assume that there are two intuitionistic fuzzy values Y_1 and Y_2, where $Y_1 = \langle y_{11}, y_{12}\rangle = [y_{11}, 1 - y_{12}]$ and $Y_2 = \langle y_{21}, y_{22}\rangle = [y_{21}, 1 - y_{22}]$. Based on Eq. (33), we define the likelihood-based comparison relation $p(Y_1 \geqslant Y_2)$ between the intuitionistic fuzzy values Y_1 and Y_2, shown as follows:

$$p(Y_1 \geqslant Y_2) = p([y_{11}, 1 - y_{12}] \geqslant [y_{21}, 1 - y_{22}]). \quad (35)$$

Example 5.1. Assume that there are two intuitionistic fuzzy values Y_1 and Y_2, where $Y_1 = \langle 0.3, 0.5\rangle = [0.3, 0.5]$ and $Y_2 = \langle 0.4, 0.3\rangle = [0.4, 0.7]$. Then, based on Eq. (35), we can get

$$p(Y_1 \geqslant Y_2) = p([0.3, 0.5] \geqslant [0.4, 0.7]) = \max\left(1 - \max\left(\frac{0.7 - 0.3}{0.5 - 0.3 + 0.7 - 0.4}, 0\right), 0\right) = 0.2,$$

$$p(Y_2 \geqslant Y_1) = p([0.4, 0.7] \geqslant [0.3, 0.5]) = \max\left(1 - \max\left(\frac{0.5 - 0.4}{0.7 - 0.4 + 0.5 - 0.3}, 0\right), 0\right) = 0.8.$$

Proposition 5.1. *For any intuitionistic fuzzy values Y_1 and Y_2, $p(Y_1 \geqslant Y_2) = 1 - p(Y_2 \geqslant Y_1)$.*

Proof. Based on Eq. (35), we can get
$$p(Y_1 \geqslant Y_2) = p([y_{11}, 1 - y_{12}] \geqslant [y_{21}, 1 - y_{22}]) = 1 - p([y_{21}, 1 - y_{22}] \geqslant [y_{11}, 1 - y_{12}]) = 1 - p(Y_2 \geqslant Y_1). \quad \square$$

In the following, we propose a fuzzy ranking method for intuitionistic fuzzy values based on likelihood-based comparison relations between intervals. Assume that there are m intuitionistic fuzzy values $Y_1, Y_2, \ldots,$ and Y_m to be ranked. The proposed fuzzy ranking method for intuitionistic fuzzy values is now presented as follows:

Step 1: Construct the judgment matrix P as follows:

$$P = (p_{ij})_{m\times m} = \begin{array}{c} \\ Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{array} \begin{array}{c} \begin{array}{cccc} Y_1 & Y_2 & \cdots & Y_m \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}_{m\times m} \end{array} \quad (36)$$

where $p_{ij} = p(Y_i \geqslant Y_j)$ is obtained by Eq. (33), $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant m$.

Step 2: Calculate the ranking value $RV(Y_i)$ of Y_i, where $1 \leqslant i \leqslant m$, shown as follows:

$$RV(Y_i) = \frac{2}{m^2}\sum_{j=1}^{m} p_{ij}, \quad (37)$$

where $\sum_{i=1}^{m} RV(Y_i) = 1$.

Step 3: Rank the values $Y_1, Y_2, \ldots,$ and Y_m according to their corresponding ranking values $RV(Y_1), RV(Y_2), \ldots,$ and $RV(Y_m)$, respectively. The larger the value of $RV(Y_i)$, the better the ranking order of Y_i, where $1 \leqslant i \leqslant m$.

Example 5.2. Assume that there are three intuitionistic fuzzy values Y_1, Y_2 and Y_3 to be ranked, where $Y_1 = \langle 0.4, 0.3\rangle = [0.4, 0.7]$, $Y_2 = \langle 0.4, 0.5\rangle = [0.4, 0.5]$ and $Y_3 = \langle 0.3, 0.1\rangle = [0.3, 0.9]$. The ranking process based on the proposed fuzzy ranking method is shown as follows:

[**Step 1**] Based on Eq. (36), we can construct the judgment matrix P, shown as follows:

$$P = \begin{array}{c} \\ Y_1 \\ Y_2 \\ Y_3 \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 & Y_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.75 & 0.4444 \\ 0.25 & 0.5 & 0.2857 \\ 0.5556 & 0.7143 & 0.5 \end{bmatrix} \end{array}$$

[**Step 2**] Based on Eq. (37), we can get

$$RV(Y_1) = \frac{2}{3^2}(0.5 + 0.75 + 0.4444) = 0.3765,$$
$$RV(Y_2) = \frac{2}{3^2}(0.25 + 0.5 + 0.2857) = 0.2302,$$
$$RV(Y_3) = \frac{2}{3^2}(0.5556 + 0.7143 + 0.5) = 0.3933.$$

[Step 3] Because $RV(Y_3) > RV(Y_1) > RV(Y_2)$, we can see that the ranking order of Y_1, Y_2 and Y_3 is "$Y_3 > Y_1 > Y_2$".

Proposition 5.2. *Assume that there is a judgment matrix $P = (p_{ij})_{m\times m}$, then the ranking value $RV(Y_i)$ of Y_i is defined as follows:*

$$RV(Y_i) = \frac{2}{m^2}\sum_{j=1}^{m} p_{ij},$$

where Y_i is an intuitionistic fuzzy value, $1 \leqslant i \leqslant m$ and $\sum_{i=1}^{m} RV(Y_i) = 1$.

Proof. Let us consider the judgment matrix $P = (p_{ij})_{m\times m}$, shown as follows:

$$P = (p_{ij})_{m\times m} = \begin{array}{c} \\ Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{array} \begin{array}{c} \begin{array}{cccc} Y_1 & Y_2 & \cdots & Y_m \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}_{m\times m} \end{array}.$$

Because $p_{ij} + p_{ji} = 1$ and because $p_{ii} = 0.5$, we can get

$$\sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij} = \sum_{i=1}^{m} p_{ii} + \sum_{i=1}^{m}\sum_{j=1, j\neq i}^{m} p_{ij} = \frac{m}{2} + \frac{m(m-1)}{2} = \frac{m^2}{2}.$$

Let $RV(Y_i) = x\sum_{j=1}^{m} p_{ij}$, where x is an unknown value and $1 \leqslant i \leqslant m$, and let $\sum_{i=1}^{m} RV(Y_i) = 1$. Then, we can get

$$\sum_{i=1}^{m} RV(Y_i) = x\sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij} = 1.$$

That is,

$$x = \frac{1}{\sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}}.$$

Because we have known that $\sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij} = \frac{m^2}{2}$, we can get

$$x = \frac{1}{\sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}} = \frac{2}{m^2}.$$

Therefore, we can get $RV(Y_i) = \frac{2}{m^2}\sum_{j=1}^{m} p_{ij}$. □

6. The proposed multiattribute decision making method based on interval-valued intuitionistic fuzzy values

In this section, we propose a new multiattribute decision making method based on interval-valued intuitionistic fuzzy values. Assume that there is a set X of alternatives, a set A of attributes and a set W of the weights of attributes, where $X = \{x_1, x_2, \ldots, x_m\}$, $A = \{a_1, a_2, \ldots, a_n\}$, $W = \{w_1, w_2, \ldots, w_n\}$, w_j denotes the weight of attribute a_j and $1 \leqslant j \leqslant n$. Assume that the decision matrix Y is shown as follows:

$$Y = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \end{array} \\ \begin{bmatrix} \langle[\underline{\mu}_{11}, \bar{\mu}_{11}], [\underline{\upsilon}_{11}, \bar{\upsilon}_{11}]\rangle & \langle[\underline{\mu}_{12}, \bar{\mu}_{12}], [\underline{\upsilon}_{12}, \bar{\upsilon}_{12}]\rangle & \cdots & \langle[\underline{\mu}_{1n}, \bar{\mu}_{1n}], [\underline{\upsilon}_{1n}, \bar{\upsilon}_{1n}]\rangle \\ \langle[\underline{\mu}_{21}, \bar{\mu}_{21}], [\underline{\upsilon}_{21}, \bar{\upsilon}_{21}]\rangle & \langle[\underline{\mu}_{22}, \bar{\mu}_{22}], [\underline{\upsilon}_{22}, \bar{\upsilon}_{22}]\rangle & \cdots & \langle[\underline{\mu}_{2n}, \bar{\mu}_{2n}], [\underline{\upsilon}_{2n}, \bar{\upsilon}_{2n}]\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle[\underline{\mu}_{m1}, \bar{\mu}_{m1}], [\underline{\upsilon}_{m1}, \bar{\upsilon}_{m1}]\rangle & \langle[\underline{\mu}_{m2}, \bar{\mu}_{m2}], [\underline{\upsilon}_{m2}, \bar{\upsilon}_{m2}]\rangle & \cdots & \langle[\underline{\mu}_{mn}, \bar{\mu}_{mn}], [\underline{\upsilon}_{mn}, \bar{\upsilon}_{mn}]\rangle \end{bmatrix}_{m\times n} \end{array},$$

where $\langle[\underline{\mu}_{ij}, \bar{\mu}_{ij}], [\underline{\upsilon}_{ij}, \bar{\upsilon}_{ij}]\rangle$ is an interval-valued intuitionistic fuzzy value denoting the decision-maker's evaluating value of alternative x_i with respect to attribute a_j, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. That is, the membership degree of alternative x_i with respect to attribute a_j evaluated by the decision-maker is at least $\underline{\mu}_{ij}$ and at most $\bar{\mu}_{ij}$, i.e., $[\underline{\mu}_{ij}, \bar{\mu}_{ij}]$, where $0 \leqslant \underline{\mu}_{ij} \leqslant \bar{\mu}_{ij} \leqslant 1$; the non-membership degree of alternative x_i with respect to attribute a_j evaluated by the decision-maker is at least $\underline{\upsilon}_{ij}$ and at most $\bar{\upsilon}_{ij}$, i.e., $[\underline{\upsilon}_{ij}, \bar{\upsilon}_{ij}]$, where $0 \leqslant \underline{\upsilon}_{ij} \leqslant \bar{\upsilon}_{ij} \leqslant 1$. Thus, the evaluating value of alternative x_i with respect to attribute a_j evaluated by the decision-maker can be represented by an interval-valued intuitionistic fuzzy value $\langle[\underline{\mu}_{ij}, \bar{\mu}_{ij}], [\underline{\upsilon}_{ij}, \bar{\upsilon}_{ij}]\rangle$, where $0 \leqslant \bar{\mu}_{ij} + \bar{\upsilon}_{ij} \leqslant 1$. Therefore, we can construct the decision matrix Y for multiattribute decision making using interval-valued intuitionistic fuzzy values, where $Y = (\langle[\underline{\mu}_{ij}, \bar{\mu}_{ij}], [\underline{\upsilon}_{ij}, \bar{\upsilon}_{ij}]\rangle)_{m\times n}$.

Assume that the degree of importance of attribute a_j evaluated by the decision-maker is at least $\underline{\omega}_j$ and at most $\bar{\omega}_j$, i.e., $[\underline{\omega}_j, \bar{\omega}_j]$, where $0 \leqslant \underline{\omega}_j \leqslant \bar{\omega}_j \leqslant 1$; the degree of unimportance of attribute a_j evaluated by the decision-maker is at least $\underline{\rho}_j$ and at most $\bar{\rho}_j$, i.e.,$[\underline{\rho}_j, \bar{\rho}_j]$, where $0 \leqslant \underline{\rho}_j \leqslant \bar{\rho}_j \leqslant 1$. Thus, the weight of attribute a_j evaluated by the decision-maker can be represented by an interval-valued intuitionistic fuzzy value $\langle[\underline{\omega}_j, \bar{\omega}_j], [\underline{\rho}_j, \bar{\rho}_j]\rangle$, where $0 \leqslant \bar{\omega}_j + \bar{\rho}_j \leqslant 1$. Therefore, we can construct weighting vector W of the attributes $a_1, a_2, \ldots,$ and a_n, where

$$W = \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ \langle[\underline{\omega}_1, \bar{\omega}_1], [\underline{\rho}_1, \bar{\rho}_1]\rangle & \langle[\underline{\omega}_2, \bar{\omega}_2], [\underline{\rho}_2, \bar{\rho}_2]\rangle & \cdots & \langle[\underline{\omega}_n, \bar{\omega}_n], [\underline{\rho}_n, \bar{\rho}_n]\rangle \end{array}.$$

In the following, based on the proposed interval-valued intuitionistic fuzzy weighted average operator and the proposed fuzzy ranking method for intuitionistic fuzzy values, we propose a new multiattribute decision making method based on interval-valued intuitionistic fuzzy values. The proposed multiattribute decision making method based on interval-valued intuitionistic fuzzy values is now presented as follows:

Step 1: If $w_1, w_2, \ldots,$ and w_n are interval-valued intuitionistic fuzzy values, where w_j denotes the weight of attribute $a_j, w_j = \langle[\underline{\omega}_j, \bar{\omega}_j], [\underline{\rho}_j, \bar{\rho}_j]\rangle = [[\underline{\omega}_j, \bar{\omega}_j], [1 - \bar{\rho}_j, 1 - \underline{\rho}_j]], 0 \leqslant \underline{\omega}_j \leqslant \bar{\omega}_j \leqslant 1,\ 0 \leqslant \underline{\rho}_j \leqslant \bar{\rho}_j \leqslant 1, 0 \leqslant \bar{\omega}_j + \bar{\rho}_j \leqslant 1$ and $1 \leqslant j \leqslant n$, then based on Eq. (32), calculate the interval-valued intuitionistic fuzzy weighted average Y_i of alternative x_i, where $1 \leqslant i \leqslant m$, shown as follows:

$$\begin{aligned} Y_i &= \left[\frac{\sum_{j=1}^{n}[\underline{\mu}_{ij}, \bar{\mu}_{ij}][\underline{\omega}_j, \bar{\omega}_j]}{\sum_{j=1}^{n}[\underline{\omega}_j, \bar{\omega}_j]}, \frac{\sum_{j=1}^{n}[1 - \bar{\upsilon}_{ij}, 1 - \underline{\upsilon}_{ij}][1 - \bar{\rho}_j, 1 - \underline{\rho}_j]}{\sum_{j=1}^{n}[1 - \bar{\rho}_j, 1 - \underline{\rho}_j]}\right] \\ &= [[\underline{y}_{i1}, \bar{y}_{i1}], [\underline{y}_{i2}, \bar{y}_{i2}]], \end{aligned} \tag{38}$$

where $Y_i = [[\underline{y}_{i1}, \bar{y}_{i1}], [\underline{y}_{i2}, \bar{y}_{i2}]] = \langle[\underline{y}_{i1}, \bar{y}_{i1}], [1 - \bar{y}_{i2}, 1 - \underline{y}_{i2}]\rangle$ is an interval-valued intuitionistic fuzzy value and $1 \leqslant i \leqslant m; \underline{y}_1, \bar{y}_1, \underline{y}_2$ and $\bar{y}_2$ are calculated by the Karnik–Mendel algorithms (Karnik & Mendel, 2001), respectively. Otherwise, if $w_1, w_2, \ldots,$ and w_n are crisp values, where w_j denotes the weight of attribute a_j and $1 \leqslant j \leqslant n$, then based on Eq. (29), calculate the interval-valued intuitionistic fuzzy weighted average Y_i of alternative x_i, where $1 \leqslant i \leqslant m$, shown as follows:

$$\begin{aligned} Y_i &= \frac{\sum_{j=1}^{n}[[\underline{\mu}_{ij}, \bar{\mu}_{ij}], [1 - \bar{\upsilon}_{ij}, 1 - \underline{\upsilon}_{ij}]] \times w_j}{\sum_{j=1}^{n} w_j} \\ &= \left[\frac{\sum_{j=1}^{n}[\underline{\mu}_{ij}, \bar{\mu}_{ij}]w_j}{\sum_{j=1}^{n} w_j}, \frac{\sum_{j=1}^{n}[1 - \bar{\upsilon}_{ij}, 1 - \underline{\upsilon}_{ij}]w_j}{\sum_{j=1}^{n} w_j}\right] \\ &= \left[\left[\frac{\sum_{j=1}^{n}\underline{\mu}_{ij}w_j}{\sum_{j=1}^{n} w_j}, \frac{\sum_{j=1}^{n}\bar{\mu}_{ij}w_j}{\sum_{j=1}^{n} w_j}\right], \left[\frac{\sum_{j=1}^{n}(1 - \bar{\upsilon}_{ij})w_j}{\sum_{j=1}^{n} w_i}, \frac{\sum_{j=1}^{n}(1 - \underline{\upsilon}_{ij})w_j}{\sum_{j=1}^{n} w_j}\right]\right] \\ &= [[\underline{y}_{i1}, \bar{y}_{i1}], [\underline{y}_{i2}, \bar{y}_{i2}]], \end{aligned} \tag{39}$$

where $Y_i = [[\underline{y}_{i1}, \bar{y}_{i1}], [\underline{y}_{i2}, \bar{y}_{i2}]] = \langle[\underline{y}_{i1}, \bar{y}_{i1}], [1-\bar{y}_{i2}, 1-\underline{y}_{i2}]\rangle$ is an interval-valued intuitionistic fuzzy value and $1 \leqslant i \leqslant m$.

Step 2: Transform each interval-valued intuitionistic fuzzy value Y_i obtained in **Step 1** into an intuitionistic fuzzy value $\overline{Y}_i$, where $1 \leqslant i \leqslant m$, shown as follows:

$$\overline{Y}_i = \left\langle [\underline{y}_{i1}, \bar{y}_{i1}], [1-\bar{y}_{i2}, 1-\underline{y}_{i2}] \right\rangle = \left\langle \frac{\underline{y}_{i1}+\bar{y}_{i1}}{2}, \frac{1-\bar{y}_{i2}+1-\underline{y}_{i2}}{2} \right\rangle, \tag{40}$$

where $1 \leqslant i \leqslant m$.

Step 3: Based on Eq. (36), construct the judgment matrix P.

Step 4: Based on Eq. (37), calculate the ranking value $RV(Y_i)$ of each alternative x_i, where $1 \leqslant i \leqslant m$. The larger the ranking value of $RV(Y_i)$ of Y_i, the better the ranking order of alternative x_i, where $1 \leqslant i \leqslant m$.

Example 6.1 (Li, 2010b). Assume that there is a set X of alternatives, where X= {Car Company, Food Company, Computer Company, Arms Company}, and assume that there is a set A of attributes, where A = {Risk Analysis, Growth Analysis, Environmental Impact Analysis}. Assume that the weights w_1, w_2 and w_3 of the attributes "Risk Analysis", "Growth Analysis" and "Environmental Impact Analysis" are $\langle[0.1,0.4],[0.2,0.55]\rangle$, $\langle[0.2,0.5],[0.15,0.45]\rangle$ and $\langle[0.25,0.6],[0.15,0.38]\rangle$, respectively. Assume that the investment company wants to evaluate the four alternatives "Car Company", "Food Company", "Computer Company" and "Arms Company" with respect to the attributes "Risk Analysis", "Growth Analysis" and "Environmental Impact Analysis", respectively. Assume that the decision matrix Y is shown as follows:

$$Y = \begin{array}{c} \\ \text{Car Company} \\ \text{Food Company} \\ \text{Computer Company} \\ \text{Arms Company} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Risk Analysis} & \text{Growth Analysis} & \text{Environmental Impact Analysis} \end{array} \\ \begin{bmatrix} \langle[0.4,0.5],[0.3,0.4]\rangle & \langle[0.4,0.6],[0.2,0.4]\rangle & \langle[0.1,0.3],[0.5,0.6]\rangle \\ \langle[0.6,0.7],[0.2,0.3]\rangle & \langle[0.6,0.7],[0.2,0.3]\rangle & \langle[0.4,0.7],[0.1,0.2]\rangle \\ \langle[0.3,0.6],[0.3,0.4]\rangle & \langle[0.5,0.6][0.3,0.4]\rangle & \langle[0.5,0.6],[0.1,0.3]\rangle \\ \langle[0.7,0.8],[0.1,0.2]\rangle & \langle[0.6,0.7],0.1,0.3]\rangle & \langle[0.3,0.4],[0.1,0.2]\rangle \end{bmatrix} \end{array}.$$

[**Step 1**] Based on Eq. (38), we can get the interval-valued intuitionistic fuzzy weighted average Y_s of alternative x_s, where $s \in$ {Car Company, Food Company, Computer Company, Arms Company}, shown as follows:

$$Y_{\text{Car Company}} = [[0.2, 0.5], [0.5081, 0.6828]] = \langle[0.2, 0.5], [0.3172, 0.4919]\rangle,$$

$$Y_{\text{Food Company}} = [[0.4667, 0.7], [0.7273, 0.8459]] = \langle[0.4667, 0.7], [0.1541, 0.2727]\rangle,$$

$$Y_{\text{Computer Company}} = [[0.4059, 0.6], [0.6273, 0.7919]] = \langle[0.4059, 0.6], [0.2081, 0.3727]\rangle,$$

$$Y_{\text{Arms Company}} = [[0.4111, 0.6696], [0.7557, 0.9]] = \langle[0.4111, 0.6696], [0.1, 0.2443]\rangle.$$

[**Step 2**] Based on Eq. (40), we can get $\overline{Y}_{\text{Car Company}}$, $\overline{Y}_{\text{Food Company}}$, $\overline{Y}_{\text{Computer Company}}$ and $\overline{Y}_{\text{Arms Company}}$, respectively, shown as follows:

$$\overline{Y}_{\text{Car Company}} = \left\langle \frac{0.2+0.5}{2}, \frac{0.3172+0.4919}{2} \right\rangle = \langle 0.35, 0.40455\rangle,$$

$$\overline{Y}_{\text{Food Company}} = \left\langle \frac{0.4667+0.7}{2}, \frac{0.1541+0.2727}{2} \right\rangle = \langle 0.58335, 0.2134\rangle,$$

$$\overline{Y}_{\text{Computer Company}} = \left\langle \frac{0.4059+0.6}{2}, \frac{0.2081+0.3727}{2} \right\rangle = \langle 0.50295, 0.2904\rangle,$$

$$\overline{Y}_{\text{Arms Company}} = \left\langle \frac{0.4111+0.6696}{2}, \frac{0.1+0.2443}{2} \right\rangle = \langle 0.54035, 0.17215\rangle.$$

[**Step 3**] Based on Eq. (36), we can construct the judgment matrix P, shown as follows:

$$P = \begin{array}{c} \\ Y_{\text{Car Company}} \\ Y_{\text{Food Company}} \\ Y_{\text{Computer Company}} \\ Y_{\text{Arms Company}} \end{array} \begin{array}{c} \begin{array}{cccc} Y_{\text{Car Company}} & Y_{\text{Food Company}} & Y_{\text{Computer Company}} & Y_{\text{Arms Company}} \end{array} \\ \begin{bmatrix} 0.5 & 0.027 & 0.2046 & 0.1034 \\ 0.973 & 0.5 & 0.692 & 0.5018 \\ 0.7954 & 0.308 & 0.5 & 0.3425 \\ 0.8966 & 0.4982 & 0.6575 & 0.5 \end{bmatrix} \end{array}.$$

[**Step 4**] Based on Eq. (37), we can get

$$RV(\text{CarCompany}) = \frac{2}{3^4}(0.5+0.027+0.2046+0.1034) = 0.1044,$$

$$RV(\text{FoodCompany}) = \frac{2}{3^4}(0.973+0.5+0.692+0.5018) = 0.3333,$$

$$RV(\text{ComputerCompany}) = \frac{2}{3^4}(0.7954+0.308+0.5+0.3425) = 0.2432,$$

$$RV(\text{ArmsCompany}) = \frac{2}{3^4}(0.8966+0.4982+0.6575+0.5) = 0.319.$$

Because RV(Food Company) > RV(Arms Company) > RV(Computer Company) > RV (Car Company), we can see that the ranking order of the alternatives "Car Company", "Food Company", "Computer Company" and "Arms Company" is: "Food Company" > "Arms Company" > "Computer Company" > 'Car Company", where the best alternative is "Food Company", which coincides with the result of Li's method (2010b).

Example 6.2 (Li, 2010b). With the same assumption as *Example 6.1*. Assume that the weights w_1, w_2 and w_3 of the attributes "Risk Analysis", "Growth Analysis" and "Environmental Impact Analysis" are crisp values, where w_1 = 0.35, w_2 = 0.25 and w_3 = 0.40.

[**Step 1**] Based on Eq. (39), we can get the interval-valued intuitionistic fuzzy weighted average Y_s of alternative x_s, where $s \in$ {Car Company, Food Company, Computer Company, Arms Company}, shown as follows:

$$Y_{\text{Car Company}} = \left[\left[\frac{0.4\times0.35+0.4\times0.25+0.1\times0.4}{0.35+0.25+0.4}, \frac{0.5\times0.35+0.6\times0.25+0.3\times0.4}{0.35+0.25+0.4}\right], \left[\frac{0.6\times0.35+0.6\times0.25+0.4\times0.4}{0.35+0.25+0.4}, \frac{0.7\times0.35+0.8\times0.25+0.5\times0.4}{0.35+0.25+0.4}\right]\right]$$
$$= [[0.28, 0.445], [0.52, 0.645]] = \langle[0.28, 0.445], [0.355, 0.48]\rangle,$$

$$Y_{\text{Food Company}} = \left[\left[\frac{0.6\times0.35+0.6\times0.25+0.4\times0.4}{0.35+0.25+0.4}, \frac{0.7\times0.35+0.7\times0.25+0.7\times0.4}{0.35+0.25+0.4}\right], \left[\frac{0.7\times0.35+0.7\times0.25+0.8\times0.4}{0.35+0.25+0.4}, \frac{0.8\times0.35+0.8\times0.25+0.9\times0.4}{0.35+0.25+0.4}\right]\right]$$
$$= [[0.52, 0.7], [0.74, 0.84]] = \langle[0.52, 0.7], [0.16, 0.26]\rangle,$$

$$Y_{\text{Computer Company}} = \left[\left[\frac{0.3\times0.35+0.5\times0.25+0.5\times0.4}{0.35+0.25+0.4}, \frac{0.6\times0.35+0.6\times0.25+0.6\times0.4}{0.35+0.25+0.4}\right], \left[\frac{0.6\times0.35+0.6\times0.25+0.7\times0.4}{0.35+0.25+0.4}, \frac{0.7\times0.35+0.7\times0.25+0.9\times0.4}{0.35+0.25+0.4}\right]\right]$$
$$= [[0.43, 0.6], [0.64, 0.78]] = \langle[0.43, 0.6], [0.22, 0.36]\rangle,$$

$$Y_{\text{Arms Company}} = \left[\left[\frac{0.7\times0.35+0.6\times0.25+0.3\times0.4}{0.35+0.25+0.4}, \frac{0.8\times0.35+0.7\times0.25+0.4\times0.4}{0.35+0.25+0.4}\right], \left[\frac{0.8\times0.35+0.7\times0.25+0.8\times0.4}{0.35+0.25+0.4}, \frac{0.9\times0.35+0.9\times0.25+0.9\times0.4}{0.35+0.25+0.4}\right]\right]$$
$$= [[0.515, 0.615], [0.775, 0.9]] = [0.515, 0.615], [0.1, 0.225]\rangle.$$

[**Step 2**] Based on Eq. (40), we can get $\overline{Y}_{\text{Car Company}}$, $\overline{Y}_{\text{Food Company}}$, $\overline{Y}_{\text{Computer Company}}$ and $\overline{Y}_{\text{Arms Company}}$, respectively, shown as follows:

$$\overline{Y}_{\text{Car Company}} = \left\langle \frac{0.28+0.445}{2}, \frac{0.355+0.48}{2} \right\rangle = \langle 0.3625, 0.4175 \rangle,$$

$$\overline{Y}_{\text{Food Company}} = \left\langle \frac{0.52+0.7}{2}, \frac{0.16+0.26}{2} \right\rangle = \langle 0.61, 0.21 \rangle,$$

$$\overline{Y}_{\text{Computer Company}} = \left[\frac{0.43+0.6}{2}, \frac{0.22+0.36}{2} \right] = \langle 0.515, 0.29 \rangle,$$

$$\overline{Y}_{\text{Arms Company}} = \left[\frac{0.515+0.615}{2}, \frac{0.1+0.225}{2} \right] = \langle 0.565, 0.1625 \rangle.$$

[Step 3] Based on Eq. (36), we can get the judgment matrix P, shown as follows:

$$P = \begin{array}{c} \\ Y_{\text{Car Company}} \\ Y_{\text{Food Company}} \\ Y_{\text{Computer Company}} \\ Y_{\text{Arms Company}} \end{array} \begin{array}{c} \begin{array}{cccc} Y_{\text{Car Company}} & Y_{\text{Food Company}} & Y_{\text{Computer Company}} & Y_{\text{Arms Company}} \end{array} \\ \begin{bmatrix} 0.5 & 0 & 0.1627 & 0.0355 \\ 1 & 0.5 & 0.7333 & 0.4972 \\ 0.8373 & 0.2667 & 0.5 & 0.3102 \\ 0.9645 & 0.5028 & 0.6898 & 0.5 \end{bmatrix} \end{array}.$$

[Step 4] Based on Eq. (37), we can get

$$RV(\text{Car Company}) = \frac{2}{3^4}(0.5+0+0.1627+0.0355) = 0.0873,$$

$$RV(\text{Food Company}) = \frac{2}{3^4}(1+0.5+0.7333+0.4972) = 0.3413,$$

$$RV(\text{Computer Company}) = \frac{2}{3^4}(0.8373+0.2667+0.5+0.3102) = 0.2393,$$

$$RV(\text{Arms Company}) = \frac{2}{3^4}(0.9645+0.5028+0.6898+0.5) = 0.3321.$$

Because RV(Food Company) > RV(Arms Company) > RV(Computer Company) > RV(Car Company), we can see that the ranking order of the alternatives "Car Company", "Food Company", "Computer Company" and "Arms Company" is: "Food Company" > "Arms Company" > "Computer Company" > "Car Company", where the best alternative is "Food Company", which coincides with the result of Li's method (2010b).

7. Conclusions

In this paper, we have presented a new multiattribute decision making method based on the proposed interval-valued intuitionistic fuzzy weighted average operator and the proposed fuzzy ranking method for intuitionistic fuzzy values. Li (2010b) pointed out that it is very difficult to solve the nonlinear programming models with $2n$ unknown variables and it will cost a large amount of computation time, where n is the number of attributes. He also pointed out that the choice of the interval-valued intuitionistic fuzzy positive ideal solution and the interval-valued intuitionistic fuzzy negative ideal solution is a sensitive problem and is not easy to determine. Because the proposed method uses the proposed interval-valued intuitionistic fuzzy weighted average operator and the Karnik–Mendel algorithms, which are known to converge monotonically and super-exponentially fast (Mendel & Liu, 2007), it can deal with multiattribute decision making problems based on interval-valued intuitionistic fuzzy values more efficient than the method presented in (Li, 2010b).

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