Linear Algebra and Its Applications

Fourth Edition

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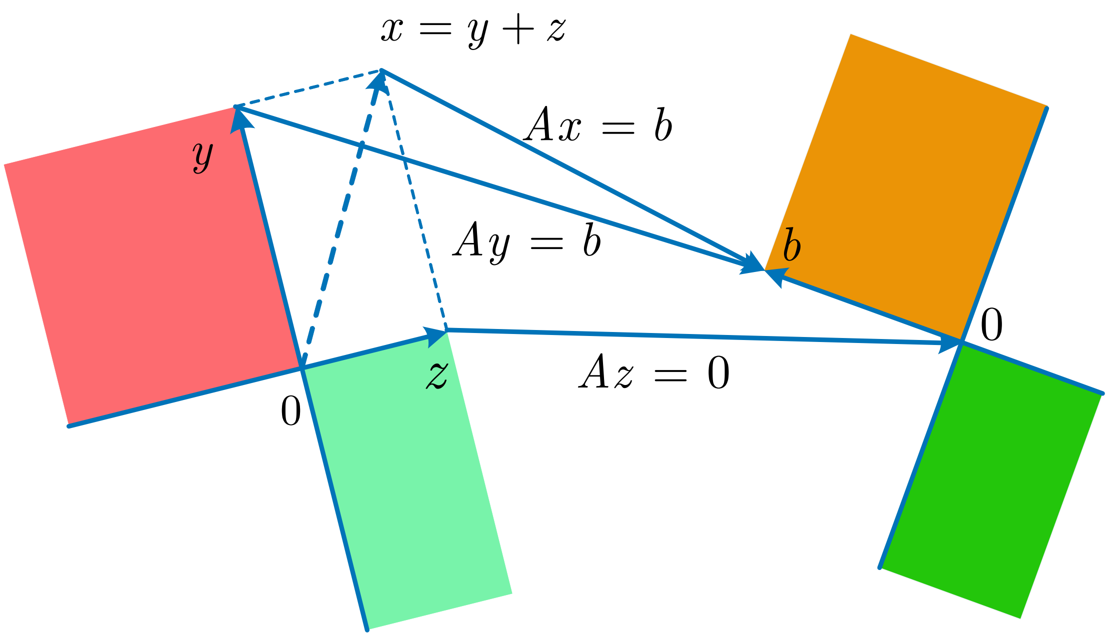


That could seem a little mysterious, unless you already know about 2 by 2 determinants. They gave the same answer $y = 2$, coming from the same ratio of -6 to -3. If we stay with determinants (which we don't plan to do), there will be a similar formula to compute the other unknown, x:

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3 \cdot 5 - 2 \cdot 6}{1 \cdot 5 - 2 \cdot 4} = \frac{3}{-3} = -1. \tag{5}$$

Let me compare those two approaches, looking ahead to real problems when n is much larger ($n = 1000$ is a very moderate size in scientific computing). The truth is that direct use of the determinant formula for 1000 equations would be a total disaster. It would use the million numbers on the left sides correctly, but not efficiently. We will find that formula (Cramer's Rule) in Chapter 4, but we want a good method to solve 1000 equations in Chapter 1.

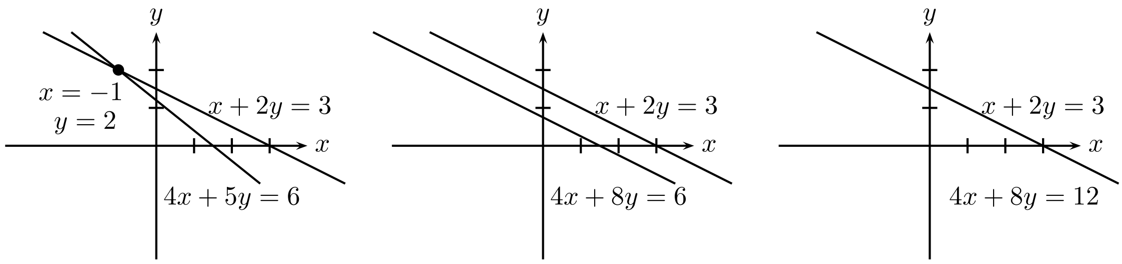
That good method is *Gaussian Elimination*. This is the algorithm that is constantly used to solve large systems of equations. From the examples in a textbook ($n = 3$ is close to the upper limit on the patience of the author and reader) too might not see much difference. Equations (2) and (4) used essentially the same steps to find $y = 2$. Certainly x came faster by the back-substitution in equation (3) than the ratio in (5). For larger n there is absolutely no question. Elimination wins (and this is even the best way to compute determinants).

The idea of elimination is deceptively simple—you will master it after a few examples. It will become the basis for half of this book, simplifying a matrix so that we can understand it. Together with the mechanics of the algorithm, we want to explain four deeper aspects in this chapter. They are:

1. Linear equations lead to ***geometry of planes***. It is not easy to visualize a nine-dimensional plane in ten-dimensional space. It is harder to see ten of those planes, intersecting at the solution to ten equations—but somehow this is almost possible. Our example has two lines in Figure 1.1, meeting at the point $(x, y) = (-1, 2)$. Linear algebra moves that picture into ten dimensions, where the intuition has to imagine the geometry (and gets it right)

2. We move to ***matrix notation***, writing the n unknowns as a vector x and the n equations as $Ax = b$. We multiply A by "elimination matrices" to reach an upper triangular matrix U. Those steps factor A into L times U, where L is lower triangular. I will write down A and its factors for our example, and explain them at the right time:

$$\textbf{Factorization} \qquad A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = L \textbf{ times } U. \tag{6}$$



One solution $(x, y) = (-1, 2)$ **Parallel: No solution** **Whole line of solutions**

Figure 1.1: The example has one solution. Singular cases have none or too many.

First we have to introduce matrices and vectors and the rules for multiplication. Every matrix has a ***transpose*** A^{T}. This matrix has an ***inverse*** A^{-1}.

3. In most cases elimination goes forward without difficulties. The matrix has an inverse and the system $Ax = b$ has one solution. In exceptional cases the method will *break down*—either the equations were written in the wrong order, which is easily fixed by exchanging them, or the equations don't have a unique solution.

   That ***singular case*** will appear if 8 replaces 5 in our example:

$$\begin{array}{lrcrcl} \textbf{Singular case} & 1x & + & 2y & = & 3 \\ \textbf{Two parallel lines} & 4x & + & 8y & = & 6. \end{array} \tag{7}$$

   Elimination still innocently subtracts 4 times the first equation from the second. But look at the result!

$$(\textbf{equation 2}) - 4(\textbf{equation 1}) \qquad 0 = -6.$$

   This singular case has ***no solution***. Other singular cases have ***infinitely many solutions***. (Change 6 to 12 in the example, and elimination will lead to $0 = 0$. Now y can have *any value*,) When elimination breaks down, we want to find every possible solution.

4. We need a rough count of the ***number of elimination steps*** required to solve a system of size n. The computing cost often determines the accuracy in the model. A hundred equations require a third of a million steps (multiplications and subtractions). The computer can do those quickly, but not many trillions. And already after a million steps, roundoff error could be significant. (Some problems are sensitive; others are not.) Without trying for full detail, we want to see large systems that arise in practice, and how they are actually solved.

The final result of this chapter will be an elimination algorithm that is about as efficient as possible. It is essentially the algorithm that is in constant use in a tremendous variety of applications. And at the same time, understanding it in terms of *matrices*—the coefficient matrix A, the matrices E for elimination and P for row exchanges, and the final factors L and U—is an essential foundation for the theory. I hope you will enjoy this book and this course.

1.2 The Geometry of Linear Equations

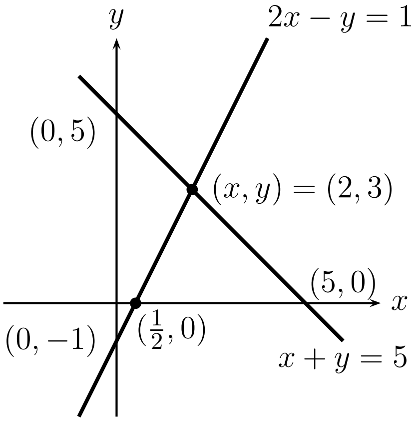
The way to understand this subject is by example. We begin with two extremely humble equations, recognizing that you could solve them without a course in linear algebra. Nevertheless I hope you will give Gauss a chance:

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5. \end{aligned}$$

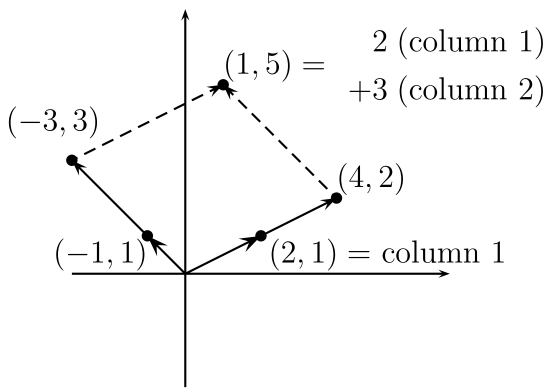
We can look at that system *by rows* or *by columns*. We want to see them both.

The first approach concentrates on the separate equations (the ***rows***). That is the most familiar, and in two dimensions we can do it quickly. The equation $2x-y=1$ is represented by a *straight line* in the x-y plane. The line goes through the points $x=1$, $y=1$ and $x=\frac{1}{2}$, $y=0$ (and also through $(2,3)$ and all intermediate points). The second equation $x+y=5$ produces a second line (Figure 1.2a). Its slope is $dy/dx=-1$ and it crosses the first line at the solution.

The point of intersection lies on both lines. It is the only solution to both equations. That point $x=2$ and $y=3$ will soon be found by "elimination."



(a) Lines meet at $x=2$, $y=3$

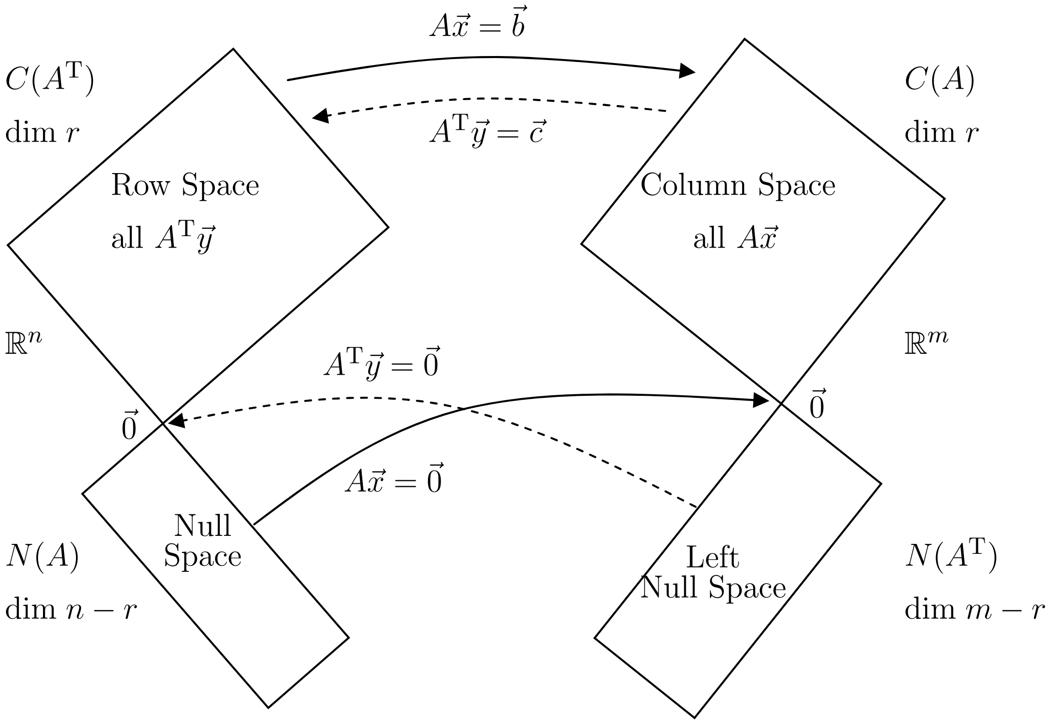


(b) Columns combine with 2 and 3

Figure 1.2: Row picture (two lines) and column picture (combine columns).

The second approach looks at the ***columns*** of the linear system. The two separate equations are really ***one vector equation***:

$$\textbf{Column form} \qquad x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}.$$


$A\vec{x} = \vec{b}$
$C(A^{\mathrm{T}})$
dim r
$A^{\mathrm{T}}\vec{y} = \vec{c}$
$C(A)$
dim r
Row Space
all $A^{\mathrm{T}}\vec{y}$
Column Space
all $A\vec{x}$
$\mathbb{R}^n$
$\mathbb{R}^m$
$A^{\mathrm{T}}\vec{y} = \vec{0}$
$\vec{0}$
$\vec{0}$
$A\vec{x} = \vec{0}$
Null
Space
$N(A)$
dim $n - r$
Left
Null Space
$N(A^{\mathrm{T}})$
dim $m - r$