When the elimination is down to $k$ equations, only $k^2 - k$ operations are needed to clear out the column below the pivot—by the same reasoning that applied to the first stage, when $k$ equaled $n$. Altogether, the total number of operations is the sum of $k^2 - k$ over all values of $k$ from 1 to $n$:

$$\textbf{Left side} \qquad (1^2 + \cdots + n^2) - (1 + \cdots + n) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n^3 - n}{3}.$$

Those are standard formulas for the sums of the first $n$ numbers and the first $n$ squares. Substituting $n = 1$ and $n = 2$ and $n = 100$ into the formula $\frac{1}{3}(n^3 - n)$, forward elimination can take no steps or two steps or about a third of a million steps:

> If $n$ is at all large, ***a good estimate for the number of operations is*** $\frac{1}{3}n^3$.

If the size is doubled, and few of the coefficients are zero, the cost is multiplied by 8.

Back-substitution is considerably faster. The last unknown is found in only one operation (a division by the last pivot). The second to last unknown requires two operations, and so on. Then the total for back-substitution is $1 + 2 + \cdots + n$.

Forward elimination also acts on the right-hand side (subtracting the same multiples as on the left to maintain correct equations). This starts with $n - 1$ subtractions of the first equation. Altogether ***the right-hand side is responsible for*** $n^2$ ***operations***—much less than the $n^3/3$ on the left. The total for forward and back is

$$\textbf{Right side} \qquad [(n-1) + (n-2) + \cdots + 1] + [1 + 2 + \cdots + n] = n^2.$$

Thirty years ago, almost every mathematician would have guessed that a general system of order $n$ could not be solved with much fewer than $n^3/3$ multiplications. (There were even theorems to demonstrate it, but they did not allow for all possible methods.) Astonishingly, that guess has been proved wrong. *There now exists a method that requires only* $Cn^{\log_2 7}$ *multiplications!* It depends on a simple fact: Two combinations of two vectors in two-dimensional space would seem to take 8 multiplications, but they can be done in 7. That lowered the exponent from $\log_2 8$, which is 3, to $\log_2 7 \approx 2.8$. This discovery produced tremendous activity to find the smallest possible power of $n$. The exponent finally fell (at IBM) below 2.376. Fortunately for elimination, the constant $C$ is so large and the coding is so awkward that the new method is largely (or entirely) of theoretical interest. The newest problem is the cost with *many processors in parallel.*

## Problem Set 1.3

**Problems 1–9 are about elimination on 2 by 2 systems.**

(c) A matrix whose entries are 0s and 1s has determinant 1, 0, or $-1$.

**4.** (a) Find the $LU$ factorization, the pivots, and the determinant of the 4 by 4 matrix whose entries are $a_{ij} =$ smaller of $i$ and $j$. (Write out the matrix.)

(b) Find the determinant if $a_{ij} =$ smaller of $n_i$ and $n_j$, where $n_1 = 2$, $n_2 = 6$, $n_3 = 8$, $n_4 = 10$. Can you give a general rule for any $n_1 \leq n_2 \leq n_3 \leq n_4$?

**5.** Let $F_n$ be the determinant of the 1, 1, $-1$ tridiagonal matrix ($n$ by $n$):

$$F_n = \det \begin{bmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & 1 & 1 & -1 & \\ & & \cdot & \cdot & \cdot \\ & & & 1 & 1 \end{bmatrix}.$$

By expanding in cofactors along row 1, show that $F_n = F_{n-1} + F_{n-2}$. This yields the *Fibonacci sequence* $1, 2, 3, 5, 8, 13, \ldots$ for the determinants.

**6.** Suppose $A_n$ is the $n$ by $n$ tridiagonal matrix with is on the three diagonals:

$$A_1 = \begin{bmatrix} 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \ldots$$

Let $D_n$ be the determinant of $A_n$; we want to find it.

(a) Expand in cofactors along the first row to show that $D_n = D_{n-1} - D_{n-2}$.

(b) Starting from $D_1 = 1$ and $D_2 = 0$, find $D_3, D_4, \ldots, D_8$. By noticing how these numbers cycle around (with what period?) find $D_{1000}$.

**7.** (a) Evaluate this determinant by cofactors of row 1:

$$\begin{vmatrix} 4 & 4 & 4 & 4 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{vmatrix}.$$

(b) Check by subtracting column 1 from the other columns and recomputing.

**8.** Compute the determinants of $A_2, A_3, A_4$. Can you predict $A_n$?

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Use row operations to produce zeros, or use cofactors of row 1.

**9.** How many multiplications to find an $n$ by $n$ determinant from

(a) the big formula (6)?

(b) the cofactor formula (10), building from the count for $n-1$?

(c) the product of pivots formula (including the elimination steps)?

**10.** In a 5 by 5 matrix, does a $+$ sign or $-$ sign go with $a_{15}a_{24}a_{33}a_{42}a_{51}$ down the reverse diagonal? In other words, is $P = (5,4,3,2,1)$ even or odd? The checkerboard pattern of $\pm$ signs for cofactors does *not* give $\det P$.

**11.** If $A$ is $m$ by $n$ and $B$ is $n$ by $m$, explain why

$$\det \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \det AB. \qquad \left( \textit{Hint:} \text{ Postmultiply by } \begin{bmatrix} I & 0 \\ B & I \end{bmatrix}. \right)$$

Do an example with $m < n$ and an example with $m > n$. Why does your second example automatically have $\det AB = 0$?

**12.** Suppose the matrix $A$ is fixed, except that $a_{11}$ varies from $-\infty$ to $+\infty$. Give examples in which $\det A$ is always zero or never zero. Then show from the cofactor expansion (8) that otherwise $\det A = 0$ for exactly *one value* of $a_{11}$.

**Problems 13–23 use the big formula with $n!$ terms: $|A| = \sum \pm a_{1\alpha}a_{2\beta}\cdots a_{n\nu}$.**

**13.** Compute the determinants of $A$, $B$, $C$ from six terms. Independent rows?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

**14.** Compute the determinants of $A$, $B$, $C$. Are their columns independent?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

**15.** Show that $\det A = 0$, regardless of the five nonzeros marked by $x$'s:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}. \qquad \text{(What is the rank of } A\text{?)}$$

**16.** This problem shows in two ways that $\det A = 0$ (the $x$'s are any numbers):

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}. \qquad \begin{array}{l} \text{5 by 5 matrix} \\ \text{3 by 3 zero matrix} \\ \text{Always singular} \end{array}$$

(a) How do you know that the rows are linearly dependent?

(b) Explain why all 120 terms are zero in the big formula for $\det A$.

**17.** Find two ways to choose nonzeros from four different rows and columns:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}. \quad (B \text{ has the same zeros as } A.)$$

Is $\det A$ equal to $1+1$ or $1-1$ or $-1-1$? What is $\det B$?

**18.** Place the smallest number of zeros in a 4 by 4 matrix that will guarantee $\det A = 0$. Place as many zeros as possible while still allowing $\det A \neq 0$.

**19.** (a) If $a_{11} = a_{22} = a_{33} = 0$, how many of the six terms in $\det A$ will be zero?

(b) If $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 products $a_{1j}a_{2k}a_{3\ell}a_{4m}$ are sure to be zero?

**20.** How many 5 by 5 permutation matrices have $\det P = +1$? Those are even permutations. Find one that needs four exchanges to reach the identity matrix.

**21.** If $\det A \neq 0$, at least one of the $n!$ terms in the big formula (6) is not zero. Deduce that some ordering of the rows of $A$ leaves no zeros on the diagonal. (Don't use $P$ from elimination; that $PA$ can have zeros on the diagonal.)

**22.** Prove that 4 is the largest determinant for a 3 by 3 matrix of 1s and $-1$s.

**23.** How many permutations of $(1,2,3,4)$ are even and what are they? Extra credit: What are all the possible 4 by 4 determinants of $I + P_{\text{even}}$?

**Problems 24–33 use cofactors $C_{ij} = (-1)^{i+j} \det M_{ij}$. Delete row $i$, column $j$.**

**24.** Find cofactors and then transpose. Multiply $C_A^{\mathrm{T}}$ and $C_B^{\mathrm{T}}$ by $A$ and $B$!

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.$$

**25.** Find the cofactor matrix $C$ and compare $AC^{\mathrm{T}}$ with $A^{-1}$:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

pivots. This is easy to correct. **Divide out of $U$ a diagonal pivot matrix $D$:**

$$\textbf{Factor out } D \qquad U = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \vdots \\ & 1 & u_{23}/d_2 & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}. \tag{9}$$

In the last example all pivots were $d_i = 1$. In that case $D = I$. But that was very exceptional, and normally $LU$ is different from $LDU$ (also written $LDV$).

***The triangular factorization can be written $A = LDU$, where $L$ and $U$ have $1$s on the diagonal and $D$ is the diagonal matrix of pivots.***

Whenever you see $LDU$ or $LDV$, it is understood that $U$ or $V$ has is on the diagonal—each row was divided by the pivot in $D$. Then $L$ and $U$ are treated evenly. An example of $LU$ splitting into $LDU$ is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & -2 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} = LDU.$$

That has the 1s on the diagonals of $L$ and $U$, and the pivots 1 and $-2$ in $D$.

***Remark 2.*** We may have given the impression in describing each elimination step, that the calculations must be done in that order. This is wrong. There is *some* freedom, and there is a "Crout algorithm" that arranges the calculations in a slightly different way. *There is no freedom in the final L, D, and U*. That is our main point:

> **1I** If $A = L_1D_1U_1$ and also $A = L_2D_2U_2$, where the $L$'s are lower triangular with unit diagonal, the $U$'s are upper triangular with unit diagonal, and the $D$'s are diagonal matrices with no zeros on the diagonal, then $L_1 = L_2$, $D_1 = D_2$, $U_1 = U_2$. The $LDU$ factorization and the $LU$ factorization are uniquely determined by $A$.

The proof is a good exercise with inverse matrices in the next section.

## Row Exchanges and Permutation Matrices

We now have to face a problem that has so far been avoided: The number we expect to use as a pivot might be zero. This could occur in the middle of a calculation. It will happen at the very beginning if $a_{11} = 0$. A simple example is

$$\textbf{Zero in the pivot position} \qquad \begin{bmatrix} \mathbf{0} & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The difficulty is clear; no multiple of the first equation will remove the coefficient 3.