Preface

Revising this textbook has been a special challenge, for a very nice reason. So many people have read this book, and taught from it, and even loved it. The spirit of the book could never change. This text was written to help our teaching of linear algebra keep up with the enormous importance of this subject—which just continues to grow.

One step was certainly possible and desirable—*to add new problems*. Teaching for all these years required hundreds of new exam questions (especially with quizzes going onto the web). I think you will approve of the extended choice of problems. The questions are still a mixture of *explain and compute*—the two complementary approaches to learning this beautiful subject.

I personally believe that many more people need linear algebra than calculus. Isaac Newton might not agree! But he isn't teaching mathematics in the 21st century (and maybe he wasn't a great teacher, but we will give him the benefit of the doubt). Certainly the laws of physics are well expressed by differential equations. Newton needed calculus—quite right. But the scope of science and engineering and management (and life) is now so much wider, and linear algebra has moved into a central place.

May I say a little more, because many universities have not yet adjusted the balance toward linear algebra. Working with curved lines and curved surfaces, the first step is always to *linearize*. Replace the curve by its tangent line, fit the surface by a plane, and the problem becomes linear. The power of this subject comes when you have ten variables, or 1000 variables, instead of two.

You might think I am exaggerating to use the word "beautiful" for a basic course in mathematics. Not at all. This subject begins with two vectors v and w, pointing in different directions. The key step is to *take their linear combinations*. We multiply to get $3v$ and $4w$, and we add to get the particular combination $3v+4w$. That new vector is in the *same plane* as v and w. When we take all combinations, we are filling in the whole plane. If I draw v and w on this page, their combinations $cv+dw$ fill the page (and beyond), but they *don't go up* from the page.

In the language of linear equations, I can solve $cv+dw=b$ exactly when the vector b lies in the same plane as v and w.

The remedy is equally clear. ***Exchange the two equations***, moving the entry 3 up into the pivot. In this example the matrix would become upper triangular:

$$\textbf{Exchange rows} \qquad \begin{aligned} 3u+4v &= b_2 \\ 2v &= b_1 \end{aligned}$$

To express this in matrix terms, we need the ***permutation matrix*** P that produces the row exchange. It comes from exchanging the rows of I:

$$\textbf{Permutation} \qquad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}.$$

P has the same effect on b, exchanging b_1 and b_2. *The new system is* $PAx = Pb$. The unknowns u and v are *not* reversed in a row exchange.

A permutation matrix P has the same rows as the identity (in some order). There is a single "1" in every row and column. The most common permutation matrix is $P = I$ (it exchanges nothing). The product of two permutation matrices is another permutation—the rows of I get reordered twice.

After $P = I$, the simplest permutations exchange two rows. Other permutations exchange more rows. **There are** $n! = (n)(n-1)\cdots(1)$ **permutations of size** n. Row 1 has n choices, then row 2 has $n-1$ choices, and finally the last row has only one choice. We can display all 3 by 3 permutations (there are $3! = (3)(2)(1) = 6$ matrices):

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \qquad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \qquad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \qquad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

There will be 24 permutation matrices of order $n = 4$. There are only two permutation matrices of order 2, namely

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

When we know about inverses and transposes (the next section defines A^{-1} and A^{T}), we discover an important fact: P^{-1} ***is always the same as*** P^{T}.

A zero in the pivot location raises two possibilities: ***The trouble may be easy to fix, or it may be serious***. This is decided by looking *below the zero*. If there is a nonzero entry lower down in the same column, then a row exchange is carried out. The nonzero entry becomes the needed pivot, and elimination can get going again:

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix} \qquad \begin{aligned} d = 0 &\implies \text{no first pivot} \\ a = 0 &\implies \text{no second pivot} \\ c = 0 &\implies \text{no third pivot.} \end{aligned}$$

If $d = 0$, the problem is incurable and this matrix is ***singular***. There is no hope for a unique solution to $Ax = b$. If d is *not* zero, an exchange P_{13} of rows 1 and 3 will move d into the pivot. However the next pivot position also contains a zero. The number a is now below it (the e above it is useless). If a is not zero then another row exchange P_{23} is called for:

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P_{23}P_{13}A = \begin{bmatrix} \boldsymbol{d} & e & f \\ 0 & \boldsymbol{a} & b \\ 0 & 0 & \boldsymbol{c} \end{bmatrix}$$

One more point: The permutation $P_{23}P_{13}$ will do both row exchanges at once:

$$P_{13}\ \textbf{acts first} \qquad P_{23}P_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P.$$

If we had known, we could have multiplied A by P in the first place. With the rows in the right order PA, any nonsingular matrix is ready for elimination.

Elimination in a Nutshell: $PA = LU$

The main point is this: If elimination can be completed with the help of row exchanges, then we can imagine that those exchanges are done first (by P). *The matrix PA will not need row exchanges.* In other words, PA allows the standard factorization into L times U. The theory of Gaussian elimination can be summarized in a few lines:

> **1J** In the ***nonsingular*** case, there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. Then $Ax = b$ has a *unique solution*:
>
> **With the rows reordered in advance, PA can be factored into LU.**
>
> In the ***singular*** case, no P can produce a full set of pivots: elimination fails.

In practice, we also consider a row exchange when the original pivot is *near* zero—even if it is not exactly zero. Choosing a larger pivot reduces the roundoff error.

You have to be careful with L. Suppose elimination subtracts row 1 from row 2, creating $\ell_{21} = 1$. Then suppose it exchanges rows 2 and 3. If that exchange is done in advance, the multiplier will change to $\ell_{31} = 1$ in $PA = LU$.

Example 7.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & \mathbf{0} & 2 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U. \tag{10}$$

That row exchange recovers LU—but now $\ell_{31} = 1$ and $\ell_{21} = 2$:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad PA = LU. \tag{11}$$

In MATLAB, A([r k] :) exchanges row k with row r below it (where the kth pivot has been found). We update the matrices L and P the same way. At the start, $P = I$ and sign $= +1$:

```
A([r k], :) = A([k r], :);
L([r k], 1:k-1) = L([k r], 1:k-1);
P([r k], :) = P([k r], :);
sign = -sign
```

The "**sign**" of P tells whether the number of row exchanges is even (sign $= +1$) or odd (sign $= -1$). A row exchange reverses sign. The final value of sign is the **determinant of** P and it does not depend on the order of the row exchanges.

To summarize: A good elimination code saves L and U and P. Those matrices carry the information that originally came in A—and they carry it in a more usable form. $Ax = b$ reduces to two triangular systems. This is the practical equivalent of the calculation we do next—*to find the inverse matrix* A^{-1} *and the solution* $x = A^{-1}b$.

Problem Set 1.5

1. When is an upper triangular matrix nonsingular (a full set of pivots)?

2. What multiple ℓ_{32} of row 2 of A will elimination subtract from row 3 of A? Use the factored form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}.$$

What will be the pivots? Will a row exchange be required?

3. Multiply the matrix $L = E^{-1}F^{-1}G^{-1}$ in equation (6) by GFE in equation (3):

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{times} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Multiply also in the opposite order. *Why are the answers what they are?*

Acknowledgments

I enjoyed writing this book, and I certainly hope you enjoy reading it. A big part of the pleasure comes from working with friends. I had wonderful help from Brett Coonley and Cordula Robinson and Erin Maneri. They created the LaTeX files and drew all the figures. Without Brett's steady support I would never have completed this new edition.

Earlier help with the Teaching Codes came from Steven Lee and Cleve Moler. Those follow the steps described in the book; MATLAB and Maple and Mathematica are faster for large matrices. All can be used (*optionally*) in this course. I could have added "Factorization" to that list above, as a fifth avenue to the understanding of matrices:

[L, U, P] = lu(A)	for linear equations
[Q, R] = qr(A)	to make the columns orthogonal
[S, E] = eig(A)	to find eigenvectors and eigenvalues.

In giving thanks, I never forget the first dedication of this textbook, years ago. That was a special chance to thank my parents for so many unselfish gifts. Their example is an inspiration for my life.

And I thank the reader too, hoping you like this book.

Gilbert Strang