Quantum Algo Track - 2025

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Reaching a Goal (Qu)bit by (Qu)bit

Meet Bob — a curious little character who lives on the integer line. Bob wants to reach a certain target, but there's a catch: he doesn't know where it is. All he can do is explore.

Being a simple creature, Bob picks up a fair coin. At each step, he flips it: heads, he moves right; tails, he moves left. It's an unbiased and memoryless process — what we call a *classical random walk*. Over time, Bob explores the space, but progress is slow and unpredictable.

Problem 0: The Classical Random Walk

Can you compute the root-mean-squared (RMS) displacement (distance from $x = 0$) of Bob's journey, and plot it against the number of steps taken?

Enter the Quantum World

So far, Bob has been flipping a fair coin and moving step by step. But he starts to wonder: *Is randomness the only way to explore? Could there be a smarter — or weirder — method?*

That's when Bob stumbles upon the world of quantum mechanics. This time, instead of an ordinary coin, he has access to a *qubit*. With it comes a new set of rules based on superposition and interference.

Now, Bob's walk behaves very differently. His position isn't a single point anymore, but a quantum superposition of possibilities. Some paths reinforce each other, others cancel out. This new kind of walk — a *quantum walk* — spreads faster, with the RMS displacement growing linearly with time, rather than diffusively as in the classical case.

Problem 1: A Quantum Coin Flip

To start simple, Bob replaces his classical coin flip with a quantum gate — the Pauli-X gate:

$$X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle.$$

Bob starts in the state

$$|x = 0\rangle \otimes |0\rangle,$$

that is, at position $x = 0$ with his qubit in the state $|0\rangle$.

At each step, Bob:

1. Applies the X gate to the qubit (flipping $|0\rangle \leftrightarrow |1\rangle$) — this acts as the *coin operation*.
2. Moves left if the qubit is in state $|0\rangle$, or right if it's $|1\rangle$ — this is the *shift operation*.

Simulate Bob's walk for a few steps (say 5). What path does he follow? Does this behavior look familiar?

Problem 2: The Superposed Walker

Now that Bob is comfortable flipping qubits, he tries something stranger — superposition.

Instead of being in just one state like $|0\rangle$ (heads) or $|1\rangle$ (tails), the qubit can be in a superposition:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle,$$

where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.

Bob decides:

- If the qubit is in $|0\rangle$, he steps left.
- If it's in $|1\rangle$, he steps right.

Since the qubit is now in a superposition, Bob's position also becomes a superposition. His combined state evolves as:

$$|\Psi\rangle = \alpha |x-1\rangle |0\rangle + \beta |x+1\rangle |1\rangle.$$

Each time step, Bob:

1. Applies a Hadamard gate[1] to the qubit.
2. Moves left or right depending on the qubit's state.

Simulate a few steps of this quantum walk starting from

$$|x = 0\rangle \otimes |0\rangle.$$

Compute the resulting position amplitudes after $t = 1, 2, 3$ steps. Then, write a program to compute the RMS displacement as a function of t. Does this spread faster than the classical random walk? Why?

Problem 3: Graph-Based Computation

After mastering 1D walks, Bob decides to explore a more abstract graph. Unfortunately, he gets lost among the vertices, and you need to help find him.

Formally, you are given a graph $G = (V, E)$ and a target vertex t.

Implement a discrete-time quantum walk with a coin operator of your choice. After T steps, measure the state and report the success probability of finding Bob at vertex t.

For simplicity, assume the entire graph is known. Compare the result against a classical random walk with the same number of steps.

Hint: You may use a Hadamard or Grover coin. Try simple graph structures like a cycle, a 2D grid, or a hypercube. Note: There is no need for IO here, hard-coded graphs are accepted.

State Estimation

You are given access to a hidden quantum state (represented as a density matrix) ρ of **two qubits**. It is guaranteed that the hidden state is **pure**.

Your only means of interacting with the state is by performing general measurements (POVMs) on copies of ρ. You are allowed at most **500** total single-copy measurements (shots) per target state. Each shot consists of preparing an independent copy of ρ, performing a single POVM measurement on it, and recording the outcome.

Your goal is to return an estimate $\hat{\rho} = |\hat{\psi}\rangle\langle\hat{\psi}|$ of the hidden pure state.

Evaluation: Submissions are ranked by the **average fidelity** $F(\rho, \hat{\rho}) = |\langle\psi|\hat{\psi}\rangle|^2$ across a fixed benchmark set of target pure states.

Note: You must take the input from a file which is in the following format:

[1] The Hadamard gate creates a 50-50 superposition: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

```
number_of_tests
a b c d // test1, where the pure state is a|00> + b|01> + c|10> + d|11>
...
```

Quantum Oscillator Search

After mastering his discrete quantum walk, Bob now decides to explore a more physical landscape — the quantum harmonic oscillator.

Instead of integer positions $\{|x\rangle\}$, Bob's world now consists of *number states* $\{|n\rangle : n = 0, 1, 2, \ldots\}$, representing quantized energy levels.

Bob starts at the ground state $|n = 0\rangle$, and in each step, he uses an ancilla qubit (the "coin") to decide whether to move up or down the energy ladder.

Problem 4

At each step:

1. Apply a coin unitary C (for example, the Hadamard gate) on the ancilla qubit.
2. If the coin is $|0\rangle$, apply the creation operator $a^\dagger$ (move up one level).
3. If the coin is $|1\rangle$, apply the annihilation operator a (move down one level, unless at $|0\rangle$, in which case remain there).

Tasks:

a. Write out the joint map for one time-step acting on $|n\rangle \otimes |q\rangle$.

b. Compute (analytically or numerically) the joint state after $t = 1, 2, 3$ steps and the corresponding probability distribution $P(n, t)$.

c. Define and compute the root-mean-square energy level:

$$\mathrm{RMS}(t) = \sqrt{\sum_n n^2\, P(n, t)}.$$

Compare the scaling of $\mathrm{RMS}(t)$ with a classical random walk on number states.

d. Introduce a phase potential whose time evolution operator (for the quantum harmonic oscillator) is specified by:

$$U_V = \sum_n e^{i\phi(n)} |n\rangle \langle n| ,$$

before each step with $\phi(n) = \alpha n$ or $\phi(n) = \alpha n^2$. Study its effect on interference and the spreading rate.

e. (*Bonus*) Truncate the Hilbert space to $n = 0, 1, \ldots, N$ and analyze boundary reflections or localization effects.

Hint: You may simulate this with a truncated oscillator basis and visualize the amplitude distribution as a heatmap over n and time steps.

Slow and Steady Wins the Quantum Race

After his many adventures with randomness and interference, Bob finally wonders: *What if I let quantum evolution itself do the work — slowly, adiabatically?*

This brings him to the world of *Adiabatic Quantum Computation (AQC)*.

Problem 6

Bob is given a problem Hamiltonian H_P whose ground state encodes the solution. Instead of preparing it directly, he starts from an easy Hamiltonian H_0 and interpolates:

$$H(s) = (1-s)H_0 + sH_P, \qquad s = \frac{t}{T}, \quad 0 \le t \le T.$$

Example: Find the satisfying assignment for the Boolean function $f(x_1, x_2) = x_1 \wedge x_2$.

$$H_P = I - |11\rangle\langle 11|, \qquad H_0 = I - |\psi_0\rangle\langle\psi_0|, \quad |\psi_0\rangle = \frac{1}{2}\sum_{x_1,x_2\in\{0,1\}} |x_1x_2\rangle.$$

Tasks:

a. Write down $H(s)$ as a 4×4 matrix in the computational basis.

b. Diagonalize $H(s)$ for $s \in [0,1]$ and plot the instantaneous eigenvalues. Identify the minimum spectral gap $\Delta_{\min}$.

c. Use the adiabatic condition
$$T \gg \frac{\max_s |\langle E_1(s)| \frac{dH}{ds} |E_0(s)\rangle|}{\Delta_{\min}^2}$$
to estimate the required runtime T.

d. Numerically solve the Schrödinger equation
$$i\frac{d}{dt}|\psi(t)\rangle = H\left(\frac{t}{T}\right)|\psi(t)\rangle$$
and plot the instantaneous ground-state fidelity $|\langle E_0(s)|\psi(s)\rangle|^2$.

e. Analyze what happens for small T (non-adiabatic evolution). How does the success probability scale with T?

Bonus: Show that, for suitable choices of H_0 and H_P, adiabatic evolution reduces to a continuous-time quantum walk. Identify such a mapping for the 2-qubit case.

Bridging QAOA and Adiabatic Paths

Bob now learns a new trick from the city of variational circuits: the **Quantum Approximate Optimization Algorithm (QAOA)**. He wonders how his slow, adiabatic journeys relate to short, layered variational circuits — and whether a few shallow layers can imitate an adiabatic evolution.

Problem 7: QAOA — A Discrete Adiabatic Shortcut

Consider a classical cost function $C(z)$ defined on n qubits (bitstrings $z \in \{0,1\}^n$), encoded as the diagonal problem Hamiltonian

$$H_P = \sum_z C(z)\, |z\rangle\langle z|.$$

Let the mixing Hamiltonian be

$$H_M = \sum_{j=1}^{n} X_j,$$

where X_j is the Pauli-X on qubit j.

The p-layer QAOA state is

$$|\psi(\boldsymbol{\gamma},\boldsymbol{\beta})\rangle \;=\; \left(\prod_{k=1}^{p} e^{-i\beta_k H_M} e^{-i\gamma_k H_P}\right) \; |+\rangle^{\otimes n}.$$

Tasks:

a. (*Connection*) Show that in the limit $p \to \infty$ with appropriately chosen parameters, the QAOA circuit can approximate the continuous adiabatic evolution from H_M to H_P. Explain the intuition: Trotterization of the adiabatic path and the identification of γ_k, β_k with small time slices.

b. (*Toy instance*) Consider the MaxCut problem on a simple 3-node triangle graph with nodes $1, 2, 3$. The cost Hamiltonian is

$$C(z) = \frac{1}{2}\Big[(1 - Z_1 Z_2) + (1 - Z_2 Z_3) + (1 - Z_3 Z_1)\Big],$$

where each Z_i is the Pauli-Z operator acting on qubit i. In the computational basis $|z_1 z_2 z_3\rangle$, with $z_i \in \{\pm 1\}$, $C(z)$ counts how many of the three edges have opposite spins (i.e., are cut by the partition).

Implement the QAOA for this instance. For depth parameters $p = 1, 2, 3$, numerically optimize the variational angles $(\boldsymbol{\gamma}, \boldsymbol{\beta})$ to maximize the expected cost

$$\langle C \rangle = \langle \psi(\boldsymbol{\gamma},\boldsymbol{\beta}) | C | \psi(\boldsymbol{\gamma},\boldsymbol{\beta}) \rangle.$$

Plot $\langle C \rangle$ as a function of p, and compare the achieved expectation to the adiabatic evolution of the same system using a linear schedule of duration T. Discuss how the QAOA performance approaches the adiabatic limit as p increases.

c. (*Landscape*) Discuss why QAOA parameters often display structure (e.g., smoothness across layers) and how that relates to the adiabatic limit. What does this imply for parameter initialization heuristics?

d. (*Heuristic*) Implement a Trotterized adiabatic path, i.e. apply the sequence

$$U^{(p)}_{\text{adiabatic}} = \prod_{k=1}^{p} \exp\big[-i(1 - s_k)\Delta t\, H_M\big] \exp\big[-i s_k \Delta t\, H_P\big],$$

with $s_k = k/p$ and $\Delta t = T/p$. Compare its performance (circuit depth, fidelity to the adiabatic ground state, and cost $\langle C \rangle$) to that of QAOA with p layers and optimized $(\boldsymbol{\gamma}, \boldsymbol{\beta})$. Discuss convergence as p increases.

Bonus: Investigate parameter transfer: use optimal parameters found for small graph sizes as initial guesses for larger graphs. Does this warm start speed up optimization? Discuss connections to transferability and locality of the problem Hamiltonian.

Decoherence, Measurements, and Quantum Search Breakdown

All of Bob's idealized walks assume perfectly isolated quantum systems. But real devices are noisy, and measurements — sometimes frequent, sometimes adaptive — can radically change dynamics. In this problem, Bob experiences the messy world of open systems and measurement-induced phenomena.

Problem 8: When the Environment Watches

A system of qubits evolves under unitary layers (e.g., QAOA or a quantum walk), but between layers each qubit interacts with an environment modeled by one of the following noise channels:

- **Dephasing:** $\mathcal{E}_\phi(\rho) = (1-p)\rho + p\, Z\rho Z$.
- **Amplitude damping:** standard Kraus form with probability p (loss of excitation).
- **Projective measurement:** with probability p, measure a qubit in the computational basis and optionally reprepare it.

Tasks:

a. (*QAOA under noise*) For the 3-node MaxCut from Problem 7, simulate QAOA at depths $p = 1, 2$ under amplitude-damping noise of strength p_{AD}. Compare how the expected cut value changes with p_{AD} for QAOA and for a Trotterized adiabatic schedule of similar depth.

b. (*Measurement-induced entanglement*) Prepare a 1D chain in $|+\rangle^{\otimes n}$. Apply alternating layers of nearest-neighbour entangling gates (CZ or iSWAP) and random measurements on each qubit with probability p_m. Track the average half-chain entanglement entropy over time and study how it varies with p_m. Identify any crossover from volume-law to area-law scaling.

c. (*Adaptive strategy*) Implement a simple feedback rule: whenever a measurement indicates deviation from a target subspace, apply a corrective local gate. Test whether this improves success probability under noise.

Discussion:

- Relate your observations to the *quantum Zeno effect*—frequent measurements slow or freeze dynamics.
- Compare robustness of different approaches: shallow QAOA, adaptive measurement correction, or adiabatic schedules.

Hint: Use density-matrix simulation or Monte Carlo quantum trajectories.
