主成分分析 PCA(Principal Components Analysis)

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1 Chapter 1

点云法向量的估计在很多场景都会用到，比如 ICP 配准，以及曲面重建。
 基于 PCA 的点云法向量估计，其实是从最小二乘法推导出来的。假设我们要估计某一点的法向量，我们需要通过利用该点的近邻点估计出一个平面，然后我们就能计算出该点的法向量。或者可以这么说，通过最小化一个目标函数（要求的参数为法向量），使得该点与其每个近邻点所构成的向量与法向量的点乘为 0，也就是垂直

$$\underset{\mathbf{c},\mathbf{n},||\mathbf{n}||=1}{min} = \sum_{i=1}^{n}((x_i - c^T)n)^2 \tag{1}$$

2 Chapter 2

2.1 ex 2.1

$$\underset{x\sim pX(x)}{\mathbb{E}}\ \underset{x\sim pX(x)}{\mathbb{E}} \mathbf{u}^T\mathbf{v} = \sum_{i=1}^{n} u_i v_i = \operatorname{tr}\left(\mathbf{v}\mathbf{u}^T\right) \tag{2}$$

2.2 ex 2.2

$$\begin{aligned}
H\left(\mathbf{x},\mathbf{y}\right) &= -\iint p\left(\mathbf{x},\mathbf{y}\right) \ln p\left(\mathbf{x},\mathbf{y}\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= -\iint p\left(\mathbf{x}\right) p\left(\mathbf{y}\right)\left(\ln p\left(\mathbf{x}\right) + \ln p\left(\mathbf{y}\right)\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= -\int p\left(\mathbf{x}\right) \ln p\left(\mathbf{x}\right) \underbrace{\left(\int p\left(\mathbf{y}\right) \mathrm{d}\mathbf{y}\right)}_{1} \mathrm{d}\mathbf{x} - \int p\left(\mathbf{y}\right) \ln p\left(\mathbf{y}\right) \underbrace{\left(\int p\left(\mathbf{x}\right) \mathrm{d}\mathbf{x}\right)}_{1} \mathrm{d}\mathbf{y} \\
&= H\left(\mathbf{x}\right) + H\left(\mathbf{y}\right)
\end{aligned} \tag{3}$$

2.3 ex 2.3

$$
\begin{aligned}
E\left(\mathbf{x}\mathbf{x}^T\right) &= E\left(\left(\mathbf{x}-\boldsymbol{\mu}+\boldsymbol{\mu}\right)\left(\mathbf{x}-\boldsymbol{\mu}+\boldsymbol{\mu}\right)^T\right) \\
&= E\left(\underbrace{\left(\mathbf{x}-\boldsymbol{\mu}\right)\left(\mathbf{x}-\boldsymbol{\mu}\right)^T}_{\boldsymbol{\Sigma}}+\underbrace{\boldsymbol{\mu}(\mathbf{x}-\boldsymbol{\mu})^T}_{\boldsymbol{\mu}\mathbf{0}^T}+\underbrace{\left(\mathbf{x}-\boldsymbol{\mu}\right)\boldsymbol{\mu}^T}_{\mathbf{0}\boldsymbol{\mu}^T}+\boldsymbol{\mu}\boldsymbol{\mu}^T\right) \\
&= \boldsymbol{\Sigma}+\boldsymbol{\mu}\boldsymbol{\mu}^T
\end{aligned}
\tag{4}
$$

2.4 ex 2.4

The integrate of an odd function is zero in the symmetric interval.

$$
\begin{aligned}
E\left(\mathbf{x}\right) &= \int_{-\infty}^{+\infty}\mathbf{x}p\left(\mathbf{x}\right)\mathrm{d}\mathbf{x} \\
&= \int_{-\infty}^{+\infty}\frac{\mathbf{x}}{\sqrt{(2\pi)^N\det\boldsymbol{\Sigma}}}\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)\mathrm{d}\mathbf{x} \\
&= \int_{-\infty}^{+\infty}\frac{\mathbf{y}+\boldsymbol{\mu}}{\sqrt{(2\pi)^N\det\boldsymbol{\Sigma}}}\exp\left(-\frac{1}{2}\mathbf{y}^T\boldsymbol{\Sigma}^{-1}\mathbf{y}\right)\mathrm{d}\mathbf{y} \\
&= \underbrace{\int_{-\infty}^{+\infty}\frac{\mathbf{y}}{\sqrt{(2\pi)^N\det\boldsymbol{\Sigma}}}\exp\left(-\frac{1}{2}\mathbf{y}^T\boldsymbol{\Sigma}^{-1}\mathbf{y}\right)\mathrm{d}\mathbf{y}}_{0}+\int_{-\infty}^{+\infty}\frac{\boldsymbol{\mu}}{\sqrt{(2\pi)^N\det\boldsymbol{\Sigma}}}\exp\left(-\frac{1}{2}\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x}\right)\mathrm{d}\mathbf{x} \\
&= \boldsymbol{\mu}\underbrace{\int_{-\infty}^{+\infty}\frac{1}{\sqrt{(2\pi)^N\det\boldsymbol{\Sigma}}}\exp\left(-\frac{1}{2}\mathbf{y}^T\boldsymbol{\Sigma}^{-1}\mathbf{y}\right)\mathrm{d}\mathbf{y}}_{\mathbf{y}\sim\boldsymbol{N}(0,\boldsymbol{\Sigma})} \\
&= \boldsymbol{\mu}\underbrace{\int_{-\infty}^{+\infty}p\left(\mathbf{y}\right)\mathrm{d}\mathbf{y}}_{1} \\
&= \boldsymbol{\mu}
\end{aligned}
\tag{5}
$$

2.5 ex 2.5

$$
\begin{aligned}
E\left((\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T\right) &= \int_{-\infty}^{\infty}(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T\frac{1}{\sqrt{(2\pi)^N}\det\boldsymbol{\Sigma}}\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T\right)\mathrm{d}\mathbf{x} \\
&= \frac{1}{\sqrt{(2\pi)^N}\det\boldsymbol{\Sigma}}\underbrace{\int_{-\infty}^{\infty} d\left(-\boldsymbol{\Sigma}(\mathbf{x}-\boldsymbol{\mu})\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T\right)\right)}_{\text{odd},=0} \\
&\quad+\frac{1}{\sqrt{(2\pi)^N}\det\boldsymbol{\Sigma}}\int_{-\infty}^{\infty}\boldsymbol{\Sigma}\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T\right)\mathrm{d}\mathbf{x} \\
&= \mathbf{0}+\boldsymbol{\Sigma}=\boldsymbol{\Sigma}
\end{aligned} \tag{6}
$$

2.6 ex 2.6

Take ln() on both sides, the right side will be:

$$
\begin{aligned}
&\ln\prod_{k=1}^{K}\exp\left(\frac{1}{2}(\mathbf{x}_k-\boldsymbol{\mu}_k)^T\boldsymbol{\Sigma}_k^{-1}(\mathbf{x}_k-\boldsymbol{\mu}_k)\right) \\
=&\sum_{k=1}^{K}\exp\left(\frac{1}{2}(\mathbf{x}_k-\boldsymbol{\mu}_k)^T\boldsymbol{\Sigma}_k^{-1}(\mathbf{x}_k-\boldsymbol{\mu}_k)\right) \\
=&\frac{1}{2}\left(\sum_{k=1}^{K}\mathbf{x}_k^T\boldsymbol{\Sigma}_k^{-1}\mathbf{x}_k+\sum_{k=1}^{K}2\boldsymbol{\mu}_k^T\boldsymbol{\Sigma}_k^{-1}\mathbf{x}_k+\sum_{k=1}^{K}\boldsymbol{\mu}_k^T\boldsymbol{\Sigma}_k^{-1}\boldsymbol{\mu}_k\right)
\end{aligned} \tag{7}
$$

Compare left side with the right side regarding of $\mathbf{x}$, we have:

$$
\boldsymbol{\Sigma}^{-1}=\sum_{k=1}^{K}\boldsymbol{\Sigma}_k^{-1},\quad \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}=\sum_{k=1}^{K}\boldsymbol{\Sigma}_k^{-1}\boldsymbol{\mu} \tag{8}
$$

2.7 ex 2.7

$$
\begin{aligned}
\mathrm{Cov}(\mathbf{x}) &= E((\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T) \\
&= E\left(\sum_{k=1}^{K}(w_k\mathbf{x}_k-w_k\boldsymbol{\mu}_k)\sum_{k=1}^{K}(w_k\mathbf{x}_k-w_k\boldsymbol{\mu}_k)^T\right) \\
&= E\left(\sum_{k=1}^{K}w_k^2(\mathbf{x}_k-\boldsymbol{\mu}_k)(\mathbf{x}_k-\boldsymbol{\mu}_k)^T+\sum_{m=1,n=1,m\neq n}^{K}w_mw_n(\mathbf{x}_m-\boldsymbol{\mu}_m)(\mathbf{x}_n-\boldsymbol{\mu}_n)^T\right) \\
&= \sum_{k=1}^{K}w_k^2\mathrm{Cov}(\mathbf{x}_k)+\underbrace{E\left(\sum_{m=1,n=1,m\neq n}^{K}w_mw_n(\mathbf{x}_m-\boldsymbol{\mu}_m)(\mathbf{x}_n-\boldsymbol{\mu}_n)^T\right)}_{\text{independent},=0} \\
&= \sum_{k=1}^{K}w_k^2\mathrm{Cov}(\mathbf{x}_k)
\end{aligned} \tag{9}
$$

2.8 ex 2.8

Expectation:

$$E\left(y\right)=E\left(\sum_{k=1}^{K}x_k^2\right)=K \tag{10}$$

Covariance:

$$\mathrm{Cov}\left(y\right)=E\left(\left(y-K\right)^2\right)=E\left(y^2-2Ky+K^2\right)=E\left(y^2\right)-2K^2+K^2 \tag{11}$$

For $E(y^2)$, we have:

$$E\left(y^2\right)=E\left(\left(\mathrm{x}^T\mathrm{x}\right)^2\right)=E\left(\sum_{k=1}^{K}x_k^4+\sum_{m=1,n=1,m\neq n}^{K}x_m^2x_n^2\right) \tag{12}$$

Use Isserlis to deal with the fourth-order items:

$$E\left(\sum_{k=1}^{K}x_k^4\right)=3\sum_{k=1}^{K}\underbrace{E\left(x_k^2\right)}_{1}E\left(x_k^2\right)=3K \tag{13}$$

and

$$\begin{aligned}E\left(\sum_{m=1,n=1,m\neq n}^{K}x_m^2x_n^2\right)&=\sum_{m=1,n=1,m\neq n}^{K}\underbrace{E\left(x_m^2\right)}_{1}E\left(x_n^2\right)+2\underbrace{E\left(x_mx_n\right)}_{0}E\left(x_mx_n\right)\\&=K\left(K-1\right)=K^2-K\end{aligned} \tag{14}$$

Take them together:

$$\mathrm{Cov}\left(y\right)=3K+K^2-K-2K^2+K^2=2K \tag{15}$$