Analysis of Algorithms

Dynamic Programming

Optimization Problems

- In which a set of choices must be made in order to arrive at an optimal (min/max) solution, subject to some constraints. (There may be several solutions to achieve ***an*** optimal value.)
- Two common techniques:
  - Dynamic Programming (global)
  - **Greedy Algorithms (local)**

Dynamic Programming

- Similar to divide-and-conquer, it breaks problems down into smaller problems that are solved recursively.
- In contrast, DP is applicable when the sub-problems are not independent, i.e. when sub-problems share sub-sub-problems. It solves every sub-sub-problem just once and save the results in a table to avoid duplicated computation.

Elements of DP Algorithms

- Sub-structure: decompose problem into smaller sub-problems. Express the solution of the original problem in terms of solutions for smaller problems.
- Table-structure: Store the answers to the sub-problem in a table, because sub-problem solutions may be used many times.
- Bottom-up computation: combine solutions on smaller sub-problems to solve larger sub-problems, and eventually arrive at a solution to the complete problem.

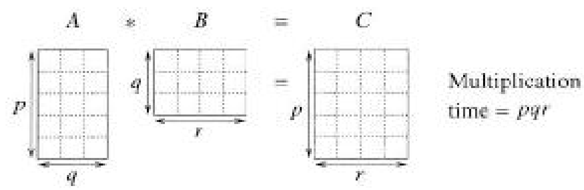
Matrix Chain Multiplication

- Determine the optimal sequence for performing a series of operations. **(the general class of the problem is important in compiler design for code optimization & in databases for query optimization)**
- For example: given a series of matrices: $A_1 \ldots A_n$, we can "parenthesize" this expression however we like, since matrix multiplication is associative (but not commutative).
- Multiply a p x q matrix A times a q x r matrix B, the result will be a p x r matrix C. (# of columns of A must be equal to # of rows of B.)

Matrix Multiplication

- In particular for $1 \le i \le p$ and $1 \le j \le r$,

  $C[i, j] = \sum_{k = 1 \text{ to } q} A[i, k]\, B[k, j]$
- Observe that there are pr total entries in C and each takes $O(q)$ time to compute, thus the total time to multiply 2 matrices is pqr.



Matrix Chain Multiplication

- Given a sequence of matrices $A_1 A_2 \ldots A_n$, and dimensions $p_0 p_1 \ldots p_n$ where A_i is of dimension p_{i-1} x p_i, determine multiplication sequence that minimizes the number of operations.
- This algorithm does not perform the multiplication, it just figures out the best order in which to perform the multiplication.

Example: MCM

- Consider 3 matrices: A_1 be 10 x 100, A_2 be 100 x 5, and A_3 be 5 x 50.

Mult[$((A_1 A_2) A_3)$] = (10x100x5) + (10x5x50) = 7500

Mult[$(A_1 (A_2 A_3))$] = (100x5x50) + (10x100x50) = 75000

Even for this small example, considerable savings can be achieved by reordering the evaluation sequence.

Fully Parenthesized

- A Product of matrices is fully parenthesized if it is either a single matrix or product of two fully parenthesized matrix products, surrounded by parentheses.

Naive Algorithm

- If we have just 1 matrix, then there is only one way to parenthesize. When n >=2, a fully parenthesized matrix product is the product of two fully parenthesized matrix products and the split between the two subproducts may occur between kth and (k+1)st matrices for any k = 1,2,….,n-1.

Cost of Naive Algorithm

- The number of different ways of parenthesizing n items is

  $P(n) = 1,$ if $n = 1$

  $P(n) = \sum_{k = 1 \text{ to } n-1} P(k)P(n-k),$ if $n \geq 2$

- Solution to this recurrence is given by *Catalan numbers* which grows as

  $\Omega(4^n / n^{3/2})$

DP Solution (I)

- Let $A_{i\ldots j}$ be the product of matrices i through j. $A_{i\ldots j}$ is a $p_{i-1} \times p_j$ matrix. At the highest level, we are multiplying two matrices together. That is, for any k, $1 \leq k \leq n-1$,

  $A_{1\ldots n} = (A_{1\ldots k})(A_{k+1\ldots n})$

- The problem of determining the optimal sequence of multiplication is broken up into 2 parts:
  - Q : How do we decide where to split the chain (what k)?
  - A : Consider all possible values of k.
  - Q : How do we parenthesize the subchains $A_{1\ldots k}$ & $A_{k+1\ldots n}$?
  - A : Solve by recursively applying the same scheme.
- Next, we store the solutions to the sub-problems in a table and build the table in a bottom-up manner.

DP Solution (II)

- For $1 \leq i \leq j \leq n$, let $m[i, j]$ denote the minimum number of multiplications needed to compute $A_{i\ldots j}$.
- Example: Minimum number of multiplies for $A_{3\ldots7}$

  $A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9$

  1 4 2 4 3

  $m[3,7]$

- In terms of p_i , the product $A_{3\ldots7}$ has dimensions ____.

DP Solution (III)

- The optimal cost can be described be as follows:
  - $i = j \Rightarrow$ the sequence contains only 1 matrix, so $m[i, j] = 0$.
  - $i < j \Rightarrow$ This can be split by considering each k, $i \leq k < j$, as $A_{i\ldots k}$ $(p_{i-1} \times p_k)$ times $A_{k+1\ldots j}$ $(p_k \times p_j)$.
- This suggests the following recursive rule for computing $m[i, j]$:

  $m[i, i] = 0$

  $m[i, j] = \min_{i \leq k < j} (m[i, k] + m[k+1, j] + p_{i-1}p_kp_j)$ for $i < j$

Computing $m[i, j]$

- For a specific k,

  $(A_i \ldots A_k)(A_{k+1} \ldots A_j)$

  =

$\mathbf{m[i, j] = \min_{i £ k < j} (m[i, k] + m[k+1, j] + p_{i-1}p_kp_j)}$

Computing $m[i, j]$

- For a specific k,

  $(A_i \ldots A_k)(A_{k+1} \ldots A_j)$

  $= A_{i\ldots k}(A_{k+1} \ldots A_j)$ ($m[i, k]$ mults)

$\mathbf{m[i, j] = \min_{i £ k < j} (m[i, k] + m[k+1, j] + p_{i-1}p_kp_j)}$

Computing $m[i, j]$

- For a specific k,

$$
\begin{aligned}
&(A_i \ldots A_k)(A_{k+1} \ldots A_j) \\
&= A_{i\ldots k}(A_{k+1} \ldots A_j) && (m[i, k] \text{ mults}) \\
&= A_{i\ldots k} A_{k+1\ldots j} && (m[k+1, j] \text{ mults})
\end{aligned}
$$

$$m[i, j] = \min_{i \le k < j} (m[i, k] + m[k+1, j] + p_{i-1} p_k p_j)$$

Computing $m[i, j]$

- For a specific k,

$$
\begin{aligned}
&(A_i \ldots A_k)(A_{k+1} \ldots A_j) \\
&= A_{i\ldots k}(A_{k+1} \ldots A_j) && (m[i, k] \text{ mults}) \\
&= A_{i\ldots k} A_{k+1\ldots j} && (m[k+1, j] \text{ mults}) \\
&= A_{i\ldots j} && (p_{i-1}\, p_k\, p_j \text{ mults})
\end{aligned}
$$

$$m[i, j] = \min_{i \le k < j} (m[i, k] + m[k+1, j] + p_{i-1} p_k p_j)$$

Computing $m[i, j]$

- For a specific k,

$$
\begin{aligned}
&(A_i \ldots A_k)(A_{k+1} \ldots A_j) \\
&= A_{i\ldots k}(A_{k+1} \ldots A_j) && (m[i, k] \text{ mults}) \\
&= A_{i\ldots k} A_{k+1\ldots j} && (m[k+1, j] \text{ mults}) \\
&= A_{i\ldots j} && (p_{i-1}\, p_k\, p_j \text{ mults})
\end{aligned}
$$

- For solution, evaluate for all k and take minimum.

$$m[i, j] = \min_{i \le k < j} (m[i, k] + m[k+1, j] + p_{i-1} p_k p_j)$$

Matrix-Chain-Order(p)

```
n ← length[p] - 1
for i ← 1 to n                          // initialization: O(n) time
    do m[i, i] ← 0
for L ← 2 to n                          // L = length of sub-chain
    do for i ← 1 to n - L+1
        do j ← i + L - 1
            m[i, j] ← ∞
            for k ← i to j - 1
                do q ← m[i, k] + m[k+1, j] + p_{i-1} p_k p_j
                    if q < m[i, j]
                        then m[i, j] ← q
                             s[i, j] ← k
return m and s
```


Analysis

- The array $s[i, j]$ is used to extract the actual sequence (see next).
- There are 3 nested loops and each can iterate at most n times, so the total running time is $\Theta(n^3)$.

Extracting Optimum Sequence

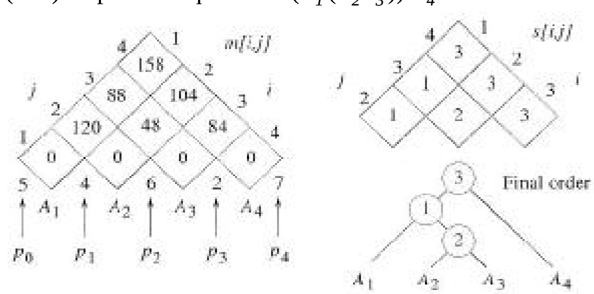
- Leave a split marker indicating where the best split is (i.e. the value of k leading to minimum values of $m[i, j]$). We maintain a parallel array $s[i, j]$ in which we store the value of k providing the optimal split.
- If $s[i, j] = k$, the best way to multiply the sub-chain $A_{i..j}$ is to first multiply the sub-chain $A_{i...k}$ and then the sub-chain $A_{k+1...j}$, and finally multiply them together. Intuitively $s[i, j]$ tells us what multiplication to perform *last*. We only need to store $s[i, j]$ if we have at least 2 matrices & $j > i$.

Mult (A, i, j)

```
if (j > i)
   then k = s[i, j]
        X = Mult(A, i, k)          // X = A[i]...A[k]
        Y = Mult(A, k+1, j)        // Y = A[k+1]...A[j]
        return X*Y                 // Multiply X*Y
   else return A[i]                // Return ith matrix
```


Example: DP for CMM

- The initial set of dimensions are <5, 4, 6, 2, 7>: we are multiplying A_1 (5x4) times A_2 (4x6) times A_3 (6x2) times A_4 (2x7). Optimal sequence is $(A_1 (A_2 A_3)) A_4$.



Finding a Recursive Solution

- Figure out the “top-level” choice you have to make (e.g., where to split the list of matrices)
- List the options for that decision
- Each option should require smaller sub-problems to be solved
- Recursive function is the minimum (or max) over all the options

$$m[i, j] = \min_{i \le k < j} (m[i, k] + m[k+1, j] + p_{i-1}p_k p_j)$$