Analysis of Algorithms

Linear-Time Sorting Algorithms

Sorting So Far

- Insertion sort:
  - Easy to code
  - Fast on small inputs (less than ~50 elements)
  - Fast on nearly-sorted inputs
  - $O(n^2)$ worst case
  - $O(n^2)$ average (equally-likely inputs) case
  - $O(n^2)$ reverse-sorted case

Sorting So Far

- Merge sort:
  - Divide-and-conquer:
    - Split array in half
    - Recursively sort subarrays
    - Linear-time merge step
  - O(n lg n) worst case
  - Doesn't sort in place

Sorting So Far

- Heap sort:
  - Uses the very useful heap data structure
    - Complete binary tree
    - Heap property: parent key > children's keys
  - O(n lg n) worst case
  - Sorts in place
  - Fair amount of shuffling memory around

Sorting So Far

- Quick sort:
  - Divide-and-conquer:
    - Partition array into two subarrays, recursively sort
    - All of first subarray < all of second subarray
    - No merge step needed!
  - O(n lg n) average case
  - Fast in practice
  - $O(n^2)$ worst case
    - Naïve implementation: worst case on sorted input
    - Address this with randomized quicksort

How Fast Can We Sort?

- We will provide a lower bound, then beat it
  - *How do you suppose we'll beat it?*
- First, an observation: all of the sorting algorithms so far are *comparison sorts*
  - The only operation used to gain ordering information about a sequence is the pairwise comparison of two elements
  - Theorem: all comparison sorts are Ω(n lg n)
    - A comparison sort must do O(n) comparisons (*why?*)
    - What about the gap between O(n) and O(n lg n)

Decision Trees

- *Decision trees* provide an abstraction of comparison sorts
  - A decision tree represents the comparisons made by a comparison sort. Every thing else ignored
  - (Draw examples on board)
- *What do the leaves represent?*
- *How many leaves must there be?*

Decision Trees

- Decision trees can model comparison sorts. For a given algorithm:
  - One tree for each n
  - Tree paths are all possible execution traces
  - *What's the longest path in a decision tree for insertion sort? For merge sort?*
- *What is the asymptotic height of any decision tree for sorting n elements?*
- Answer: $\Omega(n \lg n)$ (now let's prove it…)

Lower Bound For Comparison Sorting

- Thm: Any decision tree that sorts *n* elements has height $\Omega(n \lg n)$
- *What's the minimum # of leaves?*
- *What's the maximum # of leaves of a binary tree of height h?*
- Clearly the minimum # of leaves is less than or equal to the maximum # of leaves

Lower Bound For Comparison Sorting

- So we have…
  $n! \le 2^h$
- Taking logarithms:
  $\lg(n!) \le h$
- Stirling's approximation tells us:

$$n! > \left(\frac{n}{e}\right)^n$$

- Thus: $h \ge \lg\left(\frac{n}{e}\right)^n$

Lower Bound For Comparison Sorting

- So we have

$$h \ge \lg\left(\frac{n}{e}\right)^n$$

$$= n\lg n - n\lg e$$

$$= \Omega(n\lg n)$$

- Thus the minimum height of a decision tree is $\Omega(n \lg n)$

Lower Bound For Comparison Sorts

- Thus the time to comparison sort *n* elements is $\Omega(n \lg n)$
- Corollary: Heapsort and Mergesort are asymptotically optimal comparison sorts
- But the name of this lecture is "Sorting in linear time"!
  - *How can we do better than $\Omega(n \lg n)$?*

Sorting In Linear Time

- Counting sort
  - No comparisons between elements!
  - ***But***…depends on assumption about the numbers being sorted
    - We assume numbers are in the range *1.. k*
  - The algorithm:
    - Input: A[1..*n*], where A[j] ∈ {1, 2, 3, …, *k*}
    - Output: B[1..*n*], sorted (notice: not sorting in place)
    - Also: Array C[1..*k*] for auxiliary storage

Counting Sort

```
CountingSort(A, B, k)
      for i=1 to k
            C[i]= 0;
      for j=1 to n
            C[A[j]] += 1;
      for i=2 to k
            C[i] = C[i] + C[i-1];
      for j=n downto 1
            B[C[A[j]]] = A[j];
            C[A[j]] -= 1;
```

Work through example: A={4 1 3 4 3}, k = 4

Counting Sort

```
CountingSort(A, B, k)
      for i=1 to k
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            B[C[A[j]]] = A[j];
            C[A[j]] -= 1;
```

Takes time O(k)

Takes time O(n)

What will be the running time?

Counting Sort

- Total time: $O(n + k)$
  - Usually, $k = O(n)$
  - Thus counting sort runs in $O(n)$ time
- But sorting is $\Omega(n \lg n)$!
  - No contradiction--this is not a comparison sort (in fact, there are *no* comparisons at all!)
  - Notice that this algorithm is *stable*

Counting Sort

- Cool! *Why don't we always use counting sort?*
- Because it depends on range *k* of elements
- *Could we use counting sort to sort 32 bit integers? Why or why not?*
- Answer: no, *k* too large (2^{32} = 4,294,967,296)

Counting Sort

- *How did IBM get rich originally?*
- Answer: punched card readers for census tabulation in early 1900's.
  - In particular, a *card sorter* that could sort cards into different bins
    - Each column can be punched in 12 places
    - Decimal digits use 10 places
  - Problem: only one column can be sorted on at a time

Radix Sort

- Intuitively, you might sort on the most significant digit, then the second msd, etc.
- Problem: lots of intermediate piles of cards (read: scratch arrays) to keep track of
- Key idea: sort the *least* significant digit first

```
RadixSort(A, d)
   for i=1 to d
      StableSort(A) on digit i
```

  - Example: Fig 9.3

Radix Sort

- *Can we prove it will work?*
- Sketch of an inductive argument (induction on the number of passes):
  - Assume lower-order digits {j: j<i}are sorted
  - Show that sorting next digit i leaves array correctly sorted
    - If two digits at position i are different, ordering numbers by that digit is correct (lower-order digits irrelevant)
    - If they are the same, numbers are already sorted on the lower-order digits. Since we use a stable sort, the numbers stay in the right order

Radix Sort

- *What sort will we use to sort on digits?*
- Counting sort is obvious choice:
  - Sort *n* numbers on digits that range from 1..*k*
  - Time: O(n + k)
- Each pass over *n* numbers with *d* digits takes time O($n+k$), so total time O($dn+dk$)
  - When *d* is constant and k=O(n), takes O(n) time
- *How many bits in a computer word?*

Radix Sort

- Problem: sort 1 million 64-bit numbers
  - Treat as four-digit radix 2^{16} numbers
  - Can sort in just four passes with radix sort!
- Compares well with typical O(n lg n) comparison sort
  - Requires approx lg n = 20 operations per number being sorted
- *So why would we ever use anything but radix sort?*

Radix Sort

- In general, radix sort based on counting sort is
  - Fast
  - Asymptotically fast (i.e., O(n))
  - Simple to code
  - A good choice
- To think about: *Can radix sort be used on floating-point numbers?*

The End