Applications of first order differential equations

Let N(t) denote the amount of substance (or population) that is either growing or decaying. If we assume that $\frac{dN}{dt}$,the time rate of change of this amount of substance is proportional to the amount of substance present, then $\frac{dN}{dt} = KN$, or
Exponential Growth:

$$\frac{dN}{dt} = KN \tag{1}$$

Where:

$\frac{dN}{dt}$ represents the rate of change of the quantity N with respect to time t.
K is the growth rate constant, which determines how fast the quantity gr
N is the size of the population or quantity at time t.

To solve this differential equation, you can separate variables and integrate:

$$\int \frac{dN}{N} = \int K\,dt$$

This yields the solution:

$$N(t) = N_0 \cdot e^{Kt} \tag{2}$$

Where N_0 is the initial quantity at $t = 0$.

Example 1:A person places 20000 in saving account which pays 5 percent interest rate per annum compounded continuously.
Find the amount in the account after 3 years.
Time required to double the value.

Set up the differential equation first:

$$\frac{dN}{dt} - 0.05N = 0$$

Initial condition is: N(0)=20000

Step 1: Rewrite the differential equation in the form suitable for separation of variables:

$$\frac{dN}{dt} = 0.05N$$

Step 2: Separate variables by moving N terms to one side and t terms to the other side of the equation:

$$\frac{dN}{N} = 0.05dt$$

Step 3: Integrate both sides:

$$\int \frac{1}{N}\,dN = \int 0.05\,dt$$

On the left side, integrate with respect to N, and on the right side, integrate with respect to t:

$$\ln N = 0.05t + C$$

where C is the constant of integration.

Step 4: Solve for N by removing the natural logarithm:

$$N = e^{0.05t+C}$$

$$N = Ce^{0.05t}$$

This is the general solution to the differential equation.

To find the constant of integration C and obtain the particular solution for the differential equation with the initial condition $N(0) = 20,000$, follow these steps:

You have the general solution:

$$N(t) = C \cdot e^{0.05t}$$

Now, use the initial condition $N(0) = 20,000$:

$$N(0) = C \cdot e^{0.05 \cdot 0} = C \cdot e^0 = C \cdot 1 = C$$

So, $C = 20,000$.

Now that you've found the value of the constant of integration A, you can write the particular solution:

$$N(t) = 20,000 \cdot e^{0.05t}$$

To determine the population N after three years, we use the particular solution:

$$N(t) = 20,000 \cdot e^{0.05t}$$

Now, plug in $t = 3$ years:

$$N(3) = 20,000 \cdot e^{0.05 \cdot 3}$$

Calculate the value:

$$N(3) = 20,000 \cdot e^{0.15}$$

Using a calculator or software, you can find the approximate value:

$$N(3) \approx 20,000 \cdot 1.161231$$

Rounding to the nearest whole number, we get:

$$N(3) \approx 23,225$$

So, after three years, the population N will be approximately 23,225.

To find the time t after which $N(t) = 40,000$, we use the particular solution:

$$N(t) = 20,000 \cdot e^{0.05t}$$

Now, set $N(t)$ equal to 40,000 and solve for t:

$$40,000 = 20,000 \cdot e^{0.05t}$$

Divide both sides by 20,000:

$$2 = e^{0.05t}$$

To solve for t, take the natural logarithm (ln) of both sides:

$$\ln(2) = \ln\left(e^{0.05t}\right)$$

Using the property $\ln(e^x) = x$, you get:

$$\ln(2) = 0.05t$$

Now, solve for t:

$$t = \frac{\ln(2)}{0.05}$$

Using a calculator to find the approximate value of t:

$$t \approx \frac{0.6931}{0.05} \approx 13.862$$

So, after approximately 13.862 years, the population $N(t)$ will reach 40,000.

Example 2: A person places 5000 in an account that accrues interest compounded continuously. Assuming no additional deposits or withdrawls, how much will be in the account after seven years if the rate of interest is 8.5 percent for the first four years and 9.25 percent for the last three years?
Solution:
The differential equation is:

$$\frac{dN}{dt} - 0.085N = 0$$

N(0)=5000

Step 1: Rewrite the differential equation in the form suitable for separation of variables:

$$\frac{dN}{dt} = 0.085N$$

Step 2: Separate variables by moving N terms to one side and t terms to the other side of the equation:

$$\frac{dN}{N} = 0.085dt$$

Step 3: Integrate both sides:

$$\int \frac{1}{N}\,dN = \int 0.085\,dt$$

On the left side, integrate with respect to N, and on the right side, integrate with respect to t:

$$\ln N = 0.085t + C$$

where C is the constant of integration.

Step 4: Solve for N by removing the natural logarithm:

$$N = e^{0.085t+C}$$

$$N(t) = C_1 \cdot e^{0.085t}$$

Given that $N(0) = 5,000$, we can substitute $t = 0$ into the general solution to find the value of the constant A:

$$N(0) = C_1 \cdot e^{0.085 \cdot 0} = C_1 \cdot e^0 = C_1 \cdot 1 = C_1$$

So, $C_1 = 5,000$.

$$N(t) = 5,000 \cdot e^{0.085t}$$

To calculate $N(4)$ for the differential equation with the particular solution

$$N(t) = 5,000 \cdot e^{0.085t}$$

Substitute $t = 4$ into the particular solution to find $N(4)$:

$$N(4) = 5,000 \cdot e^{0.085 \cdot 4}$$

$$N(4) = 5,000 \cdot e^{0.34}$$

$$N(4) \approx 5,000 \cdot 1.404983 \approx 7,024.92$$

If the differential equation changes to

$$\frac{dN}{dt} - 0.0925N = 0$$

at $t = 4$ with the initial condition $N(4) = 7,024.92$ (as calculated previously), we can solve for $N(t)$ using this new differential equation.

For $t \geq 4$:

The new differential equation is $\frac{dN}{dt} - 0.0925N = 0$ with the initial condition $N(4) = 7,024.92$. We will solve this equation separately for $t \geq 4$.

Using the method of separation of variables:

$$\frac{dN}{N} = 0.0925dt$$

Integrating both sides:

$$\ln N = 0.0925(t) + C$$

Now, solve for N by removing the natural logarithm:

$$N = e^{0.0925t+C}$$

$$N(t) = Ce^{0.0925(t)}$$

To find the value of the constant C, we'll use the initial condition $N(4) = 7,024.92$:

$$7,024.92 = Ce^{0.0925(4)}$$

Now, solve for C:

$$C \approx 4852.23$$

Now, we can write the particular solution for $t \geq 4$:

$$N(t) = 4852.23e^{0.0925(t)}$$

N(7)=9271.44

Example 3: What constant interest rate is required if an initial deposit placed in to an account that accrues interest compounded continuously is to double its value in six years?

Please solve this question at your own.

Example 4: A bacteria culture is known to grow at a rate proportional to the amount present. After 1 hour, 1000 strands of bacteria are obserbed in the culture; and after 4 hours, 3000 strands. Find an expression for approximate number of strands of bacteria present in the culture at any time t. Also find the approximate number of strands of bacteria originally in the culture.

Example 5: The population of a cerain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population is doubled, and after 3 years the population is 20000. Estimate the number of people initially living in the country.