Probability and Statstics

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- **Probability Demystified**, Allan G. Bluman
- **Schaum's Outline of Probability and Statistics**
- **MATLAB Primer**, Seventh Edition
- **MATLAB Demystified** by McMahon, David

References

Readings for these lecture notes:

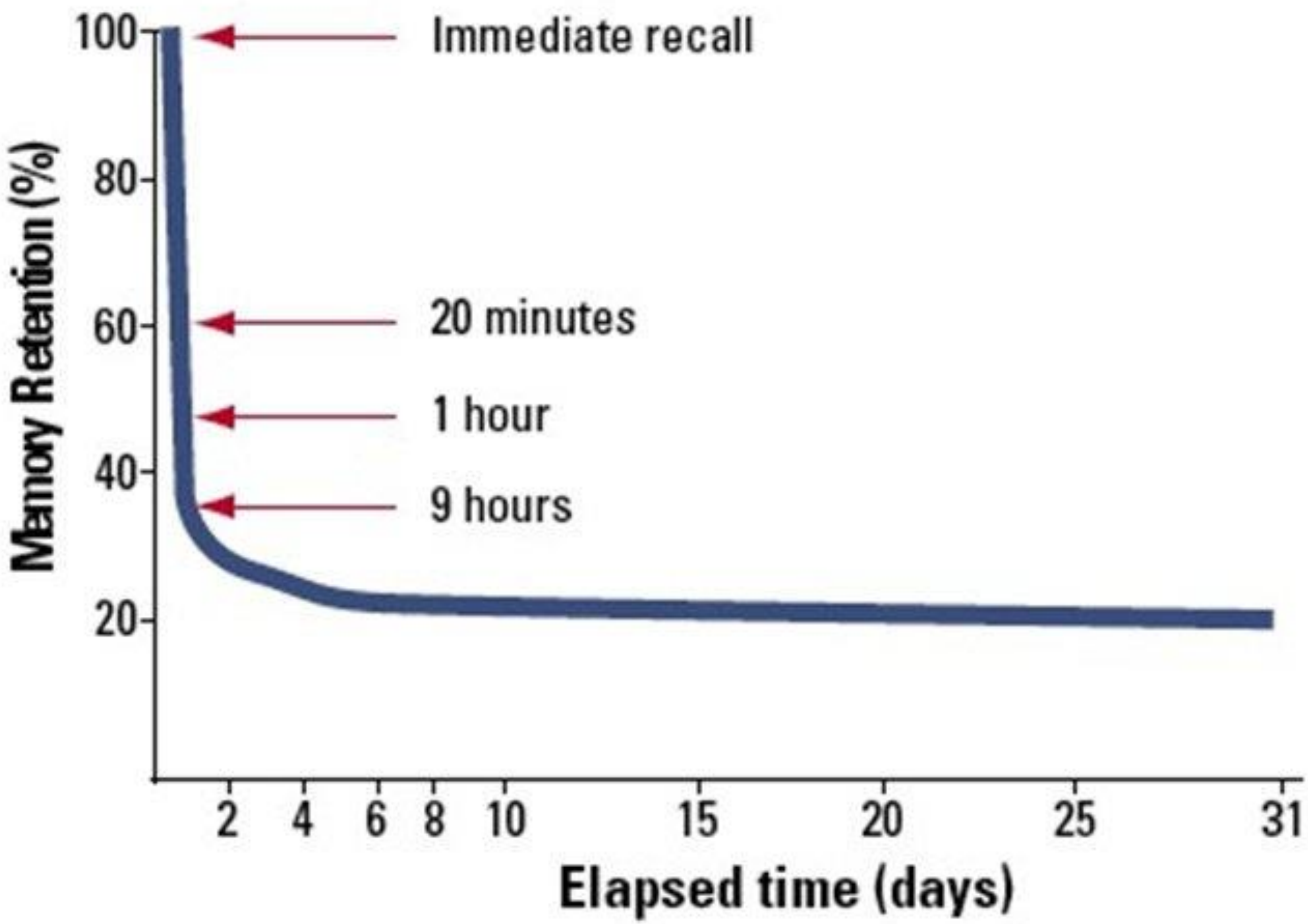
- Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer
- http://www.statisticshowto.com/geometric-distribution/
- https://peakmemory.me/category/forgetting-curve/
- https://au.mathworks.com/help/stats/prob.normaldistribution.pdf.html#d117e648466
- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

These notes contain material from the above resources.

“If you want to know what a man's like, take a good look at how he treats his inferiors, not his equals.”

— J.K. Rowling, Harry Potter and the Goblet of Fire

Forgetting curve



Poisson Distribution [1]

Example: An automobile manufacturer is concerned about a fault in the braking mechanism of a particular model. The fault can, on rare occasions, cause a catastrophe at high speed. The distribution of the number of **cars per year** that will experience the fault is a Poisson random variable with $\lambda = 5$.

(a) What is the probability that **at most 3** cars per year will experience a catastrophe?

(b) What is the probability that **more than 1** car per year will experience a catastrophe?

Solution: Here $\lambda t = (5)(1) = 5$

$$P(x;\ \lambda t) = \frac{(\lambda t)^{x} e^{-\lambda t}}{x!},\ x = 0,\ 1,\ 2, . . .$$

(a) $P(X \leq 3) = \sum_{x=0}^{x=3} p(x; 5)$

$= 0.2650$

(b) $P(X > 1) = 1 - P(x \leq 1)$

$= 1 - \sum_{x=0}^{x=1} p(x; 5)$

$= 1 - 0.0404$

$= 0.9596$

Poisson Distribution [2]

Example: Changes in airport procedures require considerable planning. Arrival rates of aircraft are important factors that must be taken into account. Suppose small aircraft arrive at a certain airport, according to a Poisson process, at the rate of **6 per hour**. Thus the Poisson parameter for arrivals for a period of hours is $\boldsymbol{\lambda = 6}$.

(a) What is the probability that **exactly 4** small aircraft arrive during a **1-hour period**?

Poisson Distribution [3]

(b) What is the probability that **at least 4** arrive during a **1-hour period**?

(c) If we define a **working day** as **12 hours**, what is the probability that at least **75 small aircraft** arrive **during a day**?

Poisson Distribution [3]

Here $\lambda t=(6)(1)=6$

$$P(x;\lambda t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}, x = 0, 1, 2, \ldots$$

(a) $P(X = 4) = \frac{(6)^4 e^{-6}}{4!}$ = **0.1339**

(b) $P(X \geq 4) = 1 - P(x < 4) = 1 - \sum_{x=0}^{x=3} p(x; 6) = 1 - 0.1512 =$ **0.8488**

Here **$\lambda t = (6)(12) = 72$**

(c) $P(X \geq 75) = 1 - P(x < 75) = 1 - \sum_{x=0}^{x=74} p(x; 72) =$ **0.3773.**

Poisson Distribution using MATLAB

poisspdf is Poisson probability density function in Matlab.

Y = poisspdf(X, LAMBDA) returns the Poisson probability density function with parameter LAMBDA at the values in X

Poisson Distribution using MATLAB

a)

```
x = 4
reqprob = poisspdf(x, 6)
% 0.1339
```

b)

```
x = 0:3
prob = poisspdf(x, 6)
Reqprob = 1 - sum(prob)
%0.8488
```

Poisson Distribution using MATLAB

c)

```
%  vector notation
x = 0:74
% sum of probability

prob = sum(poisspdf(x, 72))

% required probability
reqprob = 1 - prob

%  0.3773
```

Poisson approximation

The **Binomial distribution** converges towards the **Poisson distribution** as the number of trials goes to **infinity** while the product **np** remains fixed. Therefore the Poisson distribution with parameter **$\lambda = np$** can be used as an approximation to b(n, p) of the binomial distribution if n is sufficiently large and p is sufficiently small.

According to two rules of thumb, this approximation is good if

$n \geq 20$ and $p \leq 0.05$, or if $n \geq 100$ and $np \leq 10$.

Poisson Distribution [4]

Formula:

$$P(x; \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0, 1, 2, \ldots$$

where, λ is an average rate of value, x is a Poisson random variable and e is the base of logarithm(e = 2.718).

Example:

Consider, in an office on **average 2 customers** arrived per day. Calculate the possibilities for exactly 3 customers to be arrived on today.

Step1: Find $e^{-\lambda t}$.

where, $\lambda t = 2$ and $e = 2.718$, $e^{-\lambda t} = (2.718)^{-2} = 0.135$.

Step2: Find $(\lambda t)^x$

where, $t = 1$, $\lambda t = 2$ and $x = 3$, $(\lambda t)^x = 2^3 = 8$.

Step3: Find $P(x; \lambda)$

$$P(x; \lambda) = \frac{e^{-\lambda t}(\lambda t)^x}{x!},\ x = 0, 1, 2, \ldots$$

$$P(3; 2) = \frac{(0.135)(8)}{3!} = 0.18.$$

Hence there are 18% possibilities for 3 customers to be arrived today

Geometric Distribution

- Many actions in life are **repeated until** a success occurs.

For instance, you might have to send an **e-mail several** times before it is **successfully sent**. A situation such as this can be represented by a **geometric distribution.**

Geometric Distribution

A **geometric distribution** is a discrete probability distribution of a random variable x that satisfies these conditions.

1. A trial is **repeated** until a **success occurs**.

2. The repeated trials are **independent** of each other.

3. The probability of success p is the same for each trial.

4. The random variable x represents the number of the trial in which the **first success occurs**.

.

Geometric Distribution

- The probability that the first success will occur on trial number *x* is **$g(x; p) = p\,q^{x-1}, x = 1, 2, 3, \cdots$**

In other words, when the first success occurs on the third trial, the outcome is **FFS**, and the probability is $P(3) = q \times q \times p$, or $P(3) = p \times q^2$.

Geometric Distribution [1]

- Suppose we have a sequence of Bernoulli trials, each with a probability **p** of success and a probability **q = 1-p** of failure. How many trials occur **before we obtain a success?**

Example

- A **search engine** goes through a list of sites looking for a **given key phrase**. Suppose the **search terminates** as soon as the **key phrase is found**. The number of sites visited is **Geometric**.

•

Let the random variable X be the number of trials needed to obtain a success. Then X has values in the range {1,2,…}, and for k ≥1,

$g(x; p) = p\, q^{x-1}, x = 1, 2, 3, \cdots$

Alternative form

$g(x; p) = p\, q^{x}, x = 0, 1, 2, 3, \cdots$

Geometric Distribution [2]

Mean = $1/p$ and Variance = q/p^2

In the theory of **probability and statistics**, a **Bernoulli trial** is an experiment whose outcome is random and can be either of **two possible** outcomes, **“success” and “failure”.**

Geometric Distribution [3]

Conditions:

An experiment consists of repeating trials **until first success.**

Each trial has **two possible outcomes**.

A success with probability **p**.

A failure with probability **q** = 1 − p.

Repeated trials are **independent.**

x = number of trials to first success

x is a **Geometric Random Variable.**

$$g(x; p) = q^{x-1}p,\ x = 1, 2, 3, \cdots$$

Assumptions for the Geometric Distribution

The three assumptions are:

- There are **two possible outcomes** for each trial (success or failure).
- The trials are **independent**.
- The **probability of success** is the same for each trial.

Example Basketball player LeBron James makes a **free throw** shot about **75%** of the time. Find the probability that the **first free** throw shot he makes occurs on the **third or fourth attempt**. (Source: National Basketball Association)

Solution

Let x denotes number of attempts to get **first free** throw

$g(x; p) = p\, q^{x-1}, x = 1, 2, 3, \cdots$

p = 0.75 and q = 0.25

$g(3, 0.75) = (0.75)(0.25)^{3-1}$

$= 0.046875.$

$g(4, 0.75) = (0.75)(0.25)^{4-1}$

$= 0.011719.$

Since events are independent

$P(X = 3\ or\ X = 4) = 0.046875 + 0.011719$

$= .059$

- Even though theoretically a **success may never occur**, the **geometric distribution** is a discrete probability distribution because the values of x can be listed: 1, 2, 3, ...
- **Notice** that as ***x* becomes larger**, $P(x)$ gets **closer to zero**.

For instance, $P(15) = g(15, 0.75)$

$$= (0.75)(0.25)^{15-1}$$

$$= 0.000000028.$$

Example From past experience it is known that **3%** of accounts in a large accounting population are in **error**.

What is the probability that **5 accounts** are audited **before** an account in **error** is found?

Solution:

$$P(X = 5) = P(\text{1st 4 correctly stated})\, P(\text{5th in error})$$
$$= (0.97^4)(0.03)$$
$$= 0.0266$$

Example: In a certain manufacturing process it is known that, on the average, **1** in every **100**, items is defective. What is the probability that the **fifth item** inspected is the **first defective** item found?

Solution: Using the geometric distribution with $x = 5$ and

$p = 1/100 = 0.01$, $q = 0.99$, we have

$$g(x; p) = p\, q^{x-1}, \quad x = 1, 2, 3, \cdots$$

$$g(5; 0.01) = (0.01)(0.99)^{5-1}$$

$$= 0.0096$$

Geometric Distribution

Syntax

Y = geopdf(X,P)

Description

Y = geopdf(X,P) computes the geometric pdf at each of the values in X using the corresponding probabilities in P.

X and P can be vectors, matrices, or multidimensional arrays that all have the same size. A scalar input is expanded to a constant array with the same dimensions as the other input. The parameters in P must lie on the interval **[0 1].**

Example: At "busy time" a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the number of attempts necessary in order to gain a connection. Suppose that we let **$p = 0.05$** be the probability of a connection during busy time. We are interested in knowing the probability that **5 attempts** are necessary for a successful call.

Solution:

Using the geometric distribution with **$x = 5$** and **$p = 0.05$** yields

$$g(x; p) = p\, q^{x-1},\ x = 1, 2, 3, \cdots$$

$$P(X = x) = g(5; 0.05)$$

$$= (0.05)\,(0.95)^{5-1}$$

$$= 0.041.$$

Matlab code

```
p = 0.05
x = 4
prob = geopdf(x, p)

display(prob)

 % 0.0407
```

Discrete Uniform Distribution [1]

If a random variable has any of n possible values that are **equally probable**, then it has a discrete uniform distribution. The probability of any outcome k_i **is 1/n**.

A simple example of the discrete uniform distribution is throwing a fair die. The possible values of k are **1, 2, 3, 4, 5, 6**; and each time the die is thrown, the probability of a given score is **1/6**.

Discrete Uniform Distribution [2]

Generating random numbers are the prime application of uniform distribution. The basic random numbers are **0, 1, 2, 3, 4, 5, 6, 7, 8, 9.** Each with probability equal to **1/10**.

For **two digit random numbers** the probability of selecting a particular random variable will be **1/100**.

Discrete Uniform Distribution [3]

If the random variable X assumes the values x_1, x_1, x_2, …, x_k with equal probabilities, then the discrete uniform distribution is given by

$$\mathbf{P(x;\, k) = \frac{1}{k}}\,, \qquad \mathbf{x_1, x_2, x_3, \ldots, x_k}$$

Discrete Uniform Distribution [4]

When a light bulb is selected at random from a box that contains a 40-watt bulb, a 60-watt bulb, a 75-watt bulb, and a 100-watt bulb, each element of the sample1 space ***S* = {40, 60, 75, 100}** occurs with probability 1/4. Therefore, we have a uniform distribution, with probability

$$\mathbf{P(x;\ k)} = \frac{1}{4}\ , \qquad \mathbf{x = 40, 60, 75, 100}$$

Discrete Uniform Distribution using MATLAB [1]

Syntax

Y = unidpdf(X,N)

Description

Y = unidpdf(X,N) computes the discrete uniform pdf at each of the values in X using the corresponding maximum observable value inN. X and N can be vectors, matrices, or multidimensional arrays that have the same size. A scalar input is expanded to a constant array with the same dimensions as the other inputs. The parameters in N must be positive integers.

Discrete Uniform Distribution using MATLAB [2]

Examples

For fixed n, the uniform discrete pdf is a constant.

```
>> y = unidpdf(1:10, 10)
y =   0.1000    0.1000    0.1000    0.1000    0.1000    0.1000
      0.1000    0.1000        0.1000    0.1000

>> y = unidpdf(1:6, 6)
y =   0.1667    0.1667    0.1667    0.1667    0.1667    0.1667
```