Advanced Statistics

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer
- **Probability and Statistics for Engineers & Scientists**, Fourth Edition, Anthony Hayter
- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber
- **Elementary Statistics**, 13th Edition, Mario F. Triola

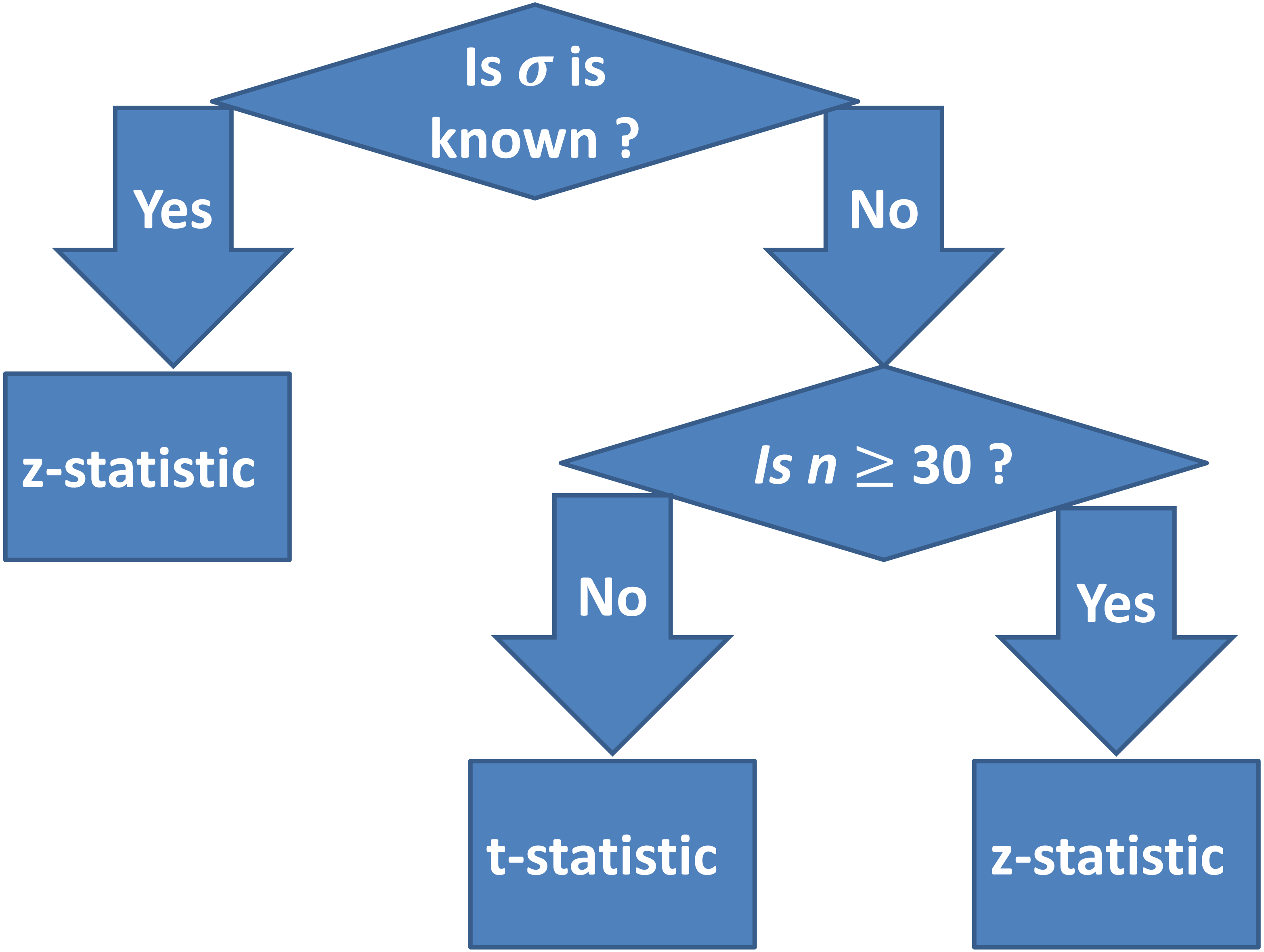
References

- **Probability & Statistics for Engineers & Scientists**, Ninth edition, Ronald E. Walpole, Raymond H. Myer

- **Probability and Statistics for Engineers & Scientists**, Fourth Edition, Anthony Hayter

- **Elementary Statistics**, Tenth Edition, Mario F. Triola

These notes contain material from the above resources.



Is σ is known ?
Yes
No
z-statistic
Is $n \geq 30$?
No
Yes
t-statistic
z-statistic

Is σ is known ?

Yes

If either the population is normally distributed or $n \geq 30$, then use the use the standard normal distribution or Z-test

No

If either the population is normally distributed or $n \geq 30$, then use the t-distribution or t-test

Inferences on a Population Mean

- **Inference methods** on a population mean based upon the *t*-procedure are appropriate for large **sample sizes $n \geq 30$** and also for **small sample sizes** as long as the data can reasonably be taken to be **approximately normally distributed**.

- **Nonparametric techniques** can be employed for **small sample sizes with data** that are clearly **not normally distributed.**

- In some circumstances an experimenter may wish to use a **"known"** value of the **population standard deviation σ** in place of the **sample standard deviation s**. In this case, the **standard normal distribution Z** is used.

$$Z_{cal} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

$$S = \sqrt{\frac{\sum(x - \bar{x})^2}{n}}$$

$$S = \sqrt{\frac{1}{n}\left\{\sum_{i=1}^{n} x^2 - \frac{(\sum_{i=1}^{n} x)^2}{n}\right\}}$$

$$t_{cal} = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n-1}}$$

$$s = \sqrt{\frac{1}{n(n-1)}\left\{n\sum_{i=1}^{n} x^2{}_i - (\sum_{i=1}^{n} x_i)^2\right\}}$$

A ***p*-value** is the **lowest level (of significance)** at which the observed value of the test statistic is significant.

A **p-value less** than or equal to your significance level (typically ≤ 0.05) is **statistically significant**.

A p-value more than the significance level (typically $p > 0.05$) is **not statistically significant** and indicates strong evidence for the null hypothesis.

***P*-value method:**

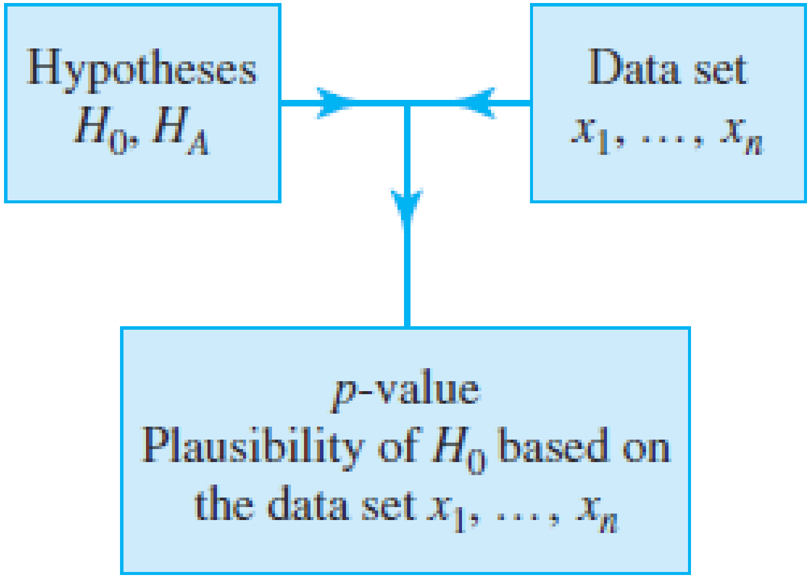
- ***Reject H_0*** if the ***P*-value $\leq \alpha$** (where α is the significance level, such as 0.05). H_0 is called **statistically significant**.

- ***Fail to reject H_0*** if the ***P*-value > α**. H_0 **is not statistically significant**.

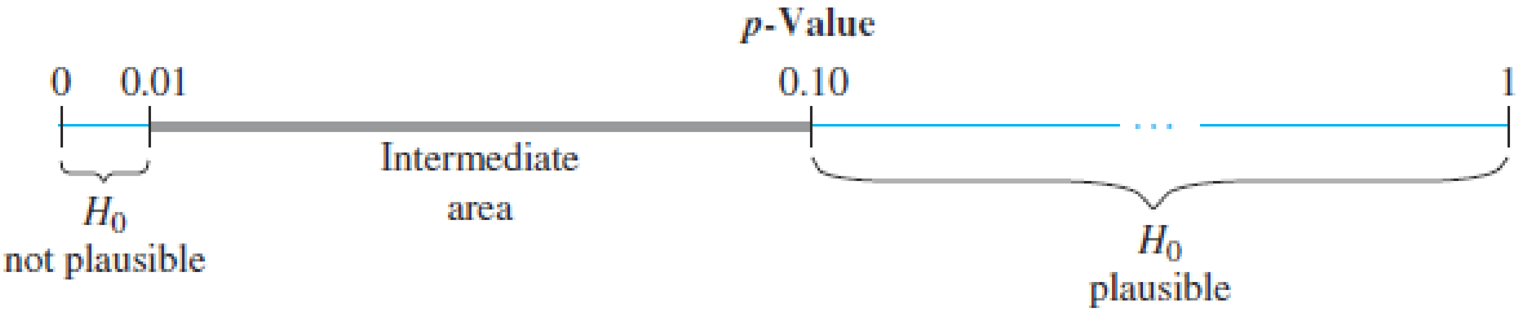
Interpretation of *p*-Values

- The **plausibility** of a null hypothesis is measured with a ***p*-value**, which is a probability that takes a value between **0 and 1**.
- The ***p*-value** is sometimes referred to as the ***observed level of significance***. A *p*-value is constructed from a **data set** as illustrated in the Figure on next slide.
- A useful way of interpreting a ***p*-value** is to consider it as the ***plausibility* or credibility of the null hypothesis**.
- The *p*-value is directly proportional to the plausibility of the null hypothesis, so that *the **smaller the p-value, the less plausible is the null hypothesis***.

P-value construction



P-value interpretation



Rejection of the Null Hypothesis

A *p*-value smaller than 0.01 is generally taken to indicate that the null hypothesis ***H_0* is not a plausible statement**. The null hypothesis ***H_0* can then be *rejected*** in favor of the **alternative hypothesis *H_1***.

Acceptance of the Null Hypothesis

- **A *p*-value larger than 0.10** is generally taken to indicate that the null hypothesis ***H_0* is a plausible statement**. The null hypothesis ***H_0* is therefore *accepted***.

- However, this does not mean that the null hypothesis H_0 has been **proven to be true**.

- The **acceptance of a null hypothesis** therefore indicates that the data **set does not provide enough evidence to reject** the null hypothesis, but it does **not indicate that the null hypothesis has been proven to be true**.

Intermediate *p*-Values

- A *p*-value in the range **1%–10%** is generally taken to indicate that the data analysis is **inconclusive**. There is some evidence that the null hypothesis is not **plausible (or credible),** but the evidence is not overwhelming.

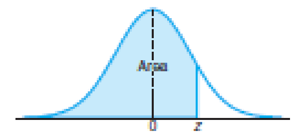


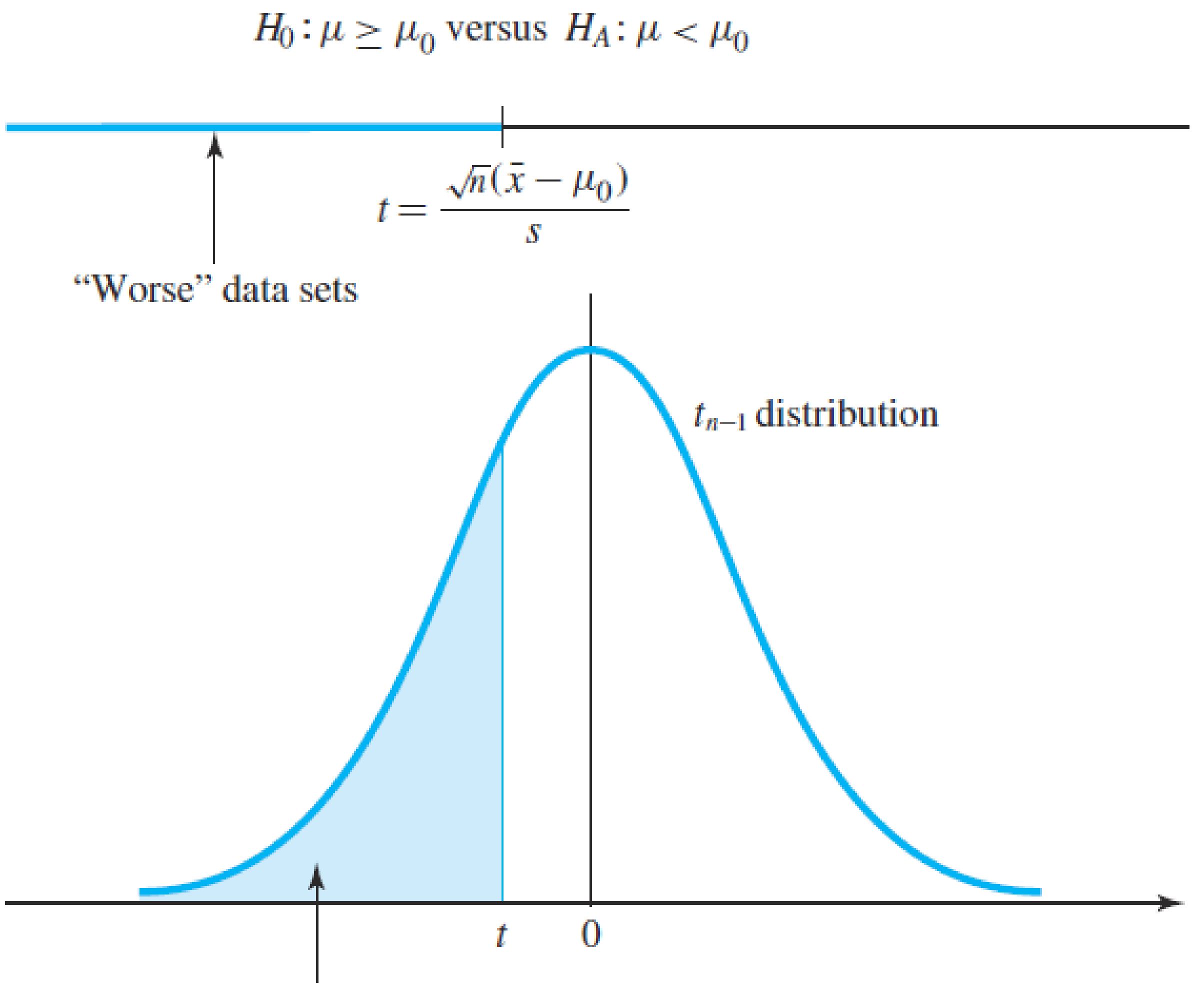
Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

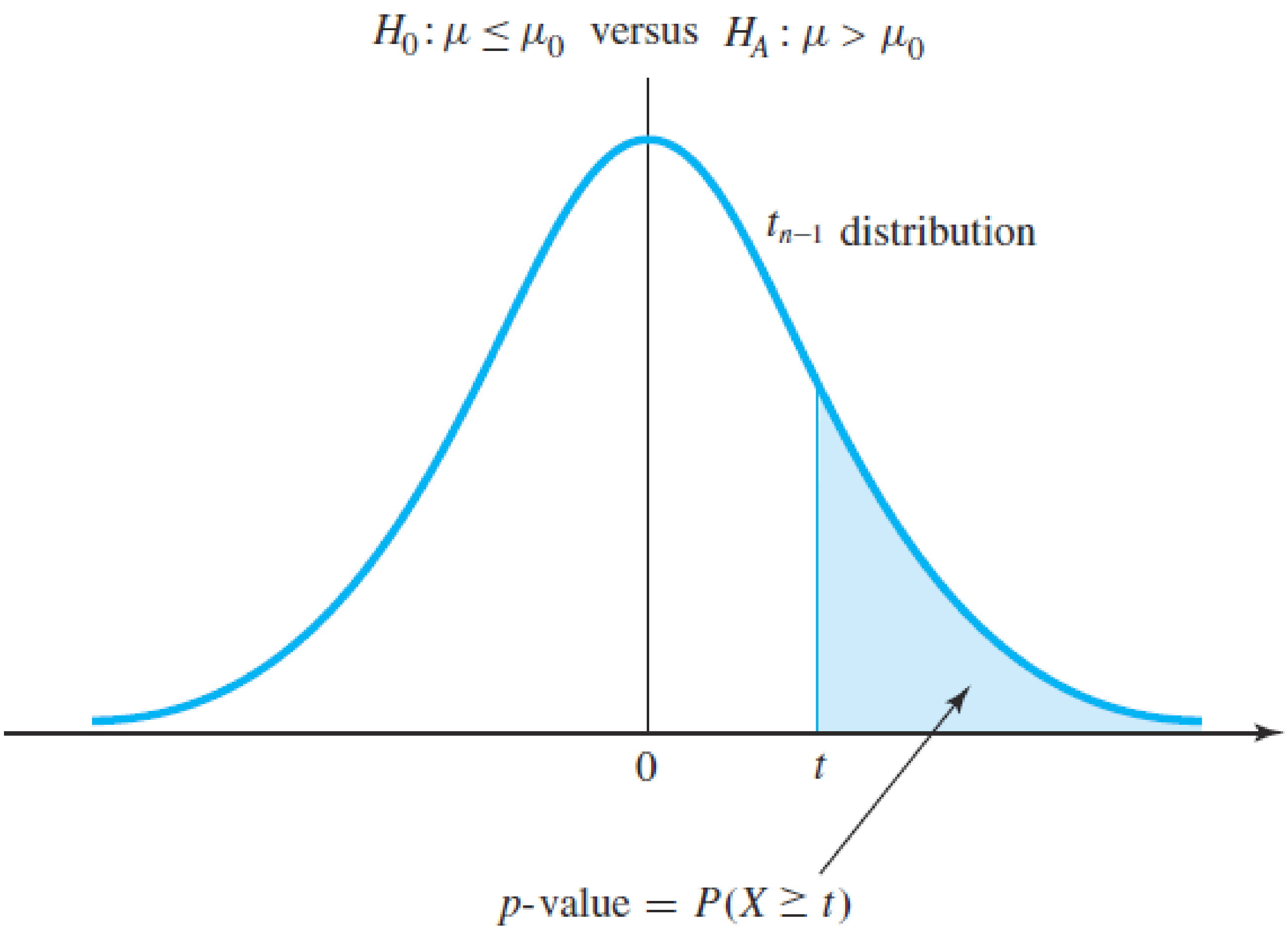
Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

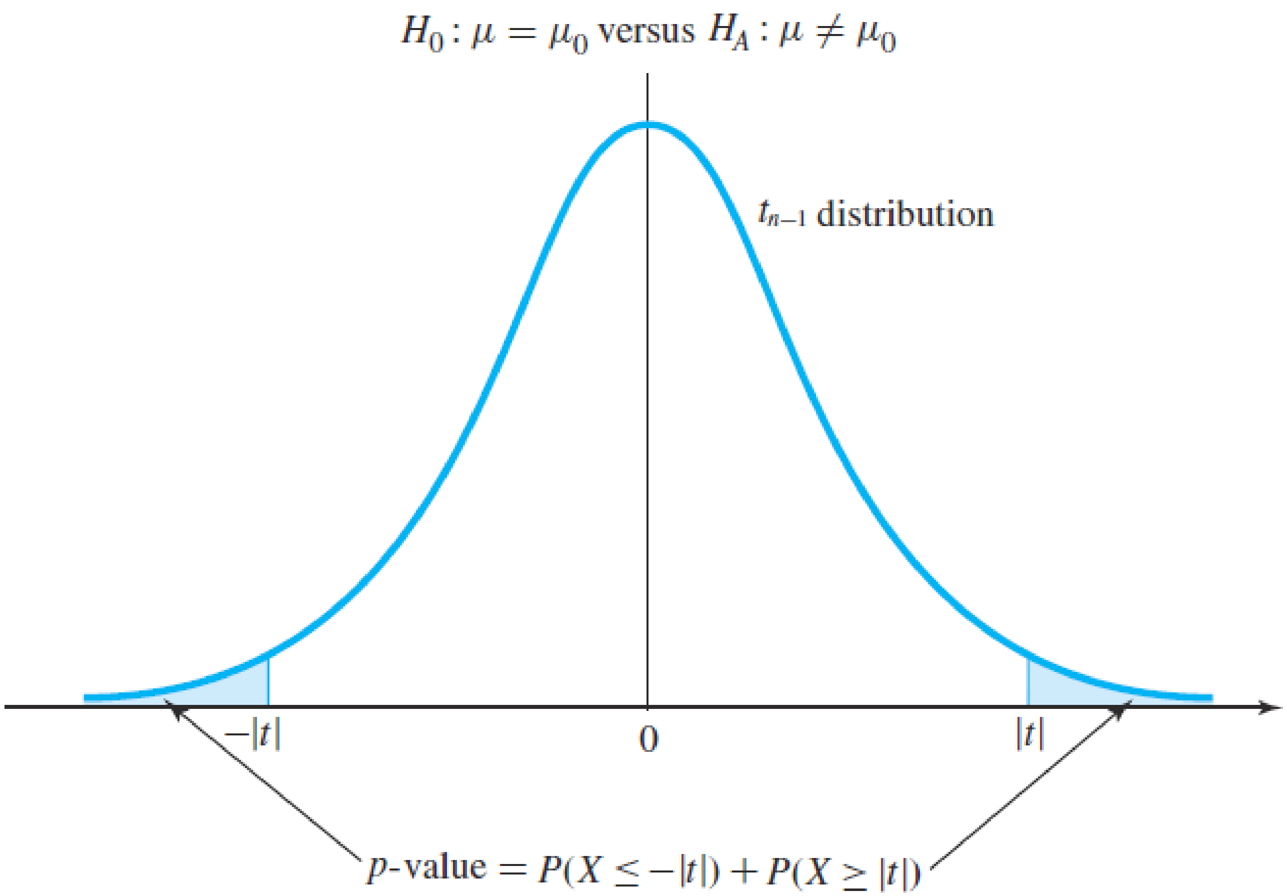
P-value calculation for a one-sided problem



P-value calculation for a one-sided problem



P-value for two-sided *t*-test



Procedure for Finding *P*-Values

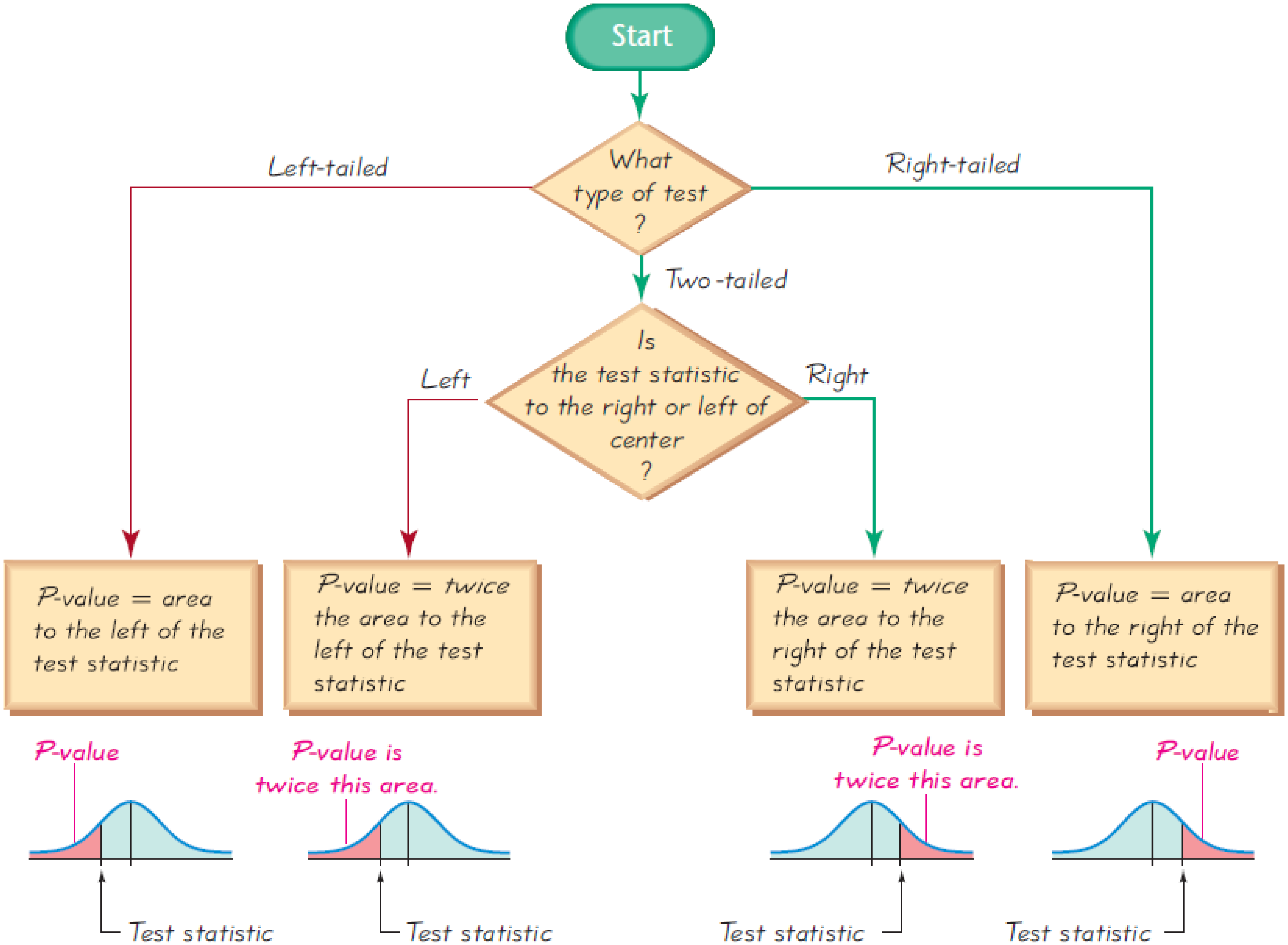


Figure 1

Testing a Proportion

1. $H_0: p = p_0$.

One of the alternatives $H_1: p < p_0$, $p > p_0$, or $p \neq p_0$.

2. Choose a level of significance equal to α.

3. Test statistic: Binomial variable X with $p = p_0$.

$$Z_{cal} = \frac{x - np_0}{\sqrt{np_0q_0}} \quad \text{or} \quad Z_{cal} = \frac{\hat{p} - p_0}{\sqrt{p_0q_{0} / n}}$$

Where $\mathbf{p_0}$ is the **population proportion** and $\hat{\mathbf{p}}$ is the **sample proportion**

4. Computations: Find x, the number of successes, and compute the appropriate P-value.

5. Decision: Draw appropriate conclusions based on the *P*-value.

EXAMPLE Finding *P*-Values First determine whether the given conditions result in a **right-tailed test,** a **left-tailed test**, or a **two-tailed test**, then use Figure 1 in the previous slide to find the *P*-value, then state a conclusion about the null hypothesis.

a. A significance level of $\alpha = 0.05$ is used in testing the claim that $p > 0.25$, and the sample data result in a test statistic of $z_{cal} = 1.18$.

b. A significance level of $\alpha = 0.05$ is used in testing the claim that $p \neq 0.25$ and the sample data result in a test statistic of $z_{cal} = 2.34$.

a. With a claim of $p > 0.25$, the test is right-tailed.

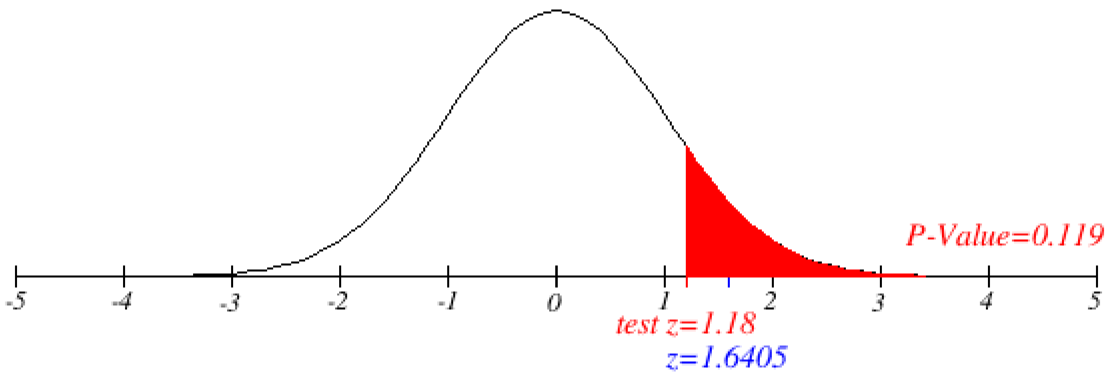
$\mathbf{Z_{cal} = 1.18}$

$P(Z_{cal} > 1.18) = 1 - P(Z_{cal} < 1.18)$

$= 1 - .8810$

$= 0.1190$

P-value = 0.1190



***P*-value $\leq \alpha$**

$0.1190 \leq 0.05$ (false)

We fail to reject the null hypothesis.

- **The *P*-value of 0.1190** is **relatively large**, indicating that the sample results **could easily occur by chance**.

b With a claim of the test is two-tailed. Using Figure 1 for a two tailed test, we see that the *P*-value is *twice* the area to the right of $z_{cal} = 2.34$.

$P(|Z_{cal}| > 2.34) = 1 – P(|Z_{cal}| < 2.34)$

$= 1 – 0.9904$

$= 0.0096$

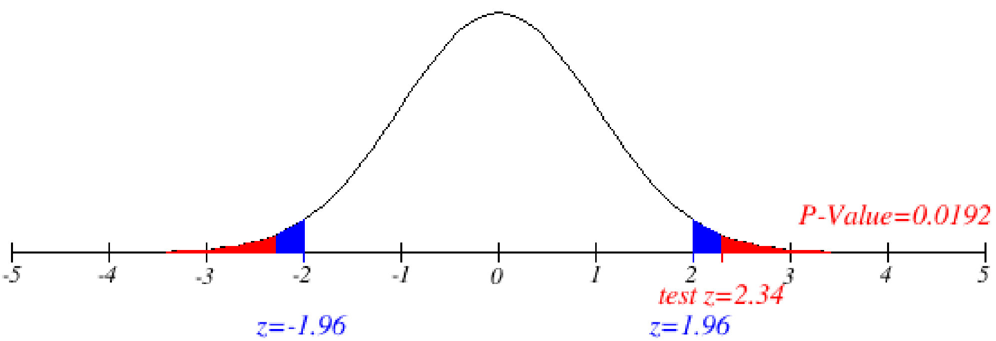
$\text{P-value} = 2 \times P(|Z_{cal}| > 2.34)$

P-value $= 2 \times 0.0096$

$= 0.0192$

***P*-value** $\leq \alpha$

$0.0192 \leq 0.05$ **(true)**



We reject the null hypothesis.

- The **small *P*-value** of **0.0192** shows that the sample results are not likely to occur by chance.

Single Sample: Tests Concerning a Single Mean

Example: A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance.

Solution:

$n = 100$ (sample size)

$\bar{x} = 71.8$ (sample mean)

$\sigma = 8.9$ (population standard deviation)

$\alpha = 0.05$ (level of significance)

1. **We state our hypothesis as**:

H_0: μ = 70 years

H_1: μ > 70 years (one sided test)

2. **The level of significance is set** $\alpha = 0.05$.

3. **Test statistic to be used is**

$$Z_{cal} = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

4. **Calculations:**

$$Z_{cal} = \frac{71.8 - 70}{8.9/\sqrt{100}} = 2.02.$$

5. **P-value**

$\mathbf{Z_{cal} = 2.02}$

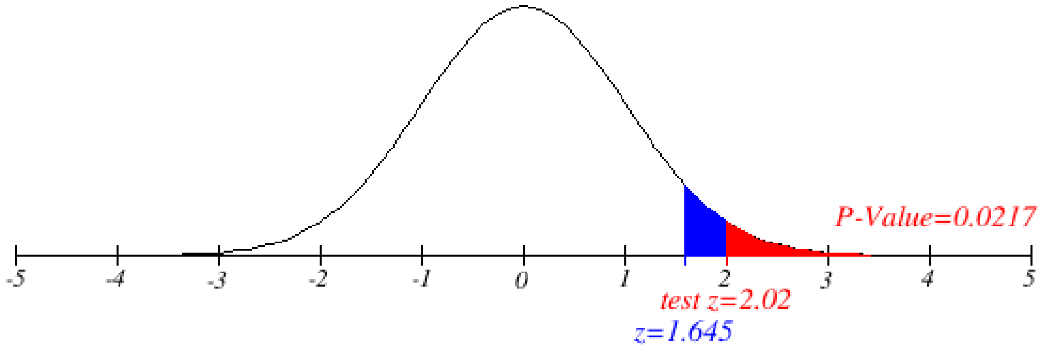
$P(Z_{cal} > 2.02) = 1 - P(Z_{cal} < 2.02)$

$= 1 - 0.9783$

$= 0.0217$

$P\text{-value} \leq \alpha$

$0.0217 \leq \mathbf{0.05}$ **(true)**



6. **Conclusion:** We reject $\mathbf{H_o}$

Example: A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of **8 kilograms** with a **standard deviation of 0.5** kilogram. Test the hypothesis that $\mu = 8$ kilograms against the alternative that $\mu \neq 8$ kilograms if a random sample of **50 lines** is tested and found to have a mean breaking strength of **7.8 kilograms**. Use a 0.01 level of significance.

Solution:

$\mu = 8$ (Population mean)

$n = 50$ (Sample size)

$\sigma = 0.5$ (Population standard deviation)

$\bar{x} = 7.8$ (Sample mean)

$\alpha = 0.01$ (Level of significance)

1. **We state our hypothesis as**:

$H_0: \mu = 8$

$H_1: \mu \neq 8$ (Two sided test)

2. **The level of significance is set** $\alpha = 0.01$.

3. **Test statistic to be used is**

$$Z_{cal} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

4. **Calculations:**

$$Z_{cal} = \frac{7.8 - 8}{0.5/\sqrt{50}} = -2.83.$$

$Z_{tab} = 2.575$

5. P-value

z_{cal} **= 2.83**.

$\mathbf{P(|Z_{cal}| > 2.83) = 1 - P(|Z_{cal}| < 2.83)}$

$= 1 - 0.9977$

$= 0.0023$

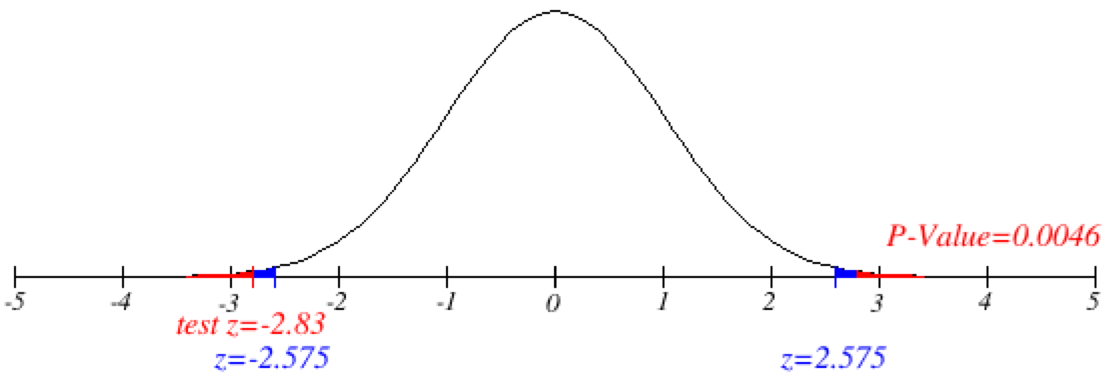
P-value $= 2 \times$ $\mathbf{P(|Z_{cal}| > 2.83)}$

P-value $= 2 \times 0.0023$

$= 0.0046$

P**-value** $\leq \alpha$

$\mathbf{0.0046 \leq 0.05}$ **(true)**



6. **Conclusion:** We reject H_O

Example: The Edison Electric Institute has published figures on the number of kilowatt hours used annually by various home appliances. It is claimed that a vacuum cleaner uses an average of **46** kilowatt hours per year. If a random sample of **12** homes included in a planned study indicates that vacuum cleaners use an average of **42** kilowatt hours per year with a standard deviation of **11.9** kilowatt hours, does this suggest at the **0.05** level of significance that vacuum cleaners use, on average, less than **46** kilowatt hours annually? Assume the population of kilowatt hours to be normal.

Solution

$\mu = 46$	(Population mean)
$n = 12$	(Sample size)
$s = 11.9$	(Sample standard deviation)
$\bar{x} = 42$	(Sample mean)
$\alpha = 0.05$	(Level of significance)

1. **We state our hypothesis as**:

$H_0: \mu = 46$

$H_1: \mu < 46$ (One tailed test)

2. **The level of significance is set** $\alpha = 0.05$.

3. **Test statistic to be used is**

$$t_{cal} = \frac{\bar{x}-\mu}{s/\sqrt{n}}$$

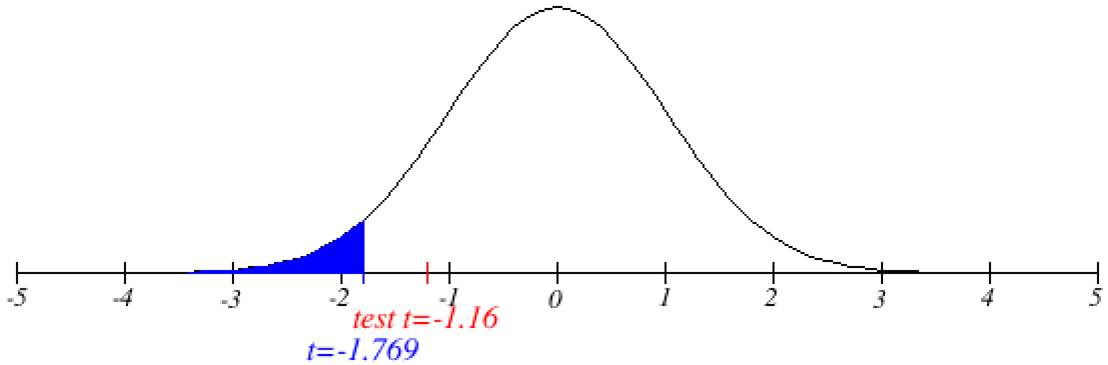
4. **Calculations:**

$$t_{cal} = \frac{42 - 46}{11.9/\sqrt{12}} = -1.16$$

5. **Critical region:**

$t_{cal} < t_{tab}$

Where $-t_{tab} = -t_{(\alpha,\, n-1)} = -t_{(0.05,\, 11)} = -1.769$



6. **Conclusion:** Since calculated value of t_{cal} is greater than the tabulate value of t, so we accept H_O

P-value

n - 1 = 12 - 1 = 11

t_{cal} **=** -1.16

P(t_{cal} **<** -1.16**) = 0.15** (Searching the t-table corresponindg to 11 d.f and choosing the closest value of α corresponding to t_{cal} ***= 1.16*** is 0.15)

p-value = 0.15

***P*-value** $\leq \alpha$

0.15 $\leq$ **0.05 (false)**

Conclusion:

so we accept H_O

Table A.4 Critical Values of the t-Distribution

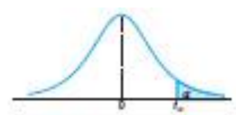


Table A.4 Critical Values of the t-Distribution

v	α = 0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

Table A.4 (continued) Critical Values of the t-Distribution

	α						
v	0.02	0.015	0.01	0.0075	0.005	0.0025	0.0005
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.689
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.660
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.290