The Master Theorem

- Given: a *divide and conquer* algorithm
  - An algorithm that divides the problem of size n into a subproblems, each of size n/b
  - Let the cost of each stage (i.e., the work to divide the problem + combine solved subproblems) be described by the function f(n)
- Then, the Master Theorem gives us a cookbook for the algorithm's running time:

The Master Theorem

- Assumptions:
  - a ≥ 1 and b ≥ 1 are constants
  - f(n) is an asymptotically positive function
  - T(n) is defined for nonnegative integers
  - We interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$

The Master Theorem

With the recurrence T(n) = a T(n/b) + f(n) as in the previous slide, T(n) can be bounded asymptotically as follows:

1. If $f(n)=O(n^{\log_b a-e})$ for some constant $e > 0$, then $T(n)= \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = W(n^{\log_b a+e})$ for some constant $e > 0$, and if $a\, f(n/b) \le c\, f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n)= \Theta(f(n))$.

Using The Master Method

- T(n) = 9T(n/3) + n
  - a=9, b=3, f(n) = n
  - $n^{\log_b a} = n^{\log_3 9} = \Theta(n^2)$
  - Since $f(n) = O(n^{\log_3 9 - \varepsilon})$, where ε=1, case 1 applies:

$$T(n) = \Theta\left(n^{\log_b a}\right) \text{when } f(n) = O\left(n^{\log_b a - e}\right)$$

  - Thus the solution is $T(n) = \Theta(n^2)$

Examples

- $T(n) = 16T(n/4) + n$
  - $a = 16$, $b = 4$, thus $n^{\log_b a} = n^{\log_4 16} = \Theta(n^2)$
  - $f(n) = n = O(n^{\log_4 16 - e})$ where $e = 1 \Rightarrow$ case 1.
  - Therefore, $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$
- $T(n) = T(3n/7) + 1$
  - $a = 1$, $b=7/3$, and $n^{\log_b a} = n^{\log_{7/3} 1} = n^0 = 1$
  - $f(n) = 1 = \Theta(n^{\log_b a}) \Rightarrow$ case 2.
  - Therefore, $T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(\lg n)$

Examples (Cont.)

- $T(n) = 3T(n/4) + n \lg n$
  - $a = 3$, $b=4$, thus $n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$
  - $f(n) = n \lg n = \Omega(n^{\log_4 3 + e})$ where $e \approx 0.2 \Rightarrow$ case 3.
  - Therefore, $T(n) = \Theta(f(n)) = \Theta(n \lg n)$
- $T(n) = 2T(n/2) + n \lg n$
  - $a = 2$, $b=2$, $f(n) = n \lg n$, and $n^{\log_b a} = n^{\log_2 2} = n$
  - $f(n)$ is asymptotically larger than $n^{\log_b a}$, but not polynomially larger. The ratio lg n is asymptotically less than n^e for any positive e. Thus, the Master Theorem *doesn't* apply here.