

# **Probability and Statistics**

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# Textbooks

❑ **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

❑ **Elementary Statistics: Picturing the World**, 6<sup>th</sup> Edition, Ron Larson and Betsy Farber

❑ **Elementary Statistics**, 13<sup>th</sup> Edition, Mario F. Triola

# Reference books

- ❑ **Probability Demystified**, Allan G. Bluman
- ❑ **Schaum's Outline of Probability and Statistics**
- ❑ **MATLAB Primer**, Seventh Edition
- ❑ **MATLAB Demystified** by McMahan, David

# References

Readings for these lecture notes:

- ❑ Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer
- ❑ Probability Demystified, Allan G. Bluman
- ❑ Elementary Statistics, 10<sup>th</sup> Edition, Mario F. Triola

These notes contain material from the above three books.

# Subjective Probability

A third type of probability is called **subjective probability**. Subjective probability is based upon an **educated guess, estimate, opinion, or inexact information**.

# Sample Spaces

There are two **specific devices** that will be used to find sample spaces for probability experiments. They are **tree diagrams** and **tables**.

A **tree diagram** consists of **branches** corresponding to the outcomes of two or more probability experiments that are done in sequence.

# Sample Spaces

- ❑ In order to construct a tree diagram, use branches corresponding to the outcomes of the **first experiment**. These branches will emanate from a single point.
- ❑ Then from each branch of the first experiment draw branches that represent the outcomes of the **second experiment**.
- ❑ You can continue the process for further experiments of the sequence if necessary.

# Tree Diagram [1]

**Example:** A coin is tossed and a die is rolled. Draw a tree diagram and find the sample space.



## Solution:

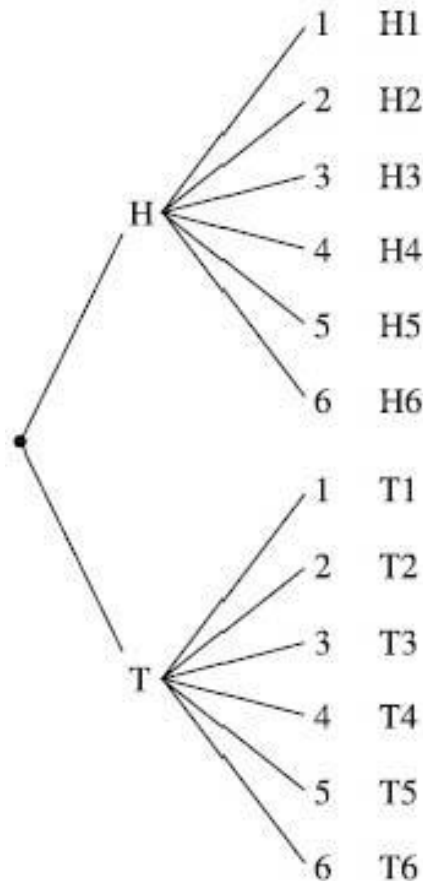
Since there are two outcomes (heads and tails for the coin), draw two branches from a single point and label one **H** for head and the other one **T** for tail.

From each one of these outcomes, draw and label six branches representing the outcomes **1, 2, 3, 4, 5,** and **6** for the die.

Trace through each branch to find the outcomes of the experiment.

# Tree Diagram [2]

**Example:** A coin is tossed and a die is rolled. Draw a tree diagram and find the sample space.



# Tree Diagram [3]

**Example:** A coin is tossed and a die is rolled. Find the probability of getting

a. A head on the coin and a 3 on the die.

b. A head on the coin.

c. A 4 on the die.

## Solution:

a.  $P(H3) = \frac{1}{12} = \mathbf{0.0833 \text{ (or 8.33\%)}$

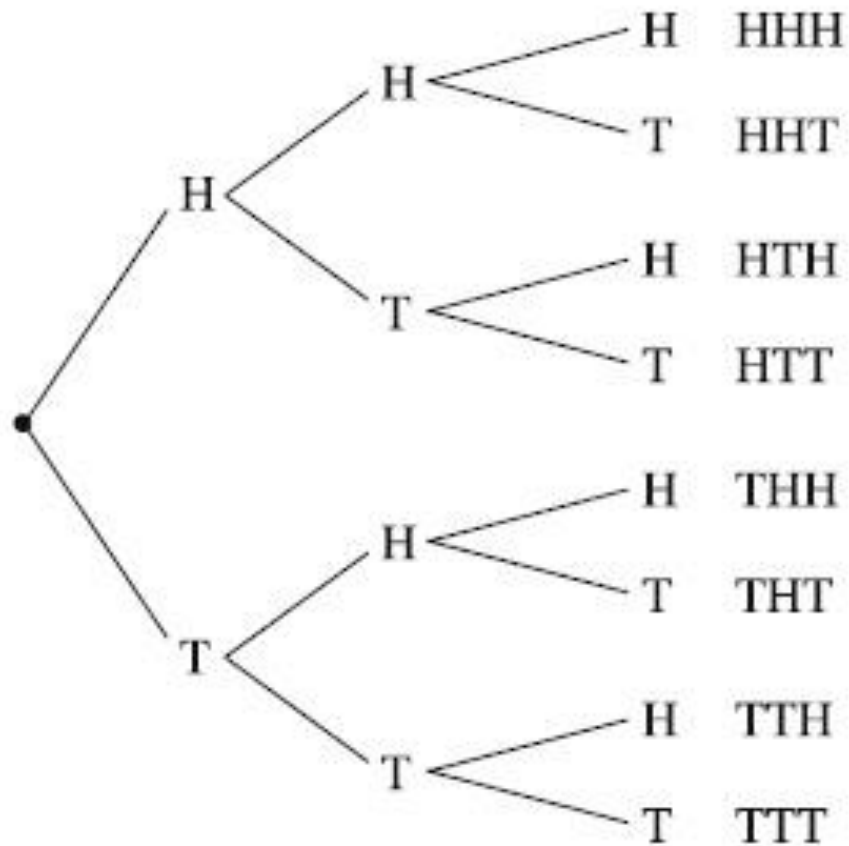
b.  $P(\text{head on the coin}) = \frac{6}{12} = \frac{1}{2} = \mathbf{0.5 \text{ (or 50\%)}$

c.  $P(4 \text{ on the die}) = \frac{2}{12} = \frac{1}{6} = \mathbf{0.1667 \text{ (16.67\%)}$

# Tree Diagram [4]

**Example:** Three coins are tossed. Draw a tree diagram and find the sample space.

# Solution



# Tree Diagram [5]

**Example:** Three coins are tossed. Find the probability of getting

a. Two heads and a tail in any order.

b. Three heads.

c. No heads.

d. **At least** two tails.

e. **At most** two tails.

## Solution:

a.  $P(2 \text{ heads and a tail}) = 3/8 = \mathbf{0.375 \text{ (or 37.5 \%)}}$

b.  $P(\text{HHH}) = 1/8 = \mathbf{0.125 \text{ (or 12.5 \%)}}$

c.  $P(\text{TTT}) = 1/8 = \mathbf{0.125 \text{ (or 12.5 \%)}}$

d.  $P(\text{at least two tails}) = 4/8 = 1/2 = \mathbf{0.5 \text{ (or 50 \%)}}$

e.  $P(\text{at most two tails}) = 7/8 = \mathbf{0.875 \text{ (or 87.5 \%)}}$

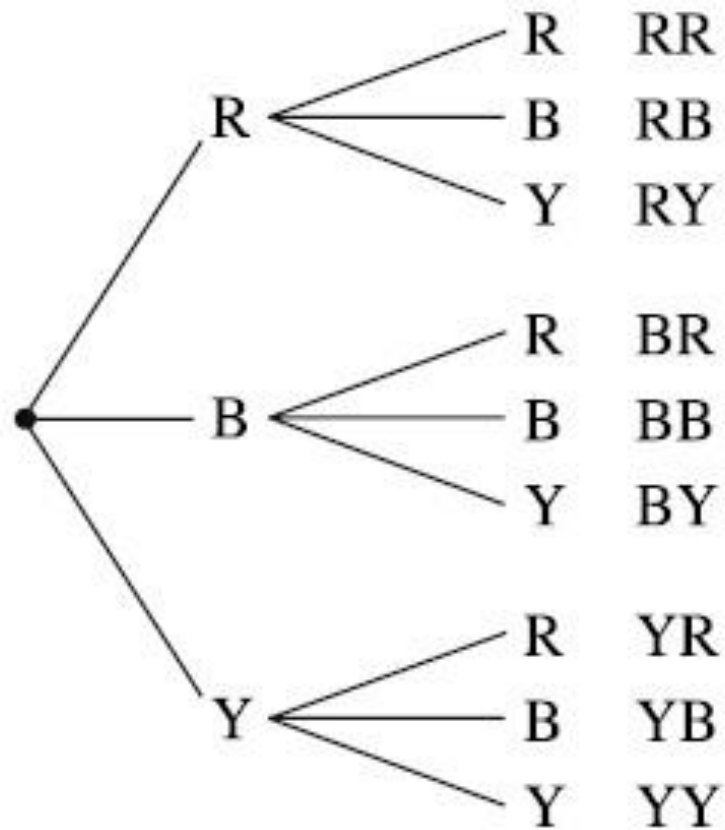
***$\Rightarrow$  At most two tails mean no three tails***



# Tree Diagram [6]

**Example:** A box contains a **red ball (R)**, a **blue ball (B)**, and a **yellow ball (Y)**. **Two balls** are selected at random in succession. Draw a **tree diagram** and find the sample space if the first ball is **replaced** before the second ball is selected.

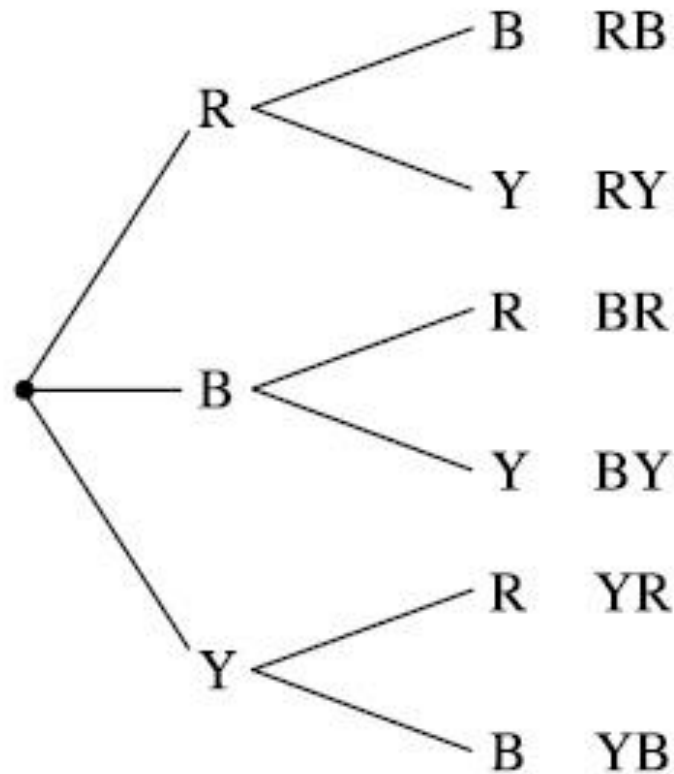
# Solution



# Tree Diagram [7]

**Example:** A box contains a **red ball (R)**, a **blue ball (B)**, and a **yellow ball (Y)**. Two balls are selected at random in succession. Draw a **tree diagram** and find the sample space if the first ball is **not replaced** before the second ball is selected.

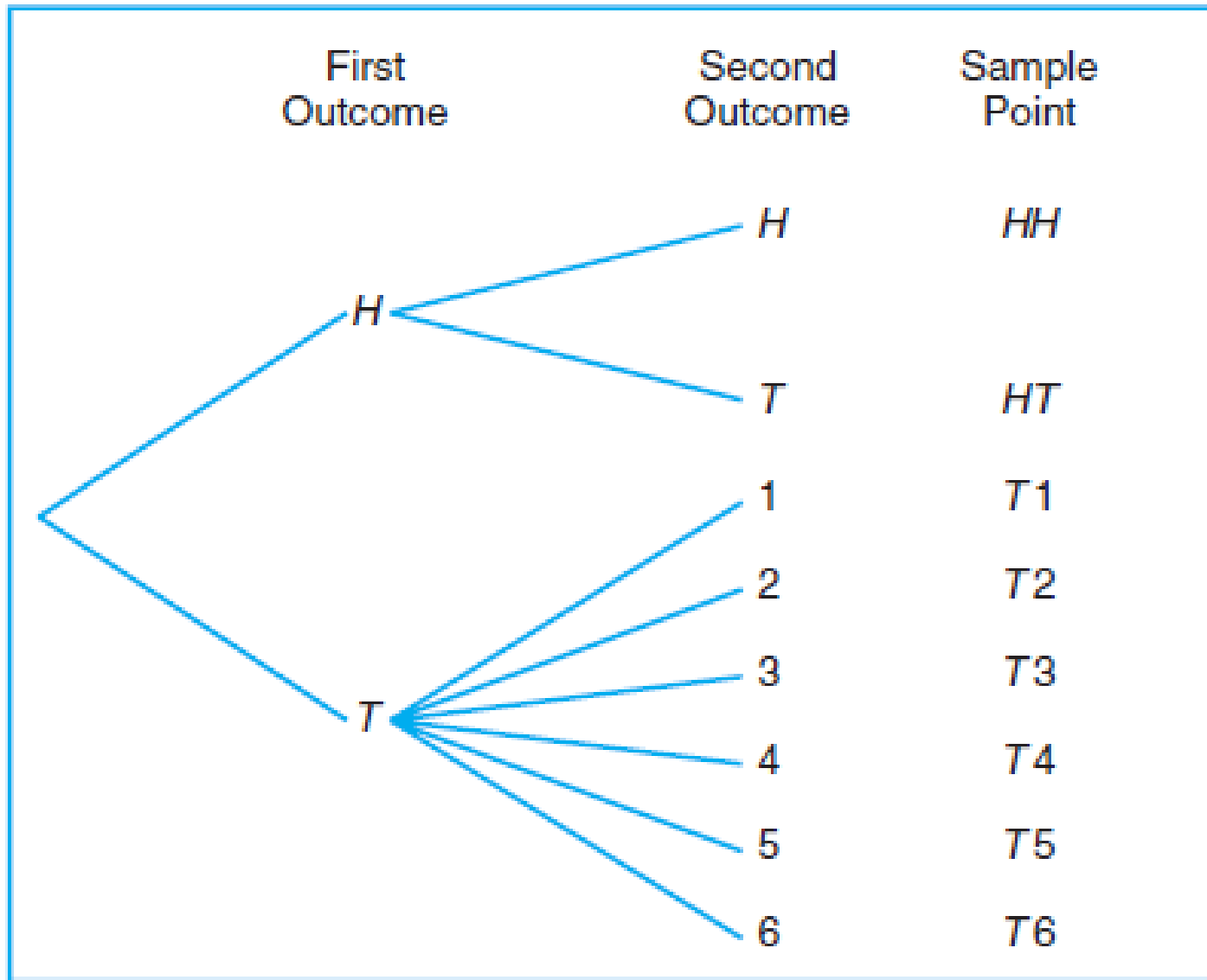
# Solutions



# Tree Diagram [9]

**Example** An experiment consists of **flipping a coin** and then flipping it a **second time** if a **head occurs**. If a **tail** occurs on the **first flip**, then a **die is tossed** once. To list the elements of the sample space providing the most information, construct the **tree diagram**

# Solution

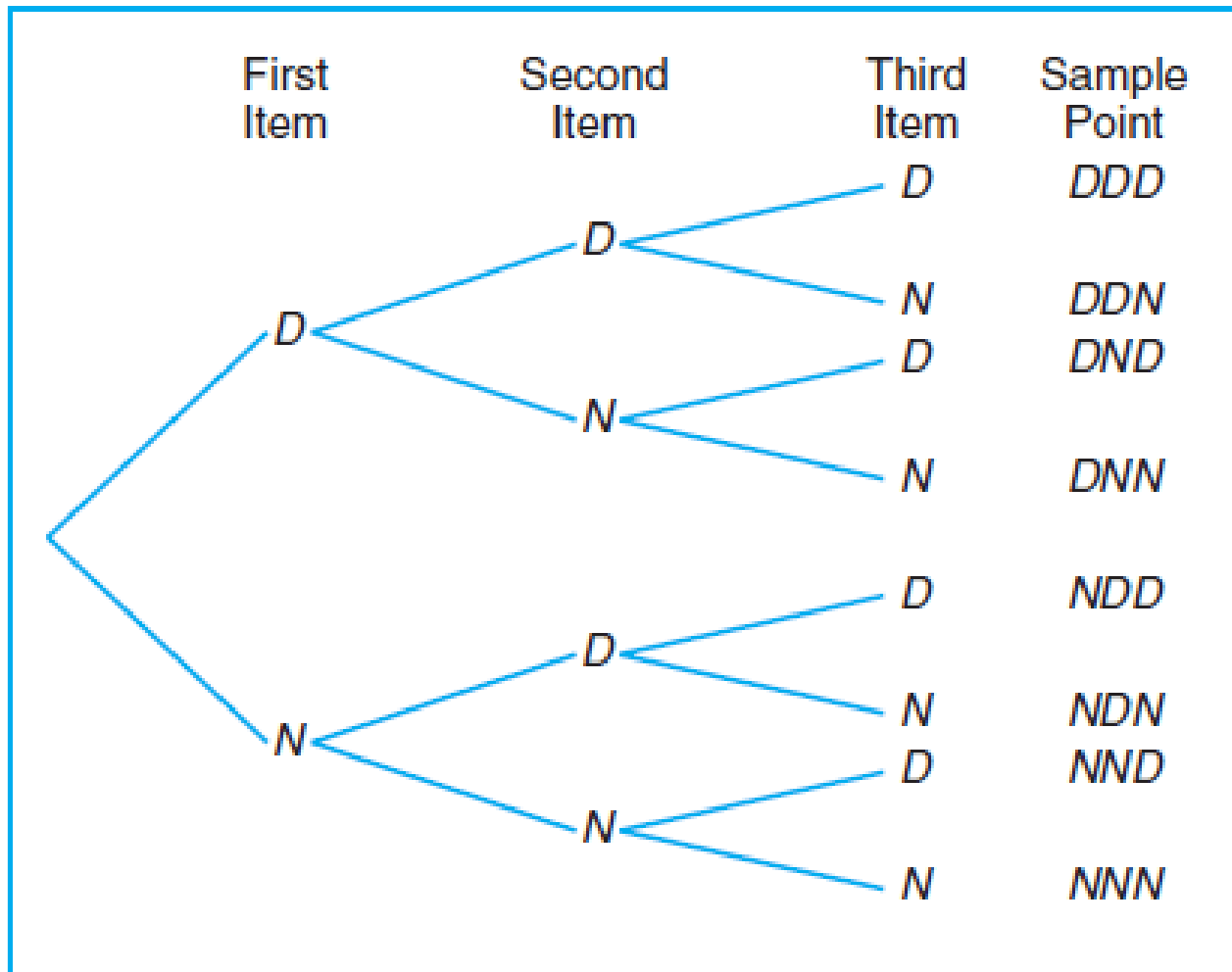


# Tree Diagram [10]

**Example** Suppose that **three items** are selected at random from a manufacturing process. Each item is inspected and classified **defective, D**, or **nondefective, N**. To list the elements of the sample space providing the most information, construct the **tree diagram**.

# Solution

□  $S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$ .





# Tables [1]

Another way to find a sample space is to use a **table**.

**Example:** Find the sample space for selecting a card from a standard deck of 52 cards.

- ❑ There are four suits—**hearts** and **diamonds**, which are **red**, and **spades** and **clubs**, which are **black**.
- ❑ Each suit consists of 13 cards—ace through king. **Face cards** are kings, queens, and jacks.

# A standard deck of 52 cards

Heart	A	2	3	4	5	6	7	8	9	10	J	Q	K
Diamond	A	2	3	4	5	6	7	8	9	10	J	Q	K
Spade	A	2	3	4	5	6	7	8	9	10	J	Q	K
Club	A	2	3	4	5	6	7	8	9	10	J	Q	K

## Tables [2]

**Example:** A single card is drawn at random from a standard deck of cards. Find the probability that it is

a. The 4 of diamonds.

b. A queen.

**Solution:**

a.  $P(\text{The 4 of diamonds}) = \frac{1}{52} = \mathbf{0.0192 \text{ (or 1.9231\%)}$

b.  $P(\text{A queen}) = \frac{4}{52} = \frac{1}{13} = \mathbf{0.0769 \text{ (or 7.6923 \% )}$

## Tables [3]

A table can be used for the sample space when two dice are rolled.

	Die 2					
Die 1	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

## Tables [4]

**Example:** When two dice are rolled, find the probability of getting a sum of nine.

**Solution:**

**Let A be the event of getting a “sum of 9”**

$$P(A) = \frac{4}{36} = \frac{1}{9} = \mathbf{0.1111 \text{ (or 11.11 \% )}}$$

# Intersection [1]

**Intersection:** The **intersection** of two events A and B, denoted by the symbol  $A \cap B$ , is the event containing all elements that are **common to A and B**.

**Example:** Let **E** be the event that a person selected at random in a classroom is majoring in **engineering**, and let **F** be the event that the person is **female**. Then  $E \cap F$  is the event of all female engineering students in the classroom.

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# Intersection [2]

**Example:** Let  $V = \{a, e, i, o, u\}$  and  $C = \{l, r, s, t\}$ ; then it follows that  $V \cap C = \emptyset$ . That is,  $V$  and  $C$  have no elements in common and, therefore, **cannot both simultaneously occur**.

# Union

**Union:** The **union** of the two events  $A$  and  $B$ , denoted by the symbol  **$A \cup B$** , is the event containing all the elements that belong to  $A$  or  $B$  or both.

## Notation for Addition Rule

**$P(A \text{ or } B)$**  =  **$P$** (in a single trial, event  $A$  occurs or event  $B$  occurs or they both occur)



# Mutually Exclusive or Disjoint

Two events A and B are **mutually exclusive, or disjoint**, if  **$A \cap B = \{ \}$**  or  **$\emptyset$**

OR

Two events A and B are **mutually exclusive, or disjoint**, if  **$A \cap B = \emptyset$** , that is, if A and B have no elements in common.

OR

Events **A** and **B** are **disjoint** (or **mutually exclusive**) if they **cannot occur at the same time**. (That is, disjoint events do not overlap.)

# Addition Rule I

**Addition Rule I:** When two events are **mutually exclusive** or **disjoint events**

$$P(A \text{ or } B) = P(A) + P(B)$$

**OR**

$$P(A \cup B) = P(A) + P(B)$$

**Example:** When a die is rolled, find the probability of getting a **2** or a **3**.

## Solution:

$$S = \{1, 2, 3, 4, 5, 6\}; n(S) = 6$$

Let **A** be the event of getting a “2”

$$A = \{2\}; n(A) = 1$$

$$P(A) = n(A)/n(S) = \frac{1}{6} = \mathbf{0.1667 \text{ (or 16.67\%)}}$$

Let **B** be the event of getting a “3”

$$B = \{3\}; n(B) = 1$$

$$P(B) = n(B)/n(S) = \frac{1}{6} = \mathbf{0.1667 \text{ (or 16.67\%)}}$$

Since events **A** and **B** are **mutually exclusive**, so

$$P(A \cup B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} = \mathbf{0.3333 \text{ (or 33.33\%)}}$$

# Addition Rule I [2]

A cable television company offers programs on **eight** different channels, **three** of which are affiliated with **ABC**, **two** with **NBC**, and **one** with **CBS**. The other **two** are an **educational channel** and the **ESPN sports** channel. Suppose that a person subscribing to this service turns on a television set without first selecting the channel. Let **A** be the event that the program belongs to the **NBC network** and **B** the event that it belongs to the **CBS network**. Since a television program cannot belong to more than one network, the events  $A$  and  $B$  have no programs in common.

Therefore, the intersection  $A \cap B$  contains no programs, and consequently the events  $A$  and  $B$  are **mutually exclusive**.

# Addition Rule I [3]

**Example:** In a committee meeting, there were **5** freshmen, **6** sophomores, **3** juniors, and **2** seniors. If a student is selected at random to be the chairperson, find the probability that the chairperson is a **sophomore** or a **junior**.

# Addition Rule I [4]

## Solution:

Let A be the event of selecting a chairperson as a **“sophomore”**

$$P(A) = \frac{6}{16} = \frac{3}{8} = \mathbf{0.3750}$$

Let B be the event of selecting a chairperson as a **“junior”**

$$P(B) = \frac{3}{16} = \mathbf{0.1875}$$

Since **A** and **B** are **mutually exclusive or disjoint events**

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{6}{16} + \frac{3}{16} = \frac{9}{16} = \mathbf{0.5625 \text{ (or 56.25\%)}} \end{aligned}$$

# Addition Rule I [5]

**Example:** A card is selected at random from a deck.  
Find the probability that the card is an **ace or** a **king**.

Heart	A	2	3	4	5	6	7	8	9	10	J	Q	K
	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥
Diamond	A	2	3	4	5	6	7	8	9	10	J	Q	K
	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦
Spade	A	2	3	4	5	6	7	8	9	10	J	Q	K
	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠
Club	A	2	3	4	5	6	7	8	9	10	J	Q	K
	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣



## Solution:

Let **A** be the event of selecting an **ace**

$$P(A) = \frac{4}{52} = \frac{1}{13} = \mathbf{0.0769 \text{ (or 7.69%)}}$$

Let **B** be the even of selecting a **king**

$$P(B) = \frac{4}{52} = \frac{1}{13} = \mathbf{0.0769 \text{ (or 7.69%)}}$$

Since **A** and **B** are **mutually exclusive or disjoint events**

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{4}{52} + \frac{4}{52} = \mathbf{0.1538 \text{ (or 15.38%)}} \end{aligned}$$

# Addition Rule II [1]

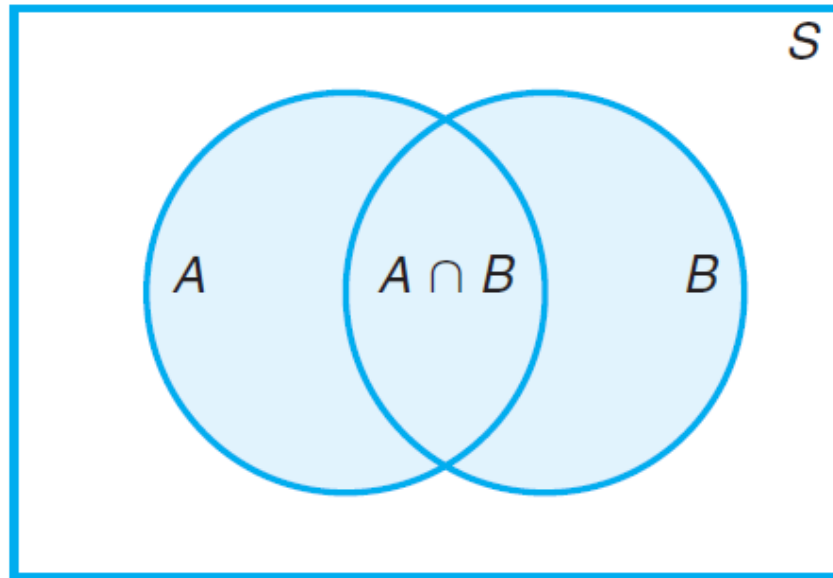
❑ When two events are **not mutually exclusive**, you need to add the probabilities of each of the two events and **subtract the probability of the outcomes that are common** to both events. In this case, addition **rule II** can be used.

❑ **Addition Rule II:** If A and B are two events that are not mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$

# When $A$ and $B$ are two events that are not mutually exclusive

If  $A$  and  $B$  are two events, then  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .



**Additive rule of probability**

# Addition Rule II [2]

**Example:** A card is selected at random from a deck of 52 cards. Find the probability that it is a **6** or a **diamond**.

## Addition Rule II [3]

### Solution:

Let A be the event of getting a “6”.

$$P(A) = \frac{4}{52} = \frac{1}{13} = \mathbf{0.0769 \text{ (or 7.69%)}}$$

Let B be the event of getting a “diamond”.

$$P(B) = \frac{13}{52} = \frac{1}{4} = \mathbf{0.2500 \text{ (or 25%)}}$$

## Addition Rule II [4]

Let  $A \cap B$  be the event of getting a “6” and a “diamond”

$$P(A \cap B) = \frac{1}{52} = 0.0192 \text{ (or 1.9231\%)}$$

Since A and B are **not mutually exclusive**, so

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} \\ &= \frac{4}{13} = 0.3077 \text{ (or 30.77\%)} \end{aligned}$$

# Addition Rule II [5]

**Example:** A die is rolled. Find the probability of getting an even number or a number less than 4.

# Addition Rule II [6]

**Solution:**

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$n(S) = 6$$

Let A be the event of getting an even number

$$A = \{2, 4, 6\}, n(A) = 3$$

$$P(A) = \frac{3}{6} = \frac{1}{2} = 0.50 \text{ (or 50\%)}$$

Let B be the event of getting a number less than 4

$$B = \{1, 2, 3\}, n(B) = 3$$

$$P(B) = \frac{3}{6} = \frac{1}{2} = 0.50 \text{ (or 50\%)}$$



## Addition Rule II [7]

Let  $A \cap B$  be the event of getting an “even number” and a “number less than 4”

$$A \cap B = \{2\}$$

$$P(A \cap B) = \frac{1}{6} = 0.1667 \text{ or } (16.67\%)$$

Since A and B are **not mutually exclusive**, so

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6} = 0.8333 \text{ (or } 83.3333 \%)$$

# Table

A table can be used for the sample space when two dice are rolled.

	Die 2					
Die 1	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

# Addition Rule II [8]

**Example:** Two dice are rolled; find the probability of getting **doubles** or a **sum of 8**.

## Addition Rule II [9]

### Solution:

Let A be the event of getting doubles

$$A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}, n(A) = 6$$

$$P(A) = \frac{6}{36} = \frac{1}{6} = \mathbf{0.1667 \text{ (or 16.67\%)}}$$

Let B be the event of getting a sum of 8

$$B = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}, n(A) = 5$$

$$P(B) = \frac{5}{36} = \mathbf{0.1389 \text{ (or 13.89\%)}}$$

## Addition Rule II [10]

Let  $A \cap B$  be the event of getting a 'doubles' and a 'sum of 8'

$$A \cap B = \{(4, 4)\}$$

$$P(A \cap B) = \frac{1}{36} = 0.0277 \text{ (or 2.7777 \%)}$$

Since A and B are **not mutually exclusive**, so

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{36} + \frac{5}{36} - \frac{1}{36} = \frac{10}{36}$$

$$= \frac{5}{18} = 0.2777 \text{ or (27.7777\%)}$$

# Addition Rule II [11]

Let **P** be the event that an employee selected at random from an oil drilling company **smokes cigarettes**.

Let **Q** be the event that the employee selected drinks **alcoholic beverages**.

Then the event  **$P \cup Q$**  is the set of all employees who either **drink** or **smoke** or do **both**.

**Example :** A coin is tossed twice. What is the probability that at **least 1 head** occurs?

**Solution :** The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}$$

$$n(S) = 4$$

Let **A** be the event of getting at **least 1 head**

$$A = \{HH, HT, TH\}$$

$$n(A) = 3$$

$$\begin{aligned}\therefore P(A) &= \frac{n(A)}{n(s)} \\ &= \frac{3}{4} = \mathbf{(0.75 \text{ or } 75\%)}\end{aligned}$$



**Example :** A die is loaded in such a way that **an even number** is **twice** as likely to occur as an **odd number**. If  **$E$**  is the event that a **number less than 4** occurs on a single toss of the die, find  **$P(E)$** .

## Solution

$$P(\text{Even number}) = 2p$$

$$P(\text{Odd number}) = p$$

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$\because \text{Sum of probability} = 1$$

$$\therefore p + 2p + p + 2p + p + 2p = 1$$

$$\Rightarrow 9p = 1$$

$$\Rightarrow p = \frac{1}{9}$$

$$\Rightarrow P(\text{Even number}) = \frac{2}{9}$$

$$\Rightarrow P(\text{Odd number}) = \frac{1}{9}$$

$$E = \{1, 2, 3\}$$

$$P(E) = P(1) + P(2) + P(3)$$

$$= \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9} \text{ (or 0.4444 or 44\%)}$$

**Example** A die is loaded in such a way that **an even number** is **twice** as likely to occur as an odd number.

Let  **$A$**  be the event that an **even number** turns up and let  **$B$**  be the event that a **number divisible by 3** occurs. Find  **$P(A \cup B)$**  and  **$P(A \cap B)$** .

## Solution

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$P(\text{Even number}) = 2p$$

$$P(\text{Odd number}) = p$$

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$\because \text{Sum of probability} = 1$$

$$\therefore p + 2p + p + 2p + p + 2p = 1$$

$$\Rightarrow 9p = 1$$

$$\Rightarrow p = \frac{1}{9}$$

$$\Rightarrow P(\text{Even number}) = \frac{2}{9}$$

$$\Rightarrow P(\text{Odd number}) = \frac{1}{9}$$

$$A = \{2, 4, 6\}$$

$$P(A) = \frac{2}{9} + \frac{2}{9} + \frac{2}{9} = \frac{6}{9} = \frac{2}{3}$$

$$B = \{3, 6\}$$

$$P(B) = \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$$

$$A \cap B = \{6\}$$

$$P(A \cap B) = \frac{2}{9} \qquad \because P(\text{Even number}) = \frac{2}{9}$$

Since A and B are **not mutually exclusive**, so

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{6}{9} + \frac{3}{9} - \frac{2}{9} = \frac{7}{9} \quad (\text{or } 0.7778 \text{ or } 77.7778\%) \end{aligned}$$

- ❑ To find  $P(A \text{ or } B)$ , begin by associating use of the word “or” with addition.
- ❑ Consider whether **events  $A$**  and  **$B$**  are **disjoint**; that is, can they happen at the same time?
- ❑ If they are **not disjoint** (that is, they can happen at the same time), be sure to avoid (or at least compensate for) **double-counting** when adding the relevant probabilities.
- ❑ If you understand the importance of not double counting when you find  $P(A \text{ or } B)$ , you don't necessarily have to calculate the value of  **$P(A) + P(B) - P(A \cap B)$**

# Errors made when applying the addition rule

- ❑ Errors made when applying the addition rule often involve **double-counting**; that is, events that **are not disjoint** are treated as if they were. One indication of such an error is a total probability that **exceeds 1**.
- ❑ However, errors involving the addition rule do not **always cause** the total probability to **exceed 1**.

# Suggested Readings

**Probability & Statistics for Engineers & Scientists,**  
Ninth Edition, Ronald E. Walpole, Raymond H. Myer

2.1 Sample space

2.2 Events

2.3 Counting Sample Points

2.4 Probability of an Event

2.5 Additive Rules