# Probability and Statstics

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# Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

- **Elementary Statistics**, 13th Edition, Mario F. Triola

# Reference books

- **Probability Demystified**, Allan G. Bluman
- **Schaum's Outline of Probability and Statistics**
- **MATLAB Primer**, Seventh Edition
- **MATLAB Demystified** by McMahon, David

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- **Probability Demystified**, Allan G. Bluman
- **Schaum's Outline of Probability and Statistics**
- **MATLAB  Primer**, Seventh Edition
- **MATLAB Demystified** by McMahon, David

# References

Readings for these lecture notes:

- **Probability & Statistics for Engineers & Scientists**, Ninth edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics**, 10th Edition, Mario F. Triola

- **Probability Demystified**, Allan G. Bluman

These notes contain material from the above three books.

**“A goal is a dream with a deadline.”**

**— Napoleon Hill**

# Expected Value

The **expected value** of a discrete random variable is equal to the mean of the random variable.

**Expected Value** = $\mu = E(x) = \sum xP(x)$

**Note:** Although **probabilities** can never be **negative**, the expected value of a random variable can be negative

**Example:** An industrial psychologist administered a personality inventory test for passive-aggressive traits to 150 employees. Each individual was given a score from 1 to 5, where 1 is extremely passive and 5 is extremely aggressive. A score of 3 indicated neither trait. The results are shown at the left. Construct a probability distribution for the random variable *x*. Find mean and variance of x.

## Frequency Distribution

| Score, *x* | frequency, f |
|---|---|
| 1 | 24 |
| 2 | 33 |
| 3 | 42 |
| 4 | 30 |
| 5 | 21 |

# Probability Distribution

| $x$ | P(x) | xP(x) | $x^2$P(x) |
|---|---|---|---|
| 1 | $\frac{24}{150} = 0.1600$ | 0.1600 | 0.1600 |
| 2 | $\frac{33}{150} = 0.2200$ | 0.4400 | 0.8800 |
| 3 | $\frac{42}{150} = 0.2800$ | 0.8400 | 2.5200 |
| 4 | $\frac{30}{150} = 0.2000$ | 0.8000 | 3.2000 |
| 5 | $\frac{21}{150} = 0.1400$ | 0.7000 | 3.5000 |
| | $\sum_{i=1}^{5} P_i = 1.0000$ | $\sum xP(x) = 2.9400$ | $\sum x^2P(x) = 10.2600$ |

**Note:**

**1. $0 \le P(x) \le 1$**

**2. $\sum P(x) = 1$**

$$\mu = E(x) = \sum xP(x) = 2.9400$$

$$E(x^2) = \sum x^2P(x) = 10.2600$$

$$\sigma^2 = E(x^2) - [E(x)]^2 = 10.2600 - (2.9400)^2$$

$$\sigma^2 = 1.6164$$

# Mean and Variance for discrete probability distribution [1]

- **Expected value or mathematical expectation or expectation** for discrete probability distribution is denoted by E(X) and is defined as

$$\mathbf{E(X)} = x_1 P(X_1 = x_1) + x_2 P(X_2 = x_2) + \ldots + x_n P(X_n = x_n)$$

$$= \sum_{i=1}^{n} x_i P(Xi = x_i)$$

Here E(X) is mean or expected value of X.

# Mean and Variance for discrete probability distribution [2]

**$E(X^2)$** $= x_1^2 P(X_1 = x_1) + x_2^2 P(X_2 = x_2) + \ldots + x_n^2 P(X_n = x_n)$

$= \sum_{i=1}^{n} x_i^2 P(X_i = x_i)$

If X is a discrete random variable taking the values $x_1$, $x_2, \ldots, x_n$, and having probability function P(x), then the **variance** is given by

$$Var(X) = E(X^2) - \{E(X)\}^2$$

# Binomial Distribution using Matlab

binopdf(x, n, p)

**Matlab code:**

```
x = 0:10;                  % vector notation
y = binopdf(x,10,0.5);     % binomial probability distribution
plot(x,y,'+')
hold on
bar(x,y)                   % Matlab code to draw bar chart
xlabel('Number of tosses')
ylabel('Probability')
```

Find the probability distribution, when $n = 10$. Also find its mean.

$p = 0.50.$

$q = 1 - p = 0.4$

$X = 0, 1, 2, 3, 4, 5, 6, \ldots, 10$

# Hypergeometric Distribution

- **Hypergeometric Distribution** If we sample from a small finite population **without replacement**, the binomial distribution should not be used because the events are **not independent.**

- If sampling is done **without replacement** and the outcomes belong to **one of two types**, we can use the **hypergeometric distribution**

# Hypergeometric Distribution

- The simplest way to view the **distinction** between the binomial distribution and the hypergeometric distribution is to note the **way the sampling is done**.

- The types of applications for the **hypergeometric** are very similar to those for the binomial distribution. We are interested in computing probabilities for the number of observations that **fall into a particular category**.

# Applications

- Applications for the hypergeometric distribution are found in many areas, with heavy use in **acceptance sampling, electronic testing, and quality assurance**. Obviously, in many of these fields, **testing is done at the expense of the item being tested**.

- That is, the item is **destroyed** and hence **cannot be replaced** in the sample

# Hypergeometric Distribution [1]

A **hypergeometric experiment** has the following properties:

1. Each trial of an experiment results in **an outcome** that can be classified into one of the two categories **success or failure**.
2. The successive trials are **dependent.**
3. The probability of success **changes** from trial to trial.
4. The experiment is repeated **a fixed number** of times.

# Hypergeometric Distribution

- In general, we are interested in the probability of selecting ***x* successes** from the ***k* items** labeled successes and ***n* − *x* failures** from the ***N* − *k* items** labeled **failures** when a random sample of size *n* is selected from ***N* items**.
- This is known as a **hypergeometric experiment**, that is, one that possesses the following two properties:
  1. A random sample of size *n* is selected **without replacement** from ***N* items**.
  2. Of the ***N* items**, ***k*** may be classified as **successes** and ***N* − *k*** are classified as **failures**.

# Hypergeometric Distribution

- The number **X of successes** of a hypergeometric experiment is called a **hypergeometric random variable**.

- Accordingly, the probability distribution of the hypergeometric variable is called the **hypergeometric distribution**, and its values are denoted by $h(x; N, n, k)$, since they depend on the **number of successes $k$** in the set $N$ from which we **select $n$ items**.

# Hypergeometric Distribution [2]

This distribution is the case of sampling **without replacement**. The formula to calculate probabilities is given by

**P(X = x)= h(x; N, n, k)**

$$= ({}_kC_x)\ ({}_{N-k}C_{n-x})\ /\ ({}_NC_n),$$

**max{0, n-(N-k)} ≤ x ≤ min{n, k}**

**OR**

$$h(x; N, n, k) = \frac{({}_kC_x)\ ({}_{N-k}C_{n-x})}{{}_NC_n},\ \max\{0, n - (N-k)\} \leq x \leq \min\{n, k\}$$

# Hypergeometric Distribution [3]

- It has **three** parameters i.e., N, n, and k
- **N**: The number of items in the **population**
- **k**: The number of items in the **population** that are classified as **successes.**
- **n**: The number of items in the sample
- **x**: The number of items in the **sample** that are **classified as successes**.

# Hypergeometric Distribution [4]

**Example1:** Suppose we randomly select **5** cards **without replacement** from an ordinary deck of playing cards. What is the probability of getting exactly **2 red cards**?

**Solution:** This is a hypergeometric experiment in which we know the following:

**N = 52**; since there are 52 cards in a deck.

**k = 26**; since there are 26 red cards in a deck.

**n = 5**; since we randomly select 5 cards from the deck.

**x = 2**; since 2 of the cards we select are red.

$h(x; N, n, k) = ({}_{k}C_{x})({}_{N-k}C_{n-x}) / {}_{N}C_{n}$

$h(2; 52, 5, 26) = ({}_{26}C_{2})({}_{26}C_{3}) / ({}_{52}C_{5})$

$h(2; 52, 5, 26) = (325)(2600)/(2{,}598{,}960) = 0.32513$

# Hypergeometric Distribution [5]

**Example:** A committee of **4** people is selected at random without replacement from a group of **6** men and **4** women. Find the probability that the committee consists of **2** men and **2** women.

**Solution:**

**$P(X = x) = h(x; N, n, k) = ({}_kC_x)({}_{N-k}C_{n-x})/({}_NC_n)$, $\max\{0, n-(N-k)\} \le x \le \min\{n, k\}$**

**$N = 10$; $k = 6$; $n = 4$; $x = 2$ (let x denotes number of men)**

$h(x; N, n, k) = ({}_6C_x)({}_4C_{4-x})/({}_{10}C_4)$

$h(2; 10, 4, 6) = ({}_6C_2)({}_4C_2) /({}_{10}C_4)$

$h(2; 10, 4, 6) = (15)(6)/(210) = 0.429$

# Hypergeometric Distribution [6]

**Example:** A lot of **12 oxygen tanks** contains **3** defective ones. If **4 tanks** are randomly selected and tested, find the probability that exactly **one will be defective**.

**Solution:**

**$P(X = x) = h(x; N, n, k) = ({}_kC_x)({}_{N-k}C_{n-x})/({}_NC_n)$, $\max\{0, n-(N-k)\} \le x \le \min\{n, k\}$**

**OR**

**$h(x; N, n, k) = \dfrac{({}_kC_x)\,({}_{N-k}C_{n-x})}{{}_NC_n}$, $\max\{0, n - (N-k)\} \le x \le \min\{n, k\}$**

**$N = 12$; $k = 3$; $n = 4$; $x = 1$ (let x denotes defective tanks)**

$P(X = 1) = ({}_3C_1)({}_9C_3)/({}_{12}C_4)$

$P(X = 1) = (3)(84)/(495) = 0.509$

# Hypergeometric Distribution [1]

**Example:** In a box of **12** shirts there are **5** defective ones. If **5** shirts are sold at random, find the probability that exactly two are defective.

**Solution:**

**$h(x; N, n, k) = \frac{({}_kC_x)\,({}_{N-k}C_{n-x})}{{}_NC_n}$, $\max\{0, n-(N-k)\} \le x \le \min\{n, k\}$**

| Defective shirts | Non-detective shirts | Total |
|---|---|---|
| **5** | **7** | **12** |

N = 12, k = 5, n = 5, and x = 2

Let **X** denotes the number of **defective shirts**

$P(X = 2) = ({}_5C_2)({}_7C_3)/{}_{12}C_5 = 0.442$

# Hypergeometric Distribution [2]

**Example:** In a fitness club of **18** members, **10** prefer the exercise bicycle and **8** prefer the aerobic stepper. If **6** members are selected at random, find the probability that exactly **3** use the bicycle.

**Solution:**

$$h(x; N, n, k) = \frac{({}_kC_x)\ ({}_{N-k}C_{n-x})}{{}_NC_n},\ \max\{0, n-(N-k)\} \le x \le \min\{n, k\}$$

| Exercise Bicycle | Aerobic Stepper | Total |
|---|---|---|
| 10 | 8 | 18 |

$N = 18$, $k = 10$, $n = 6$, and $x = 3$

Let **X** denotes the number of **bicycles**

$P(X=3) = ({}_{10}C_3)({}_8C_3)/{}_{18}C_6 = 0.362$

# Hypergeometric Distribution [3]

**Example:** In a shipment of **10** lawn chairs, **6** are brown and **4** are blue. If **3** chairs are sold at random, find the probability that all are **brown**.

**Solution:**

$$h(x; N, n, k) = \frac{({}_kC_x)\,({}_{N-k}C_{n-x})}{{}_NC_n}, \ \max\{0, n-(N-k)\} \leq x \leq \min\{n, k\}$$

| Brown | Blue | Total |
|---|---|---|
| 6 | 4 | 10 |

N = 10, k = **6**, n = 3, and x = **3**

Let **X** denotes the number of **brown** chairs

$P(X = 3) = ({}_6C_3)({}_4C_0)/{}_{10}C_3 = 0.167$

# Hypergeometric Distribution [4]

**Example:** A class consists of 5 women and 4 men. If a committee of 3 people is selected at random, find the probability that all 3 are women.

## Solution:

$$h(x; N, n, k) = \frac{({}_kC_x)\,({}_{N-k}C_{n-x})}{{}_NC_n},\ \max\{0, n-(N-k)\} \leq x \leq \min\{n, k\}$$

| Men | Women | Total |
|---|---|---|
| 4 | 5 | 9 |

N = 9, k = 5, n = 3, and x = 3

Let **X** denotes the number of **women**

$P(X=3) = ({}_5C_3)({}_4C_0)/{}_9C_3 = 0.119$

# Hypergeometric Distribution [5]

**Example:** A box contains **3 red** balls and **3 white balls**. If **two balls** are selected at random without replacement, find the probability that both are **red**.

**Solution:**

**$h(x; N, n, k) = \frac{({}_kC_x)\,({}_{N-k}C_{n-x})}{{}_NC_n}$, $\max\{0, n-(N-k)\} \le x \le \min\{n, k\}$**

| Red | White | Total |
|---|---|---|
| 3 | 3 | 6 |

N = 6, k = 3, n = 2, and x = 2

Let **X** denotes the number of **red balls**

$P(X = 2) = ({}_3C_2)({}_3C_0) / {}_6C_2 = 0.2$

Table A.1 Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

| | | $p$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | 0.10 | 0.20 | 0.25 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
| 1 | 0 | 0.9000 | 0.8000 | 0.7500 | 0.7000 | 0.6000 | 0.5000 | 0.4000 | 0.3000 | 0.2000 | 0.1000 |
| | 1 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 2 | 0 | 0.8100 | 0.6400 | 0.5625 | 0.4900 | 0.3600 | 0.2500 | 0.1600 | 0.0900 | 0.0400 | 0.0100 |
| | 1 | 0.9900 | 0.9600 | 0.9375 | 0.9100 | 0.8400 | 0.7500 | 0.6400 | 0.5100 | 0.3600 | 0.1900 |
| | 2 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 3 | 0 | 0.7290 | 0.5120 | 0.4219 | 0.3430 | 0.2160 | 0.1250 | 0.0640 | 0.0270 | 0.0080 | 0.0010 |
| | 1 | 0.9720 | 0.8960 | 0.8438 | 0.7840 | 0.6480 | 0.5000 | 0.3520 | 0.2160 | 0.1040 | 0.0280 |
| | 2 | 0.9990 | 0.9920 | 0.9844 | 0.9730 | 0.9360 | 0.8750 | 0.7840 | 0.6570 | 0.4880 | 0.2710 |
| | 3 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 4 | 0 | 0.6561 | 0.4096 | 0.3164 | 0.2401 | 0.1296 | 0.0625 | 0.0256 | 0.0081 | 0.0016 | 0.0001 |
| | 1 | 0.9477 | 0.8192 | 0.7383 | 0.6517 | 0.4752 | 0.3125 | 0.1792 | 0.0837 | 0.0272 | 0.0037 |
| | 2 | 0.9963 | 0.9728 | 0.9492 | 0.9163 | 0.8208 | 0.6875 | 0.5248 | 0.3483 | 0.1808 | 0.0523 |
| | 3 | 0.9999 | 0.9984 | 0.9961 | 0.9919 | 0.9744 | 0.9375 | 0.8704 | 0.7599 | 0.5904 | 0.3439 |
| | 4 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 5 | 0 | 0.5905 | 0.3277 | 0.2373 | 0.1681 | 0.0778 | 0.0313 | 0.0102 | 0.0024 | 0.0003 | 0.0000 |
| | 1 | 0.9185 | 0.7373 | 0.6328 | 0.5282 | 0.3370 | 0.1875 | 0.0870 | 0.0308 | 0.0067 | 0.0005 |
| | 2 | 0.9914 | 0.9421 | 0.8965 | 0.8369 | 0.6826 | 0.5000 | 0.3174 | 0.1631 | 0.0579 | 0.0086 |
| | 3 | 0.9995 | 0.9933 | 0.9844 | 0.9692 | 0.9130 | 0.8125 | 0.6630 | 0.4718 | 0.2627 | 0.0815 |
| | 4 | 1.0000 | 0.9997 | 0.9990 | 0.9976 | 0.9898 | 0.9688 | 0.9222 | 0.8319 | 0.6723 | 0.4095 |
| | 5 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 6 | 0 | 0.5314 | 0.2621 | 0.1780 | 0.1176 | 0.0467 | 0.0156 | 0.0041 | 0.0007 | 0.0001 | 0.0000 |
| | 1 | 0.8857 | 0.6554 | 0.5339 | 0.4202 | 0.2333 | 0.1094 | 0.0410 | 0.0109 | 0.0016 | 0.0001 |
| | 2 | 0.9842 | 0.9011 | 0.8306 | 0.7443 | 0.5443 | 0.3438 | 0.1792 | 0.0705 | 0.0170 | 0.0013 |
| | 3 | 0.9987 | 0.9830 | 0.9624 | 0.9295 | 0.8208 | 0.6563 | 0.4557 | 0.2557 | 0.0989 | 0.0159 |
| | 4 | 0.9999 | 0.9984 | 0.9954 | 0.9891 | 0.9590 | 0.8906 | 0.7667 | 0.5798 | 0.3446 | 0.1143 |
| | 5 | 1.0000 | 0.9999 | 0.9998 | 0.9993 | 0.9959 | 0.9844 | 0.9533 | 0.8824 | 0.7379 | 0.4686 |
| | 6 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 7 | 0 | 0.4783 | 0.2097 | 0.1335 | 0.0824 | 0.0280 | 0.0078 | 0.0016 | 0.0002 | 0.0000 | |
| | 1 | 0.8503 | 0.5767 | 0.4449 | 0.3294 | 0.1586 | 0.0625 | 0.0188 | 0.0038 | 0.0004 | 0.0000 |
| | 2 | 0.9743 | 0.8520 | 0.7564 | 0.6471 | 0.4199 | 0.2266 | 0.0963 | 0.0288 | 0.0047 | 0.0002 |
| | 3 | 0.9973 | 0.9667 | 0.9294 | 0.8740 | 0.7102 | 0.5000 | 0.2898 | 0.1260 | 0.0333 | 0.0027 |
| | 4 | 0.9998 | 0.9953 | 0.9871 | 0.9712 | 0.9037 | 0.7734 | 0.5801 | 0.3529 | 0.1480 | 0.0257 |
| | 5 | 1.0000 | 0.9996 | 0.9987 | 0.9962 | 0.9812 | 0.9375 | 0.8414 | 0.6706 | 0.4233 | 0.1497 |
| | 6 | | 1.0000 | 0.9999 | 0.9998 | 0.9984 | 0.9922 | 0.9720 | 0.9176 | 0.7903 | 0.5217 |
| | 7 | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

Table A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

| | | *p* | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **n** | **r** | **0.10** | **0.20** | **0.25** | **0.30** | **0.40** | **0.50** | **0.60** | **0.70** | **0.80** | **0.90** |
| 8 | 0 | 0.4305 | 0.1678 | 0.1001 | 0.0576 | 0.0168 | 0.0039 | 0.0007 | 0.0001 | 0.0000 | |
| | 1 | 0.8131 | 0.5033 | 0.3671 | 0.2553 | 0.1064 | 0.0352 | 0.0085 | 0.0013 | 0.0001 | |
| | 2 | 0.9619 | 0.7969 | 0.6785 | 0.5518 | 0.3154 | 0.1445 | 0.0498 | 0.0113 | 0.0012 | 0.0000 |
| | 3 | 0.9950 | 0.9437 | 0.8862 | 0.8059 | 0.5941 | 0.3633 | 0.1737 | 0.0580 | 0.0104 | 0.0004 |
| | 4 | 0.9996 | 0.9896 | 0.9727 | 0.9420 | 0.8263 | 0.6367 | 0.4059 | 0.1941 | 0.0563 | 0.0050 |
| | 5 | 1.0000 | 0.9988 | 0.9958 | 0.9887 | 0.9502 | 0.8555 | 0.6846 | 0.4482 | 0.2031 | 0.0381 |
| | 6 | | 0.9999 | 0.9996 | 0.9987 | 0.9915 | 0.9648 | 0.8936 | 0.7447 | 0.4967 | 0.1869 |
| | 7 | | 1.0000 | 1.0000 | 0.9999 | 0.9993 | 0.9961 | 0.9832 | 0.9424 | 0.8322 | 0.5695 |
| | 8 | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 9 | 0 | 0.3874 | 0.1342 | 0.0751 | 0.0404 | 0.0101 | 0.0020 | 0.0003 | 0.0000 | | |
| | 1 | 0.7748 | 0.4362 | 0.3003 | 0.1960 | 0.0705 | 0.0195 | 0.0038 | 0.0004 | 0.0000 | |
| | 2 | 0.9470 | 0.7382 | 0.6007 | 0.4628 | 0.2318 | 0.0898 | 0.0250 | 0.0043 | 0.0003 | 0.0000 |
| | 3 | 0.9917 | 0.9144 | 0.8343 | 0.7297 | 0.4826 | 0.2539 | 0.0994 | 0.0253 | 0.0031 | 0.0001 |
| | 4 | 0.9991 | 0.9804 | 0.9511 | 0.9012 | 0.7334 | 0.5000 | 0.2666 | 0.0988 | 0.0196 | 0.0009 |
| | 5 | 0.9999 | 0.9969 | 0.9900 | 0.9747 | 0.9006 | 0.7461 | 0.5174 | 0.2703 | 0.0856 | 0.0083 |
| | 6 | 1.0000 | 0.9997 | 0.9987 | 0.9957 | 0.9750 | 0.9102 | 0.7682 | 0.5372 | 0.2618 | 0.0530 |
| | 7 | | 1.0000 | 0.9999 | 0.9996 | 0.9962 | 0.9805 | 0.9295 | 0.8040 | 0.5638 | 0.2252 |
| | 8 | | | 1.0000 | 1.0000 | 0.9997 | 0.9980 | 0.9899 | 0.9596 | 0.8658 | 0.6126 |
| | 9 | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 10 | 0 | 0.3487 | 0.1074 | 0.0563 | 0.0282 | 0.0060 | 0.0010 | 0.0001 | 0.0000 | | |
| | 1 | 0.7361 | 0.3758 | 0.2440 | 0.1493 | 0.0464 | 0.0107 | 0.0017 | 0.0001 | 0.0000 | |
| | 2 | 0.9298 | 0.6778 | 0.5256 | 0.3828 | 0.1673 | 0.0547 | 0.0123 | 0.0016 | 0.0001 | |
| | 3 | 0.9872 | 0.8791 | 0.7759 | 0.6496 | 0.3823 | 0.1719 | 0.0548 | 0.0106 | 0.0009 | 0.0000 |
| | 4 | 0.9984 | 0.9672 | 0.9219 | 0.8497 | 0.6331 | 0.3770 | 0.1662 | 0.0473 | 0.0064 | 0.0001 |
| | 5 | 0.9999 | 0.9936 | 0.9803 | 0.9527 | 0.8338 | 0.6230 | 0.3669 | 0.1503 | 0.0328 | 0.0016 |
| | 6 | 1.0000 | 0.9991 | 0.9965 | 0.9894 | 0.9452 | 0.8281 | 0.6177 | 0.3504 | 0.1209 | 0.0128 |
| | 7 | | 0.9999 | 0.9996 | 0.9984 | 0.9877 | 0.9453 | 0.8327 | 0.6172 | 0.3222 | 0.0702 |
| | 8 | | 1.0000 | 1.0000 | 0.9999 | 0.9983 | 0.9893 | 0.9536 | 0.8507 | 0.6242 | 0.2639 |
| | 9 | | | | 1.0000 | 0.9999 | 0.9990 | 0.9940 | 0.9718 | 0.8926 | 0.6513 |
| | 10 | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 11 | 0 | 0.3138 | 0.0859 | 0.0422 | 0.0198 | 0.0036 | 0.0005 | 0.0000 | | | |
| | 1 | 0.6974 | 0.3221 | 0.1971 | 0.1130 | 0.0302 | 0.0059 | 0.0007 | 0.0000 | | |
| | 2 | 0.9104 | 0.6174 | 0.4552 | 0.3127 | 0.1189 | 0.0327 | 0.0059 | 0.0006 | 0.0000 | |
| | 3 | 0.9815 | 0.8389 | 0.7133 | 0.5696 | 0.2963 | 0.1133 | 0.0293 | 0.0043 | 0.0002 | |
| | 4 | 0.9972 | 0.9496 | 0.8854 | 0.7897 | 0.5328 | 0.2744 | 0.0994 | 0.0216 | 0.0020 | 0.0000 |
| | 5 | 0.9997 | 0.9883 | 0.9657 | 0.9218 | 0.7535 | 0.5000 | 0.2465 | 0.0782 | 0.0117 | 0.0003 |
| | 6 | 1.0000 | 0.9980 | 0.9924 | 0.9784 | 0.9006 | 0.7256 | 0.4672 | 0.2103 | 0.0504 | 0.0028 |
| | 7 | | 0.9998 | 0.9988 | 0.9957 | 0.9707 | 0.8867 | 0.7037 | 0.4304 | 0.1611 | 0.0185 |
| | 8 | | 1.0000 | 0.9999 | 0.9994 | 0.9941 | 0.9673 | 0.8811 | 0.6873 | 0.3826 | 0.0896 |
| | 9 | | | 1.0000 | 1.0000 | 0.9993 | 0.9941 | 0.9698 | 0.8870 | 0.6779 | 0.3026 |
| | 10 | | | | | 1.0000 | 0.9995 | 0.9964 | 0.9802 | 0.9141 | 0.6862 |
| | 11 | | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

| | | *p* | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *n* | *r* | 0.10 | 0.20 | 0.25 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
| 12 | 0 | 0.2824 | 0.0687 | 0.0317 | 0.0138 | 0.0022 | 0.0002 | 0.0000 | | | |
| | 1 | 0.6590 | 0.2749 | 0.1584 | 0.0850 | 0.0196 | 0.0032 | 0.0003 | 0.0000 | | |
| | 2 | 0.8891 | 0.5583 | 0.3907 | 0.2528 | 0.0834 | 0.0193 | 0.0028 | 0.0002 | 0.0000 | |
| | 3 | 0.9744 | 0.7946 | 0.6488 | 0.4925 | 0.2253 | 0.0730 | 0.0153 | 0.0017 | 0.0001 | |
| | 4 | 0.9957 | 0.9274 | 0.8424 | 0.7237 | 0.4382 | 0.1938 | 0.0573 | 0.0095 | 0.0006 | 0.0000 |
| | 5 | 0.9995 | 0.9806 | 0.9456 | 0.8822 | 0.6652 | 0.3872 | 0.1582 | 0.0386 | 0.0039 | 0.0001 |
| | 6 | 0.9999 | 0.9961 | 0.9857 | 0.9614 | 0.8418 | 0.6128 | 0.3348 | 0.1178 | 0.0194 | 0.0005 |
| | 7 | 1.0000 | 0.9994 | 0.9972 | 0.9905 | 0.9427 | 0.8062 | 0.5618 | 0.2763 | 0.0726 | 0.0043 |
| | 8 | | 0.9999 | 0.9996 | 0.9983 | 0.9847 | 0.9270 | 0.7747 | 0.5075 | 0.2054 | 0.0256 |
| | 9 | | 1.0000 | 1.0000 | 0.9998 | 0.9972 | 0.9807 | 0.9166 | 0.7472 | 0.4417 | 0.1109 |
| | 10 | | | | 1.0000 | 0.9997 | 0.9968 | 0.9804 | 0.9150 | 0.7251 | 0.3410 |
| | 11 | | | | | 1.0000 | 0.9998 | 0.9978 | 0.9862 | 0.9313 | 0.7176 |
| | 12 | | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 13 | 0 | 0.2542 | 0.0550 | 0.0238 | 0.0097 | 0.0013 | 0.0001 | 0.0000 | | | |
| | 1 | 0.6213 | 0.2336 | 0.1267 | 0.0637 | 0.0126 | 0.0017 | 0.0001 | 0.0000 | | |
| | 2 | 0.8661 | 0.5017 | 0.3326 | 0.2025 | 0.0579 | 0.0112 | 0.0013 | 0.0001 | | |
| | 3 | 0.9658 | 0.7473 | 0.5843 | 0.4206 | 0.1686 | 0.0461 | 0.0078 | 0.0007 | 0.0000 | |
| | 4 | 0.9935 | 0.9009 | 0.7940 | 0.6543 | 0.3530 | 0.1334 | 0.0321 | 0.0040 | 0.0002 | |
| | 5 | 0.9991 | 0.9700 | 0.9198 | 0.8346 | 0.5744 | 0.2905 | 0.0977 | 0.0182 | 0.0012 | 0.0000 |
| | 6 | 0.9999 | 0.9930 | 0.9757 | 0.9376 | 0.7712 | 0.5000 | 0.2288 | 0.0624 | 0.0070 | 0.0001 |
| | 7 | 1.0000 | 0.9988 | 0.9944 | 0.9818 | 0.9023 | 0.7095 | 0.4256 | 0.1654 | 0.0300 | 0.0009 |
| | 8 | | 0.9998 | 0.9990 | 0.9960 | 0.9679 | 0.8666 | 0.6470 | 0.3457 | 0.0991 | 0.0065 |
| | 9 | | 1.0000 | 0.9999 | 0.9993 | 0.9922 | 0.9539 | 0.8314 | 0.5794 | 0.2527 | 0.0342 |
| | 10 | | | 1.0000 | 0.9999 | 0.9987 | 0.9888 | 0.9421 | 0.7975 | 0.4983 | 0.1339 |
| | 11 | | | | 1.0000 | 0.9999 | 0.9983 | 0.9874 | 0.9363 | 0.7664 | 0.3787 |
| | 12 | | | | | 1.0000 | 0.9999 | 0.9987 | 0.9903 | 0.9450 | 0.7458 |
| | 13 | | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 14 | 0 | 0.2288 | 0.0440 | 0.0178 | 0.0068 | 0.0008 | 0.0001 | 0.0000 | | | |
| | 1 | 0.5846 | 0.1979 | 0.1010 | 0.0475 | 0.0081 | 0.0009 | 0.0001 | | | |
| | 2 | 0.8416 | 0.4481 | 0.2811 | 0.1608 | 0.0398 | 0.0065 | 0.0006 | 0.0000 | | |
| | 3 | 0.9559 | 0.6982 | 0.5213 | 0.3552 | 0.1243 | 0.0287 | 0.0039 | 0.0002 | | |
| | 4 | 0.9908 | 0.8702 | 0.7415 | 0.5842 | 0.2793 | 0.0898 | 0.0175 | 0.0017 | 0.0000 | |
| | 5 | 0.9985 | 0.9561 | 0.8883 | 0.7805 | 0.4859 | 0.2120 | 0.0583 | 0.0083 | 0.0004 | |
| | 6 | 0.9998 | 0.9884 | 0.9617 | 0.9067 | 0.6925 | 0.3953 | 0.1501 | 0.0315 | 0.0024 | 0.0000 |
| | 7 | 1.0000 | 0.9976 | 0.9897 | 0.9685 | 0.8499 | 0.6047 | 0.3075 | 0.0933 | 0.0116 | 0.0002 |
| | 8 | | 0.9996 | 0.9978 | 0.9917 | 0.9417 | 0.7880 | 0.5141 | 0.2195 | 0.0439 | 0.0015 |
| | 9 | | 1.0000 | 0.9997 | 0.9983 | 0.9825 | 0.9102 | 0.7207 | 0.4158 | 0.1298 | 0.0092 |
| | 10 | | | 1.0000 | 0.9998 | 0.9961 | 0.9713 | 0.8757 | 0.6448 | 0.3018 | 0.0441 |
| | 11 | | | | 1.0000 | 0.9994 | 0.9935 | 0.9602 | 0.8392 | 0.5519 | 0.1584 |
| | 12 | | | | | 0.9999 | 0.9991 | 0.9919 | 0.9525 | 0.8021 | 0.4154 |
| | 13 | | | | | 1.0000 | 0.9999 | 0.9992 | 0.9932 | 0.9560 | 0.7712 |
| | 14 | | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

Table A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

| | | p | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n | r | 0.10 | 0.20 | 0.25 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
| 15 | 0 | 0.2059 | 0.0352 | 0.0134 | 0.0047 | 0.0005 | 0.0000 | | | | |
| | 1 | 0.5490 | 0.1671 | 0.0802 | 0.0353 | 0.0052 | 0.0005 | 0.0000 | | | |
| | 2 | 0.8159 | 0.3980 | 0.2361 | 0.1268 | 0.0271 | 0.0037 | 0.0003 | 0.0000 | | |
| | 3 | 0.9444 | 0.6482 | 0.4613 | 0.2969 | 0.0905 | 0.0176 | 0.0019 | 0.0001 | | |
| | 4 | 0.9873 | 0.8358 | 0.6865 | 0.5155 | 0.2173 | 0.0592 | 0.0093 | 0.0007 | 0.0000 | |
| | 5 | 0.9978 | 0.9389 | 0.8516 | 0.7216 | 0.4032 | 0.1509 | 0.0338 | 0.0037 | 0.0001 | |
| | 6 | 0.9997 | 0.9819 | 0.9434 | 0.8689 | 0.6098 | 0.3036 | 0.0950 | 0.0152 | 0.0008 | |
| | 7 | 1.0000 | 0.9958 | 0.9827 | 0.9500 | 0.7869 | 0.5000 | 0.2131 | 0.0500 | 0.0042 | 0.0000 |
| | 8 | | 0.9992 | 0.9958 | 0.9848 | 0.9050 | 0.6964 | 0.3902 | 0.1311 | 0.0181 | 0.0003 |
| | 9 | | 0.9999 | 0.9992 | 0.9963 | 0.9662 | 0.8491 | 0.5968 | 0.2784 | 0.0611 | 0.0022 |
| | 10 | | 1.0000 | 0.9999 | 0.9993 | 0.9907 | 0.9408 | 0.7827 | 0.4845 | 0.1642 | 0.0127 |
| | 11 | | | 1.0000 | 0.9999 | 0.9981 | 0.9824 | 0.9095 | 0.7031 | 0.3518 | 0.0556 |
| | 12 | | | | 1.0000 | 0.9997 | 0.9963 | 0.9729 | 0.8732 | 0.6020 | 0.1841 |
| | 13 | | | | | 1.0000 | 0.9995 | 0.9948 | 0.9647 | 0.8329 | 0.4510 |
| | 14 | | | | | | 1.0000 | 0.9995 | 0.9953 | 0.9648 | 0.7941 |
| | 15 | | | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 16 | 0 | 0.1853 | 0.0281 | 0.0100 | 0.0033 | 0.0003 | 0.0000 | | | | |
| | 1 | 0.5147 | 0.1407 | 0.0635 | 0.0261 | 0.0033 | 0.0003 | 0.0000 | | | |
| | 2 | 0.7892 | 0.3518 | 0.1971 | 0.0994 | 0.0183 | 0.0021 | 0.0001 | | | |
| | 3 | 0.9316 | 0.5981 | 0.4050 | 0.2459 | 0.0651 | 0.0106 | 0.0009 | 0.0000 | | |
| | 4 | 0.9830 | 0.7982 | 0.6302 | 0.4499 | 0.1666 | 0.0384 | 0.0049 | 0.0003 | | |
| | 5 | 0.9967 | 0.9183 | 0.8103 | 0.6598 | 0.3288 | 0.1051 | 0.0191 | 0.0016 | 0.0000 | |
| | 6 | 0.9995 | 0.9733 | 0.9204 | 0.8247 | 0.5272 | 0.2272 | 0.0583 | 0.0071 | 0.0002 | |
| | 7 | 0.9999 | 0.9930 | 0.9729 | 0.9256 | 0.7161 | 0.4018 | 0.1423 | 0.0257 | 0.0015 | 0.0000 |
| | 8 | 1.0000 | 0.9985 | 0.9925 | 0.9743 | 0.8577 | 0.5982 | 0.2839 | 0.0744 | 0.0070 | 0.0001 |
| | 9 | | 0.9998 | 0.9984 | 0.9929 | 0.9417 | 0.7728 | 0.4728 | 0.1753 | 0.0267 | 0.0005 |
| | 10 | | 1.0000 | 0.9997 | 0.9984 | 0.9809 | 0.8949 | 0.6712 | 0.3402 | 0.0817 | 0.0033 |
| | 11 | | | 1.0000 | 0.9997 | 0.9951 | 0.9616 | 0.8334 | 0.5501 | 0.2018 | 0.0170 |
| | 12 | | | | 1.0000 | 0.9991 | 0.9894 | 0.9349 | 0.7541 | 0.4019 | 0.0684 |
| | 13 | | | | | 0.9999 | 0.9979 | 0.9817 | 0.9006 | 0.6482 | 0.2108 |
| | 14 | | | | | 1.0000 | 0.9997 | 0.9967 | 0.9739 | 0.8593 | 0.4853 |
| | 15 | | | | | | 1.0000 | 0.9997 | 0.9967 | 0.9719 | 0.8147 |
| | 16 | | | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

Table A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

| | | $p$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | 0.10 | 0.20 | 0.25 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
| 17 | 0 | 0.1668 | 0.0225 | 0.0075 | 0.0023 | 0.0002 | 0.0000 | | | | |
| | 1 | 0.4818 | 0.1182 | 0.0501 | 0.0193 | 0.0021 | 0.0001 | 0.0000 | | | |
| | 2 | 0.7618 | 0.3096 | 0.1637 | 0.0774 | 0.0123 | 0.0012 | 0.0001 | | | |
| | 3 | 0.9174 | 0.5489 | 0.3530 | 0.2019 | 0.0464 | 0.0064 | 0.0005 | 0.0000 | | |
| | 4 | 0.9779 | 0.7582 | 0.5739 | 0.3887 | 0.1260 | 0.0245 | 0.0025 | 0.0001 | | |
| | 5 | 0.9953 | 0.8943 | 0.7653 | 0.5968 | 0.2639 | 0.0717 | 0.0106 | 0.0007 | 0.0000 | |
| | 6 | 0.9992 | 0.9623 | 0.8929 | 0.7752 | 0.4478 | 0.1662 | 0.0348 | 0.0032 | 0.0001 | |
| | 7 | 0.9999 | 0.9891 | 0.9598 | 0.8954 | 0.6405 | 0.3145 | 0.0919 | 0.0127 | 0.0005 | |
| | 8 | 1.0000 | 0.9974 | 0.9876 | 0.9597 | 0.8011 | 0.5000 | 0.1989 | 0.0403 | 0.0026 | 0.0000 |
| | 9 | | 0.9995 | 0.9969 | 0.9873 | 0.9081 | 0.6855 | 0.3595 | 0.1046 | 0.0109 | 0.0001 |
| | 10 | | 0.9999 | 0.9994 | 0.9968 | 0.9652 | 0.8338 | 0.5522 | 0.2248 | 0.0377 | 0.0008 |
| | 11 | | 1.0000 | 0.9999 | 0.9993 | 0.9894 | 0.9283 | 0.7361 | 0.4032 | 0.1057 | 0.0047 |
| | 12 | | | 1.0000 | 0.9999 | 0.9975 | 0.9755 | 0.8740 | 0.6113 | 0.2418 | 0.0221 |
| | 13 | | | | 1.0000 | 0.9995 | 0.9936 | 0.9536 | 0.7981 | 0.4511 | 0.0826 |
| | 14 | | | | | 0.9999 | 0.9988 | 0.9877 | 0.9226 | 0.6904 | 0.2382 |
| | 15 | | | | | 1.0000 | 0.9999 | 0.9979 | 0.9807 | 0.8818 | 0.5182 |
| | 16 | | | | | | 1.0000 | 0.9998 | 0.9977 | 0.9775 | 0.8332 |
| | 17 | | | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 18 | 0 | 0.1501 | 0.0180 | 0.0056 | 0.0016 | 0.0001 | 0.0000 | | | | |
| | 1 | 0.4503 | 0.0991 | 0.0395 | 0.0142 | 0.0013 | 0.0001 | | | | |
| | 2 | 0.7338 | 0.2713 | 0.1353 | 0.0600 | 0.0082 | 0.0007 | 0.0000 | | | |
| | 3 | 0.9018 | 0.5010 | 0.3057 | 0.1646 | 0.0328 | 0.0038 | 0.0002 | | | |
| | 4 | 0.9718 | 0.7164 | 0.5187 | 0.3327 | 0.0942 | 0.0154 | 0.0013 | 0.0000 | | |
| | 5 | 0.9936 | 0.8671 | 0.7175 | 0.5344 | 0.2088 | 0.0481 | 0.0058 | 0.0003 | | |
| | 6 | 0.9988 | 0.9487 | 0.8610 | 0.7217 | 0.3743 | 0.1189 | 0.0203 | 0.0014 | 0.0000 | |
| | 7 | 0.9998 | 0.9837 | 0.9431 | 0.8593 | 0.5634 | 0.2403 | 0.0576 | 0.0061 | 0.0002 | |
| | 8 | 1.0000 | 0.9957 | 0.9807 | 0.9404 | 0.7368 | 0.4073 | 0.1347 | 0.0210 | 0.0009 | |
| | 9 | | 0.9991 | 0.9946 | 0.9790 | 0.8653 | 0.5927 | 0.2632 | 0.0596 | 0.0043 | 0.0000 |
| | 10 | | 0.9998 | 0.9988 | 0.9939 | 0.9424 | 0.7597 | 0.4366 | 0.1407 | 0.0163 | 0.0002 |
| | 11 | | 1.0000 | 0.9998 | 0.9986 | 0.9797 | 0.8811 | 0.6257 | 0.2783 | 0.0513 | 0.0012 |
| | 12 | | | 1.0000 | 0.9997 | 0.9942 | 0.9519 | 0.7912 | 0.4656 | 0.1329 | 0.0064 |
| | 13 | | | | 1.0000 | 0.9987 | 0.9846 | 0.9058 | 0.6673 | 0.2836 | 0.0282 |
| | 14 | | | | | 0.9998 | 0.9962 | 0.9672 | 0.8354 | 0.4990 | 0.0982 |
| | 15 | | | | | 1.0000 | 0.9993 | 0.9918 | 0.9400 | 0.7287 | 0.2662 |
| | 16 | | | | | | 0.9999 | 0.9987 | 0.9858 | 0.9009 | 0.5497 |
| | 17 | | | | | | 1.0000 | 0.9999 | 0.9984 | 0.9820 | 0.8499 |
| | 18 | | | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

| n | r | p = 0.10 | 0.20 | 0.25 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 19 | 0 | 0.1351 | 0.0144 | 0.0042 | 0.0011 | 0.0001 | | | | | |
| | 1 | 0.4203 | 0.0829 | 0.0310 | 0.0104 | 0.0008 | 0.0000 | | | | |
| | 2 | 0.7054 | 0.2369 | 0.1113 | 0.0462 | 0.0055 | 0.0004 | 0.0000 | | | |
| | 3 | 0.8850 | 0.4551 | 0.2631 | 0.1332 | 0.0230 | 0.0022 | 0.0001 | | | |
| | 4 | 0.9648 | 0.6733 | 0.4654 | 0.2822 | 0.0696 | 0.0096 | 0.0006 | 0.0000 | | |
| | 5 | 0.9914 | 0.8369 | 0.6678 | 0.4739 | 0.1629 | 0.0318 | 0.0031 | 0.0001 | | |
| | 6 | 0.9983 | 0.9324 | 0.8251 | 0.6655 | 0.3081 | 0.0835 | 0.0116 | 0.0006 | | |
| | 7 | 0.9997 | 0.9767 | 0.9225 | 0.8180 | 0.4878 | 0.1796 | 0.0352 | 0.0028 | 0.0000 | |
| | 8 | 1.0000 | 0.9933 | 0.9713 | 0.9161 | 0.6675 | 0.3238 | 0.0885 | 0.0105 | 0.0003 | |
| | 9 | | 0.9984 | 0.9911 | 0.9674 | 0.8139 | 0.5000 | 0.1861 | 0.0326 | 0.0016 | |
| | 10 | | 0.9997 | 0.9977 | 0.9895 | 0.9115 | 0.6762 | 0.3325 | 0.0839 | 0.0067 | 0.0000 |
| | 11 | | 1.0000 | 0.9995 | 0.9972 | 0.9648 | 0.8204 | 0.5122 | 0.1820 | 0.0233 | 0.0003 |
| | 12 | | | 0.9999 | 0.9994 | 0.9884 | 0.9165 | 0.6919 | 0.3345 | 0.0676 | 0.0017 |
| | 13 | | | 1.0000 | 0.9999 | 0.9969 | 0.9682 | 0.8371 | 0.5261 | 0.1631 | 0.0086 |
| | 14 | | | | 1.0000 | 0.9994 | 0.9904 | 0.9304 | 0.7178 | 0.3267 | 0.0352 |
| | 15 | | | | | 0.9999 | 0.9978 | 0.9770 | 0.8668 | 0.5449 | 0.1150 |
| | 16 | | | | | 1.0000 | 0.9996 | 0.9945 | 0.9538 | 0.7631 | 0.2946 |
| | 17 | | | | | | 1.0000 | 0.9992 | 0.9896 | 0.9171 | 0.5797 |
| | 18 | | | | | | | 0.9999 | 0.9989 | 0.9856 | 0.8649 |
| | 19 | | | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 20 | 0 | 0.1216 | 0.0115 | 0.0032 | 0.0008 | 0.0000 | | | | | |
| | 1 | 0.3917 | 0.0692 | 0.0243 | 0.0076 | 0.0005 | 0.0000 | | | | |
| | 2 | 0.6769 | 0.2061 | 0.0913 | 0.0355 | 0.0036 | 0.0002 | | | | |
| | 3 | 0.8670 | 0.4114 | 0.2252 | 0.1071 | 0.0160 | 0.0013 | 0.0000 | | | |
| | 4 | 0.9568 | 0.6296 | 0.4148 | 0.2375 | 0.0510 | 0.0059 | 0.0003 | | | |
| | 5 | 0.9887 | 0.8042 | 0.6172 | 0.4164 | 0.1256 | 0.0207 | 0.0016 | 0.0000 | | |
| | 6 | 0.9976 | 0.9133 | 0.7858 | 0.6080 | 0.2500 | 0.0577 | 0.0065 | 0.0003 | | |
| | 7 | 0.9996 | 0.9679 | 0.8982 | 0.7723 | 0.4159 | 0.1316 | 0.0210 | 0.0013 | 0.0000 | |
| | 8 | 0.9999 | 0.9900 | 0.9591 | 0.8867 | 0.5956 | 0.2517 | 0.0565 | 0.0051 | 0.0001 | |
| | 9 | 1.0000 | 0.9974 | 0.9861 | 0.9520 | 0.7553 | 0.4119 | 0.1275 | 0.0171 | 0.0006 | |
| | 10 | | 0.9994 | 0.9961 | 0.9829 | 0.8725 | 0.5881 | 0.2447 | 0.0480 | 0.0026 | 0.0000 |
| | 11 | | 0.9999 | 0.9991 | 0.9949 | 0.9435 | 0.7483 | 0.4044 | 0.1133 | 0.0100 | 0.0001 |
| | 12 | | 1.0000 | 0.9998 | 0.9987 | 0.9790 | 0.8684 | 0.5841 | 0.2277 | 0.0321 | 0.0004 |
| | 13 | | | 1.0000 | 0.9997 | 0.9935 | 0.9423 | 0.7500 | 0.3920 | 0.0867 | 0.0024 |
| | 14 | | | | 1.0000 | 0.9984 | 0.9793 | 0.8744 | 0.5836 | 0.1958 | 0.0113 |
| | 15 | | | | | 0.9997 | 0.9941 | 0.9490 | 0.7625 | 0.3704 | 0.0432 |
| | 16 | | | | | 1.0000 | 0.9987 | 0.9840 | 0.8929 | 0.5886 | 0.1330 |
| | 17 | | | | | | 0.9998 | 0.9964 | 0.9645 | 0.7939 | 0.3231 |
| | 18 | | | | | | 1.0000 | 0.9995 | 0.9924 | 0.9308 | 0.6083 |
| | 19 | | | | | | | 1.0000 | 0.9992 | 0.9885 | 0.8784 |
| | 20 | | | | | | | | 1.0000 | 1.0000 | 1.0000 |

# The mean and variance of the Hypergeometric Distribution [1]

The mean and variance of the hypergeometric distribution h(x; N, n, k) are:

Mean = $\frac{\mathbf{nk}}{\mathbf{N}}$

Variance = $\left(\frac{\mathbf{N-n}}{\mathbf{N-1}}\right) * \frac{\mathbf{nk}}{\mathbf{N}} * \left(\frac{\mathbf{N-k}}{\mathbf{N}}\right)$

# The mean and variance of the Hypergeometric Distribution [2]

**Example:** Calculate the mean and variance of a hypergeometric random variable for parameters N = 700, k = 35, and n = 20.

## Solution:

$$\text{Mean} = \frac{nk}{N}$$

$$\text{Mean} = \frac{(20)(35)}{700}$$

$$= 1$$

$$\text{Variance} = \left(\frac{N-n}{N-1}\right) * \frac{nk}{N} * \left(\frac{N-k}{N}\right)$$

$$\text{Variance} = \left(\frac{700-20}{700-1}\right) * \frac{(20)(35)}{700} * \left(\frac{700-35}{700}\right)$$

$$= 0.9242$$

# Relationship to the Binomial Distribution [1]

- There is an interesting relationship between the: **hypergeometric** and the **binomial distribution**. As one might expect, if **n is small compared to N**, the nature of the **N** items changes **very little** in each draw.

- So a **binomial distribution** can be used to approximate the **hypergeometric distribution** when **n is small**, **compared** to **N**.

- In fact, as a **rule of thumb** the **approximation** is good when $\frac{n}{N} \leq 0.05$ **or 5 %**.

# Relationship to the Binomial Distribution [2]

- As a result, the **binomial distribution** may be viewed as a **large population** edition of the **hypergeometric distributions**

The mean and variance then come from the formulas

Mean = **np** = $\frac{nk}{N}$

Variance = **npq** = $\frac{nk}{N} * (\frac{N-k}{N})$

$\frac{\mathbf{N-n}}{\mathbf{N-1}}$ is **negligible** when **n is small** relative to **N**

# Relationship to the Binomial Distribution [3]

**Example:** A manufacturer of automobile tires reports that among a shipment of **5000** sent to a local distributor, **1000** are slightly blemished. If one purchases **10** of these tires at random from the distributor, what, is the probability that exactly **3** are blemished?

## Solution:

| Blemished | Non-blemished | Total |
|---|---|---|
| **1000** | **4000** | **5000** |

Rule of thumb: $\frac{\mathbf{n}}{\mathbf{N}} \leq \mathbf{0.05} = \frac{\mathbf{10}}{\mathbf{5000}}$ = 0.002 or 2% (true)

Here N = 5000

k = 1000

n = 10

p = k/N = 1000/5000 = 1/5 = 0.2 (probability of blemished)

X = 3 (Let X denotes number of blemished tires)

h(3; 5000,10,1000) = b(3; 10,0.2) = $\sum_{x=0}^{x=3} b(x; 10{,}0.2)$ - $\sum_{x=0}^{x=2} b(x: 10{,}0.2)$ = 0.8791 - 0.6778 = 0.2013.

# Hypergeometric Distribution

**Example:** Suppose that a shipment contains 5 defective items and 10 non defective items. If 7 items are selected at random without replacement, what is the probability that at least 3 defective items will be obtained?

**Solution:**

| Defective | Non defective | Total |
|---|---|---|
| 5 | 10 | 15 |

**Here N = 15, n =7**

**k = 5 (defective items in the population)**

**Let X denotes number of defective items**

$P(X \geq 3) = 1 – P(X < 3) = 1- \{P(X = 0) + P(X = 1) + P(X = 2)\}$

$$h(x; N, n, k) = \frac{({}_kC_x)\,({}_{N-k}C_{n-x})}{{}_NC_n}, \quad \max\{0, n - (N-k)\} \leq x \leq \min\{n, k\}$$

$\therefore \max\{0, n - (N-k)\} = \max\{0, 7- (15 – 5)\} = 0$

$$P(0) = \frac{({}^{5}_{0}C)({}^{10}_{7}C)}{({}^{15}_{7}C)} = 0.0186$$

$$P(1) = \frac{({}^{5}_{1}C)({}^{10}_{6}C)}{({}^{15}_{7}C)} = 0.1631$$

$$P(2) = \frac{({}^{5}_{2}C)({}^{10}_{5}C)}{({}^{15}_{7}C)} = 0.3916$$

$P(X \geq 3) = 1 - P(X<3) = 1 - (0.0186 + 0.1631 + 0.3916)$

$= 0.4267$

**Example** A purchaser of electrical components buys them in **lots of size 10**. It is his policy to inspect **3 components** randomly from a lot and to **accept the lot** only if all **3 are nondefective**. If **30 percent** of the lots have **4 defective components** and **70 percent** have only **1**, what proportion of lots does the **purchaser reject**?

**30% of the lot:** Let x denotes the defective items from the **30% of the lot**. N = 10, n = 3, x = 0, **k = 4 (# of defectives items in 30% of the lot)**

$$h(x; N, n, k) = \frac{({}_{k}C_{x})\,(({}_{N-k}C_{n-x})}{{}_{N}C_{n}}$$

$$h(0; 10, 3, 4) = \frac{({}_{4}C_{0})\,({}_{6}C_{3})}{{}_{10}C_{3}}$$

**70% of the lot :** Let y denotes the defective items from the remaining **70% of lot**. N = 10, n = 3, y = 0, **k = 1 (# of defectives items in 70% of the lot)**

$$h(x; N, n, k) = \frac{({}_{k}C_{x})\,(({}_{N-k}C_{n-x})}{{}_{N}C_{n}}$$

$$h(0; 10, 3, 1) = \frac{({}_{1}C_{0})\,({}_{9}C_{3})}{{}_{10}C_{3}}$$

Let ***A*** denote the event that the purchaser **accepts a lot**.

$\therefore$ P(A) = **P(A |lot has 4 defectives)($\frac{3}{10}$)** + **P(A |lot has 1 defective)( $\frac{7}{10}$ )**

$$P(A) = \frac{({}_4C_0)\,({}_6C_3)}{{}_{10}C_3}\left(\frac{3}{10}\right) + \frac{({}_1C_0)\,({}_9C_3)}{{}_{10}C_3}\left(\frac{7}{10}\right)$$

$$= \frac{54}{10} \text{ or } 54\ \%$$

P($A^c$) = 1- P(A) = 1- 0.54 = 0.46 or 46%

Hence, 46 percent of the lots are rejected.

# Mean and Variance for discrete probability distribution [1]

- **Expected value or mathematical expectation or expectation** for discrete probability distribution is denoted by E(X) and is defined as

$$\mathbf{E(X)} = x_1P(X_1 = x_1) + x_2P(X_2 = x_2) + \ldots + x_nP(X_n = x_n)$$
$$= \sum_{i=1}^{n} x_i P(Xi = x_i)$$

Here E(X) is mean or expected value of X.

# Mean and Variance for discrete probability distribution [2]

**$E(X^2)$** = $x_1^2 P(X_1 = x_1) + x_2^2 P(X_2 = x_2) + \ldots + x_n^2 P(X_n = x_n)$

= $\sum_{i=1}^{n} x_i^2 P(Xi = x_i)$

If X is a discrete random variable taking the values $x_1$, $x_2, \ldots, x_n$, and having probability function P(x), then the **variance** is given by

**$Var(X) = E(X^2) - \{E(X)\}^2$**

**Example** Suppose that a shipment contains **5 defective items** and 10 non defective items. If 7 items are selected at random without replacement, what is the **probability distribution of defective items**? Implement it in Matlab.

# Multivariate Hypergeometric Distribution [1]

If ***N* items** can be partitioned into the ***k*** cells $A_1, A_2, \ldots, A_k$ with $a_1, a_2, \ldots, a_k$ elements, respectively, then the probability distribution of the random variables $X_1, X_2, \ldots, X_k$, representing the number of elements selected from $A_1, A_2, \ldots, A_k$ in a random sample of size $n$, is

$$f(x_1, x_2, \ldots, x_k; a_1, a_2, \ldots, a_k, N, n) = \{({}^{a_1}C_{x_1})\,({}^{a_2}C_{x_2}) \ldots ({}^{a_n}C_{x_n})\} / {}_{N}C_{n}$$

# Multivariate Hypergeometric Distribution [2]

**Example:** A group of **10** individuals is used for a biological case study. The group contains **3** people with blood type **O**, **4** with blood type **A**, and **3** with blood type **B**. What is the probability that a random sample of **5** will contain **1** person with blood type **O**, **2** people with blood type **A**, and **2** people with blood type **B**?

# Multivariate Hypergeometric Distribution [3]

**Solution :** $a_1$ (type **O**)= 3, $a_2$ = 4(type **A**), $a_3$ = 3 (type **B**)

$x_1 = 1$, $x_2 = 2$, $x_3 = 2$,

N = 10

n = 5

$f(x_1, x_2, \ldots, x_k; a_1, a_2, \ldots, a_k, N, n)$= $\{({}^{a1}C_{x1})\ ({}^{a2}C_{x2}) \ldots ({}^{an}C_{xn})\} / {}_{N}C_{n}$

$f(1, 2, 2; 3, 4, 3, 10, 5)$= $\{({}^{3}C_{1})\ ({}^{4}C_{2})\ ({}^{3}C_{2})\} / {}_{10}C_{5} = 3/14$
**= 0.2143**