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Bachelorarbeit

A Tool for the Estimation of Lattice Parameters

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Abstract

<Short summary of the thesis>

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1 Introduction

- rise of quantum computing (short history)

* conceptual

* reality

- problem: some hard classical problems no longer hard

* Shor's Algorithm (Peter Shor, 1994) => quantum computers can solve the factoring and the discrete logarithm problem in polynomial time

* application to encryption

* overview of current encryption methods that will become insecure

- one solution (among hash-based, code-based, isogeny-based, and multivariate): lattice crypto

* overview over history and capability of lattice crypto

* advantages: good (quasilinear) asymptotic key sized, good concrete runtimes and key sizes, worst-case secure instantiations, advanced cryptographic primitives previously infeasible

* including intro to LWE/SIS and applications to build crypto systems

. SIS: signature schemes, hash functions

. LWE: "cryptomania" applications (PKE, ...), signature schemes, lines:

- cryptographic applications

- establishing theoretical and asymptotic hardness [Reg05] [BLP+13; MP13] - concrete hardness of LWE: attacks, runtime estimates,

* briefly outline concept and benefits of hard-case to average-case reductions

- purpose of this thesis

* building schemes: need realistic hardness estimates of schemes for given parameter settings

* lack in the past: no unified/easy to use tool => thesis aims to solve this problem tool we call *Lattice Parameter Estimation* LWE instances are estimated by calling various estimation functions from the LWE Estimator [APS15], which we will refer to as *Estimator*.

- overview of chapters/how to read

2 Preliminaries

2.1 Notation

In the following, we denote vectors by bold lower-case letters like $\mathbf{v}$ and matrices by bold upper-case letters $\mathbf{M}$. Unless specified otherwise, $\|\cdot\|$ is the Euclidean norm. By $[n]$ we denote the set $\{1, \ldots, n\}$ for $n \in \mathbb{Z}^+$. With a slight abuse of notation, we interchangably use matrices and sets of column vectors, i.e. $\mathbf{M} = [\mathbf{v}_1 \cdots \mathbf{v}_n] = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

2.2 Math

2.2.1 Norms and Bounds

Let $\mathcal{R}_q$ be a ring as defined in [BDL+18] and $f \in \mathcal{R}_q$ with $f = \sum_i f_i X^i$. We define the following norms [BDL+18]:

$$\ell_1 : \||f|\|_1 = \sum_i |f_i| \tag{2.1}$$

$$\ell_2 : \||f|\|_2 = \left(\sum_i |f_i|^2\right)^{\frac{1}{2}} \tag{2.2}$$

$$\ell_\infty : \||f|\|_\infty = \max_i |f_i| \tag{2.3}$$

Then the following inequations hold [BDL+18]:

$$\|f\|_1 \leq \sqrt{n}\|f\|_2 \tag{2.4}$$

$$\|f\|_1 \leq n\|f\|_\infty \tag{2.5}$$

$$\|f\|_2 \leq \sqrt{n}\|f\|_\infty \quad (\text{since } \sqrt{n}\|f\|_2 \leq n\|f\|_\infty) \tag{2.6}$$

$$\|f\|_\infty \leq \|f\|_1 \tag{2.7}$$

Let O_K be the ring of integers of a number field $K = \mathbb{Q}(\theta)$, where θ is an algebraic number and σ denote the canonical embedding as defined in [DPSZ12]. Then, for $x, y \in O_K$ it holds the following inequations hold (we assume that C_m in [DPSZ12] is 1) [DPSZ12].

$$\|f\|_\infty \leq \|\sigma(f)\|_\infty \tag{2.8}$$

$$\|\sigma(f)\|_\infty \leq \|f\|_1 \tag{2.9}$$

From the above inequations, we obtain the following norm transformations to ℓ_p-norms:

- From Equation (2.4), it follows that $\|f\|_1 \leq \sqrt{n}\|f\|_2$ and from Equation (2.5), $\|f\|_1 \leq n\|f\|_\infty$.
- From Equation (2.6) and Equation (2.7), it follows that $\|f\|_2 \leq \sqrt{n}\|f\|_1$ and from Equation (2.6), $\|f\|_2 \leq \sqrt{n}\|f\|_\infty$.
- From Equation (2.7), it follows that $\|f\|_\infty \leq \|f\|_1$ and from Equation (2.4) and Equation (2.7), $\|f\|_\infty \leq \sqrt{n}\|f\|_2$.
- From Equation (2.9), it follows that $\|\sigma(f)\|_\infty \leq \|f\|_1$, from Equation (2.4) and Equation (2.9), $\|\sigma(f)\|_\infty \leq \sqrt{n}\|f\|_2$, and from Equation (2.5) and Equation (2.9), $\|\sigma(f)\|_\infty \leq n\|f\|_\infty$.

Likewise, we get the following transformations to the C_∞-norm:

- From Equation (2.5) and Equation (2.8), it follows that $\|f\|_1 \leq n\|\sigma(f)\|_\infty$.
- From Equation (2.6) and Equation (2.8), it follows that $\|f\|_2 \leq \sqrt{n}\|\sigma(f)\|_\infty$.
- From Equation (2.8), it follows that $\|f\|_\infty \leq \|\sigma(f)\|_\infty$.

Let f be defined as above and let $g \in \mathcal{R}_q$ where $g = \sum_i \overline{g}_i X^i$ where $g_i \in [-(q-1)/2, (q-1)/2]$ and $\overline{g}_i = g_i \mod q$ as in [BDL+18]. Then, we can define the following inequations for multiplication according to [BDL+18]:

- If $\|f\|_\infty \leq \beta$, $\|g\|_1 \leq \gamma$ then $\|f \cdot g\|_\infty \leq \beta \cdot \gamma$.
- If $\|f\|_2 \leq \beta$, $\|g\|_2 \leq \gamma$ then $\|f \cdot g\|_\infty \leq \beta \cdot \gamma$.

Let $x, y \in \mathcal{O}_K$. Again, we assume that $C_m = 1$. Then, the following inequation holds according to [DPSZ12]:

(2.10)
$$\|x \cdot y\|_\infty \leq C_m \cdot n^2 \cdot \|x\|_\infty \cdot \|y\|_\infty$$
(2.11)
$$\|\sigma(x \cdot y)\|_\infty \leq \|\sigma(x)\|_\infty \cdot \|\sigma(y)\|_\infty.$$

2.2.2 lattice

- background and history: example from lecture -> change

* Birhoff [Bir40]

* cryptoanalysis [LLL82]

* cryptosystems [Ajt96, HPS98] SIS introduced Ajtai [Ajt96]

* [MR04]

* LWE, assumption: worst-case lattice problems are hard [Reg05]

* fully homomorphic [Gen09]

* BGV scheme [BV11, BGV12]

* tools [LPR10, LPR13] ideal latties, RLWE

Other Notes: - PKE [AD97; Reg03; Reg05], CCA security [Pei09; PW08], identity-based encryption [ABB10; CHKP10; GPV08], fully homomorphic [Gen09] - , LWE introduced by [Reg05] "provably as hard as certain lattice problems in worst case, appear to require time exponential in main security parameter to solve NTRU [HPS98] - q-ary lattice: modulus $q \geq 2$

- math * lattice Λ

- discrete additive subgroup of $\mathbb{R}^m$

- Let $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^m$ be a set of linearly independent basis vectors and $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_n] \in \mathbb{R}^{m \times n}$ be the corresponding basis with column vectors $\mathbf{b}_i$

- n is the dimension of the Lattice

- $\Lambda(\mathbf{B})$ defined by all integer combinations of elements of $\mathbf{B}$:

$$\Lambda(\mathbf{B}) = \left\{ x \in \mathbb{R}^m \mid \exists \alpha_1, \ldots, \alpha_n \in \mathbb{Z} : \mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{b}_i \right\} \tag{2.12}$$

- show example plot

- full-ranked lattice: dimension is maximal, m

- basis $\mathbf{B}$ is not unique -> let $\mathbf{U} \in \mathbb{Z}^{n \times n}$ be a modular matrix (determinant is ± 1), then $\mathbf{B} \cdot \mathbf{U}$ is also a basis of the Λ ($\mathbf{U} \cdot \mathbb{Z}^n = \mathbb{Z}^n$) -> different basis for the same lattice Λ

- lattice coset: quotient group $\mathbb{R}^n / \Lambda$ of cosets

$$\mathbf{c} + \Lambda = \mathbf{c} + \mathbf{v} \mid v \in \Lambda$$

with $\mathbf{c} \in \mathbb{R}^n$

- fundamental domain: subset of $\mathbb{R}^m$ containing exactly one representative of every coset

- (shifted) fundamental parallelipiped : $\mathcal{P}(\mathbf{B}) = \mathbf{B} \cdot [-1/2, 1/2)^n = \left\{ \mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} \sum_{i=1}^{n} \gamma_i \mathbf{x}_i, \gamma_i \in [-1/2, 1/2) \right\}$ every coset has representative

- determinant of lattice $\Lambda(\mathbf{B})$: $\sqrt{\det(\mathbf{B}^{\top}\mathbf{B})}$. For a full-ranked lattice the determinant is

$$\det(\Lambda(\mathbf{B})) = |\det(\mathbf{B})| \tag{2.13}$$

is well-defined (independent from basis) => volume of fundamental domain can be generalized to not full-ranked => $\det(\Lambda(\mathbf{A})) = \sqrt{\det(\mathbf{A}^{\perp}\mathbf{A})}$

* minimum distance of $\lambda_1(\Lambda)$ of a lattice is the length of its shortest nonzero vector, i.e. $\lambda_1(\Lambda) \min_{v \in \Lambda \setminus \{0\}}$ * ith successive minimum $\lambda_i(\Lambda)$ - Let $r \in \mathbb{R}$ and $\mathbf{c} \in \mathbb{R}^m$, then we define $\mathcal{B}(\mathbf{c}, r)$ as the ball of radius r with center $\mathbf{c}$.

- smallest radius r such that the ball $\mathcal{B}(\mathbf{0}, r)$ centered at the origin of Λ contains i linearly independent lattice vectors.

- in general hard to calculate $\lambda_i(\Lambda(\mathbf{B}))$ for a given basis

* modular integer (or q-ary) lattices

- full-ranked lattice Λ such that $q\mathbb{Z}^m \subseteq \Lambda \subseteq \mathbb{Z}^m$ given $q \in \mathbb{N}$ => if $\mathbf{x} \in \mathbb{Z}^m$ in Λ then $\mathbf{x} \mod q$ also in Λ.

- can be specified in two ways by matrix $\mathbf{A} \in \mathbb{Z}_q^{m\times n}$:

(2.14) $\Lambda_q(\mathbf{A}) = \{x \in \mathbb{Z}^m \mid \exists y \in \mathbb{Z}^n : \mathbf{x} = \mathbf{Ay} \mod q\}$

or

(2.15) $\Lambda_q^{\perp}(\mathbf{A}) = \{x \in \mathbb{Z}^m \mid \mathbf{A}^{\intercal}\mathbf{x} = 0 \mod q\}$

- finding a short vector in $\Lambda_q(\mathbf{A})$ corresponds to LWE

- finding short vectors in $\Lambda_q^{\perp}(\mathbf{A})$ corresponds to SIS

- easy to find basis of $\Lambda_q(\mathbf{A})$ [AFG13]

- with high probability determinant of q-ary lattice is $\det(\Lambda_q(\mathbf{A})) = q^{m-n}$ if $\mathbf{A} \in \mathbb{Z}_q^{m\times n}$

* Gram-Schmidt basis

- set of column vectors $\mathbf{B} \in \mathbb{Z}_q^{m\times n}$, $\pi_{\mathrm{span}(\mathbf{B})}(\mathbf{t})$ for projection of vector $\mathbf{t}$ unto span of vectors of $\mathbf{B}$

- $\pi_{\mathrm{span}(\mathbf{B})}(\mathbf{t}) = \mathbf{B}(\mathbf{B}^{\perp}\mathbf{B})^{-1}\mathbf{B}^{\intercal} \cdot \mathbf{t}$

- Gram-Schmidt orthogonalization $\tilde{\mathbf{B}} = \left[\tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_n\right]$ of basis $\mathbf{B}$: $\tilde{\mathbf{b}}_i = \mathbf{b}_i - \pi_{\mathrm{span}(\mathbf{b}_1,\ldots,\mathbf{b}_{i-1})}(\mathbf{b}_i)$ for $i \in \{1,\ldots,n\}$

- Gram-schmidt coefficients $\mu_{i,j} = \frac{\langle\tilde{\mathbf{b}}_j,\mathbf{b}_i\rangle}{\langle\tilde{\mathbf{b}}_j,\tilde{\mathbf{b}}_j\rangle}$

Alternative: Let $\mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_n], \mathbf{b}_i \in \mathbb{Z}_q^m$ be a basis. Define $\tilde{\mathbf{b}}_i$ as follows: $\tilde{\mathbf{b}}_1 = \mathbf{b}_1$. For $i \in \{2,\ldots,n\}$ let $\tilde{\mathbf{b}}_i$ be the component of $\mathbf{b}_i$ that is orthogonal to the span of $\{\mathbf{b}_1,\ldots,\mathbf{b}_{i-1}\}$. Then, $\tilde{\mathbf{B}} = \left[\tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_n\right]$ is called the Gram-Schmidt orthogonalization of basis $\mathbf{B}$ where $\|\tilde{\mathbf{b}}_i\| \leq \|\mathbf{b}_i\|$.

* dual of a lattice is "the set of points whose inner products with the vectors in the lattice are integers"Λ: $\Lambda^{\perp} := \{\mathbf{y} \in \mathbb{R}^m \mid \forall \mathbf{v} \in \Lambda : \langle\mathbf{y},\mathbf{v}\rangle \in \mathbb{Z}\}$ scaled-by-q dual lattice: $\{\mathbf{y} \in \mathbb{Z}^m \mid \forall \mathbf{v} \in \Lambda : \langle\mathbf{y},\mathbf{v}\rangle = 0 \mod q\}$

* smoothing lemma

* Voronoi region The fundamental Voronoi region $\mathcal{V}$ is defined as

(2.16) $\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^n \mid \forall \mathbf{y} \in \Lambda : \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|\}$

* Linear Code [Van12] Let $\mathbb{F}_q^n$ be the n-dimensional vector space over the field $\mathbb{F}_q$. A q-ary linear code C or $[n,k]$-code is a k-dimensional linear subspace of $\mathbb{F}_q^n$ such that

- $\mathbf{0} \in C$,
- if $\mathbf{x},\mathbf{y} \in C$, then $\mathbf{x} + \mathbf{x} \in C$,
- and if $\mathbf{x} \in C$ and $\gamma \in \mathbb{F}_q$, then $\gamma\mathbf{x} \in C$.

There are q^k different codewords in C.

Let C be a q-ary linear $[n, k]$-code. The lattice over C is defined as

(2.17) $\Lambda(C) = \{\mathbf{x} \in \mathbb{R}^n \mid \exists \mathbf{y} \in C : \mathbf{x} = \mathbf{y} \mod q\}$.

Similarly, for a lattice $\Lambda(\mathbf{B})$ a lattice code C defined by $\Lambda(\mathbf{B})$ and a shaping region $\mathcal{V} \subset \mathbb{R}^n$ (e.g. the Voronoi region) is a subspace of $\mathbb{R}^n$ such that all codewords are lattice vectors in $\Lambda(\mathbf{B})$ within the region $\mathcal{V}$ [SFS08]:

(2.18) $C' = \{x \in \Lambda(\mathbf{B}) \mid x \in \mathcal{V}\}$.

We define $\text{dist}(\mathbf{t}, \Lambda(\mathbf{B}))$ where $\Lambda(\mathbf{B}) \subset \mathbb{R}^m$ as the distance of a vector $\mathbf{t} \in \mathbb{R}^m$ to the closest lattice vector $\mathbf{v} \in \Lambda(\mathbf{B})$, i.e.

(2.19) $\text{dist}(\mathbf{t}, \Lambda(\mathbf{B})) = \min_{\mathbf{v} \in \Lambda(\mathbf{B})} \|\mathbf{t} - \mathbf{v}\|$.

- Lattice problems

* Minkowski theorem: Let Λ be a lattice of dimension n, then $\lambda_1 \leq \sqrt{n} \cdot (\det \Lambda)^{\frac{1}{n}}$

* Lattice reduction: find short basis compared to $\lambda_1(\Lambda)$...

* SVP: given a basis $\mathbf{B}$ of lattice Λ find shortest nonzero lattice vector => $v \in \Lambda$ s.t. $\|v\| = \lambda_1(\Lambda)$

Definition 2.2.1 (γ-approximate Shortest Vector Problem (SVP_γ))
Given a basis $\mathbf{B}$ *of lattice* Λ*, find a short lattice vector* $v \in \Lambda$ *such that* $0 < \|v\| \leq \gamma\lambda_1(\Lambda)$

Definition 2.2.2 (κ-approximate Hermite Shortest Vector Problem (HSVP_κ))
Given a basis $\mathbf{B}$ *of a lattice* $\Lambda(\mathbf{B}) \in \mathbb{R}^m$*, find a nonzero lattice vector* $\mathbf{v} \in \Lambda$ *such that* $\|\mathbf{v}\| \leq \kappa \cdot det(\Lambda)^{\frac{1}{n}}$.

* GapSVP_γ (decision version of SVP): "given basis $\mathbf{B}$ of n-dimensional lattice Λ with either $\lambda_1\Lambda \leq 1$ or $\lambda_1\Lambda \geq \gamma(n)$, decide which is the case"NP hard for any constant γ fastest algorithm for $1 \leq \gamma \leq \text{poly}(n)$ has runtime complexity of $2^{O(n)}$

* γ-unique Shortest Vector Problem (uSVP_γ) [LM09]: given lattice Λ such that $\lambda_2(\Lambda) > \gamma\lambda_1(\Lambda)$, find shortest nonzero vector in $\mathbf{v} \in \Lambda$ with $\|\mathbf{v}\| = \lambda_1(\Lambda)$

* CVP_γ: given basis $\mathbf{B}$ of n-dimensional lattice Λ and target $\mathbf{t} \in \mathbb{R}^n$ find point in lattice that is close to $\mathbf{t}$ => find $\mathbf{v} \in \mathbb{R}^n$ with $\|\mathbf{t} - \mathbf{v}\| < \gamma \min_{\mathbf{v}' \in \Lambda} \|\mathbf{v}' - \mathbf{v}\|$

* SIVP (shortest independent vector problem): given basis $\mathbf{B}$ of n-dimensional lattice Λ, find n linearly independent lattice vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \Lambda(\mathbf{B})$ such that $\max_i \|\mathbf{v}_i\|$ for $i \in \{1, \ldots, n\}$ is minimal

* γ-Bounded Distance Decoding (BDD_γ): Given a lattice Λ and a target vector $\mathbf{t}$ such that $\text{dist}(\mathbf{t}, \Lambda) < \gamma\lambda_1(\Lambda)$, find the closest lattice vector $\mathbf{v} \in \Lambda$, i.e. find $\mathbf{v} = \min_{\mathbf{v}' \in \Lambda} \|\mathbf{v}' - \mathbf{t}\|$ minimal [LM09]

* ideal lattice (do I need that?)

* ...?

* eher die Sachen für LWE/SIS als die Sachen für Algorithmen (analog Vorlesung), evtl.

Intuition für die anderen Sachen... Solving SVP with approximation factors: - 1 => NP-hard [Ajt98] - $\tilde{O}(n)$ => OWF [Ajt96; MR04] - $2^{n \log\log n / \log n}$ and $2^{n/2}$ in Poly-time [LLL82] => best known 2^k-approx in $2^{\tilde{O}(n/k)}$ time (even quantum!)

2.2.3 distributions

- Gaussian, def, component-wise, trafo to bound

* definition: discrete Gaussian distribution over q-ary lattice Λ with Gaussian width parameter $s > 0$ and center $\mathbf{c}$, denoted by $D_{\Lambda,s,\mathbf{c}}$: probability of sampling a vector $\mathbf{x} \in \Lambda$ is proportional to $e^{-\pi \|\mathbf{x}-\mathbf{c}\|^2/s^2}$ In order to avoid confusion, throughout this work and in the *Lattice Parameter Estimation* we use σ to denote the standard deviation, where $\sigma = \frac{s}{\sqrt{2\pi}}$, and define $\alpha := \frac{s}{q} = \frac{\sqrt{2\pi}\sigma}{q}$.

* better definition in GPV08 => different definition needed for LWE??? * how to do this? => variant of Babai's "nearest-plane" algorithm, see [GPV08]

* component-wise

For some applications, we receive a Gaussian distribution as input, but require a bound in some norm in order to estimate the hardness of SIS. Hence, we need to transform the Gaussian width parameter into a bound β given some security parameter sec. Note that a n-dimensional Gaussian can be sampled by sampling n independent 1-dimensional Gaussians.

For a Gaussian distribution, the following holds:

$$\Pr\left[|X| \geq \beta\right] \leq 2e^{-\pi\beta^2/s^2} \tag{2.20}$$

We demand $2e^{-\pi\beta^2/s^2} \approx 2^{-sec}$, hence

$$\begin{aligned} 2e^{-\pi\beta^2/s^2} &\approx 2^{-sec} \\ -\pi\frac{\beta^2}{s^2} &\approx (-sec - 1)\ln(2) \\ \beta &\approx s\sqrt{\frac{(sec+1)\ln(2)}{\pi}} \end{aligned}$$

* smoothing factor here?

* Uniform (stuff I use in tool)

2.3 LWE and SIS

Applications: SIS can be used for one-way functions and collision-resistant hasing. LWE can be used to build pseudo-random number generators, public-key encryption schemes and oblivious transfer and secure MPC. Lattice Trapdoors (trapdoor functions, digital signatures)? Punctured Trapdoors (identity-based encryption, attribute-based encryption, predicate encryption)?

2.3.1 LWE

Following based on [Reg10]:

Introduced by Regev in [Reg09] Origin: work of Ajtai and Dwork [AD97], first public-key cryptosystem based on worst-case lattice problems, simlifications/improvements [GGH97; Reg03] imply hardness result for LWE. Early work: hardness based on unique-SVP, Peikert [Pei09] and Lyubashevsky and Micciancio [LM09] show that unique-SVP is essentially equivalent to GapSVP.

- ´cryptomania´ applications: public-key encryption schemes under chosen-plaintext attacks [KTX07; PVW08; Reg05], and chosen-ciphertext attacks [Pei09; PW08], oblivious transfer protocoles [PVW08], identity-based encryption (IBE) schemes [ABB10; CHKP10; GPV08], leakage-resilient encryption [ACPS09; AGV09; DGK+10; GKPV10], and more

- most important: fully homomorphic encryption schemes [Bra12; BV11; Gen09; GSW13]

Intuition: - "recover $\mathbf{s} \in \mathbb{Z}_q^n$ given sequence of ´approximate´ random linear equations on $\mathbf{s}$ public matrix $\mathcal{A} \in \mathbb{Z}^{n \times m}$, secret vector $\int \in \mathbb{Z}^n$, given $\ddagger = \mathcal{A}^\top \int$ we can find $\mathbf{s}$ by linear algebra when we add a small error vector $\mathbf{e} \in \mathbb{Z}^m$, solving $\ddagger' = \mathcal{A}^\top \int + \mathbf{e}$ for $\mathbf{s}$ or distinguishing $\mathbf{z}'$ from uniform becomes hard

Formal Definition:

Definition 2.3.1 (LWE Distribution [Reg10])
For $n \geq 1$, modulus $q \geq 2$, error distribution χ on $\mathbb{Z}_q$, and a fixed secret vector $\mathbf{s}$, let $\mathcal{A}_{\mathbf{s},\chi}$ be the probability distribution over $\mathbb{Z}_q^n \times \mathbb{Z}_q$ by choosing a vector $\mathbf{a}_i \in \mathbb{Z}_q^n$ uniformly at random, $e_i \in \mathbb{Z}_q$ according to χ and returning pairs of $(\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \mod q) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$.

Additions are performed in $\mathbb{Z}_q$. We say that an algorithm solves LWE with modulus q and error distribution χ if, for any $\mathbf{s} \in \mathbb{Z}_q^n$, given an arbitrary number of independent samples from $\mathcal{A}_{\mathbf{s},\chi}$ it outputs $\mathbf{s}$ (with high probability). For $q = 2$ corresponds to *learning parity with noise* (LPN) problem.

Definition 2.3.2 (Search-LWE$_{n,q,m,\chi}$)
Search-LWE$_{n,q,m,\chi}$ asks for the recovery of the secret vector $\mathbf{s}$ given m independent samples $(\mathbf{a}_i, z_i) \leftarrow \mathcal{A}_{\mathbf{s},\chi}$

Definition 2.3.3 (Decision-LWE$_{n,q,m,\chi}$)
Given m samples, Search-LWE$_{n,q,m,\chi}$ asks to distinguish whether the samples were drawn from $\mathcal{A}_{\mathbf{s},\chi}$ or from a uniform distribution on $\mathbb{Z}_q^n \times \mathbb{Z}_q$.

LWE as a Decoding Problem

We request m samples $(\mathbf{a}_1, z_1), \dots, (\mathbf{a}_m, z_m)$ where $z_i = \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \in \mathbb{Z}_q$. Let $A = [\mathbf{a}_1 \cdots \mathbf{a}_m]$, $\mathbf{z} = [z_1, \dots, z_m]^\intercal$ and $e = [e_1, \dots, e_n]^\intercal$. Hence, we can reformulate LWE as a decoding problem as in [GJS15]:

(2.21) $\mathbf{z} = \mathbf{A}^\intercal \mathbf{s} + \mathbf{e}$

with generator matrix $\mathbf{A}$ for a linear code over $\mathbb{Z}_q$ and $\mathbf{z}$ as the received word. Finding the secret vector $\mathbf{z}$ is equivalent to finding the codeword $\mathbf{y} = \mathbf{A}^\intercal \mathbf{s}$ with minimum distance $\|\mathbf{y} - \mathbf{z}\|$.

An $\mathrm{LWE}_{n,q,m,\chi}$ instance with a secret vector $\mathbf{s}$ chosen according to a uniform distribution can be transformed into an $\mathrm{LWE}_{n,q,m-n,\chi}$ instance with a secret vector $\hat{\mathbf{s}}$ chosen according to the error distribution χ at a loss of n samples as follows: Let $\mathbf{A}_0 = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ where $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the first n columns of $\mathbf{A}$. We introduce new variables $\hat{\mathbf{s}} = \mathbf{A}_0^\intercal \mathbf{s} - [z_1, \dots, z_n]^\intercal = [e_0, \dots, e_n]^\intercal$ and $\hat{\mathbf{A}} = \mathbf{A}_0^{-1} \mathbf{A} = [\mathbf{I}\ \hat{\mathbf{a}}_{n+1} \cdots \hat{\mathbf{a}}_m]$ and compute $\hat{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{A}}^\intercal\, [z_1, \dots, z_n]^\intercal = [\mathbf{0}, \hat{z}_{n+1} \cdots \hat{z}_m]^\intercal$.

LWE as a BDD Problem

Solving LWE also corresponds to solving the *Bounded Distance Decoding problem* (BDD) in the lattice $\Lambda(\mathbf{A}^\intercal) = \{\mathbf{x} \in \mathbb{Z}_q^m \mid \exists \mathbf{s} \in \mathbb{Z}_q^n : \mathbf{x} = \mathbf{A}^\intercal \mathbf{s} \mod q\}$, where the m columns of $\mathbf{A}$ correspond to the vectors $\mathbf{a}_i \in \mathbb{Z}_q^n$ of m independent LWE samples $(\mathbf{a}_i, z_i) \leftarrow \mathcal{A}_{\mathbf{s},\chi}$ and the components z_i correspond to a perturbed lattice point in $\Lambda(\mathbf{A}^\intercal)$.

Best algorithm to solve LWE: Blum, Kalai, and Wasserman [BKW03] with $2^{O(n)}$ samples and time.

Hardness: best algorithm exponential, extension of LPN (LPN believed to be hard), hard assuming worst-case hardness of GapSVP and SIVP [Pei09; Reg05]. More details? Different cases for q exponential/polynomial, approximation factors... Hardness based on worst-case lattice problems => strong security guarantees, such as conjectured security against quantum computers...

Search to decision reduction => distiguishing is LWE samples from uniform samples sufficient, worst-case to average-case reduction => sufficient to solve distinguishing for uniform secret

2.3.2 Short Integer Solution (SIS)

The dual problem to LWE is the *Short Integer Solution problem* (SIS).

- principle: given a set of set of uniformly random vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{Z}_q^n$ find a subset of them or combination with small coefficients that sums to zero (modulo q).

- introduced in [MR04], origins in [Ajt96], used for ´minicrypt´ primitives: one-way functions [Ajt96], collision resistant hash functions [GGH96], digital signature schemes [CHKP10; GPV08], and identification schemes [KTX07; Lyu08; MV03]

Definition 2.3.4 (SIS Problem (Adapted from [[LS15], Definition 3.1)
)] The problem $SIS_{n,q,m,\beta}$ is defined as follows: Given a uniformly random matrix $\mathbf{A}^{n \times m}$, find a vector $\mathbf{s} \in \mathbb{Z}_q^m$ such that $\mathbf{A} \cdot \mathbf{s} = 0$ mod q and $0 < \|\mathbf{s}\| \leq \beta$.

Finding such a vector corresponds to finding a short lattice vector in costets of the lattice $\Lambda^{\perp}(\mathbf{A}) = \{y \mid \mathbf{A} \cdot y \mod q\}$

Hardness: for any poly-bounded m, β and for “large enough” prime q: $\mathrm{SIS}_{n,q,m,\beta}$ is as hard as worst-case approx-SIVP (and GapSVP) to within $\beta \cdot \tilde{O}(\sqrt{n})$ factor

2.3.3 Ring and Module Variants

- problem key sizes in LWE/SIS in $O(n^2)$ (matrix $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$), where $m \in \Omega(n)$)

- idea: introduce some sort of a structure in samples: n power of two, $\mathbf{a}$ vectors in groups of size n, for each group $\mathbf{a}_1 = [a_1, \ldots, a_n]^\intercal$, a_i are uniformly random in $\mathbb{Z}_q$, and $\mathbf{a}_i = [a_i, \ldots, a_n, -a_1, \ldots, -a_{i-1}]^\intercal$. Hence, n vectors only need $O(n)$ memory, also speedups in operations by using FFT

- formally: vectors are elements of the ring $\mathbb{Z}_q\,[x]\,/\langle x^n + 1\rangle$ which we call $\mathcal{R}_q$ instead of the group $\mathbb{Z}_q^n$, n power of two ensures that $x^n + 1$ is irreducible over the rationals

- add more?

Definition 2.3.5 (Ring-SIS Problem [[LS15], Definition 3.3)
)] The problem $RSIS_{n,q,m,\beta}$ is defined as follows: Given $a_1, \ldots, a_n \in \mathcal{R}_q$ chosen independently from the uniform distribution, find $s_1, \ldots, s_n \in \mathcal{R}$ such that $\sum_{i=1}^{m} a_i \cdot s_i = 0 \mod q$ and $0 < \|\mathbf{s}\| \le \beta$, where $\mathbf{s} = [s_1, \ldots, s_m]^\intercal \in \mathcal{R}^m$.

Definition 2.3.6 (Module-SIS Problem [[LS15], Definition 3.3)
)] The problem $MSIS_{n,d,q,m,\beta}$ is defined as follows: Given $a_1, \ldots, a_n \in \mathcal{R}_q^d$ chosen independently from the uniform distribution, find $s_1, \ldots, s_n \in \mathcal{R}$ such that $\sum_{i=1}^{m} a_i \cdot s_i = 0 \mod q$ and $0 < \|\mathbf{s}\| \le \beta$, where $\mathbf{s} = [s_1, \ldots, s_m]^\intercal \in \mathcal{R}^m$.

While there exist special cases where the Ring structure of problem instances can be exploited in an attack on LWE or SIS, in general, the hardness of Ring and Module variants is estimated by interpreting the coefficients of elements of $\mathcal{R}_q$ as vectors in $\mathbb{Z}_q^n$ [ACD+18]. We thus reduce Ring and Module instances as follows:

- $\mathrm{RLWE}_{n,q,m,\chi} \longrightarrow \mathrm{LWE}_{n,q,m\cdot n,\chi}$
- $\mathrm{MLWE}_{n,d,q,m,\chi} \longrightarrow \mathrm{LWE}_{n\cdot d,q,m\cdot n,\chi}$
- $\mathrm{RSIS}_{n,q,m,\beta} \longrightarrow \mathrm{SIS}_{n,q,m\cdot n,\beta}$
- $\mathrm{MSIS}_{n,d,q,m,\beta} \longrightarrow \mathrm{SIS}_{n\cdot d,q,m\cdot n,\beta}$

Note that in the Ring and Module variants n denotes the degree of the polnomial of the underlying Ring, while in the standard variant, n denotes the dimension of the secret.

3 Algorithms and Estimates

3.1 Lattice Basis Reduction

Problem: usually ugly basis (long vectors...), we want a better basis with shorter and more orthogonal basis vectors... - improve lattice basis quality => measure by hermite factor (compare shortest vector in basis to lattice volume) or approximation factor (compare shortest vector in basis to shortest lattice vector) - algorithm finding vector with approximation factor γ can be used to solve uSVP with gap $\lambda_2(\Lambda)/\lambda_1(\Lambda) > \gamma$ - best known theoretical bound by Slide reduction [GN08a], BKZ better in practice

- measure quality of basis: Hermite factor

* basis $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$, m-dimensional lattice $\Lambda(\mathbf{B})$ has Hermite factor δ if

(3.1) $||\mathbf{b}_1|| \approx \delta^m \det(\Lambda)^{1/m}$

* use Geometric Series Assumption (GSA) to obtain estimates for $\mathbf{b}_i$:

(3.2) $||\tilde{\mathbf{b}}_i|| \approx \alpha^{i-1}||\mathbf{b}_1||$

for $0 < \alpha < 1$ Equation (3.1) into Equation (3.2) -> $||\tilde{\mathbf{b}}_i|| \approx \alpha^{i-1}\delta^m \det(\Lambda)^{1/m}$ with $\prod_{i-1}^{m} ||\tilde{\mathbf{b}}_i|| = \det(\Lambda)$ we get

$$
\begin{aligned}
& \prod_{i-1}^{m} ||\tilde{\mathbf{b}}_i|| \approx \prod_{i-1}^{m} \alpha^{i-1}\delta^m \det(\Lambda)^{1/m} \\
\Leftrightarrow \quad & \det(\Lambda) \approx \delta^{2m} \det(\Lambda) \prod_{i-1}^{m} \alpha^{i-1} \\
\Leftrightarrow \quad & \delta^{-m^2} \approx \alpha^{\frac{m(m-1)}{2}} \\
\Leftrightarrow \quad & \delta^{-2} \approx \alpha^{(m-1)/m}
\end{aligned}
$$

Hence, $alpha \approx \delta^{-2}$ and

(3.3) $||\tilde{\mathbf{b}}_i|| \approx \delta^{-2(i-1)+m} \det(\Lambda)^{1/m}$

* good basis -> first Gram-Schmidt vectors become shorter (latter longer)

* $\delta = 1.01$ feasible, $\delta = 1.007$ seems infeasible for now

* gap between provable and experimental cost estimate to reach some hermite δ => provable results only give upper bounds, for practical security we need lower bound => combine theoretical results with experimental results

* well-established estimate [LP11]

In the following, we will focus on two related methods for lattice reduction.

3.1.1 LLL

The LLL algorithm was proposed by Lenstra, Lenstra and Lovász [LLL82] and can be considered as a generalization of the two dimensional Lagrange reduction. The lagrange reduction reduces a basis of two basis vectors such that output basis satisfies $\|\mathbf{b}_1\| \leq \|\mathbf{b}_2\|$ and $\frac{|\langle \mathbf{b}_1, \mathbf{b}_2 \rangle|}{\|\mathbf{b}_1\|} = |\mu_{2,1}| \leq \frac{1}{2}$). Intuitively, a multiple of the shorter vector $\mathbf{b}_1$ is subtracted from the longer vector $\mathbf{b}_2$ such that the resulting vector $\mathbf{b}_2'$ is as orthogonal to $\mathbf{b}_0$ as possible, i.e. $\mathbf{b}_1' = \mathbf{b}_1 - \lfloor \mu_{1,0} \rceil \mathbf{b}_0$. We set $\mathbf{b}_2 = \mathbf{b}_2'$ and repeat until nothing changes.

A δ-LLL reduced basis ensures two criterias [LLL82]:

1. Size reduced: $|\mu_{i,j}| \leq \frac{1}{2}$ for $1 \leq i \leq n$ and $j < i$
2. Lovász condition: $\delta \|\tilde{\mathbf{b}}_i\|^2 > \|\mu_{i+1,i} \tilde{\mathbf{b}}_i + \tilde{\mathbf{b}}_{i+1}\|^2$ for $1 \leq i < n$

Recall the definition of the Gram-Schmidt coefficients $\mu_{i,j} = \frac{\langle \tilde{\mathbf{b}}_j, \mathbf{b}_i \rangle}{\langle \tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j \rangle}$. The LLL algorithm shown in Algorithm 1 follows the notation in [Reg04]. We start by computing the Gram-Schmidt orthogonalization of the input basis (Line 2) and continue with a reduction step in which we update every basis vector $\mathbf{b}_i$ by pairwisely comparing and subtracting lower indexed basis vector just as in the Lagrange reduction (Line 5) to ensure Criteria 1. Finally, vectors violating the Lovász condition are swapped (Line 6ff) and the process is repeated until nothing changes. The LLL algorithm can be used to find short vectors of at most $2^{n/2}\lambda_1(\Lambda)$ in polynomial time. Several floating-point variants have been suggested that can significantly speed up the runtime of LLL. For example, L^2 runs in $OO(n^2 \log^2 B)$, where B is a bound on the norm of the input basis vectors [NS05].

Algorithm 1: The δ-LLL Algorithm

1 **function** δ-LLL($\mathbf{B} \in \mathbb{Z}^{m \times n}$)
2   Compute $\tilde{\mathbf{B}}$
3   **for** $i = 2, \ldots, n$ **do**
4   **for** $j = i - 1, \ldots, 1$ **do**
5             $\mathbf{b}_i = \mathbf{b}_i - \lfloor \mu_{i,j} \rceil \mathbf{b}_j$
6   **if** $\exists i$ such that $\delta \|\tilde{\mathbf{b}}_i\|^2 > \|\mu_{i+1,i} \tilde{\mathbf{b}}_i + \tilde{\mathbf{b}}_{i+1}\|^2$ **then**
7   tmp = $\mathbf{b}_i$
8         $\mathbf{b}_i = \mathbf{b}_{i+1}$
9         $\mathbf{b}_{i+1} = \mathbf{b}_i$
10   Return δ-LLL($\mathbf{B}$)
11   **else**
12   Return $\mathbf{B}$

3.1.2 BKZ

The Block Korkin-Zolotarev (BKZ) algorithm was proposed by Schnorr in 1987 and adapted by Schnorr and Euchner in [SE91] and represents a family of lattice reduction algorithm. Essentially, BKZ iteratively divides the input basis into blocks of a lower dimension k and calling an SVP oracle on each block. The output of the oracle is then used to obtain a basis of improved quality.

Algorithm 2 presents the main concept of BKZ and follows the description in [CN11] with some adjustments. Initially, we run an LLL reduction on the input basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and update the basis. In each jth iteration, we consider a block of k basis vectors $\mathbf{b}_j, \ldots, \mathbf{b}_{j+k-1}$. The vectors of the current block are projected onto the orthogonal complement of the span of vectors from previous iterations span $(\{\mathbf{b}_i | i \in [j-1]\})$ (**????**, we skip this step if the span is empty). Note that the orthogonal complement $A^\perp$ of a subspace A is defined as the set of all vectors that are orthogonal to every vector in A. We then run an SVP oracle on the projected block to obtain a shortest vector $\mathbf{b}'_{\text{new}}$ in the projected lattice (Line 12) and reconstruct a lattice vector $\mathbf{b}_{\text{new}}$ of which $\mathbf{b}'_{\text{new}}$ is a projection Line 13. Note that in practice, the SVP oracle should include this step. If $\mathbf{b}_{\text{new}}$ is a new vector we insert it in our list of basis vectors before $\mathbf{b}_j$. Otherwise as nothing changed, we increment a counter z. Finally, we run LLL on all basis vectors up to index $j + i$ (including the possibly newly added vector). If no new lattice vectors can be found in n iterations, the reduction terminates. After n iterations, j is reset to start over at the first block. The ouput of the algorithm is a BKZ$_k$-reduced basis. For $k = 2$ we obtain an LLL-reduced basis in polynomial time and for $k = n$ an optimally HKZ-reduced basis in at least exponential time.

Algorithm 2: The BKZ Algorithm

1 **function** BKZ($\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}, k \in [n]\backslash\{1\}$)
2   z = 1; j = 0
3     $\mathbf{B}$ = LLL($\mathbf{B}$)
4   **while** $z < n - 1$ **do**
5         $j = (j \mod (n-1)) + 1; l = \min(j + k - 1, n); h = \min(l + 1, n)$
6         $A = \text{span}\left(\{\mathbf{b}_i | i \in [j-1]\}\right)$
7   **for** $i \in \{j, ..., l\}$ **do**
8   **if** $A \neq \emptyset$ **then**
9                 $\mathbf{b}'_i = \pi_{A^\perp}(\mathbf{b_i})$
10   **else**
11                $\mathbf{b}'_i = \mathbf{b}_i$
12        $\mathbf{b}'_{\text{new}} = \text{SVP-Oracle}(\mathbf{b}'_j, \ldots, \mathbf{b}'_l)$
13   Reconstruct $\mathbf{b}_{\text{new}} = \sum_{i=j}^{l} \alpha_i \mathbf{b}_i$ with $\alpha_i \in \mathbb{Z}$ such that $\mathbf{b}'_{\text{new}} = \pi_{\left(\text{span}\left(\mathbf{b}_j, \ldots, \mathbf{b}_l\right)\right)^\perp}(\mathbf{b}_{\text{new}})$
14   **if** $\mathbf{b}'_{new} \neq \tilde{\mathbf{b}}_j$ **then**
15            $z = 0; \left\{\mathbf{b}_j, \ldots, \mathbf{b}_h\right\} = \text{LLL}(\left\{\mathbf{b}_j, \ldots, \mathbf{b}_{j-1}, \mathbf{b}_{\text{new}}, \mathbf{b}_j, \ldots, \mathbf{b}_h\right\})$
16   **else**
17            $z = z + 1; \left\{\mathbf{b}_j, \ldots, \mathbf{b}_h\right\} = \text{LLL}(\left\{\mathbf{b}_j, \ldots, \mathbf{b}_h\right\})$

Several improvements have been suggested (see **??**). The total number of rounds until termination is unknown and can be quite large. Hanrot *et al.* [HPS11] show an early termination of BKZ still yields a very good output basis quality and propose $\frac{n^2}{k^2} \log n$ rounds as a bound. Th Local preprocessing increases the quality of the current block basis by recursively calling BKZ with smaller block size. A variant known as progressive BKZ applies the recursion globally [AWHT16]. If enumeration is used as an SVP oracle (see Section 3.1.3), extreme pruning can be applied to significantly reduce the search space. Gamma *et al.* show that applying such a bounding function on the search tree reduces the running time by a much larger factor than the success probability. Repeating the search yields the desired speedup [GNR10]. In addition, [CN11] optimizes the enumeration radius by using experimental results. BKZ 2.0 incorporates a number of these techniques [CN11].

It is difficult to find hard runtime bounds for BKZ. The upper bound on the number of rounds is superexponential in the dimension n for a fixed block size [GN08b; HPS11] before BKZ terminates by itself. In addition, calls to the SVP oracle is called in all dimensions $k' \leq k$ must be taken into account. Albrecht et al. [APS15] ignore these intricacies and estimate the cost of BKZ in clock cycles as $\rho \cdot n \cdot t_k$ where ρ is the number of rounds needed and t_k is the cost (in block cycles) of calling the SVP oracle on a block of dimension k. The value ρ is set to 8 in the *Estimator* based on experiments in [Che13] that indicate that the most significant progress happens in the first $7 - 9$ rounds.

- CN11, APS15, ADPS16 sieving [ADPS16; APS15; BDGL16; Laa16] enumeration [APS15; CN11; FP85; HPS+17; Kan87; MW15] cost models (tabular overview) [ACC+19] - number of SVP calls [ADPS16; Alb17] - upper bound on rounds is exponential [GN08b; HPS11] - lattice rule of thumbs: achievable root hermite factor $\delta_0 = k^{\frac{1}{2k}}$

3.1.3 Cost Models for Lattice Reduction

In this section, we will look at various high level ideas to realize an SVP oracle, i.e. to solve SVP, and present up-to-date cost models from the literature.

SVP is NP-complete

- enumeration => [ABF+20], [ABLR21] enumerating all lattice vectors within a bounded region low memory cost (linear in n), perform good in moderately low dimensions provide asymtotically fastest algorithms in PSPACE both in theory and practice worst-case $2^{O(n \log n)}$

- sieving => [ADH+19], [AGPS20] [ABF+20]: achieve root Hermite factor $k^{1/(2k)}$ in $k^{k/8+o(k)}$ time and polynomial memory single exponential time algorithms 2^{cn}, $c \in O(1)$ use randomization and exponential spaces

We start by sampling a list of 2^{cn} lattice points $\mathbf{v}_i, i \in [2^{cn}]$ of length smaller than some bound r. Then, we choose some center points $\mathbf{v}'_i, i \in [l]$ for some $l \ll 2^{cn}$ of the list such that all points $\mathbf{v}_i$ of the initial list are covered by spheres of a radius $r' < r$ centered at these center points $\mathbf{v}'_i$. Finally, we compute short vectors by subtracting the centers.

primarily studied for SVP in euclidean norm two main types: Classic Sieve [AKS01]: create a long list of lattice vectors, then find shorter lattice vectors and discard longer List Sieve [MV10]: start from empty list, find shorter vectors and append them to list

best provable: $2^{n+o(n)}$ (sieving by averages) heuristic state of the art: $(3/2)^{n/2+o(n)} \approx 2^{0.29n}$

combinarial: create list of short random (possibly non-lattice) vectors, try to combine vectors in list to produce short lattice vectors provable versions not very useful (exponential space, random perturbations) but basis of heuristic variants used in practice

- Micciancio-Voulgaris Algorithm uses voronoi cell [MV10] in $4^{n+o(n)}$ time and $2^{n+o(n)}$ space

- discrete Gaussian sampling [**ADRS15**]

3.2 Algorithms for Solving LWE

3.2.1 Three Approaches for Solving LWE

Distinguishing attacks (MR09, RS10): distinguish (with noticeable advantage) LWE instance from uniformly random => break semantic security of LWE-based cryptosystem with same advantage (typically), find short nonzero integral vector $\mathbf{v}$ s.t. $\mathbf{A}^\intercal \mathbf{v} = 0 \mod q$ => short vector in (scaled) dual of LWE lattice $\Lambda(\mathbf{A})$ then test whether $\langle \mathbf{v}, z\rangle$ is close to zero mod q. If uniform test accepts with prob 1/2, if LWE with parameter s, $\langle \mathbf{v}, \mathbf{z}\rangle = \langle \mathbf{v}, \mathbf{e}\rangle \mod q$, Gaussian mod q with parameter $\|\mathbf{v}\| \cdot s$. If that's not much larger than q, advantage for distinguishing very close to $\exp(-\pi(\|\mathbf{v}\|s/q)^2)$. high confidence needs $\|\mathbf{v}\| \leq q/(2s)$ advantage an computational effort need to be balanced (often inverse distinguising advantage is in total cost of attack)

SIS

rewrite LWE as the problem of finding short vector in dual lattice => SIS

BDD

lattice reduction algorithms solve SIS and BDD

Direct

- algebraic approach Arora and Ge with subexponential complexity when $\sigma \leq \sqrt{n}$, else fully exponential, mainly of asymptotic interest (higher complexity than others)

- combinatorial algorithms: BKW as basis [BKW03], resembles generalized birthday approach by Wagner, originally for solving LPN, can be analyzed => explicit complexity for different LWE instances, theoretical analysis and actual performance close, very memory expensive (often same order as time complexity)

3.2.2 BKW [BKW03]

- BKW by Blum, Kalai and Wasserman [BKW03] to solve Learning Parity with Noise problem (LPN), subproblem of LWE - applied to LWE in [ACF+15] (original paper appeared in 2013) - time and space complexity $2^{O(n)}$ for LWE with prime modulus $q \in \text{poly}(n)$ - Various improvements have been suggested since. For example, [AFFP14] and [KF15] use modulus switching for binary-LWE and other small secret variants [AFFP14] and [DTV15] applies multidimensional Furier transformations.

modified BKW step -> coded-BKW step to cancel out more positions in the **a** vectors than traditional BKW step

map part of **a** vector into nearest codeword in lattice code (linear code over $\mathcal{Z}_q$, Euclidean distance)

introduces some noise, can be kept small by appropriate parameters

pair of **a** vectors map to same codeword => add together to create new sample with part of **a** vector cancelled

samples are input to next step in BKW procedure

additional steps using discrete FFT

slightly modified for BINARY-LWE (secret vector uniformly chosen from $\{0, 1\}^n$) greatly increases performance

In the following, we present an outline of the original BKW algorithm. The steps in Algorithm 3 are inspired by the textual description in [GJS15] with minor adjustments in notation.

For the algorithm, we use the matrix notation of LWE as in Equation (2.21), i.e. $\mathbf{z} = \mathbf{A}^\intercal \mathbf{s} + \mathbf{e}$. BKW consists of a series of BKW steps that iteratively reduce the dimension of input matrix $\mathbf{A}$ by finding collisions of its column vectors in the currently examined block of b entries. We start from the last b entries of $\mathbf{A}^{(1)} = \mathbf{A}$. In every step i, we maintain a collision table $\mathbf{T}^{(i)}$ and loop over the columns $\mathbf{a}_k^{(i)}$ of $\mathbf{A}^{(i)}$ and distinguish between the following cases: (1) If $\mathbf{a}_k^{(i)}$ only has zero entries in the examined block, pass $\mathbf{a}_k^{(i)}$ and $z_k^{(i)}$ to the next step, (2) if no match of $\mathbf{a}_k^{(i)}$ or the negation of $\mathbf{a}_k^{(i)}$ can be found in the collision table, add $\mathbf{a}_k^{(i)}$ to the colission table, and (3) if a match $\mathbf{a}_l^{(i)}$ is found, compute $\mathbf{a}_l^{(i)} + \mathbf{a}_k^{(i)}$ or in the case of a negation match $\mathbf{a}_l^{(i)} - \mathbf{a}_k^{(i)}$ (in $\mathbb{Z}_q$) such that the last b nonzero entries cancel out. By exploiting the symmetry of $\mathbb{Z}_q$ in this way, in every step we obtain at most $(q^b - 1)/2$ columns with distinct coefficients in the current b entries. We also make note of "observed symbols" $z_j^{(i)}$ that represent the combination of two samples given their respective matching columns (see lines 20, 24 for more details).

In each BKW step, the number of columns (and samples) decreases by at least $(q^b - 1)/2$ (size of the colusion set) and the variance of the error distribution σ^2 increases by a factor of two. Once the number of remaining nonzero rows of $\mathbf{A}$ is small enough, the remaining part of the secret vector $\mathbf{s}$ is guessed. A hypothesis test ensures that the remaining samples follow a Gaussian with noise $2^t \cdot \sigma^2$, where t is the number of steps. Finally, back substitution is applied to obtain the complete secret vector $\mathbf{s}$.

Algorithm 3: BKW

function BKW($\mathbf{A}, \mathbf{z}, t$)
  $i = 1$
  $\mathbf{A}^{(i)} = \mathbf{A}$
  $\mathbf{z}^{(i)} = \mathbf{z}$
  while *the last t coefficients of the columns of* $\mathbf{A}^{(i)}$ *are nonzero* **do**
    // BKW step
    $j = 1$
    $\mathbf{T}^{(i)} = []$ // Collision table
    for $k = 1, \ldots, m^{(i)}$ **do**
      // $m^{(i)}$ is number of columns in $\mathbf{A}^{(i)}$
      if *last $(i \cdot b)$ coefficients of* $\mathbf{a}_k^{(i)}$ *are zero* **then**
        $\mathbf{a}_j^{(i+1)} = \mathbf{a}_k^{(i)}$
        $z_j^{(i+1)} = z_k$
        $j = j + 1$
      else if *no match for* $\mathbf{a}_k^{(i)}$ *in* $\mathbf{T}$ **then**
        $\mathbf{T} = \mathbf{T} + \left[\mathbf{a}_k^{(i)}\right]$ // append to collision set
      else if *match* $\mathbf{a}_l^{(i)}$ *for* $\mathbf{a}_k^{(i)}$ *is found* **then**
        if $\mathbf{a}_l^{(i)}$ *matches* $\mathbf{a}_k^{(i)}$ *in the last $(i \cdot b)$ components* **then**
          $\mathbf{a}_j^{(i+1)} = \mathbf{a}_k^{(i)} - \mathbf{a}_l^{(i)}$; // last $i \cdot b$ coefficients of $\mathbf{a}_j^{(i+1)}$ are now zero
          $z_j^{(i+1)} = z_k^{(i)} - z_l^{(i)} = y_j^{(i)} + e_j^{(i)}$, where $y_j^{(i)} = \left\langle \mathbf{s}, \mathbf{a}_j^{(i)} \right\rangle$ and $e_j^{(i)} = e_k^{(i)} - e_l^{(i)}$
          $j = j + 1$
        else if *the negation of* $\mathbf{a}_l^{(i)}$ *in* $\mathbb{Z}_q^n$ *matches* $\mathbf{a}_k^{(i)}$ *in the last $(i \cdot b)$ components* **then**
          $\mathbf{a}_j^{(i+1)} = \mathbf{a}_k^{(i)} + \mathbf{a}_l^{(i)}$
          $z_j^{(i+1)} = z_k^{(i)} + z_l^{(i)} = y_j^{(i)} + e_j^{(i)}$, where $y_j^{(i)} = \left\langle \mathbf{s}, \mathbf{a}_j^{(i)} \right\rangle$ and $e_j^{(i)} = e_k^{(i)} + e_l^{(i)}$
          $j = j + 1$
    $i = i + 1$
    // Calculate input for next BKW step
    $\mathbf{A}^{(i)} = (\mathbf{a}_1^{(i)} \cdots \mathbf{a}_{j-1}^{(i)})$
    $\mathbf{z} = (z_1^{(i)}, \ldots, z_{j-1}^{(i)})$

Coded-BKW [GJS15]

- change BKW step -> more column entries are removed, but additional noise - index set I, $\mathbf{x}_I$ is part of $\mathbf{x}$ with entries indexed by I - step i: I set of b positions to be removed, fix some q-ary linear $[N_i, b]$ code C_i with q^b codewords, find the closest codeword $\mathbf{c}_I \in C$ for every input vector $\mathbf{a}_I$ such that $\mathbf{a}_I = \mathbf{c}_I + \mathbf{e}_I$, where the error part $\mathbf{e}_I \in \mathbb{Z}_q^{N_i}$ is minimized by a decoding procedure.

Finally, we subtract two vectors and their correpsonding samples and pass the result to the next BKW step. Consider the inner product$\langle \mathbf{s}_I, \mathbf{a}_I \rangle = \langle \mathbf{s}_I, \mathbf{c}_I \rangle + \langle \mathbf{s}_I, \mathbf{e}_I \rangle$. In the subtraction, only the error part $\langle \mathbf{s}_I, \mathbf{e}_I \rangle$ remains.

3.2.3 Dual Attack [MicReg09]

"Gama and Nguyen [GN08b]: (in)feasibility of obtaining various Hermite factors natural distinguishing attack on LWE by finding one relatively short vector in associated lattice"

3.2.4 Decoding Attack [LP11]

combines lattice basis reduction followed by an enumeration algorithm (bounded-distance decoding with preprocessing?) => time/success tradeoff specifically for LWE, exploits structural properties of LWE on search version of LWE problem, approach preferable to distinguishing attack on decision LWE in [MR09; RS10], same or better advantage than distinguishing attack using lattice vectors of lower quality => runtime is smaller post-reduction: simple extension of Babai's "nearest-plane" algorithm [Bab85] => trade basis quality against decoding time related to Klein's (de)randomized algorithm [Kle00] for bounded-distance decoding

use entire reduced basis, post-reduction part is fully parallelizable

LLL reduction to input Lattice, integer combination of basis vectors close to target (like inner loop in reduction step of LLL), seek vector in lattice close to target, finds output that is in fundamental parallelipiped $\mathcal{P}(\mathbf{B})$ Section 2.2.2 => if error vector not in $\mathcal{P}(\mathbf{B})$, secret is not restored => basis quality has to be sufficiently good

Algorithm 4: Babai's Nearest Plane Algorithm [Bab85]

function NearestPlane($\mathbf{B} \in \mathbb{R}^{m \times n}, \mathbf{t} \in \mathbb{R}^m$)
  run δ-LLL on basis $\mathbf{B}$ with $\delta = \frac{3}{4}$
  $\mathbf{b} = \mathbf{t}$
  for $i = n, \ldots, 1$ **do**
    $c_i = \text{round}(\langle \mathbf{b}, \tilde{\mathbf{b}}_i \rangle / \langle \tilde{\mathbf{b}}_i, \tilde{\mathbf{b}}_i \rangle)$ $\mathbf{b} = \mathbf{b} - c_i \mathbf{b}_i$
  output $\mathbf{t} - \mathbf{b}$

Output is a lattice vector $\mathbf{v} \in \Lambda(\mathbf{B})$ such that $\|\mathbf{v} - \mathbf{t}\| \leq 2^{n/2}\text{dist}(\mathbf{t}, \Lambda(\mathbf{B}))$

Goal: recover lattice vector relatively close to target vector Intuition: - project $\mathbf{t}$ to $\mathrm{span}(\mathbf{B})$ - from $i = n, \ldots, 1$ find closest hyperplane $c_i\tilde{\mathbf{b}}_i + \mathrm{span}(\mathbf{b}_1, \ldots, \mathbf{b}_i)$ to the projection, subtract $c_i\mathbf{b}_i$ from the projection and continue - output vector is $\sum_{i=1}^{n} c_i\mathbf{b}_i$ for every basis vector $\mathbf{b}_i$ find c_i such that distance between target and hyperplane spanned by $\mathbf{b}_1, ..., \mathbf{b}_{i-1}$ and shifted by $c_i\tilde{\mathbf{b}}_i$ is minimal , subtract $c_i\mathbf{b}_i$ from target vector and continue for $i = n, \ldots, 1$. After the last iteration $\sum_{i=1}^{n} c_i\mathbf{b}_i$ is returned.

Application to LWE: $\mathbf{t} = \mathbf{A}^\intercal\mathbf{s} + \mathbf{e}$ => we get $\mathbf{v}$ where $\mathbf{t} - \mathbf{v} = \mathbf{e}$ is in fundamental parallelipiped of Gram-Schmidt basis

Generalized version by [LP11]: Problem: in reduced basis last Gram-Schmidt vectors of B short, first long => long and skinny parallelipiped, Gaussian e unlikely to be in it => incorrect answer from NearestPlane

=> generalized version admitting time/success tradeoff recurse on some $d_i \geq 1$ distinct planes in ith

Algorithm 5: Generalized Nearest Plane Algorithm [LP11]

function GeneralizedNearestPlane($\mathbf{B} \in \mathbb{R}^{m \times k}, \mathbf{t} \in \mathbb{R}^m, \mathbf{d} \in (\mathbb{Z}^+)^k$)
    if $k = 0$ **then**
        Return $\mathbf{0}$
    else
        Compute projection $\mathbf{v}$ of $\mathbf{t}$ onto $\mathrm{span}(\mathbf{B})$
        Compute the d_k distinct integers $c_1, \ldots, c_{d_k}$ closest to $\langle \mathbf{v}, \tilde{\mathbf{b}}_k \rangle / \langle \tilde{\mathbf{b}}_k, \tilde{\mathbf{b}}_k \rangle)$
        Return $\bigcup_{i \in \{1,\ldots,d_k\}} (c_i \cdot \mathbf{b}_k +$ GeneralizedNearestPlane$(\{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\}, (d_1, \ldots, d_{k-1}), \mathbf{v} - c_i \cdot \mathbf{b}_k))$

Instead of choosing only the nearest plane in each iteration step, Algorithm 5 selects a variable amount d_k of distinct planes in each step. As a consequence, the fundamental parallelipiped of the Gram-Schmidt basis is stretched in the direction of $\tilde{\mathbf{b}}_k$. The values of $\mathbf{d}$ should be chosen such that the covered area is approximately the same in each direction (i.e. by maximizing $\min_i(d_i \cdot \|\tilde{\mathbf{b}}_i\|)$). In particular this implies that the d_k are larger for larger k as the Gram-Schmidt vectors have a smaller length. Compared to Algorithm 4 the runtime increases by a factor $\prod_{i \in \{1,\ldots,d_k\}} d_i$, however, the recursion step can be fully parallelized.

It should be evident that a lower quality of the reduced input basis can be compensated for by increasing the values of $\mathbf{d}$. Hence we can adjust the input parameters for the lattice reduction and Algorithm 5 to minimize the runtime given a fixed required success probability.

3.2.5 Primal-uSVP [ADPS16, BaiGal14]

BKZ: reduce lattice basis using SVP oracle in smaller dimension b, known that number of calls to oracle polynomial - enumeration algorithm as oracle: in super-exponential time - sieve algorithms as oracle: exponential time but so far slower in practice for accesible dimensions $b \approx 130$

primal attack: construct unique-SVP instance from LWE instance LWE instance $(\mathbf{A}, \mathbf{z} = \mathbf{A}^\top \mathbf{s} + \mathbf{e})$ construct lattice

$$(3.4)\quad \Lambda = \left\{\mathbf{x} \in \mathbb{Z}^{m+n+1} \mid (\mathbf{A}^\top | -\mathbf{I}_m | -\mathbf{b})\mathbf{x} = \mathbf{0} \mod q\right\}$$

lattice has dimension $d = m + n + 1$, volume q^m and unique-SVP solution $\mathbf{v} = (\mathbf{s}, \mathbf{e}, 1)$

success condition: - geometric series assumption known to be optimistic from attacker's point of view => finds basis with Gram-Schmidt norms $\|mat\tilde{h}bfb_i\| = \delta^{d-2i-1} \cdot \mathrm{Vol}(\Lambda)^{1/d}$ and $\delta = ((\pi b)^{1/b} \cdot b/2\pi e)^{1/2(b-1)}$ unique short vector $\mathbf{v}$ is detected if projection of $\mathbf{v}$ onto span of last b Gram-Schmidt vectors is shorter than $mat\tilde{h}bfb_{d-b}$, norm of projection is expected to be $\gamma\sqrt{b}$ => attack successful iff $\gamma\sqrt{b} \leq \delta^{d-2i-1} \cdot q^{m/d}$

LWE as inhomogeneous-SIS (ISIS)

As in Section 3.2.4, we view the $\mathrm{LWE}_{n,q,m,\chi}$ instance $(\mathbf{A}, \mathbf{z})$ as a BDD instance in the q-ary lattice $\Lambda(\mathbf{A}^\top) = \{\mathbf{y} \mid \exists \mathbf{x} \in \mathbb{Z}_q^n : \mathbf{y} = \mathbf{A}^\top \mathbf{x} \mod q\}$ Section 2.3.1 generated by rows of LWE instance. The target vector is $\mathbf{z}$.

Recall the γ-uSVP problem. Given a lattice Λ where $\lambda_2(\Lambda) > \gamma\lambda_1(\Lambda)$, we are asked to find shortest nonzero vector in Λ. In the primal attack, instead of directly solving BDD, we reduce BDD to uSVP, i.e., we reduce a BDD instance to a γ-uSVP instance. By solving γ-uSVP, we obtain a solution to BDD. To do this we apply Kannan's embedding technique [Kan87]. Intuitively, Kannan's embedding creates a lattice with uSVP structure. We know that $\mathbf{A}^\top \mathbf{s} \mod q$ is the closest vector to the target $\mathbf{z} = \mathbf{A}^\top \mathbf{s} + \mathbf{e}^\top \mod q$ in $\Lambda(\mathbf{A}^\top)$. We now add a linearly indpenendent basis vector $(\mathbf{z}, \mu)$ and append a zero coefficient to each basis vector of the original lattice (i.e. the rows of $\mathbf{A}$). Thereby, we ensure that the new lattice contains the vector $[-\mathbf{e}, -\mu]^\top$ as $[\mathbf{A} \mid \mathbf{0}]^\top \mathbf{s} - 1 \cdot [\mathbf{z}^\top, \mu] = [-\mathbf{e}, -\mu]^\top$.

More formally, let $\mathbf{B}$ be a basis of $\Lambda(\mathbf{A}^\top)$ and an embedding factor $\mu = \mathrm{dist}(\mathbf{z}, \Lambda(\mathbf{A}^\top)) = \|\mathbf{z} - \mathbf{s}\|$ where $\mathbf{s}$ is the secret vector of the LWE instance. A relatively close approximation of μ can be guessed in polynomial time (see [LM09] for more details). We now embed $\Lambda(\mathbf{A}^\top)$ into $\Lambda(\mathbf{B}')$ with γ-uSVP structure as follows:

$$(3.5)\quad \mathbf{B}' = \begin{pmatrix} \mathbf{B} & \mathbf{z} \\ \mathbf{0}^\top & \mu \end{pmatrix}$$

If $\gamma \geq 1$ and $\mu < \frac{\lambda_1(\Lambda(\mathbf{B})}{2\gamma}$ (or equivalently, $(\Lambda(\mathbf{A}^\top), \mathbf{z})$ a $\mathrm{BDD}_{1/(2\gamma)}$-instance), then $\Lambda(\mathbf{B}')$ contains a γ-unique shortest vector $\mathbf{z}' = [(\mathbf{A}^\top \mathbf{s} - \mathbf{z})^\top, -\mu]^\top = [-\mathbf{e}^\top, -\mu]^\top$. The statement can be proven by showing by contradiction that all vectors $\mathbf{v} \in \Lambda\mathbf{B}'$ that are independent of $\mathbf{z}'$ satisfy $\|\mathbf{v}\| \geq \lambda_1\Lambda(\mathbf{B})/\sqrt{2} > \sqrt{2}\gamma\mu = \gamma\|\mathbf{z}'\|$ (see Section 4 of [LM09] for more details). Note that the reduction can be done in polynomial time (Theorem 4.1 in [LM09]).Hence, from $\mathbf{z}'$ we can recover the error vector $\mathbf{e}$ and thereby the secret vector $\mathbf{s} = \mathbf{z} - \mathbf{e} \mod q$.

A solution to γ-uSVP can be found by reducing it to κ-HSVP where $\gamma = \kappa^2$ [APS15]. Various algorithms, in particular, lattice reduction algorithms, exist to solve κ-HSVP. If we are able to solve a linear number of κ-HSVP instances that correspond to a κ^2-approximate SVP instance, we can construct a solution the latter (see Definition 2.2.1, see Section 1.2.21 in [Lov87] for more details). Consider any lattice with uSVP structure. In exactly one direction, that is, in the direction of its unique shortest vector, the lattice has vectors that are significantly smaller than in other directions. A lattice reduction algorithm that yields a sufficiently good output basis quality, therefore, must return some small vector in the desired direction. Let $\mathbf{v}$ be a solution to $\mathrm{SVP}_{\kappa}^{2}$, i.e. $\|\mathbf{v}\| \leq \kappa^2 \lambda_1(\Lambda)$. All other vectors $\mathbf{w} \in \Lambda$ that are not multiples of a shortest vector have length $\|\mathbf{w}\| \geq \lambda_2(\Lambda) > \kappa^2 \lambda_1(\Lambda)$. Thus, we obtain a solution to γ-uSVP and, as shown above, we can reconstruct the secret vector to solve LWE.

3.2.6 Meet-in-the-Middle [AlbPlaSco15]

3.2.7 Arora-Ge [AroGe11,ACFP14]

3.3 Algorithms for Solving SIS

3.3.1 Dual Attack

MR variant [MR09]

RS variant [RS10]

"concrete estimates of “symmetric bit security”, concrete runtime estimates for various Hermite factors in random q-ary lattices"permissive form of distiguishing attack in [MR09], adversarial advantage is about 2^{-72}

3.3.2 Combinatorial Attack [MR09]

3.4 Tool

class for distributions... from section this modelling, problems, generic search... Überblick, wie verwendbar, automatische norm umwandlung, sonstige features

3.4.1 Runtime and Cost Comparison

defaults... schnellste, beste => effizient, etc. parallel... problem reductions...

4 Usage Examples

4.1 Two Problem Search

basiert auf [BDLOP18]

4.2 TODO: find other schemes to apply

5 Conclusion

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