A Tool for the Estimation of Lattice Parameters

Bachelor Thesis

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A lattice is a discrete Λ additive subgroup of the vector space $\mathbb{R}^m$ and can be defined by a basis $\mathbf{B}$ of n linearly independent vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^m$ with $m \geq n$.

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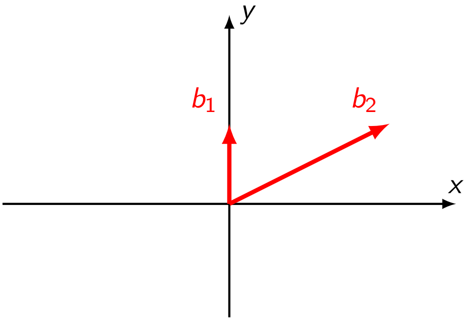

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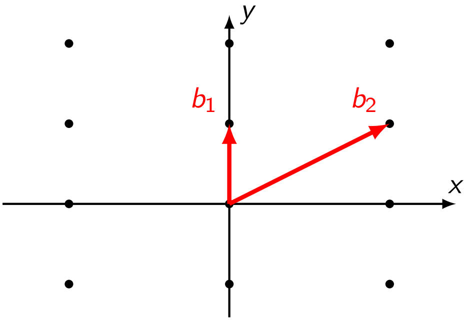

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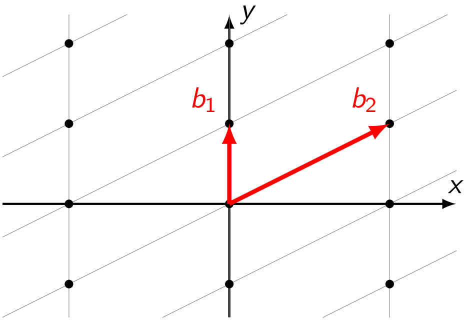

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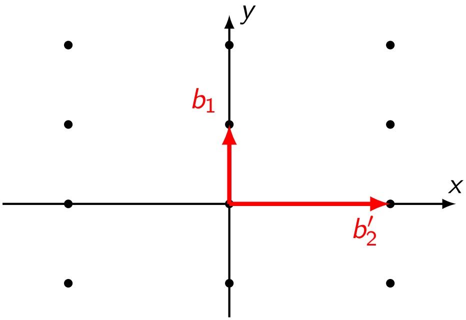

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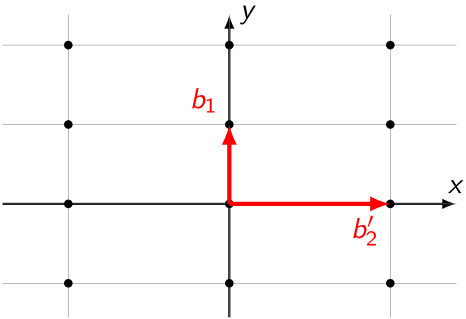

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SVP_γ and $\textsc{GapSVP}_\gamma$

Given a basis $\mathbf{B}$ of a lattice Λ, the (approximate) Shortest Vector Problem (SVP_γ) is the problem of finding a short lattice vector $\mathbf{v} \in \Lambda$ such that $0 < \|\mathbf{v}\| \leq \gamma\lambda_1(\Lambda)$. The corresponding decision version is the $\textsc{GapSVP}_\gamma$ problem, in which we are asked to decide whether $\lambda_1(\Lambda) \leq 1$ or $\lambda_1(\Lambda) \geq \gamma$ given a basis $\mathbf{B}$ of Λ. If neither is the case, any answer is accepted.

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- Worst-case to average-case reduction from SVP_γ to the Short Integer Solution (SIS) problem[2]

[2] M. Ajtai, "Generating hard instances of lattice problems (extended abstract)," in *Proceedings of the Twenty-Eighth Annual ACM Symposium on the Theory of Computing, Philadelphia, Pennsylvania, USA, May 22-24, 1996*, G. L. Miller, Ed., ACM, 1996, pp. 99–108.

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- Worst-case to average-case reduction from SVP_γ to the Short Integer Solution (SIS) problem[2]
- Similar reduction from $\textsc{GapSVP}_\gamma$ to the Learning with Errors (LWE) problem[3]

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[3] O. Regev, "On lattices, learning with errors, random linear codes, and cryptography," in *Proceedings of the 37th Annual ACM Symposium on Theory of Computing, Baltimore, MD, USA, May 22-24, 2005*, H. N. Gabow and R. Fagin, Eds., ACM, 2005, pp. 84–93.

The Learning with Errors (LWE) Problem

The $\mathrm{LWE}_{n,q,m,\chi}$ distribution

Given an integer $n \geq 1$, a modulus $q \geq 2$, an error distribution χ on $\mathbb{Z}_q$, and a fixed secret vector $\mathbf{s}$, let $\mathcal{A}_{\mathbf{s},\chi}$ be the probability distribution over $\mathbb{Z}_q^n \times \mathbb{Z}_q$ by choosing a vector $\mathbf{a}_i \in \mathbb{Z}_q^n$ uniformly at random and $e_i \in \mathbb{Z}_q$ according to χ.

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$$(\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \mod q) \in \mathbb{Z}_q^n \times \mathbb{Z}_q.$$

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- Search-LWE asks to recover $\mathbf{s}$ and Decision-LWE asks to distinguish m samples from uniformly random
- (Primal) LWE lattice

$$\Lambda_q(\mathbf{A}^\mathsf{T}) = \{\mathbf{v} \in \mathbb{Z}^m \mid \exists \mathbf{y} \in \mathbb{Z}^n : \mathbf{v} = \mathbf{A}^\mathsf{T}\mathbf{y} \mod q\}$$

The Short Integer Solution (SIS) Problem

The SIS Problem

Given a uniformly random matrix $\mathbf{A}^{n \times m}$, the $\mathsf{SIS}_{n,q,m,\beta}$ problem asks us to find a vector $\mathbf{s} \in \mathbb{Z}^m$ such that

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$$\Lambda_q^{\perp}(\mathbf{A}) = \left\{ \mathbf{v} \in \mathbb{Z}^m \mid \mathbf{A} \cdot \mathbf{v} = \mathbf{0} \mod q \right\}.$$

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- LWE Estimator[4] encapsulates attack estimates for LWE

[4] M. R. Albrecht, R. Player, and S. Scott, "On the concrete hardness of learning with errors," *J. Math. Cryptol.*, vol. 9, no. 3, pp. 169–203, 2015.

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    - ρ: number of rounds
    - t_k: cost of SVP oracle in dimension k
  - Most significant progress in first 8 rounds[7] $\Rightarrow$ LWE-Estimator chooses $\rho = 8$ with estimated output quality

$$\lim_{n \to \infty} \delta \approx \left(\frac{k(\pi k)^{\frac{1}{k}}}{2\pi e} \right)^{\frac{1}{2(k-1)}}$$

[6] C.-P. Schnorr and M. Euchner, “Lattice basis reduction: Improved practical algorithms and solving subset sum problems,” in *Fundamentals of Computation Theory, 8th International Symposium, FCT ’91, Gosen, Germany, September 9-13, 1991, Proceedings*, L. Budach, Ed., ser. Lecture Notes in Computer Science, vol. 529, Springer, 1991, pp. 68–85.

[7] Y. Chen, “Réduction de réseau et sécurité concrete du chiffrement completement homomorphe,” PhD thesis, Paris 7, 2013.

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Realizing an SVP oracle in dimension k:

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  - In $2^{\mathcal{O}(k)}$ time and exponential space

BKZ Sieving Cost Models

Name	Cost model
Q-Sieve (paranoid lower bound)[8]	$2^{0.2075k}$
Q-Sieve[9]	$2^{0.265k}$
Sieve [9]	$2^{0.292k}$

[8] E. Alkim, L. Ducas, T. Pöppelmann, *et al.*, "Post-quantum key exchange - A new hope," in *25th USENIX Security Symposium, USENIX Security 16, Austin, TX, USA, August 10-12, 2016*, T. Holz and S. Savage, Eds., USENIX Association, 2016, pp. 327–343.

[9] M. R. Albrecht, V. Gheorghiu, E. W. Postlethwaite, *et al.*, "Estimating quantum speedups for lattice sieves," in *Advances in Cryptology - ASIACRYPT 2020 - 26th International Conference on the Theory and Application of Cryptology and Information Security, Daejeon, South Korea, December 7-11, 2020, Proceedings, Part II*, S. Moriai and H. Wang, Eds., ser. Lecture Notes in Computer Science, vol. 12492, Springer, 2020, pp. 583–613.

BKZ Enumeration Cost Models

Name	Cost model
Lotus[10]	$2^{0.125k\log k - 0.755k + 2.254}$
Enum + O(1) [10]	$2^{0.187k\log k - 1.019k + 16.1}$
Q-Enum + O(1) [10]	$2^{0.0936k\log k - 0.51k + 8.05}$
BKZ2.0-Enum[11]	$2^{0.184k\log k - 0.995k + 16.25}$
ABF20-Enum [11]	$2^{0.125k\log k}$
Q-ABF20-Enum [11]	$2^{0.0625k\log k}$

[10] M. R. Albrecht, B. R. Curtis, A. Deo, *et al.*, "Estimate all the {lwe, ntru} schemes!" In *Security and Cryptography for Networks - 11th International Conference, SCN 2018, Amalfi, Italy, September 5-7, 2018, Proceedings*, D. Catalano and R. D. Prisco, Eds., ser. Lecture Notes in Computer Science, vol. 11035, Springer, 2018, pp. 351–367.

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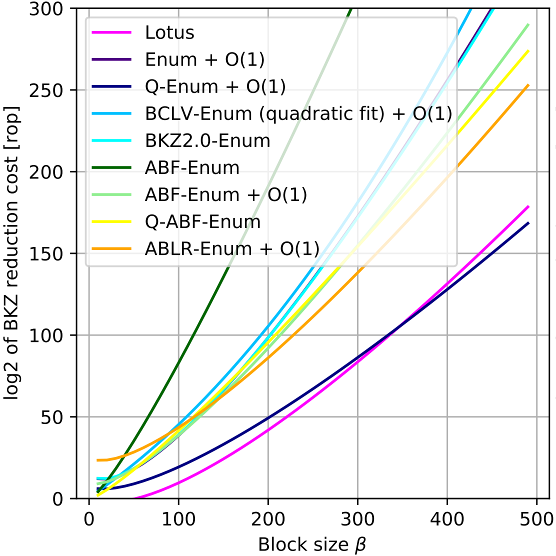
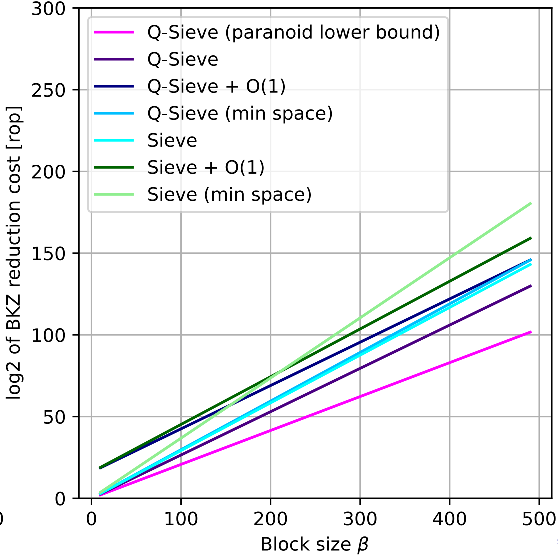
BKZ Enumeration Cost Models





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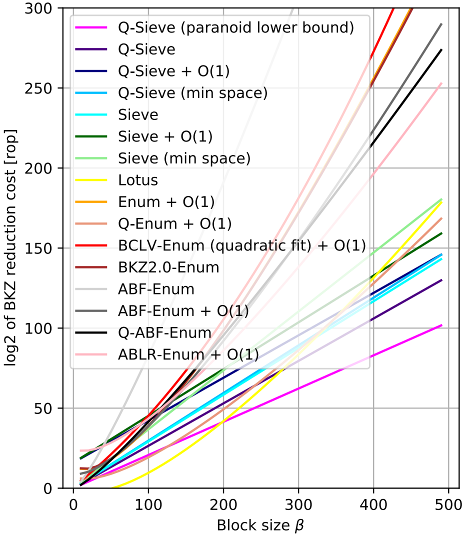
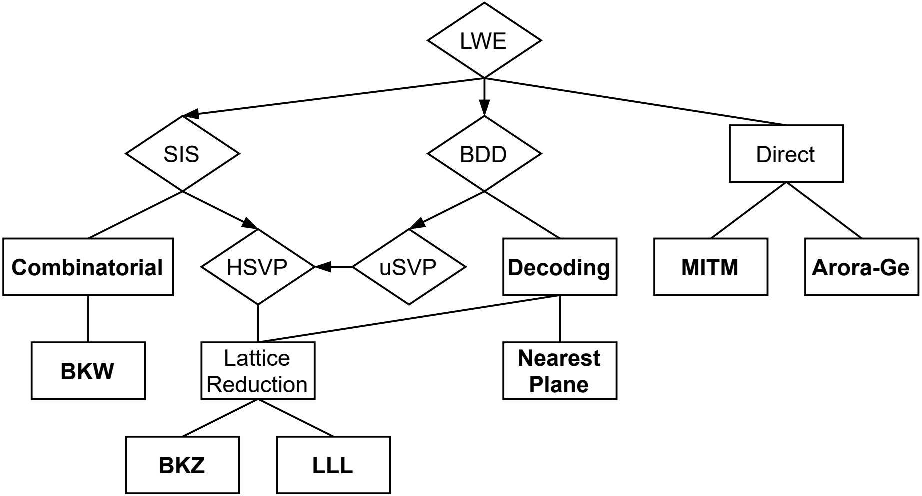


Approaches to Solving LWE



SIS Attack Estimates Comparison

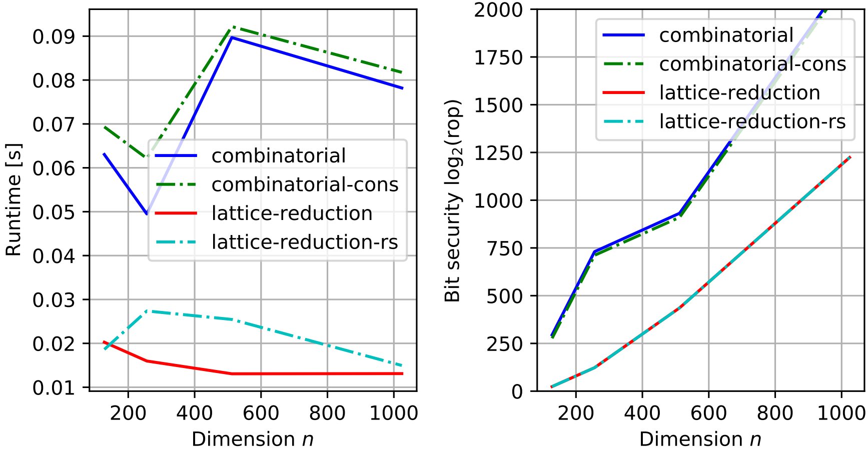


Figure: SIS with $n^2 < q < 2n^2,\ m = 2n\sqrt{n \log q},\ s = 2\sqrt{n \log q}$

SIS Attack Estimates Prioritization

Algorithm	Priority	Justification
Lattice Reduction MR	1	fastest, low cost estimates
Lattice Reduction RS	2	same results as lattice-reduction, not always applicable
Combinatorial Attack	10	fast, often higher cost results
Combinatorial Conservative	9	fast, slighly lower estimates than Combinatorial Attack

LWE Attacks Estimates Comparison

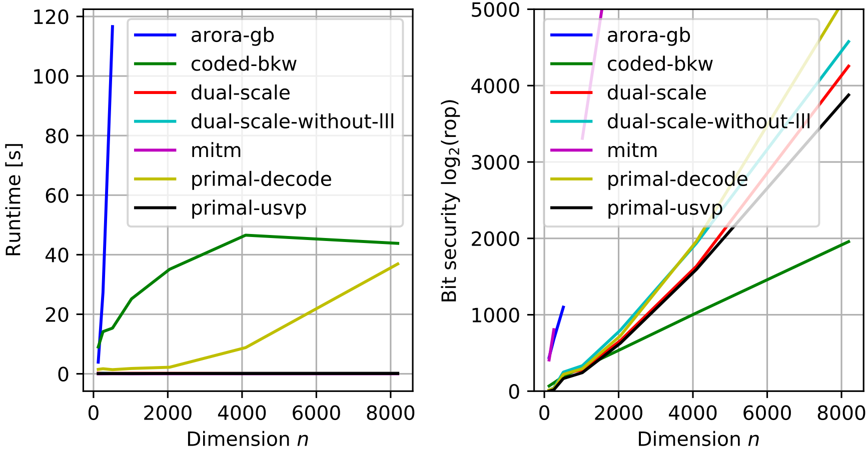


Figure: LWE with $\sigma = 2.828,\ m = \infty,\ n < q < 2n$

LWE Attacks Estimates Comparison

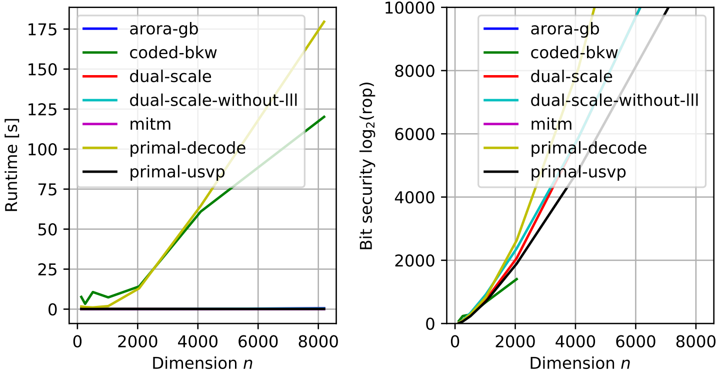


Figure: LWE with parameters chosen as in Regev (ACM 2005)[12]

[12] O. Regev, "On lattices, learning with errors, random linear codes, and cryptography," in *Proceedings of the 37th Annual ACM Symposium on Theory of Computing, Baltimore, MD, USA, May 22-24, 2005*, H. N. Gabow and R. Fagin, Eds., ACM, 2005, pp. 84–93.

LWE Attack Estimates Prioritization

Algorithm	Priority	Justification
Meet-in-the-Middle	5	fastest, high cost estimate, as a prefilter
Primal uSVP	10	fast, low cost estimatate estimates
Dual Attack	20	fast, often higher estimates than Primal uSVP
Dual Attack (no LLL)	30	fast, often higher estimates than Dual
Coded-BKW	90	slow, somtimes very low cost estimate (for small stddev), does not always yield results
Decoding Attack	100	slow, often higher estimates than faster algorithms
Arora-Ge	200	extremely slow, often higher estimates, does not always yield results

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Ring and Module Variants

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$$\mathrm{Rot}(a) = \begin{pmatrix} a_0 & -a_{n-1} & \cdots & -a_1 \\ a_1 & a_0 & \cdots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}$$

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- For module variants this becomes

$$n \cdot d \left\{ \overbrace{\begin{bmatrix} \mathrm{Rot}(\boldsymbol{a}_{1,1}) & \mathrm{Rot}(\boldsymbol{a}_{1,2}) & \cdots & \mathrm{Rot}(\boldsymbol{a}_{1,m}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Rot}(\boldsymbol{a}_{d,1}) & \mathrm{Rot}(\boldsymbol{a}_{d,2}) & \cdots & \mathrm{Rot}(\boldsymbol{a}_{d,m}) \end{bmatrix}}^{n \cdot m} \right.$$

Ring and Module Variants

Resulting mapping to standard variant:

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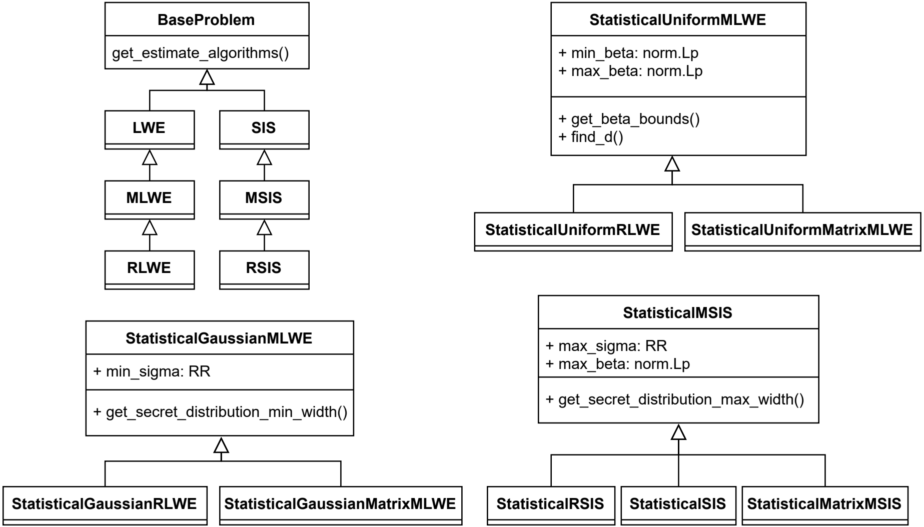
- $\mathsf{RLWE}_{n,q,m,\chi} \longrightarrow \mathsf{LWE}_{n,q,m \cdot n,\chi}$
- $\mathsf{MLWE}_{n,d,q,m,\chi} \longrightarrow \mathsf{LWE}_{n \cdot d,q,m \cdot n,\chi}$

LWE and SIS Classes Overview



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        return (result, current_params)
    else
        next_param_sets = next_parameters(current_params)
        forall param_set in next_param_sets do
            sort param_set into L according to parameter_cost function
```

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Thank You!

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Gram-Schmidt Orthogonalization

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- **if** $\exists i$ such that $\theta \|\tilde{\mathbf{b}}_i\|^2 > \|\mu_{i+1,i}\tilde{\mathbf{b}}_i + \tilde{\mathbf{b}}_{i+1}\|^2$ **then**
  - Swap $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$
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[a] O. Regev, *Lecture notes in lattices in computer science*, Fall 2004.

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[14] C.-P. Schnorr and M. Euchner, "Lattice basis reduction: Improved practical algorithms and solving subset sum problems," in *Fundamentals of Computation Theory, 8th International Symposium, FCT '91, Gosen, Germany, September 9-13, 1991, Proceedings*, L. Budach, Ed., ser. Lecture Notes in Computer Science, vol. 529, Springer, 1991, pp. 68–85.

The Block Korkin-Zolatarev (BKZ) Algorithm[14]

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- Repeat until no change in n iterations, counter j resets to 1 after $n-k+1$ iterations (one round)

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- Most significant progress in first 8 rounds[15] $\Rightarrow$ LWE-Estimator chooses $\rho = 8$

[15] Y. Chen, "Réduction de réseau et sécurité concrete du chiffrement completement homomorphe," PhD thesis, Paris 7, 2013.

Primal Attack - Reduction of LWE to BDD

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Given a lattice $\Lambda \subset \mathbb{R}^m$ and a target vector $\mathbf{t} \in \mathbb{R}^m$ such that $\mathrm{dist}(\mathbf{t}, \Lambda) < \gamma\lambda_1(\Lambda)$, the (approximate) Bounded Distance Decoding (BDD_γ) is the problem of finding the closest lattice vector $\mathbf{v} \in \Lambda$, i.e., $\mathbf{v} = \arg\min_{\mathbf{v}' \in \Lambda} \|\mathbf{v}' - \mathbf{t}\|$.

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- $\mathbf{A}^\intercal \mathbf{s} + q\mathbf{x}$
- $\mathrm{dist}(\mathbf{z}, \Lambda_q(\mathbf{A}^\intercal)) = \|\mathbf{e}\|$ and, in general, $\|\mathbf{e}\| < \gamma\lambda_1(\Lambda_q(\mathbf{A}^\intercal))$
- Solving BDD solves LWE

Primal Attack - Variants

- Decoding Attack[16]

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- Estimator uses a variant called Coded-BKW

[19] A. Blum, A. Kalai, and H. Wasserman, "Noise-tolerant learning, the parity problem, and the statistical query model," *J. ACM*, vol. 50, no. 4, pp. 506–519, 2003.

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[20] D. Micciancio and O. Regev, "Lattice-based cryptography," in *Post-Quantum Cryptography*, D. J. Bernstein, J. Buchmann, and E. Dahmen, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 147–191, ISBN: 978-3-540-88702-7.

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- Similar result in Rückert and Schneider (2010, IACR Cryptol. ePrint Arch.)

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- Overall cost dominated by list size L, total cost $\approx 2^k \cdot L \cdot \log_2(q) \cdot n$

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Ring-SIS

RSIS

Let $\mathcal{R}_q$ be the quotient ring $\mathbb{Z}_q[x] / \langle x^n + 1 \rangle$. Given $a_1, \ldots, a_m \in \mathcal{R}_q$ chosen independently from the uniform distribution, the Ring-SIS problem $\mathrm{RSIS}_{n,q,m,\beta}$ asks to find $s_1, \ldots, s_m \in \mathcal{R}$ such that $\sum_{i=1}^{m} \mathbf{a}_i \cdot s_i = 0 \mod q$ and $0 < \|\mathbf{s}\| \leq \beta$, where $\mathbf{s} = [s_1, \ldots, s_m]^\intercal \in \mathcal{R}^m$.

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- Interpret $r \in \mathcal{R}$ as an n dimensional vector s.t. $r = \sum_{i=0}^{n-1} r_i x^i$
- Each a_i in RSIS corresponds to an $n \times n$ block in the standard SIS matrix $\mathbf{A}_{\mathrm{SIS}}$ obtained by rotation:

$$\mathrm{Rot}(a) = \begin{pmatrix} a_0 & -a_{n-1} & \cdots & -a_1 \\ a_1 & a_0 & \cdots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}$$

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$\Rightarrow \mathbf{A}_{\mathrm{SIS}} = [\mathrm{Rot}(a_1) \mid \cdots \mid \mathrm{Rot}(a_m)]$

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- $\mathrm{RSIS}_{n,q,m,\beta} \longrightarrow \mathrm{SIS}_{n,q,m \cdot n,\beta}$

Module-SIS

MSIS

Let $\mathcal{R}^d$ be a module with ring dimension n and module rank d. Given $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathcal{R}_q^d$ chosen independently from the uniform distribution, the Module-SIS problem $\mathrm{MSIS}_{n,d,q,m,\beta}$ asks to find $s_1, \ldots, s_m \in \mathcal{R}$ such that $\sum_{i=1}^m \mathbf{a}_i \cdot s_i = \mathbf{0} \mod q$ and $0 < \|\mathbf{s}\| \leq \beta$, where $\mathbf{s} = [s_1, \ldots, s_m]^\mathsf{T} \in \mathcal{R}^m$.

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$$
\begin{array}{cc}
 & \overbrace{\hphantom{\mathrm{Rot}(\boldsymbol{a}_{1,1}) \; \mathrm{Rot}(\boldsymbol{a}_{1,2}) \; \cdots \; \mathrm{Rot}(\boldsymbol{a}_{1,m})}}^{n \cdot m} \\
n \cdot d & \left[\begin{array}{c|c|c|c}
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- $\mathrm{MSIS}_{n,d,q,m,\beta} \longrightarrow \mathrm{SIS}_{n\cdot d,q,m\cdot n,\beta}$

Ring-LWE and Module-LWE

RLWE Distribution

Let χ be the error distribution on $\mathbb{T}_{\mathcal{R}^\perp} = K_\mathbb{R}/\mathcal{R}^\perp$ and $s \in \mathcal{R}^\perp$ be the secret. Then, we define $\mathcal{A}^{(\mathcal{R})}_{q,s,\chi}$ as the Ring-LWE (RLWE) distribution on $\mathcal{R}_q \times \mathbb{T}_{\mathcal{R}^\perp}$ obtained by choosing $a \in \mathbb{R}_q$ uniformly at random and an error term $e \in \mathbb{T}_{\mathcal{R}^\perp}$ according to χ, and returning samples $(a, (a \cdot s)/q + e)$.

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MLWE Distribution

Let χ be the error distribution on $\mathbb{T}_{\mathcal{R}^\perp}$ and $\mathbf{s} \in (\mathcal{R}^\perp)^d$ be the secret vector. Then, we define $\mathcal{A}^{(\mathcal{M})}_{q,\mathbf{s},\chi}$ as the Module-LWE (MLWE) distribution on $(\mathcal{R}_q)^d \times \mathbb{T}_{\mathcal{R}^\perp}$ obtained by choosing $\mathbf{a} \in (\mathbb{R}_q)^d$ uniformly at random and an error term $e \in \mathbb{T}_{\mathcal{R}^\perp}$ according to χ, and returning samples $(\mathbf{a}, \frac{1}{q}\langle \mathbf{a}, \mathbf{s}\rangle + e)$.

- $\mathrm{RLWE}_{n,q,m,\chi} \longrightarrow \mathrm{LWE}_{n,q,m\cdot n,\chi}$

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- $\mathrm{RLWE}_{n,q,m,\chi} \longrightarrow \mathrm{LWE}_{n,q,m\cdot n,\chi}$
- $\mathrm{MLWE}_{n,d,q,m,\chi} \longrightarrow \mathrm{LWE}_{n\cdot d,q,m\cdot n,\chi}$

Statistically Secure MLWE (Gaussian Variant)[22]

- Given mth cyclomatic number field K of degree $n = \phi(m)$ and integer $q \geq 2$ and

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- Let $\mathbf{x} \in (\mathcal{R}_q)^{[m+d]}$ where each component is chosen from a discrete Gaussian distribution of parameter $s > 2n \cdot q^{m/(m+d)+2/(n(m+d))}$ over $\mathcal{R}$

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Statistically Secure MLWE (Uniform Variant)[23]

- Given $\mathbf{A} = [\mathbf{I}_{[m]} \mid \bar{\mathbf{A}}] \in (\mathcal{R}_q)^{[m]\times[m+d]}$ as before, $1 < d_2 < n$, where d_2 is a power of 2 and a prime q congruent to $2d_2 + 1 \pmod{4d_2}$

[23] C. Baum, I. Damgård, V. Lyubashevsky, *et al.*, "More efficient commitments from structured lattice assumptions," in *Security and Cryptography for Networks - 11th International Conference, SCN 2018, Amalfi, Italy, September 5-7, 2018, Proceedings*, D. Catalano and R. D. Prisco, Eds., ser. Lecture Notes in Computer Science, vol. 11035, Springer, 2018, pp. 368–385.

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$$\beta_{min} = \frac{q^{m/(m+d)} \cdot 2^{2\mathrm{sec}/((m+d)\cdot n)}}{2}$$

$$\beta_{max} = \frac{1}{2\sqrt{d_2}} \cdot q^{1/d_2} - 1$$

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Statistically Secure MLWE (Uniform Variant)[23]

- Given $\mathbf{A} = [\mathbf{I}_{[m]} \mid \bar{\mathbf{A}}] \in (\mathcal{R}_q)^{[m]\times[m+d]}$ as before, $1 < d_2 < n$, where d_2 is a power of 2 and a prime q congruent to $2d_2 + 1 \pmod{4d_2}$
- If $\beta \in \mathbb{R}$ such that $\beta_{min} \leq \beta \leq \beta_{max}$ with

$$\beta_{min} = \frac{q^{m/(m+d)} \cdot 2^{2\sec/((m+d)\cdot n)}}{2}$$
$$\beta_{max} = \frac{1}{2\sqrt{d_2}} \cdot q^{1/d_2} - 1$$

then any (all-powerful) algorithm $\mathcal{A}$ has advantage at most $2^{-\sec}$ in distinguishing $\mathbf{A}\mathbf{x} \in (\mathcal{R}_q)^{[m]}$ from the uniform distribution, where $\mathbf{x}$ is chosen uniformly random with $\|\mathbf{x}\|_\infty \leq \beta$

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Statistical MSIS[24]

- Given $\mathbf{A} = [\mathbf{I}_{[m]} \mid \bar{\mathbf{A}}] \in (\mathcal{R}_q)^{[m]\times[m+d]}$ as before
- It should be hard to find $\mathbf{r}, \mathbf{r}' \in \mathcal{R}_q^{m+d}$ of ℓ_2-norm $\leq B$ such that $\mathbf{A} \cdot (\mathbf{r} - \mathbf{r}') = \mathbf{0} \bmod q$

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- We demand that $\Pr[\mathbf{A} \cdot \mathbf{r} = \mathbf{0}] \leq 2^{-\mathsf{sec}}$ with non zero elements $\mathbf{r}$ in the Euclidean ball $B_m(0, 2B)$

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- Also works for RSIS and SIS

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Norm Bounding

Let $f \in \mathcal{R}_q$ with $f = \sum_i f_i X^i$ and $\sigma : K \to \mathbb{C}$ with number field K the canonical embedding[25] and $p, q \in \mathbb{N}$.

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- $\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_1$
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- $\|\sigma(x \cdot y)\|_p \leq \|\sigma(x)\|_\infty \cdot \|\sigma(y)\|_p$
- Encapsulated in `to_Lp()` and `to_Cp()` of the norm classes `Lp` and `Cp`

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Gaussian to Bound

- Classes for uniform and Gaussian distribution in the module `distributions`

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    - In all other cases `to_Lp()` bounds the value via ℓ_2-norm

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