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Bachelorarbeit

A Tool for the Estimation of Lattice Parameters

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Abstract

<Short summary of the thesis>

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1 Introduction

- rise of quantum computing (short history)
 - * conceptual
 - * reality
- problem: some hard classical problems no longer hard
 - * Shor's Algorithm (Peter Shor, 1994) => quantum computers can solve the factoring and the discrete logarithm problem in polynomial time
 - * application to encryption
 - * overview of current encryption methods that will become insecure
- one solution (among hash-based, code-based, isogeny-based, and multivariate): lattice crypto
 - * overview over history and capability of lattice crypto
 - * advantages: good (quasilinear) asymptotic key sized, good concrete runtimes and key sizes, worst-case secure instantiations, advanced cryptographic primitives previously infeasible
 - * including intro to LWE/SIS and applications to build crypto systems
 - . SIS: signature schemes, hash functions
 - . LWE: "cryptomania" applications (PKE, ...), signature schemes, lines:
- cryptographic applications
 - establishing theoretical and asymptotic hardness [Reg05] [BLP+13; MP13] - concrete hardness of LWE: attacks, runtime estimates,
 - * briefly outline concept and benefits of hard-case to average-case reductions
 - purpose of this thesis
 - * building schemes: need realistic hardness estimates of schemes for given parameter settings
 - * lack in the past: no unified/easy to use tool => thesis aims to solve this problem tool we call *Lattice Parameter Estimation* LWE instances are estimated by calling various estimation functions from the LWE Estimator [APS15], which we will refer to as *Estimator*.
- overview of chapters/how to read

2 Preliminaries

2.1 Notation

In the following, we denote vectors by bold lower-case letters like \mathbf{v} and matrices by bold upper-case letters \mathbf{M} . We interchangeably use matrix notation and sets of column vectors $[\mathbf{v}_1 \cdots \mathbf{v}_n] = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Unless specified otherwise, by $\|\cdot\|$ or simply *norm* we refer to the Euclidean norm. By $[n]$ we denote the set $\{1, \dots, n\}$ for $n \in \mathbb{Z}^+$.

2.2 Math

2.2.1 Norms and Bounds

Let \mathcal{R}_q be a ring as defined in [BDL+18] and $f \in \mathcal{R}_q$ with $f = \sum_i f_i X^i$. We define the following norms [BDL+18]:

$$(2.1) \quad \ell_1 : \|f\|_1 = \sum_i |f_i|$$

$$(2.2) \quad \ell_2 : \|f\|_2 = \left(\sum_i |f_i|^2 \right)^{\frac{1}{2}}$$

$$(2.3) \quad \ell_\infty : \|f\|_\infty = \max_i |f_i|$$

Then the following inequations hold [BDL+18]:

$$(2.4) \quad \|f\|_1 \leq \sqrt{n} \|f\|_2$$

$$(2.5) \quad \|f\|_1 \leq n \|f\|_\infty$$

$$(2.6) \quad \|f\|_2 \leq \sqrt{n} \|f\|_\infty \quad (\text{since } \sqrt{n} \|f\|_2 \leq n \|f\|_\infty)$$

$$(2.7) \quad \|f\|_\infty \leq \|f\|_1$$

Let \mathcal{O}_K be the ring of integers of a number field $K = \mathbb{Q}(\theta)$, where θ is an algebraic number and σ denote the canonical embedding as defined in [DPSZ12]. Then, for $x, y \in \mathcal{O}_K$ it holds the following inequations hold (we assume that C_m in [DPSZ12] is 1) [DPSZ12].

$$(2.8) \quad \|f\|_\infty \leq \|\sigma(f)\|_\infty$$

$$(2.9) \quad \|\sigma(f)\|_\infty \leq \|f\|_1$$

From the above inequations, we obtain the following norm transformations to ℓ_p -norms:

- From Equation (2.4), it follows that $\|f\|_1 \leq \sqrt{n}\|f\|_2$ and from Equation (2.5), $\|f\|_1 \leq n\|f\|_\infty$.
- From Equation (2.6) and Equation (2.7), it follows that $\|f\|_2 \leq \sqrt{n}\|f\|_1$ and from Equation (2.6), $\|f\|_2 \leq \sqrt{n}\|f\|_\infty$.
- From Equation (2.7), it follows that $\|f\|_\infty \leq \|f\|_1$ and from Equation (2.4) and Equation (2.7), $\|f\|_\infty \leq \sqrt{n}\|f\|_2$.
- From Equation (2.9), it follows that $\|\sigma(f)\|_\infty \leq \|f\|_1$, from Equation (2.4) and Equation (2.9), $\|\sigma(f)\|_\infty \leq \sqrt{n}\|f\|_2$, and from Equation (2.5) and Equation (2.9), $\|\sigma(f)\|_\infty \leq n\|f\|_\infty$.

Likewise, we get the following transformations to the C_∞ -norm:

- From Equation (2.5) and Equation (2.8), it follows that $\|f\|_1 \leq n\|\sigma(f)\|_\infty$.
- From Equation (2.6) and Equation (2.8), it follows that $\|f\|_2 \leq \sqrt{n}\|\sigma(f)\|_\infty$.
- From Equation (2.8), it follows that $\|f\|_\infty \leq \|\sigma(f)\|_\infty$.

Let f be defined as above and let $g \in \mathcal{R}_q$ where $g = \sum_i \bar{g}_i X^i$ where $g_i \in [-(q-1)/2, (q-1)/2]$ and $\bar{g}_i = g_i \bmod q$ as in [BDL+18]. Then, we can define the following inequations for multiplication according to [BDL+18]:

- If $\|f\|_\infty \leq \beta$, $\|g\|_1 \leq \gamma$ then $\|f \cdot g\|_\infty \leq \beta \cdot \gamma$.
- If $\|f\|_2 \leq \beta$, $\|g\|_2 \leq \gamma$ then $\|f \cdot g\|_\infty \leq \beta \cdot \gamma$.

Let $x, y \in \mathcal{O}_K$. Again, we assume that $C_m = 1$. Then, the following inequation holds according to [DPSZ12]:

$$(2.10) \quad \|x \cdot y\|_\infty \leq C_m \cdot n^2 \cdot \|x\|_\infty \cdot \|y\|_\infty$$

$$(2.11) \quad \|\sigma(x \cdot y)\|_\infty \leq \|\sigma(x)\|_\infty \cdot \|\sigma(y)\|_\infty.$$

2.2.2 Lattices

- background and history: example from lecture -> change

- * Birhoff [Bir40]
- * cryptanalysis [LLL82]
- * cryptosystems [Ajt96, HPS98] SIS introduced Ajtai [Ajt96]
- * [MR04]
- * LWE, assumption: worst-case lattice problems are hard [Reg05]
- * fully homomorphic [Gen09]
- * BGV scheme [BV11, BGV12]
- * tools [LPR10, LPR13] ideal lattices, RLWE

Other Notes: - PKE [AD97; Reg03; Reg05], CCA security [Pei09; PW08], identity-based encryption [ABB10; CHK010; GPV08], fully homomorphic [Gen09] - , LWE introduced by [Reg05] "provably as hard as certain lattice problems in worst case, appear to require time exponential in main security parameter to solve NTRU [HPS98] - q -ary lattice: modulus $q \geq 2$

- math * lattice Λ

- discrete additive subgroup of \mathbb{R}^m

- Let $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathbb{R}^m$ be a set of linearly independent basis vectors and $\mathbf{B} = [\mathbf{b}_1, \dots, \mathbf{b}_n] \in \mathbb{R}^{m \times n}$ be the corresponding basis with column vectors \mathbf{b}_i

- n is the dimension of the Lattice

- $\Lambda(\mathbf{B})$ defined by all integer combinations of elements of \mathbf{B} :

$$(2.12) \quad \Lambda(\mathbf{B}) = \left\{ \mathbf{x} \in \mathbb{R}^m \mid \exists \alpha_1, \dots, \alpha_n \in \mathbb{Z} : \mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{b}_i \right\}$$

- show example plot

- full-ranked lattice: dimension is maximal, m

- basis \mathbf{B} is not unique -> let $\mathbf{U} \in \mathbb{Z}^{n \times n}$ be a modular matrix (determinant is ± 1), then $\mathbf{B} \cdot \mathbf{U}$ is also a basis of the Λ ($\mathbf{U} \cdot \mathbb{Z}^n = \mathbb{Z}^n$) -> different basis for the same lattice Λ

- lattice coset: quotient group \mathbb{R}^n / Λ of cosets

$$\mathbf{c} + \Lambda = \mathbf{c} + \mathbf{v} \mid \mathbf{v} \in \Lambda$$

with $\mathbf{c} \in \mathbb{R}^n$

- fundamental domain: subset of \mathbb{R}^m containing exactly one representative of every coset

-(shifted) fundamental parallelepiped: $\mathcal{P}(\mathbf{B}) = \mathbf{B} \cdot [-1/2, 1/2]^n = \{ \mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} = \sum_{i=1}^n \gamma_i \mathbf{b}_i, \gamma_i \in [-1/2, 1/2] \}$
every coset has representative

- determinant of lattice $\Lambda(\mathbf{B})$: $\sqrt{\det(\mathbf{B}^\top \mathbf{B})}$. For a full-ranked lattice the determinant is

$$(2.13) \quad \det(\Lambda(\mathbf{B})) = |\det(\mathbf{B})|$$

is well-defined (independent from basis) => volume of fundamental domain can be generalized to not full-ranked => $\det(\Lambda(\mathbf{A})) = \sqrt{\det(\mathbf{A}^\top \mathbf{A})}$

* minimum distance of $\lambda_1(\Lambda)$ of a lattice is the length of its shortest nonzero vector, i.e. $\lambda_1(\Lambda) = \min_{\mathbf{v} \in \Lambda \setminus \{0\}} \|\mathbf{v}\|$ * i th successive minimum $\lambda_i(\Lambda)$ - Let $r \in \mathbb{R}$ and $\mathbf{c} \in \mathbb{R}^m$, then we define $\mathcal{B}(\mathbf{c}, r)$ as the ball of radius r with center \mathbf{c} .

- smallest radius r such that the ball $\mathcal{B}(\mathbf{0}, r)$ centered at the origin of Λ contains i linearly independent lattice vectors.

- in general hard to calculate $\lambda_i(\Lambda(\mathbf{B}))$ for a given basis

* modular integer (or q -ary) lattices

2 Preliminaries

- full-ranked lattice Λ such that $q\mathbb{Z}^m \subseteq \Lambda \subseteq \mathbb{Z}^m$ given $q \in \mathbb{N} \Rightarrow$ if $\mathbf{x} \in \mathbb{Z}^m$ in Λ then $\mathbf{x} \bmod q$ also in Λ .

- can be specified in two ways by matrix $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$:

$$(2.14) \quad \Lambda_q(\mathbf{A}) = \{\mathbf{x} \in \mathbb{Z}^m \mid \exists \mathbf{y} \in \mathbb{Z}^n : \mathbf{x} = \mathbf{A}\mathbf{y} \bmod q\}$$

or

$$(2.15) \quad \Lambda_q^\perp(\mathbf{A}) = \{\mathbf{x} \in \mathbb{Z}^m \mid \mathbf{A}^\top \mathbf{x} = \mathbf{0} \bmod q\}$$

- finding a short vector in $\Lambda_q(\mathbf{A})$ corresponds to LWE

- finding short vectors in $\Lambda_q^\perp(\mathbf{A})$ corresponds to SIS

- easy to find basis of $\Lambda_q(\mathbf{A})$ [AFG13]

- with high probability determinant of q -ary lattice is $\det(\Lambda_q(\mathbf{A})) = q^{m-n}$ if $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$

* Gram-Schmidt basis

- set of column vectors $\mathbf{B} \in \mathbb{Z}_q^{m \times n}$, $\pi_{\text{span}(\mathbf{B})}(\mathbf{t})$ for projection of vector \mathbf{t} unto span of vectors of \mathbf{B}

- $\pi_{\text{span}(\mathbf{B})}(\mathbf{t}) = \mathbf{B}(\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top \cdot \mathbf{t}$

- Gram-Schmidt orthogonalization $\tilde{\mathbf{B}} = [\tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_n]$ of basis \mathbf{B} : $\tilde{\mathbf{b}}_i = \mathbf{b}_i - \pi_{\text{span}(\mathbf{b}_1, \dots, \mathbf{b}_{i-1})}(\mathbf{b}_i)$ for $i \in \{1, \dots, n\}$

- Gram-schmidt coefficients $\mu_{i,j} = \frac{\langle \tilde{\mathbf{b}}_j, \mathbf{b}_i \rangle}{\langle \tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j \rangle}$

Alternative: Let $\mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_n]$, $\mathbf{b}_i \in \mathbb{Z}_q^m$ be a basis. Define $\tilde{\mathbf{b}}_i$ as follows: $\tilde{\mathbf{b}}_1 = \mathbf{b}_1$. For $i \in \{2, \dots, n\}$ let $\tilde{\mathbf{b}}_i$ be the component of \mathbf{b}_i that is orthogonal to the span of $\{\mathbf{b}_1, \dots, \mathbf{b}_{i-1}\}$. Then, $\tilde{\mathbf{B}} = [\tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_n]$ is called the Gram-Schmidt orthogonalization of basis \mathbf{B} where $\|\tilde{\mathbf{b}}_i\| \leq \|\mathbf{b}_i\|$.

* dual of a lattice is "the set of points whose inner products with the vectors in the lattice are integers" Λ : $\Lambda^\perp := \{\mathbf{y} \in \mathbb{R}^m \mid \forall \mathbf{v} \in \Lambda : \langle \mathbf{y}, \mathbf{v} \rangle \in \mathbb{Z}\}$ scaled-by- q dual lattice: $\{\mathbf{y} \in \mathbb{Z}^m \mid \forall \mathbf{v} \in \Lambda : \langle \mathbf{y}, \mathbf{v} \rangle = 0 \bmod q\}$ basis of the dual of a lattice with basis \mathbf{B} is $\mathbf{B}' = \mathbf{B}(\mathbf{B}^\top \mathbf{B})^{-1}$

* smoothing lemma

* Voronoi region The fundamental Voronoi region \mathcal{V} is defined as

$$(2.16) \quad \mathcal{V} = \{\mathbf{x} \in \mathbb{R}^n \mid \forall \mathbf{y} \in \Lambda : \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|\}$$

* Linear Code [Van12] Let \mathbb{F}_q^n be the n -dimensional vector space over the field \mathbb{F}_q . A q -ary linear code C or $[n, k]$ -code is a k -dimensional linear subspace of \mathbb{F}_q^n such that

- $\mathbf{0} \in C$,
- if $\mathbf{x}, \mathbf{y} \in C$, then $\mathbf{x} + \mathbf{y} \in C$,
- and if $\mathbf{x} \in C$ and $\gamma \in \mathbb{F}_q$, then $\gamma \mathbf{x} \in C$.

There are q^k different codewords in C .

Let C be a q -ary linear $[n, k]$ -code. The lattice over C is defined as

$$(2.17) \quad \Lambda(C) = \{\mathbf{x} \in \mathbb{R}^n \mid \exists \mathbf{y} \in C : \mathbf{x} = \mathbf{y} \pmod{q}\}.$$

Similarly, for a lattice $\Lambda(\mathbf{B})$ a lattice code C defined by $\Lambda(\mathbf{B})$ and a shaping region $\mathcal{V} \subset \mathbb{R}^n$ (e.g. the Voronoi region) is a subspace of \mathbb{R}^n such that all codewords are lattice vectors in $\Lambda(\mathbf{B})$ within the region \mathcal{V} [SFS08]:

$$(2.18) \quad C' = \{x \in \Lambda(\mathbf{B}) \mid x \in \mathcal{V}\}.$$

We define $\text{dist}(\mathbf{t}, \Lambda(\mathbf{B}))$ where $\Lambda(\mathbf{B}) \subset \mathbb{R}^m$ as the distance of a vector $\mathbf{t} \in \mathbb{R}^m$ to the closest lattice vector $\mathbf{v} \in \Lambda(\mathbf{B})$, i.e.

$$(2.19) \quad \text{dist}(\mathbf{t}, \Lambda(\mathbf{B})) = \min_{\mathbf{v} \in \Lambda(\mathbf{B})} \|\mathbf{t} - \mathbf{v}\|.$$

- Lattice problems

* Minkowski theorem: Let Λ be a lattice of dimension n , then $\lambda_1 \leq \sqrt{n} \cdot (\det \Lambda)^{\frac{1}{n}}$

* Lattice reduction: find short basis compared to $\lambda_1(\Lambda)$...

* SVP: given a basis \mathbf{B} of lattice Λ find shortest nonzero lattice vector $\Rightarrow v \in \Lambda$ s.t. $\|v\| = \lambda_1(\Lambda)$

Definition 2.2.1 (γ -approximate Shortest Vector Problem (SVP_γ))

Given a basis \mathbf{B} of lattice Λ , find a short lattice vector $v \in \Lambda$ such that $0 < \|v\| \leq \gamma \lambda_1(\Lambda)$

Definition 2.2.2 (κ -approximate Hermite Shortest Vector Problem (HSVP_κ))

Given a basis \mathbf{B} of a lattice $\Lambda(\mathbf{B}) \in \mathbb{R}^m$, find a nonzero lattice vector $\mathbf{v} \in \Lambda$ such that $\|\mathbf{v}\| \leq \kappa \cdot \det(\Lambda)^{\frac{1}{n}}$.

* GAPSV_γ (decision version of SVP): "given basis \mathbf{B} of n -dimensional lattice Λ with either $\lambda_1 \Lambda \leq 1$ or $\lambda_1 \Lambda \geq \gamma(n)$, decide which is the case" NP hard for any constant γ fastest algorithm for $1 \leq \gamma \leq \text{poly}(n)$ has runtime complexity of $2^{O(n)}$

* γ -unique Shortest Vector Problem (uSVP_γ) [LM09]: given lattice Λ such that $\lambda_2(\Lambda) > \gamma \lambda_1(\Lambda)$, find shortest nonzero vector in $\mathbf{v} \in \Lambda$ with $\|\mathbf{v}\| = \lambda_1(\Lambda)$

* CVP_γ : given basis \mathbf{B} of n -dimensional lattice Λ and target $\mathbf{t} \in \mathbb{R}^n$ find point in lattice that is close to $\mathbf{t} \Rightarrow$ find $\mathbf{v} \in \mathbb{R}^n$ with $\|\mathbf{t} - \mathbf{v}\| < \gamma \min_{\mathbf{v}' \in \Lambda} \|\mathbf{v}' - \mathbf{v}\|$

* SIVP (shortest independent vector problem): given basis \mathbf{B} of n -dimensional lattice Λ , find n linearly independent lattice vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \Lambda(\mathbf{B})$ such that $\max_i \|\mathbf{v}_i\|$ for $i \in \{1, \dots, n\}$ is minimal

* γ -Bounded Distance Decoding (BDD_γ): Given a lattice Λ and a target vector \mathbf{t} such that $\text{dist}(\mathbf{t}, \Lambda) < \gamma \lambda_1(\Lambda)$, find the closest lattice vector $\mathbf{v} \in \Lambda$, i.e. find $\mathbf{v} = \min_{\mathbf{v}' \in \Lambda} \|\mathbf{v}' - \mathbf{t}\|$ minimal [LM09]

* ideal lattice (do I need that?)

* ...?

* eher die Sachen für LWE/SIS als die Sachen für Algorithmen (analog Vorlesung), evtl.

Intuition für die anderen Sachen...

2.2.3 Distributions

- Gaussian, def, component-wise, trafo to bound

* definition: discrete Gaussian distribution over q -ary lattice Λ with Gaussian width parameter $s > 0$ and center \mathbf{c} , denoted by $D_{\Lambda, s, \mathbf{c}}$: probability of sampling a vector $\mathbf{x} \in \Lambda$ is proportional to $e^{-\pi \|\mathbf{x} - \mathbf{c}\|^2 / s^2}$. In order to avoid confusion, throughout this work and in the *Lattice Parameter Estimation* we use σ to denote the standard deviation, where $\sigma = \frac{s}{\sqrt{2\pi}}$, and define $\alpha := \frac{s}{q} = \frac{\sqrt{2\pi}\sigma}{q}$.

* better definition in GPV08 => different definition needed for LWE??? * how to do this? => variant of Babai's "nearest-plane" algorithm, see [GPV08]

* component-wise

For some applications, we receive a Gaussian distribution as input, but require a bound in some norm in order to estimate the hardness of SIS. Hence, we need to transform the Gaussian width parameter into a bound β given some security parameter sec . Note that a n -dimensional Gaussian can be sampled by sampling n independent 1-dimensional Gaussians.

For a Gaussian distribution, the following holds:

$$(2.20) \quad \Pr[|X| \geq \beta] \leq 2e^{-\pi\beta^2/s^2}$$

We demand $2e^{-\pi\beta^2/s^2} \approx 2^{-\text{sec}}$, hence

$$\begin{aligned} 2e^{-\pi\beta^2/s^2} &\approx 2^{-\text{sec}} \\ -\pi \frac{\beta^2}{s^2} &\approx (-\text{sec} - 1) \ln(2) \\ \beta &\approx s \sqrt{\frac{(\text{sec} + 1) \ln(2)}{\pi}} \end{aligned}$$

* smoothing factor here?

* Uniform (stuff I use in tool)

2.3 Two Important Problems

Applications: SIS can be used for one-way functions and collision-resistant hashing. LWE can be used to build pseudo-random number generators, public-key encryption schemes and oblivious transfer and secure MPC. Lattice Trapdoors (trapdoor functions, digital signatures)? Punctured Trapdoors (identity-based encryption, attribute-based encryption, predicate encryption)?

2.3.1 Learning with Errors (LWE)

Following based on [Reg10]:

Introduced by Regev in [Reg09] Origin: work of Ajtai and Dwork [AD97], first public-key cryptosystem based on worst-case lattice problems, simplifications/improvements [GGH97; Reg03] imply hardness result for LWE. Early work: hardness based on unique-SVP, Peikert [Pei09] and Lyubashevsky and Micciancio [LM09] show that unique-SVP is essentially equivalent to GapSVP .

- ‘cryptomania’ applications: public-key encryption schemes under chosen-plaintext attacks [KTX07; PVW08; Reg05], and chosen-ciphertext attacks [Pei09; PW08], oblivious transfer protocols [PVW08], identity-based encryption (IBE) schemes [ABB10; CHKP10; GPV08], leakage-resilient encryption [ACPS09; AGV09; DGK+10; GKPV10], and more

- most important: fully homomorphic encryption schemes [Bra12; BV11; Gen09; GSW13]

Intuition: - “recover $\mathbf{s} \in \mathbb{Z}_q^n$ given sequence of ‘approximate’ random linear equations on \mathbf{s} public matrix $\mathcal{A} \in \mathbb{Z}^{n \times m}$, secret vector $\mathbf{f} \in \mathbb{Z}^n$, given $\ddagger = \mathcal{A}^\top \mathbf{f}$ we can find \mathbf{s} by linear algebra when we add a small error vector $\mathbf{e} \in \mathbb{Z}^m$, solving $\ddagger' = \mathcal{A}^\top \mathbf{f} + \mathbf{e}$ for \mathbf{s} or distinguishing \mathbf{z}' from uniform becomes hard

Formal Definition:

Definition 2.3.1 (LWE Distribution [Reg10])

For $n \geq 1$, modulus $q \geq 2$, error distribution χ on \mathbb{Z}_q , and a fixed secret vector \mathbf{s} , let $\mathcal{A}_{\mathbf{s}, \chi}$ be the probability distribution over $\mathbb{Z}_q^n \times \mathbb{Z}_q$ by choosing a vector $\mathbf{a}_i \in \mathbb{Z}_q^n$ uniformly at random, $e_i \in \mathbb{Z}_q$ according to χ and returning pairs of $(\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \bmod q) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$.

Additions are performed in \mathbb{Z}_q . We say that an algorithm solves LWE with modulus q and error distribution χ if, for any $\mathbf{s} \in \mathbb{Z}_q^n$, given an arbitrary number of independent samples from $\mathcal{A}_{\mathbf{s}, \chi}$ it outputs \mathbf{s} (with high probability). For $q = 2$ corresponds to *learning parity with noise* (LPN) problem.

Definition 2.3.2 (Search-LWE $_{n,q,m,\chi}$)

Search-LWE $_{n,q,m,\chi}$ asks for the recovery of the secret vector \mathbf{s} given m independent samples $(\mathbf{a}_i, z_i) \leftarrow \mathcal{A}_{\mathbf{s}, \chi}$

Definition 2.3.3 (Decision-LWE $_{n,q,m,\chi}$)

Given m samples, Search-LWE $_{n,q,m,\chi}$ asks to distinguish whether the samples were drawn from $\mathcal{A}_{\mathbf{s}, \chi}$ or from a uniform distribution on $\mathbb{Z}_q^n \times \mathbb{Z}_q$.

LWE as a Decoding Problem

We request m samples $(\mathbf{a}_1, z_1), \dots, (\mathbf{a}_m, z_m)$ where $z_i = \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \in \mathbb{Z}_q$. Let $A = [\mathbf{a}_1 \cdots \mathbf{a}_m]$, $\mathbf{z} = [z_1, \dots, z_m]^\top$ and $\mathbf{e} = [e_1, \dots, e_m]^\top$. Hence, we can reformulate LWE as a decoding problem as in [GJS15]:

$$(2.21) \quad \mathbf{z} = \mathbf{A}^\top \mathbf{s} + \mathbf{e}$$

with generator matrix \mathbf{A} for a linear code over \mathbb{Z}_q and \mathbf{z} as the received word. Finding the secret vector \mathbf{z} is equivalent to finding the codeword $\mathbf{y} = \mathbf{A}^\top \mathbf{s}$ with minimum distance $\|\mathbf{y} - \mathbf{z}\|$.

An $\text{LWE}_{n,q,m,\chi}$ instance with a secret vector \mathbf{s} chosen according to a uniform distribution can be transformed into an $\text{LWE}_{n,q,m-n,\chi}$ instance with a secret vector $\hat{\mathbf{s}}$ chosen according to the error distribution χ at a loss of n samples as follows: Let $\mathbf{A}_0 = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ where $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the first n columns of \mathbf{A} . We introduce new variables $\hat{\mathbf{s}} = \mathbf{A}_0^\top \mathbf{s} - [z_1, \dots, z_n]^\top = [e_0, \dots, e_n]^\top$ and $\hat{\mathbf{A}} = \mathbf{A}_0^{-1} \mathbf{A} = [\mathbf{I} \ \hat{\mathbf{a}}_{n+1} \cdots \hat{\mathbf{a}}_m]$ and compute $\hat{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{A}}^\top [z_1, \dots, z_n]^\top = [\mathbf{0}, \hat{z}_{n+1} \cdots \hat{z}_m]^\top$.

LWE as a BDD Problem

Solving LWE also corresponds to solving the *Bounded Distance Decoding problem* (BDD) in the lattice $\Lambda(\mathbf{A}^\top) = \{\mathbf{x} \in \mathbb{Z}_q^m \mid \exists \mathbf{s} \in \mathbb{Z}_q^n : \mathbf{x} = \mathbf{A}^\top \mathbf{s} \bmod q\}$, where the m columns of \mathbf{A} correspond to the vectors $\mathbf{a}_i \in \mathbb{Z}_q^n$ of m independent LWE samples $(\mathbf{a}_i, z_i) \leftarrow \mathcal{A}_{\mathbf{s},\chi}$ and the components z_i correspond to a perturbed lattice point in $\Lambda(\mathbf{A}^\top)$.

Best algorithm to solve LWE: Blum, Kalai, and Wasserman [BKW03] with $2^{O(n)}$ samples and time.

Hardness: best algorithm exponential, extension of LPN (LPN believed to be hard), hard assuming worst-case hardness of GapSVP and SIVP [Pei09; Reg05]. More details? Different cases for q exponential/polynomial, approximation factors... Hardness based on worst-case lattice problems => strong security guarantees, such as conjectured security against quantum computers...

Search to decision reduction => distinguishing is LWE samples from uniform samples sufficient, worst-case to average-case reduction => sufficient to solve distinguishing for uniform secret

2.3.2 Short Integer Solution (SIS)

The dual problem to LWE is the *Short Integer Solution problem* (SIS).

- principle: given a set of set of uniformly random vectors $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{Z}_q^n$ find a subset of them or combination with small coefficients that sums to zero (modulo q).
- introduced in [MR04], origins in [Ajt96], used for ‘minicrypt’ primitives: one-way functions [Ajt96], collision resistant hash functions [GGH96], digital signature schemes [CHKP10; GPV08], and identification schemes [KTX07; Lyu08; MV03]

Definition 2.3.4 (SIS Problem (Adapted from [LS15], Definition 3.1))

)] The problem $\text{SIS}_{n,q,m,\beta}$ is defined as follows: Given a uniformly random matrix $\mathbf{A}^{n \times m}$, find a vector $\mathbf{s} \in \mathbb{Z}_q^m$ such that $\mathbf{A} \cdot \mathbf{s} = \mathbf{0} \bmod q$ and $0 < \|\mathbf{s}\| \leq \beta$.

Finding such a vector corresponds to finding a short lattice vector in cosets of the lattice $\Lambda^\perp(\mathbf{A}) = \{y \mid \mathbf{A} \cdot y \bmod q\}$

Hardness: for any poly-bounded m, β and for “large enough” prime q : $\text{SIS}_{n,q,m,\beta}$ is as hard as worst-case approx-SIVP (and GAP SVP) to within $\beta \cdot \tilde{O}(\sqrt{n})$ factor

2.3.3 Ring and Module Variants

- problem key sizes in LWE/SIS in $O(n^2)$ (matrix $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$), where $m \in \Omega(n)$
- idea: introduce some sort of a structure in samples: n power of two, \mathbf{a} vectors in groups of size n , for each group $\mathbf{a}_1 = [a_1, \dots, a_n]^\top$, a_i are uniformly random in \mathbb{Z}_q , and $\mathbf{a}_i = [a_i, \dots, a_n, -a_1, \dots, -a_{i-1}]^\top$. Hence, n vectors only need $O(n)$ memory, also speedups in operations by using FFT
- formally: vectors are elements of the ring $\mathbb{Z}_q[x] / \langle x^n + 1 \rangle$ which we call \mathcal{R}_q instead of the group \mathbb{Z}_q^n , n power of two ensures that $x^n + 1$ is irreducible over the rationals
- add more?

Definition 2.3.5 (Ring-SIS Problem [[LS15], Definition 3.3])

)] The problem $\text{RSIS}_{n,q,m,\beta}$ is defined as follows: Given $a_1, \dots, a_n \in \mathcal{R}_q$ chosen independently from the uniform distribution, find $s_1, \dots, s_n \in \mathcal{R}$ such that $\sum_{i=1}^m a_i \cdot s_i = 0 \bmod q$ and $0 < \|\mathbf{s}\| \leq \beta$, where $\mathbf{s} = [s_1, \dots, s_m]^\top \in \mathcal{R}^m$.

Definition 2.3.6 (Module-SIS Problem [[LS15], Definition 3.3])

)] The problem $\text{MSIS}_{n,d,q,m,\beta}$ is defined as follows: Given $a_1, \dots, a_n \in \mathcal{R}_q^d$ chosen independently from the uniform distribution, find $s_1, \dots, s_n \in \mathcal{R}$ such that $\sum_{i=1}^m a_i \cdot s_i = 0 \bmod q$ and $0 < \|\mathbf{s}\| \leq \beta$, where $\mathbf{s} = [s_1, \dots, s_m]^\top \in \mathcal{R}^m$.

While there exist special cases where the Ring structure of problem instances can be exploited in an attack on LWE or SIS , in general, the hardness of Ring and Module variants is estimated by interpreting the coefficients of elements of \mathcal{R}_q as vectors in \mathbb{Z}_q^n [ACD+18]. We thus reduce Ring and Module instances as follows:

- $\text{RLWE}_{n,q,m,\chi} \longrightarrow \text{LWE}_{n,q,m \cdot n,\chi}$
- $\text{MLWE}_{n,d,q,m,\chi} \longrightarrow \text{LWE}_{n \cdot d,q,m \cdot n,\chi}$
- $\text{RSIS}_{n,q,m,\beta} \longrightarrow \text{SIS}_{n,q,m \cdot n,\beta}$
- $\text{MSIS}_{n,d,q,m,\beta} \longrightarrow \text{SIS}_{n \cdot d,q,m \cdot n,\beta}$

Note that in the Ring and Module variants n denotes the degree of the polynomial of the underlying Ring, while in the standard variant, n denotes the dimension of the secret.

3 Algorithms and Estimates

3.1 Lattice Basis Reduction

Problem: usually ugly basis (long vectors...), we want a better basis with shorter and more orthogonal basis vectors... - improve lattice basis quality => measure by hermite factor (compare shortest vector in basis to lattice volume) or approximation factor (compare shortest vector in basis to shortest lattice vector) - algorithm finding vector with approximation factor γ can be used to solve uSVP with gap $\lambda_2(\Lambda)/\lambda_1(\Lambda) > \gamma$ - best known theoretical bound by Slide reduction [GN08a], BKZ better in practice

- measure quality of basis: Hermite factor

* basis $\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$, m -dimensional lattice $\Lambda(\mathbf{B})$ has root Hermite factor δ if

$$(3.1) \quad \|\mathbf{b}_1\| \approx \delta^m \det(\Lambda)^{1/m}$$

* use Geometric Series Assumption (GSA) [Sch03] to obtain estimates for \mathbf{b}_i :

$$(3.2) \quad \|\tilde{\mathbf{b}}_i\| \approx \alpha^{i-1} \|\mathbf{b}_1\|$$

for $0 < \alpha < 1$ Equation (3.1) into Equation (3.2) -> $\|\tilde{\mathbf{b}}_i\| \approx \alpha^{i-1} \delta^m \det(\Lambda)^{1/m}$ with $\prod_{i=1}^m \|\tilde{\mathbf{b}}_i\| = \det(\Lambda)$ we get

$$\begin{aligned} \prod_{i=1}^m \|\tilde{\mathbf{b}}_i\| &\approx \prod_{i=1}^m \alpha^{i-1} \delta^m \det(\Lambda)^{1/m} \\ \Leftrightarrow \det(\Lambda) &\approx \delta^{2m} \det(\Lambda) \prod_{i=1}^m \alpha^{i-1} \\ \Leftrightarrow \delta^{-m^2} &\approx \alpha^{\frac{m(m-1)}{2}} \\ \Leftrightarrow \delta^{-2} &\approx \alpha^{(m-1)/m} \end{aligned}$$

Hence, $\alpha \approx \delta^{-2}$ and

$$(3.3) \quad \|\tilde{\mathbf{b}}_i\| \approx \delta^{-2(i-1)+m} \det(\Lambda)^{1/m}$$

* good basis -> first Gram-Schmidt vectors become shorter (latter longer)

* $\delta = 1.01$ feasible, $\delta = 1.007$ seems infeasible for now

* gap between provable and experimental cost estimate to reach some hermite $\delta \Rightarrow$ provable results only give upper bounds, for practical security we need lower bound \Rightarrow combine theoretical results with experimental results

* well-established estimate [LP11]

In the following, we will focus on two related methods for lattice reduction.

3.1.1 The LLL Algorithm

The LLL algorithm was proposed by Lenstra, Lenstra and Lovász [LLL82] and can be considered as a generalization of the two dimensional Lagrange reduction. The lagrange reduction reduces a basis of two basis vectors such that output basis satisfies $\|\mathbf{b}_1\| \leq \|\mathbf{b}_2\|$ and $\frac{|\langle \mathbf{b}_1, \mathbf{b}_2 \rangle|}{\|\mathbf{b}_1\|} = |\mu_{2,1}| \leq \frac{1}{2}$. Intuitively, a multiple of the shorter vector \mathbf{b}_1 is subtracted from the longer vector \mathbf{b}_2 such that the resulting vector \mathbf{b}'_2 is as orthogonal to \mathbf{b}_0 as possible, i.e. $\mathbf{b}'_1 = \mathbf{b}_1 - \lfloor \mu_{1,0} \rfloor \mathbf{b}_0$. We set $\mathbf{b}_2 = \mathbf{b}'_2$ and repeat until nothing changes.

A δ -LLL reduced basis ensures two criterias [LLL82]:

1. Size reduced: $|\mu_{i,j}| \leq \frac{1}{2}$ for $1 \leq i \leq n$ and $j < i$
2. Lovász condition: $\delta \|\tilde{\mathbf{b}}_i\|^2 > \|\mu_{i+1,i} \tilde{\mathbf{b}}_i + \tilde{\mathbf{b}}_{i+1}\|^2$ for $1 \leq i < n$

Recall the definition of the Gram-Schmidt coefficients $\mu_{i,j} = \frac{\langle \tilde{\mathbf{b}}_j, \mathbf{b}_i \rangle}{\langle \tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j \rangle}$. The LLL algorithm shown in Algorithm 1 follows the notation in [Reg04]. We start by computing the Gram-Schmidt orthogonalization of the input basis (Line 2) and continue with a reduction step in which we update every basis vector \mathbf{b}_i by pairwise comparing and subtracting lower indexed basis vector just as in the Lagrange reduction (Line 5) to ensure Criteria 1. Finally, vectors violating the Lovász condition are swapped (Line 6ff) and the process is repeated until nothing changes. The LLL algorithm can be used to find short vectors of at most $2^{n/2} \lambda_1(\Lambda)$ in polynomial time. Several floating-point variants have been suggested that can significantly speed up the runtime of LLL. For example, L^2 runs in $O(n^2 \log^2 B)$, where B is a bound on the norm of the input basis vectors [NS05].

Algorithm 1: The δ -LLL Algorithm [LLL82]

```

1 function  $\delta$ -LLL( $\mathbf{B} \in \mathbb{Z}^{m \times n}$ )
2   Compute  $\tilde{\mathbf{B}}$ 
3   for  $i = 2, \dots, n$  do
4     for  $j = i - 1, \dots, 1$  do
5        $\mathbf{b}_i = \mathbf{b}_i - \lfloor \mu_{i,j} \rfloor \mathbf{b}_j$ 
6   if  $\exists i$  such that  $\delta \|\tilde{\mathbf{b}}_i\|^2 > \|\mu_{i+1,i} \tilde{\mathbf{b}}_i + \tilde{\mathbf{b}}_{i+1}\|^2$  then
7      $\text{tmp} = \mathbf{b}_i$ 
8      $\mathbf{b}_i = \mathbf{b}_{i+1}$ 
9      $\mathbf{b}_{i+1} = \text{tmp}$ 
10    Return  $\delta$ -LLL( $\mathbf{B}$ )
11 else
12   Return  $\mathbf{B}$ 

```

3.1.2 The BKZ Algorithm

The Block Korkin-Zolotarev (BKZ) algorithm was proposed by Schnorr in 1987 and adapted by Schnorr and Euchner in [SE91] and represents a family of lattice reduction algorithm. Essentially, BKZ iteratively divides the input basis into blocks of a lower dimension k and calling an SVP oracle on each block. The output of the oracle is then used to obtain a basis of improved quality.

Algorithm 2 presents the main concept of BKZ and follows the description in [CN11] with some adjustments. Initially, we run an LLL reduction on the input basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and update the basis. In each j th iteration, we consider a block of k basis vectors $\mathbf{b}_j, \dots, \mathbf{b}_{j+k-1}$. The vectors of the current block are projected onto the orthogonal complement of the span of vectors from previous iterations $\text{span}(\{\mathbf{b}_i \mid i \in [j-1]\})$ (Line 6 - 9, we skip this step if the span is empty). Note that the orthogonal complement A^\perp of a subspace A is defined as the set of all vectors that are orthogonal to every vector in A . We then run an SVP oracle on the projected block to obtain a shortest vector \mathbf{b}'_{new} in the projected lattice (Line 12) and reconstruct a lattice vector \mathbf{b}_{new} of which \mathbf{b}'_{new} is a projection Line 13. Note that in practice, the SVP oracle should include this step. If \mathbf{b}_{new} is a new vector we insert it in our list of basis vectors before \mathbf{b}_j . Otherwise as nothing changed, we increment a counter z . Finally, we run LLL on all basis vectors up to index $j + i$ (including the possibly newly added vector). If no new lattice vectors can be found in n iterations, the reduction terminates. After n iterations, j is reset to start over at the first block. The output of the algorithm is a BKZ_k -reduced basis. For $k = 2$ we obtain an LLL-reduced basis in polynomial time and for $k = n$ an optimally HKZ-reduced basis in at least exponential time.

Algorithm 2: The BKZ Algorithm [SE91]

```

1 function BKZ( $\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}, k \in [n] \setminus \{1\}$ )
2    $z = 1; j = 0$ 
3    $\mathbf{B} = \text{LLL}(\mathbf{B})$ 
4   while  $z < n - 1$  do
5      $j = (j \bmod (n - 1)) + 1; l = \min(j + k - 1, n); h = \min(l + 1, n)$ 
6      $A = \text{span}(\{\mathbf{b}_i \mid i \in [j - 1]\})$ 
7     for  $i \in \{j, \dots, l\}$  do
8       if  $A \neq \emptyset$  then
9          $\mathbf{b}'_i = \pi_{A^\perp}(\mathbf{b}_i)$ 
10      else
11         $\mathbf{b}'_i = \mathbf{b}_i$ 
12       $\mathbf{b}'_{\text{new}} = \text{SVP-Oracle}(\mathbf{b}'_j, \dots, \mathbf{b}'_l)$ 
13      Reconstruct  $\mathbf{b}_{\text{new}} = \sum_{i=j}^l \alpha_i \mathbf{b}_i$  with  $\alpha_i \in \mathbb{Z}$  such that  $\mathbf{b}'_{\text{new}} = \pi_{(\text{span}(\mathbf{b}_j, \dots, \mathbf{b}_l))^\perp}(\mathbf{b}_{\text{new}})$ 
14      if  $\mathbf{b}'_{\text{new}} \neq \tilde{\mathbf{b}}_j$  then
15         $z = 0; \{\mathbf{b}_j, \dots, \mathbf{b}_h\} = \text{LLL}(\{\mathbf{b}_j, \dots, \mathbf{b}_{j-1}, \mathbf{b}_{\text{new}}, \mathbf{b}_j, \dots, \mathbf{b}_h\})$ 
16      else
17         $z = z + 1; \{\mathbf{b}_j, \dots, \mathbf{b}_h\} = \text{LLL}(\{\mathbf{b}_j, \dots, \mathbf{b}_h\})$ 

```

Several improvements have been suggested. The total number of rounds until termination is unknown and can be quite large. Hanrot *et al.* [HPS11] show an early termination of BKZ still yields a very good output basis quality and propose $\frac{n^2}{k^2} \log n$ rounds as a bound. The Local preprocessing increases the quality of the current block basis by recursively calling BKZ with smaller block size. A variant known as progressive BKZ applies the recursion globally [AWHT16]. If enumeration is used as an SVP oracle (see Section 3.1.3), extreme pruning can be applied to significantly reduce the search space. Gamma *et al.* show that applying such a bounding function on the search tree reduces the running time by a much larger factor than the success probability. Repeating the search yields the desired speedup [GNR10]. In addition, [CN11] optimizes the enumeration radius by using experimental results. BKZ 2.0 incorporates a number of these techniques [CN11].

It is difficult to find hard runtime bounds for BKZ. The upper bound on the number of rounds is superexponential in the dimension n for a fixed block size [GN08b; HPS11] before BKZ terminates by itself. In addition, calls to the SVP oracle in all dimensions $k' \leq k$ must be taken into account. Albrecht *et al.* ignore these intricacies and estimate the cost of BKZ in clock cycles as $\rho \cdot n \cdot t_k$ where ρ is the number of rounds needed and t_k is the cost (in block cycles) of calling the SVP oracle on a block of dimension k Albrecht *et al.* [APS15]. The value ρ is set to 8 in the *Estimator* derived from experiments in [Che13] that indicate that the most significant progress happens in the first 7 – 9 rounds.

3.1.3 Cost Models for Lattice Reduction

In this section, we will look at various high level ideas to realize an SVP solver that can be used as a subroutine in BKZ and present up-to-date cost models from the literature. SVP is known to be NP-complete even for large constant approximation factors [Ajt98; Kho05]. An exponential approximation factor can be achieved in polynomial time but is mostly insufficient for practical purposes [LLL82]. We will mainly focus on two classes of (nearly) exact SVP solvers, namely, enumeration algorithms and sieving algorithms. Enumeration algorithms can solve SVP in a lattice of dimension k in $2^{O(k \log k)}$ time and polynomial space. Sieving algorithms only need $2^{O(k)}$ time, however, at the cost of exponential memory complexity. Only recently, progress in sieving strategies has given rise to BKZ implementations relying on sieving (e.g. the General Sieve Kernel (G6K) implementation [ADH+19; DSW21]) that outperform enumeration implementations already in relatively small dimensions $\gtrsim 70$ in the classical setting [ABLR21]. On the other hand, quantum speedups for enumeration are greater than for sieving. Aono *et al.* [ANS18] show a quadratic cost reduction for enumeration, while the cost sieving only decreases by a factor of $2^{0.027}$ with idealized assumptions [Laa16].

Enumeration

Enumeration aims to find the shortest vector by enumerating all lattice vectors within some bounded region. In general, we start with reducing the lattice basis to improve the basis quality. We then define a bound and iteratively project the lattice to the span of its Gram-Schmidt vectors beginning from $\tilde{\mathbf{b}}_n$ until we arrive at the lowest level of a one-dimensional subspace. We continue by enumerating all vectors of norm less than r in the projected lattice and “lift” each of these vectors to the level above and repeat this process until we arrive at the level from which we started. The search space can be thought of as a large tree of (projected) vectors on which we apply depth-first

search. Note that the root of the tree here is at the lowest level and the leafs are the lattice vectors in our target lattice. The low memory cost of enumeration is due to its similarities to depth-first search.

A very early but very efficient variant was suggested by Kannan [Kan83] with a worst-case runtime of $2^{O(k \log k)}$. BKZ_k using Kannan’s enumeration algorithm as SVP oracle yields a short lattice vector of norm $\approx (k^{1/(2k)})^n \cdot \text{Vol}(\Lambda)^{1/n}$ [ABF+20; HS07].

In [ABF+20; Che13] we find a more concrete cost model of $\text{poly}(n) \cdot 2^{1/(2e)k \log k + -0.995k + 16.25}$ for BKZ 2.0 (see Section 3.1.2), where $\text{poly}(n)$ is the number of calls to the enumeration subroutine.

BKZ 2.0 achieves a root Hermite factor of $\left(\frac{k}{2\pi e} \cdot (\pi k)^{1/k} \right)^{\frac{1}{2(k-1)}}$.

The FastEnum algorithm in Albrecht et al. [ABF+20] incorporates an idea called “extended preprocessing” and achieves a root Hermite factor of $k^{\frac{1}{2k}(1+o(1))}$ in $\text{poly}(n) \cdot 2^{0.125k \log k - 0.050k + 56}$ time. The corresponding quantum algorithm reduces the runtime to $k^{k/16+o(k)}$. In extended preprocessing, instead of preprocessing the current projected basis block of size k , the BKZ-reduction is applied to a block of higher dimension $\lceil (1+c) \cdot k \rceil$ for some constant c . Enumeration is faster on the first basis vectors as their Gram-Schmidt norms closely follow the Geometric Series Assumption [MW16].

A tradeoff of runtime and success probability for “relaxing” the approximation and extreme pruning turns out to exponentially speed up the search [LN20] and was combined with extended preprocessing in [ABF+20] to further reduce the runtime of BKZ to $\text{poly}(n) \cdot 2^{0.125k \log k - 0.654k + 25.84}$ for a root Hermite factor of $k^{1/(2k)}$.

Sieving

The second group of SVP solvers are sieving algorithms. In sieving, initially, we create a long list of randomly selected lattice points. The points in the list are then combined or “reduced” in some way to find points of smaller length. One way to achieve this is by finding a minimal sublist of “center” points in the initial list such that spheres centered at these points cover all points list points. Subtracting the center points yields short lattice points. ListSieve [MV10] uses a smaller initial list to divide the space into two half-spaces, one closer to the center and one closer to the respective point. The list is then used to reduce the length of newly sampled points as much as possible by subtracting each list vectors such that the result is located in the half-space closer to the center respectively. Once two points with a distance less than the target distance are found, they are subtracted and the result is returned.

Table 3.1 presents a list of currently best sieving algorithms. In the Nguyen-Vidick sieve [NV08], we iteratively reduce a pair of list points whose combined length is smaller than the longest list vector. The longest vector is then replaced by the result. The list length is fixed. In the Gauss sieve [MV10], we start with an empty list and a stack. In each step, a new point is either sampled or taken from the stack. We then attempt to reduce the new point with all points in the list. If a reduction is successful, the longer vector of the pair is replaced. If the longer vector was the list point, the replacement is inserted in the stack. If no reduction is possible, the stack points are moved back to the list. If the stack is empty, all list points are reduced pairwise. In practice, the Gauss sieve outperforms Nguyen-Vidick sieve. The Becker-Gama-Joux sieve [BGJ15] exploits coding theory to find vectors that are likely to be nearest neighbors. Similar vectors are stored

Name	Cost Estimate
List Sieve [MV10]	$2^{0.3199n+o(n)}$ time, $2^{0.1325+o(n)}$ memory
NV-sieve [ADH+19; NV08]	$2^{0.415n+o(n)}$ time, $2^{0.2075n+o(n)}$ memory
NV-sieve (quantum) [ADH+19; NV08]	$2^{0.311n+o(n)}$ time, $2^{0.2075n+o(n)}$ memory
Gauss sieve [HK17; MV10]	$2^{0.415n+o(n)}$ time, $2^{0.2075n+o(n)}$ memory
BGJ-sieve [BGJ15]	$2^{0.311n+o(n)}$ time, $2^{0.2075n+o(n)}$ memory
3-sieve [BLS16; HK17]	$2^{0.3962n+o(n)}$ time, $2^{0.1887n+o(n)}$ memory
BDGL-sieve [BDGL16]	$2^{0.292n+o(n)}$ time
BDGL-sieve (quantum) [BDGL16]	$2^{0.265n+o(n)}$ time

Table 3.1: Sieving Algorithms

Name	Reference	Cost model
Q-Core-Sieve	[ADPS16; AGPS20; Laa16]	$2^{0.265n}$
Q-Core-Sieve + O(1)	[SAL+17]	$2^{0.265n+16}$
Q-Core-Sieve (min space)	[SHRS17]	$2^{0.2975n}$
Q- β -Sieve	[NAB+17]	$n \cdot 2^{0.265n}$
Q- $8d$ -Sieve	[BAA+17]	$8d \cdot 2^{0.265n+16.4}$
Core-Sieve	[ADPS16; AGPS20; BDGL16]	$2^{0.292n}$
Core-Sieve + O(1)	[SAL+17]	$2^{0.292n+16}$
Core-Sieve (min space)	[SHRS17]	$2^{0.368n}$
β -Sieve	[NAB+17]	$n \cdot 2^{0.292n}$
$8d$ -Sieve	[DTGW17]	$8d \cdot 2^{0.292n+16.4}$
Q-Core-Enum + O(1)	[ACD+18; Che13; SHRS17]	$2^{\frac{0.18728n \log n - 1.0192n + 16.1}{2}}$
Lotus (Enum)	[ACD+18; PHAM17]	$2^{0.125n \log n - 0.755n + 2.254}$
Core-Enum + O(1)	[ACD+18; Che13; SHRS17]	$2^{0.18728n \log n - 1.0192n + 16.1}$
$8d$ -Enum (quadratic fit) + O(1)	[BCLV17]	$8d \cdot 2^{0.000784n^2 + 0.366n + 0.875}$
BKZ 2.0 Core-Enum	[ABF+20; Che13; CN11]	$2^{0.184k \log k - 0.995k + 16.25}$
ABF+ Core-Enum	[ABF+20]	$2^{0.125k \log k}$
ABF+ Core-Enum + O(1)	[ABF+20]	$2^{0.125k \log k - 0.547k + 10.4}$
ABF Q-Core-Enum	[ABF+20]	$2^{0.0625k \log k}$
ABLR Core-Enum + O(1)	[ABLR21]	$2^{0.125k \log k - 0.654k + 25.84}$

Table 3.2: Cost Models Overview

in the same bucket to speed up the search for reduction candidates. The 3-sieve [BLS16; HK17] reduces the required list size by using triples instead of pairs of points for combination. Finally, the Becker-Ducas-Gama-Laarhoven sieve [BDGL16] applies locality sensitive hashing to create buckets of points in near neighborhood similar to the Becker-Gama-Joux sieve.

TODO: small section about use in tool... Table 3.2 provides an overview of the cost models that are built into the Lattice Parameter Estimation.

3.2 Algorithms for Solving LWE

3.2.1 Overview

Distinguishing attacks (MR09, RS10): distinguish (with noticeable advantage) LWE instance from uniformly random \Rightarrow break semantic security of LWE-based cryptosystem with same advantage (typically), find short nonzero integral vector \mathbf{v} s.t. $\mathbf{A}^\top \mathbf{v} = \mathbf{0} \pmod{q} \Rightarrow$ short vector in (scaled) dual of LWE lattice $\Lambda(\mathbf{A})$ then test whether $\langle \mathbf{v}, \mathbf{z} \rangle$ is close to zero mod q . If uniform test accepts with prob $1/2$, if LWE with parameter s , $\langle \mathbf{v}, \mathbf{z} \rangle = \langle \mathbf{v}, \mathbf{e} \rangle \pmod{q}$, Gaussian mod q with parameter $\|\mathbf{v}\| \cdot s$. If that's not much larger than q , advantage for distinguishing very close to $\exp(-\pi(\|\mathbf{v}\|s/q)^2)$. high confidence needs $\|\mathbf{v}\| \leq q/(2s)$ advantage an computational effort need to be balanced (often inverse distinguishing advantage is in total cost of attack)

Dual Attacks

reduce LWE to SIS recover secret vector by finding a short vector in the dual lattice $\Lambda(\mathbf{A}^\top)^\perp = \{\mathbf{y} \in \mathbb{Z}^m \mid \mathbf{A}\mathbf{y} = \mathbf{0} \pmod{q}\}$ generated by the rows of \mathbf{A} and scaled by q .

Primal Attacks

lattice reduction algorithms solve SIS and BDD

Direct

- algebraic approach Arora and Ge with subexponential complexity when $\sigma \leq \sqrt{n}$, else fully exponential, mainly of asymptotic interest (higher complexity than others)
- combinatorial algorithms: BKW as basis [BKW03], resembles generalized birthday approach by Wagner, originally for solving LPN, can be analyzed \Rightarrow explicit complexity for different LWE instances, theoretical analysis and actual performance close, very memory expensive (often same order as time complexity)

3.2.2 BKW [BKW03]

The Blum, Kalai and Wasserman (BKW) algorithm was originally designed to solve the Learning Parity with Noise problem (LPN) [BKW03]. In Section 2.3.1 we pointed out that LPN is a subproblem of LWE and Albrecht *et al.* adapted BKW to LWE in [ACF+15]. The runtime and memory complexity of BKW is in $2^{O(n)}$ for an LWE instance with secret dimension n prime modulus $q \in \text{poly}(n)$. The number of samples m must be sufficiently large (in $O(n \log n)$).

BKW falls into the regime of dual attacks, that is, it solves LWE by finding a short vector \mathbf{s} in the scaled dual lattice $\Lambda(\mathbf{A}^\top)^\perp$.

three stages [ACF+15]: sample reduction, hypothesis testing and back substitution

Sample Reduction. In the following, we present an outline of the main BKW algorithm. The steps in Algorithm 3 are inspired by the textual description in [GJS15] with minor adjustments in notation.

Algorithm 3: BKW (Sample Reduction)

```

1 function BKW( $\mathbf{A} \in \mathbb{Z}^{n \times m}, \mathbf{z} \in \mathbb{Z}^m, b \in \mathbb{Z}, d \in \mathbb{Z}$ )
2    $i = 1$ 
3    $\mathbf{A}^{(i)} = \mathbf{A}$ 
4    $\mathbf{z}^{(i)} = \mathbf{z}$ 
5   while the last  $n - d$  coefficients of the columns of  $\mathbf{A}^{(i)}$  are nonzero do
6     // BKW step
7      $j = 1$ 
8      $\mathbf{T}^{(i)} = []$  // Collision table
9     for  $k = 1, \dots, m^{(i)}$  do
10      //  $m^{(i)}$  is number of columns in  $\mathbf{A}^{(i)}$ 
11      if last  $(i \cdot b)$  coefficients of  $\mathbf{a}_k^{(i)}$  are zero then
12         $\mathbf{a}_j^{(i+1)} = \mathbf{a}_k^{(i)}$ 
13         $z_j^{(i+1)} = z_k$ 
14         $j = j + 1$ 
15      else if no match for  $\mathbf{a}_k^{(i)}$  in  $\mathbf{T}$  then
16         $\mathbf{T} = \mathbf{T} + [\mathbf{a}_k^{(i)}]$  // append to collision set
17      else if match  $\mathbf{a}_l^{(i)}$  for  $\mathbf{a}_k^{(i)}$  is found then
18        if  $\mathbf{a}_l^{(i)}$  matches  $\mathbf{a}_k^{(i)}$  in the last  $(i \cdot b)$  components then
19           $\mathbf{a}_j^{(i+1)} = \mathbf{a}_k^{(i)} - \mathbf{a}_l^{(i)}$ ; // last  $i \cdot b$  coefficients of  $\mathbf{a}_j^{(i+1)}$  are now zero
20           $z_j^{(i+1)} = z_k^{(i)} - z_l^{(i)} = y_j^{(i)} + e_j^{(i)}$ , where  $y_j^{(i)} = \langle \mathbf{s}, \mathbf{a}_j^{(i)} \rangle$  and  $e_j^{(i)} = e_k^{(i)} - e_l^{(i)}$ 
21           $j = j + 1$ 
22        else if the negation of  $\mathbf{a}_l^{(i)}$  in  $\mathbb{Z}_q^n$  matches  $\mathbf{a}_k^{(i)}$  in the last  $(i \cdot b)$  components then
23           $\mathbf{a}_j^{(i+1)} = \mathbf{a}_k^{(i)} + \mathbf{a}_l^{(i)}$ 
24           $z_j^{(i+1)} = z_k^{(i)} + z_l^{(i)} = y_j^{(i)} + e_j^{(i)}$ , where  $y_j^{(i)} = \langle \mathbf{s}, \mathbf{a}_j^{(i)} \rangle$  and  $e_j^{(i)} = e_k^{(i)} + e_l^{(i)}$ 
25           $j = j + 1$ 
26       $i = i + 1$ 
27      // Calculate input for next BKW step
28       $\mathbf{A}^{(i)} = (\mathbf{a}_1^{(i)} \dots \mathbf{a}_{j-1}^{(i)})$ 
29       $\mathbf{z}^{(i)} = (z_1^{(i)}, \dots, z_{j-1}^{(i)})$ 
30  Return  $(\mathbf{A}^{(i)}, \mathbf{z}^{(i)})$ 

```

For the algorithm, we use the matrix notation of LWE as in Equation (2.21), i.e. $\mathbf{z} = \mathbf{A}^\top \mathbf{s} + \mathbf{e}$. BKW consists of a series of BKW steps that iteratively reduce the dimension of input matrix \mathbf{A} by finding collisions of its column vectors in the currently examined block of b entries. We start from the last b entries of $\mathbf{A}^{(1)} = \mathbf{A}$. In every step i , we maintain a collision table $\mathbf{T}^{(i)}$ and loop over the columns $\mathbf{a}_k^{(i)}$ of $\mathbf{A}^{(i)}$ and distinguish between the following cases: (1) If $\mathbf{a}_k^{(i)}$ only has zero entries

in the examined block, pass $\mathbf{a}_k^{(i)}$ and $z_k^{(i)}$ to the next step, (2) if no match of $\mathbf{a}_k^{(i)}$ or the negation of $\mathbf{a}_k^{(i)}$ can be found in the collision table, add $\mathbf{a}_k^{(i)}$ to the collision table, and (3) if a match $\mathbf{a}_l^{(i)}$ is found, compute $\mathbf{a}_l^{(i)} + \mathbf{a}_k^{(i)}$ or in the case of a negation match $\mathbf{a}_l^{(i)} - \mathbf{a}_k^{(i)}$ (in \mathbb{Z}_q) such that the last b nonzero entries cancel out. By exploiting the symmetry of \mathbb{Z}_q in this way, in every step we obtain at most $(q^b - 1)/2$ columns with distinct coefficients in the current b entries. We also make note of “observed symbols” $z_j^{(i)}$ that represent the combination of two samples given their respective matching columns (see lines 20, 24 for more details).

In each BKW step, the number of columns (and samples) decreases by at least $(q^b - 1)/2$ (size of the collision set) and the variance of the error distribution σ^2 increases by a factor of two. The algorithm terminates after $t = \lceil b/(n - d) \rceil$ steps returns a set of observed symbols $\mathbf{z}^{(t)}$ and a corresponding reduced matrix $\mathbf{A}^{(t)}$ in which only the first d rows have nonzero entries. Parameter d should be set to 1, as in the original BKW algorithm, or 2 for the best performance [ACF+15].

The remaining part \mathbf{s}' of the secret vector \mathbf{s} is then guessed by means of hypothesis testing. After t steps the error term $\left(\mathbf{z}_j^{(t)} - \langle \mathbf{s}', \mathbf{a}_j^{(t)} \rangle\right)$ with $j \in [m']$ of the m' remaining observed symbols follows a Gaussian distribution χ with noise $\sigma'^2 = 2^t \cdot \sigma^2$ (see Lemma 1 in [ACF+15]). We can test the noise of the error term for all $\mathbf{s}'' \in \mathbb{Z}_q^d$ against the hypothesized noise σ'^2 by means of the log-likelihood ratio (for details we again refer to [ACF+15]) and are thus able to determine \mathbf{s}' given sufficiently many samples m' .

Finally, we can apply back substitution to recover all elements of \mathbf{s} . We again apply a similar procedure as in Algorithm 3 to reduce a number of columns from the collision tables computed in the Sample Reduction step and obtain m' columns with $d + d'$ nonzero entries and their corresponding “observed symbols”. We then substitute the part of \mathbf{s} that was recovered in the previous steps and recover the next part of \mathbf{s} by hypothesis testing and repeat the process until we have found \mathbf{s} .

Theorem 1 (BKZ Complexity [ACF+15], Corollary 2)

Let (\mathbf{a}_i, z_i) be samples following $\mathcal{A}_{\mathbf{s}, \chi}$, set $a = \lfloor \log_2(1/(2\alpha)^2) \rfloor$, $b = n/a$ and q a prime. Let d be a small constant $0 < d < \log_2(n)$. Assume α is such that $q^b = q^{n/a} = q^{\lfloor \log_2(1/(2\alpha)^2) \rfloor}$ is superpolynomial in n . Then, given these parameters the cost of the BKW algorithm to solve Search-LWE is

$$(3.4) \quad \left(\frac{q^b - 1}{2}\right) \cdot \left(\frac{a(a-1)}{2} \cdot (n+1)\right) + \left\lceil \frac{q^b}{2} \right\rceil \cdot \left(\left\lceil \frac{n}{d} \right\rceil + 1\right) \cdot d \cdot a + \text{poly}(n) \approx (a^2 n) \cdot \frac{q^b}{2}$$

operations in \mathbb{Z}_q . Furthermore,

$$(3.5) \quad a \cdot \left\lceil \frac{q^b}{2} \right\rceil + \text{poly}(n) \text{ calls to } \mathcal{A}_{\mathbf{s}, \chi} \text{ and storage of } \left(a \cdot \left\lceil \frac{q^b}{2} \right\rceil \cdot n\right) \text{ elements in } \mathbb{Z}_q \text{ are needed.}$$

The first summand of Equation (3.4) roughly corresponds to the cost of creating the collision tables and the second summand is the cost of backsubstitution. For a more detailed cost analysis, see Theorem 2 in [ACF+15].

Coded-BKW [GJS15]

- change BKW step -> more column entries are removed, but additional noise - index set I , \mathbf{x}_I is part of \mathbf{x} with entries indexed by I - step i : I set of b positions to be removed, fix some q -ary linear $[N_i, b]$ code C_i with q^b codewords, find the closest codeword $\mathbf{c}_I \in C$ for every input vector \mathbf{a}_I such that $\mathbf{a}_I = \mathbf{c}_I + \mathbf{e}_I$, where the error part $\mathbf{e}_I \in \mathbb{Z}_q^{N_i}$ is minimized by a decoding procedure.

Finally, we subtract two vectors and their corresponding samples and pass the result to the next BKW step. Consider the inner product $\langle \mathbf{s}_I, \mathbf{a}_I \rangle = \langle \mathbf{s}_I, \mathbf{c}_I \rangle + \langle \mathbf{s}_I, \mathbf{e}_I \rangle$. In the subtraction, only the error part $\langle \mathbf{s}_I, \mathbf{e}_I \rangle$ remains.

3.2.3 Dual Attack [MR09]

"Gama and Nguyen [GN08b]: (in)feasibility of obtaining various Hermite factors natural distinguishing attack on LWE by finding one relatively short vector in associated lattice"

3.2.4 Decoding Attack [LP11]

combines lattice basis reduction followed by an enumeration algorithm (bounded-distance decoding with preprocessing?) => time/success tradeoff specifically for LWE, exploits structural properties of LWE on search version of LWE problem, approach preferable to distinguishing attack on decision LWE in [MR09; RS10], same or better advantage than distinguishing attack using lattice vectors of lower quality => runtime is smaller post-reduction: simple extension of Babai's "nearest-plane" algorithm [Bab85] => trade basis quality against decoding time related to Klein's (de)randomized algorithm [Kle00] for bounded-distance decoding

use entire reduced basis, post-reduction part is fully parallelizable

LLL reduction to input Lattice, integer combination of basis vectors close to target (like inner loop in reduction step of LLL), seek vector in lattice close to target, finds output that is in fundamental parallelepiped $\mathcal{P}(\mathbf{B})$ Section 2.2.2 => if error vector not in $\mathcal{P}(\mathbf{B})$, secret is not restored => basis quality has to be sufficiently good

Algorithm 4: Babai's Nearest Plane Algorithm [Bab85]

```

1 function NearestPlane( $\mathbf{B} \in \mathbb{R}^{m \times n}, \mathbf{t} \in \mathbb{R}^m$ )
2   run  $\delta$ -LLL on basis  $\mathbf{B}$  with  $\delta = \frac{3}{4}$ 
3    $\mathbf{b} = \mathbf{t}$ 
4   for  $i = n, \dots, 1$  do
5      $c_i = \text{round}(\langle \mathbf{b}, \tilde{\mathbf{b}}_i \rangle / \langle \tilde{\mathbf{b}}_i, \tilde{\mathbf{b}}_i \rangle)$   $\mathbf{b} = \mathbf{b} - c_i \mathbf{b}_i$ 
6   output  $\mathbf{t} - \mathbf{b}$ 

```

Output is a lattice vector $\mathbf{v} \in \Lambda(\mathbf{B})$ such that $\|\mathbf{v} - \mathbf{t}\| \leq 2^{n/2} \text{dist}(\mathbf{t}, \Lambda(\mathbf{B}))$

Goal: recover lattice vector relatively close to target vector Intuition: - project \mathbf{t} to $\text{span}(\mathbf{B})$ - from $i = n, \dots, 1$ find closest hyperplane $c_i \tilde{\mathbf{b}}_i + \text{span}(\mathbf{b}_1, \dots, \mathbf{b}_i)$ to the projection, subtract $c_i \mathbf{b}_i$ from the projection and continue - output vector is $\sum_{i=1}^n c_i \mathbf{b}_i$ for every basis vector \mathbf{b}_i find c_i such that distance between target and hyperplane spanned by $\mathbf{b}_1, \dots, \mathbf{b}_{i-1}$ and shifted by $c_i \tilde{\mathbf{b}}_i$ is minimal, subtract $c_i \mathbf{b}_i$ from target vector and continue for $i = n, \dots, 1$. After the last iteration $\sum_{i=1}^n c_i \mathbf{b}_i$ is returned.

Application to LWE: $\mathbf{t} = \mathbf{A}^\top \mathbf{s} + \mathbf{e} \Rightarrow$ we get \mathbf{v} where $\mathbf{t} - \mathbf{v} = \mathbf{e}$ is in fundamental parallelepiped of Gram-Schmidt basis

Generalized version by [LP11]: Problem: in reduced basis last Gram-Schmidt vectors of \mathbf{B} short, first long \Rightarrow long and skinny parallelepiped, Gaussian \mathbf{e} unlikely to be in it \Rightarrow incorrect answer from NearestPlane

\Rightarrow generalized version admitting time/success tradeoff recurse on some $d_i \geq 1$ distinct planes in i th

Algorithm 5: Generalized Nearest Plane Algorithm [LP11]

```

1 function GeneralizedNearestPlane( $\mathbf{B} \in \mathbb{R}^{m \times k}, \mathbf{t} \in \mathbb{R}^m, \mathbf{d} \in (\mathbb{Z}^+)^k$ )
2   if  $k = 0$  then
3     Return  $\mathbf{0}$ 
4   else
5     Compute projection  $\mathbf{v}$  of  $\mathbf{t}$  onto  $\text{span}(\mathbf{B})$ 
6     Compute the  $d_k$  distinct integers  $c_1, \dots, c_{d_k}$  closest to  $\langle \mathbf{v}, \tilde{\mathbf{b}}_k \rangle / \langle \tilde{\mathbf{b}}_k, \tilde{\mathbf{b}}_k \rangle$ 
7     Return  $\bigcup_{i \in \{1, \dots, d_k\}} (c_i \cdot \mathbf{b}_k +$ 
      GeneralizedNearestPlane( $\{\mathbf{b}_1, \dots, \mathbf{b}_{k-1}\}, (d_1, \dots, d_{k-1}), \mathbf{v} - c_i \cdot \mathbf{b}_k)$ )

```

Instead of choosing only the nearest plane in each iteration step, Algorithm 5 selects a variable amount d_k of distinct planes in each step. As a consequence, the fundamental parallelepiped of the Gram-Schmidt basis is stretched in the direction of $\tilde{\mathbf{b}}_k$. The values of \mathbf{d} should be chosen such that the covered area is approximately the same in each direction (i.e. by maximizing $\min_i (d_i \cdot \|\tilde{\mathbf{b}}_i\|)$). In particular this implies that the d_k are larger for larger k as the Gram-Schmidt vectors have a smaller length. Compared to Algorithm 4 the runtime increases by a factor $\prod_{i \in \{1, \dots, d_k\}} d_i$, however, the recursion step can be fully parallelized.

It should be evident that a lower quality of the reduced input basis can be compensated for by increasing the values of \mathbf{d} . Hence we can adjust the input parameters for the lattice reduction and Algorithm 5 to minimize the runtime given a fixed required success probability.

3.2.5 Primal-uSVP [ADPS16; BG14]

BKZ: reduce lattice basis using SVP oracle in smaller dimension b , known that number of calls to oracle polynomial - enumeration algorithm as oracle: in super-exponential time - sieve algorithms as oracle: exponential time but so far slower in practice for accessible dimensions $b \approx 130$

primal attack: construct unique-SVP instance from LWE instance LWE instance $(\mathbf{A}, \mathbf{z} = \mathbf{A}^\top \mathbf{s} + \mathbf{e})$
 construct lattice

$$(3.6) \quad \Lambda = \{ \mathbf{x} \in \mathbb{Z}^{m+n+1} \mid (\mathbf{A}^\top - \mathbf{I}_m - \mathbf{b})\mathbf{x} = \mathbf{0} \pmod{q} \}$$

lattice has dimension $d = m + n + 1$, volume q^m and unique-SVP solution $\mathbf{v} = (\mathbf{s}, \mathbf{e}, 1)$

success condition: - geometric series assumption known to be optimistic from attacker's point of view \Rightarrow finds basis with Gram-Schmidt norms $\|\mathbf{g}_i\| = \delta^{d-2i-1} \cdot \text{Vol}(\Lambda)^{1/d}$ and $\delta = ((\pi b)^{1/b} \cdot b/2\pi e)^{1/2(b-1)}$ unique short vector \mathbf{v} is detected if projection of \mathbf{v} onto span of last b Gram-Schmidt vectors is shorter than \mathbf{g}_{d-b} , norm of projection is expected to be $\gamma\sqrt{b} \Rightarrow$ attack successful iff $\gamma\sqrt{b} \leq \delta^{d-2i-1} \cdot q^{m/d}$

LWE as inhomogeneous-SIS (ISIS)

As in Section 3.2.4, we view the $\text{LWE}_{n,q,m,\chi}$ instance (\mathbf{A}, \mathbf{z}) as a BDD instance in the q -ary lattice $\Lambda(\mathbf{A}^\top) = \{ \mathbf{y} \mid \exists \mathbf{x} \in \mathbb{Z}_q^n : \mathbf{y} = \mathbf{A}^\top \mathbf{x} \pmod{q} \}$ Section 2.3.1 generated by rows of LWE instance. The target vector is \mathbf{z} .

Recall the γ -uSVP problem. Given a lattice Λ where $\lambda_2(\Lambda) > \gamma\lambda_1(\Lambda)$, we are asked to find shortest nonzero vector in Λ . In the primal attack, instead of directly solving BDD, we reduce BDD to uSVP, i.e., we reduce a BDD instance to a γ -uSVP instance. By solving γ -uSVP, we obtain a solution to BDD. To do this we apply Kannan's embedding technique [Kan87]. Intuitively, Kannan's embedding creates a lattice with uSVP structure. We know that $\mathbf{A}^\top \mathbf{s} \pmod{q}$ is the closest vector to the target $\mathbf{z} = \mathbf{A}^\top \mathbf{s} + \mathbf{e}^\top \pmod{q}$ in $\Lambda(\mathbf{A}^\top)$. We now add a linearly independent basis vector (\mathbf{z}, μ) and append a zero coefficient to each basis vector of the original lattice (i.e. the rows of \mathbf{A}). Thereby, we ensure that the new lattice contains the vector $[-\mathbf{e}, -\mu]^\top$ as $[\mathbf{A} \mid \mathbf{0}]^\top \mathbf{s} - 1 \cdot [\mathbf{z}^\top, \mu] = [-\mathbf{e}, -\mu]^\top$.

More formally, let \mathbf{B} be a basis of $\Lambda(\mathbf{A}^\top)$ and an embedding factor $\mu = \text{dist}(\mathbf{z}, \Lambda(\mathbf{A}^\top)) = \|\mathbf{z} - \mathbf{s}\|$ where \mathbf{s} is the secret vector of the LWE instance. A relatively close approximation of μ can be guessed in polynomial time (see [LM09] for more details). We now embed $\Lambda(\mathbf{A}^\top)$ into $\Lambda(\mathbf{B}')$ with γ -uSVP structure as follows:

$$(3.7) \quad \mathbf{B}' = \begin{pmatrix} \mathbf{B} & \mathbf{z} \\ \mathbf{0}^\top & \mu \end{pmatrix}$$

If $\gamma \geq 1$ and $\mu < \frac{\lambda_1(\Lambda(\mathbf{B}))}{2\gamma}$ (or equivalently, $(\Lambda(\mathbf{A}^\top), \mathbf{z})$ a $\text{BDD}_{1/(2\gamma)}$ -instance), then $\Lambda(\mathbf{B}')$ contains a γ -unique shortest vector $\mathbf{z}' = [(\mathbf{A}^\top \mathbf{s} - \mathbf{z})^\top, -\mu]^\top = [-\mathbf{e}^\top, -\mu]^\top$. The statement can be proven by showing by contradiction that all vectors $\mathbf{v} \in \Lambda(\mathbf{B}')$ that are independent of \mathbf{z}' satisfy $\|\mathbf{v}\| \geq \lambda_1(\Lambda(\mathbf{B}'))/\sqrt{2} > \sqrt{2}\gamma\mu = \gamma\|\mathbf{z}'\|$ (see Section 4 of [LM09] for more details). Note that the reduction can be done in polynomial time (Theorem 4.1 in [LM09]). Hence, from \mathbf{z}' we can recover the error vector \mathbf{e} and thereby the secret vector $\mathbf{s} = \mathbf{z} - \mathbf{e} \pmod{q}$.

A solution to γ -uSVP can be found by reducing it to κ -HSVP where $\gamma = \kappa^2$ [APS15]. Various algorithms, in particular, lattice reduction algorithms, exist to solve κ -HSVP. If we are able to solve a linear number of κ -HSVP instances that correspond to a κ^2 -approximate SVP instance, we can construct a solution the latter (see Definition 2.2.1, see Section 1.2.21 in [Lov87] for more details). Consider any lattice with uSVP structure. In exactly one direction, that is, in the direction of its unique shortest vector, the lattice has vectors that are significantly smaller than in other directions. A lattice reduction algorithm that yields a sufficiently good output basis quality, therefore, must return

some small vector in the desired direction. Let \mathbf{v} be a solution to SVP_κ^2 , i.e. $\|\mathbf{v}\| \leq \kappa^2 \lambda_1(\Lambda)$. All other vectors $\mathbf{w} \in \Lambda$ that are not multiples of a shortest vector have length $\|\mathbf{w}\| \geq \lambda_2(\Lambda) > \kappa^2 \lambda_1(\Lambda)$. Thus, we obtain a solution to γ -uSVP and, as shown above, we can reconstruct the secret vector to solve LWE.

3.2.6 Meet-in-the-Middle [APS15]

3.2.7 Arora-Ge [AG11]

3.3 Algorithms for Solving SIS

Recall that the $\text{SIS}_{n,q,m,\beta}$ problem asks to find a short vector $\mathbf{s} \in \mathbb{Z}_q^m$ of norm $\|\mathbf{s}\| \leq \beta$ such that $\mathbf{A} \cdot \mathbf{s} = 0 \pmod q$ for some uniformly distributed matrix $\mathbf{A}^{n \times m}$. Solving SIS is equivalent to finding a short vector in the dual lattice $\Lambda(\mathbf{A}^\top)^\perp = \{\mathbf{y} \in \mathbb{Z}^m \mid \mathbf{A}\mathbf{y} = \mathbf{0} \pmod q\}$.

3.3.1 Dual Attack [MR09; RS10]

Our first approach to solving SIS follows quite naturally. Given \mathbf{A} , we can efficiently compute the basis $\mathbf{B}(\mathbf{B}^\top \mathbf{B})^{-1}$ of the dual lattice $\Lambda(\mathbf{A}^\top)^\perp$ in polynomial time using Gauss-Jordan elimination or other more modern algorithms.

We can then apply a lattice reduction algorithm and obtain a basis with root Hermite factor δ . The first basis \mathbf{b}_1 vector of the reduced basis has length $\|\mathbf{b}_1\| = \delta^m \det(\Lambda(\mathbf{A}^\top)^\perp)^{1/m}$. We can see that δ depends on the subdimension m which we want to be ideal in order to minimize the cost of the lattice reduction by relaxing δ .

We further assume that $\det(\Lambda(\mathbf{A}^\top)^\perp) = \text{Vol}(\Lambda(\mathbf{A}^\top)^\perp) = q^n$. For q prime and m much larger than n we have that the rank of \mathbf{A} is n as the rows of \mathbf{A} are with high probability linearly independent. As a result of the rank-nullity theorem, the nullity or the dimension of the kernel of \mathbf{A} is $m - n$. Hence the number of lattice vectors in \mathbb{Z}_q^m is q^{m-n} and the assumption follows immediately.

We now have our first equation

$$(3.8) \quad \|\mathbf{b}_1\| = \delta^m q^{\frac{n}{m}},$$

which becomes minimal for $m = \sqrt{n \log q / \log \delta}$.

Theorem 2 (Optimal subdimension m [MR09])

Given a q -ary scaled dual lattice $\Lambda(\mathbf{A}^\top)^\perp$ defined by a matrix $\mathbf{A} \in \mathbb{Z}^{n \times m}$ with m sufficiently larger than n and a prime q . Then a lattice reduction algorithm yields an optimal output if performed in subdimension

$$(3.9) \quad m' = \sqrt{\frac{n \log q}{\log \delta}}.$$

Higher dimension increase the complexity of the reduction algorithms and lower dimensions may cause a lack of sufficiently short lattice vectors [MR09]. In contexts in which Equation (3.8) does not hold, we may still choose m as in Equation (3.9) heuristically. Removing columns from \mathbf{A} does not greatly impact our results since we can just set the corresponding components of the secret vector s to zero. We reformulate Equation (3.8) a bit:

(3.10)

$$\|\mathbf{b}_1\| = \delta^m q^{\frac{n}{m}} \iff \log \beta = m \log \delta + \frac{n \log q}{m}$$

(3.11)

$$\iff \log \delta = \frac{\log \beta}{m} - \frac{n \log q}{m^2}$$

We continue by plugging Equation (3.9) into Equation (3.11):

(3.12)

$$\log \delta = \frac{\log \beta}{\sqrt{\frac{n \log q}{\log \delta}}} - \frac{n \log q}{\left(\sqrt{\frac{n \log q}{\log \delta}}\right)^2} \iff \log \delta = \frac{\log \beta}{\sqrt{\frac{n \log q}{\log \delta}}} - \log \delta$$

(3.13)

$$\iff 2 \log \delta = \frac{\log \beta}{\sqrt{\frac{n \log q}{\log \delta}}}$$

(3.14)

$$\iff \log \delta = \frac{\log^2 \beta}{4n \log q}$$

Theorem 3 (Optimal subdimension m [MR09])

Given a q -ary scaled dual lattice $\Lambda(\mathbf{A}^\top)^\perp$ defined by a matrix $\mathbf{A} \in \mathbb{Z}^{n \times m}$ with m sufficiently larger than n and a prime q . Then a lattice reduction algorithm performed in its optimal subdimension achieves a log root Hermite factor of

$$(3.15) \quad \log \delta = \frac{\log^2 \beta}{4n \log q}.$$

MR variant [MR09]

RS variant [RS10]

"concrete estimates of "symmetric bit security", concrete runtime estimates for various Hermite factors in random q -ary lattices" permissive form of distinguishing attack in [MR09], adversarial advantage is about 2^{-72}

3.3.2 Combinatorial Attack [MR09]

3.4 Tool

class for distributions... from section this modelling, problems, generic search... Überblick, wie verwendbar, automatische norm umwandlung, sonstige features

3.4.1 Runtime and Cost Comparison

defaults... schnellste, beste => effizient, etc. parallel... problem reductions...

4 Usage Examples

4.1 Two Problem Search

two problems Iwe sis, what they are, how to solve it by the tool

4.2 TODO: find other schemes to apply

5 Conclusion

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