# Multivariate density estimation from privatised data: universal consistency and minimax rates

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October 15, 2021

#### Abstract

We revisit the classical problem of nonparametric density estimation, but impose local differential privacy constraints. Under such constraints, the original multivariate data  $X_1, \ldots, X_n \in \mathbb{R}^d$  cannot be directly observed, and all estimators are functions of the randomised output of a suitable privacy mechanism. The statistician is free to choose the form of the privacy mechanism, and in this work we propose to add Laplace distributed noise to a discretisation of the location of a vector  $X_i$ . Based on these randomised data, we design a novel estimator of the density function, which can be viewed as a privatised version of the well-studied histogram density estimator. Our theoretical results include universal pointwise consistency and strong universal  $L_1$ -consistency. In addition, a convergence rate over classes of Lipschitz functions is derived, which is complemented by a matching minimax lower bound. We illustrate the trade-off between data utility and privacy by means of a small simulation study.

Keywords: nonparametric multivariate density estimation, local differential privacy, density-free consistency, universal pointwise consistency, rate of convergence, minimax rate AMS Subject classification: 62G08, 62G20.

#### 1 Introduction

Let X be a random vector taking values in  $\mathbb{R}^d$ . We denote by  $\mu$  the distribution of the vector X, that is, for all Borel sets  $A \subset \mathbb{R}^d$ , we have

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 $\mu(A) = \mathbb{P}\{X \in A\}$ . Assume that  $\mu$  has a density denoted by f. In the classical, non-private, setup the density estimation problem is to estimate f based on data consisting of independent, identically distributed (i.i.d.) copies of the vector X:

$$\mathcal{D}_n = \{X_1, \dots, X_n\}.$$

In this context, two questions are relevant:

- (i) How to estimate the density?
- (ii) Why to estimate a density?

Concerning question (i), which is usually the only one addressed in the literature, we exclusively consider histogram estimators in this paper. To formulate such an estimator, let  $\{A_{h_n,1}, A_{h_n,2}, \ldots\}$  be a cubic partition of  $\mathbb{R}^d$ , such that the cells  $A_{h_n,j}$  are cubes of volume  $h_n^d$ . A natural estimator for  $\mu(A)$  is given by the empirical distribution

$$\mu_n(A) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{\{X_i \in A\}},$$

where  $\mathbb I$  denotes the indicator function. Then, the standard histogram estimator is defined by

$$f_n(oldsymbol{x}) = rac{\mu_n(A_{h_n,j})}{h_n^d} \qquad ext{if } oldsymbol{x} \in A_{h_n,j}.$$

Concerning question (ii) stated above, the obvious aim is to derive probability estimates from density estimates. For example, if we estimate the probability

$$\mu(A) := \int_A f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

by

$$\check{\mu}_n(A) := \int_A \check{f}_n(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},$$

for some density estimator  $\check{f}_n$ , then Scheffé's theorem yields

$$TV(\mu, \check{\mu}_n) := \sup_{A \in \mathscr{B}} |\mu(A) - \check{\mu}_n(A)| = \frac{1}{2} \int_{\mathbb{R}^d} |f(\boldsymbol{x}) - \check{f}_n(\boldsymbol{x})| d\boldsymbol{x}, \qquad (1)$$

where  $\mathscr{B}$  is the Borel  $\sigma$ -algebra of  $\mathbb{R}^d$ . This suggests to measure the performance of an arbitrary density estimator  $\check{f}_n$  by the  $L_1$ -error

$$\int_{\mathbb{D}_d} |f(\boldsymbol{x}) - \check{f}_n(\boldsymbol{x})| \mathrm{d}\boldsymbol{x}.$$

We refer the interested reader to Devroye and Györfi [8] for a comprehensive introduction to density estimation from the  $L_1$ -point of view. Using (1), one can immediately derive uniformly convergent probability estimates from  $L_1$ -consistent density estimators.

For the standard case of non-private inference, where estimators can be expressed in terms of the original data  $\mathcal{D}_n$ , strong universal consistency of the histogram can be established under appropriate conditions on the smoothing parameter  $h_n$ . More precisely, we can cite the following well-known theorem.

**Theorem 1** (Abou-Jaoude [1], Theorem 3.2 in Devroye and Györfi [8]). If

$$\lim_{n\to\infty} h_n = 0 \ and \ \lim_{n\to\infty} nh_n^d = \infty,$$

then

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}|f(\boldsymbol{x})-f_n(\boldsymbol{x})|\mathrm{d}\boldsymbol{x}=0\quad almost\ surely.$$

If we further restrict the class of density functions under consideration, one can also obtain bounds on the weak rate of convergence, which is done in the following theorem.

**Theorem 2** (Cf. Beirlant and Györfi [3], Theorems 5.5 and 5.6 in Devroye and Györfi [8]). If X has a bounded support S of Lebesgue measure  $\lambda(S)$ , and f is Lipschitz continuous with Lipschitz constant L, then

$$\mathbb{E}\left\{\int_{\mathbb{R}^d} |f(\boldsymbol{x}) - f_n(\boldsymbol{x})| d\boldsymbol{x}\right\} \le L\sqrt{d}h_n + \frac{\lambda(S)}{\sqrt{nh_n^d}}.$$

In particular, the choice  $h_n = c \cdot n^{-1/(d+2)}$  yields

$$\mathbb{E}\left\{\int_{\mathbb{R}^d}|f(\boldsymbol{x})-f_n(\boldsymbol{x})|\mathrm{d}\boldsymbol{x}\right\}=O\left(n^{-1/(d+2)}\right).$$

Here, the last theorem has been formulated in a slightly more general form (the given references cover only the case where f is supposed to be continuously differentiable with compact support, which implies that f is Lipschitz continuous).

	non-private histogram	privatised histogram
(UPC)	$h_n \to 0,  nh_n^d/\log n \to \infty$	$h_n \to 0,  r_n \to \infty,  nh_n^{2d}/\log n \to \infty$
(SUC)	$h_n \to 0, nh_n^d \to \infty$	$(\mathrm{UPC}) + nh_n^d/(r_n^d \log n) \to \infty$
(LIP)	$n^{-\frac{1}{d+2}}$	$n^{-\frac{1}{2d+2}}$

Table 1: Comparison of classical and (modified) privatised histogram estimator. The rows indicate: (UPC) assumptions for universal pointwise consistency, (SUC) assumptions for strong universal consistency, and (LIP) minimax optimal rate over Lipschitz classes.

The point of origin of the present paper is to derive results in the spirit of the cited Theorems 1 and 2 in the framework of local differential privacy (LDP), where one does not have access to the original data  $\mathcal{D}_n$  but only to some appropriately anonymised surrogates. As the method of choice we suggest a privatised version of the histogram. Anonymisation is achieved by publishing only a Laplace perturbed version of the random variable  $\mathbb{I}_{\{X_i \in A_{h_n,j}\}}$  for a finite number of indices j, namely those for which the cell  $A_{h_n,j}$  has a non-empty intersection with the closed ball of radius  $r_n$  centred at the origin. In addition to LDP analogues of Theorems 1 and 2, we also derive a lower bound for the rate of convergence over Lipschitz classes under LDP showing that the private histogram attains the optimal rate on this class. Our findings can be boiled down to the statement that known results from the scenario with non-private observations can be transferred to the LDP scenario, but the conditions on the smoothing sequence  $h_n$  are harder and the attainable rates of convergence worse under LDP; cf. Table 1 for a condensed comparison of the two histogram estimators.

The remaining part of the paper is organised as follows. In Section 2 we introduce a privatised version of the multivariate histogram estimator. Section 3 contains consistency results, namely universal pointwise consistency and strong universal  $L_1$ -consistency of the private histogram. In Section 4 we obtain the rate of convergence of the private histogram for Lipschitz continuous densities, which is complemented by a lower bound in Section 5. Section 6 provides a simulation study illustrating our findings. All proofs are deferred to Section 7.

# 2 Histogram estimator under LDP

In order to define the privatised histogram estimator we first choose a sequence of closed balls  $(S_n)_{n\geq 1}\subset \mathbb{R}^d$  centred at the origin. The radius of

 $S_n$  is denoted by  $r_n > 0$ . As in the introduction, let  $\{A_{h_n,1}, A_{h_n,2}, \ldots\}$  be a cubic partition of  $\mathbb{R}^d$ . Without loss of generality we can assume that the cells  $A_{h_n,j}$  are numbered such that  $A_{h_n,j} \cap S_n \neq \emptyset$  when  $j \leq N_n$  for some positive integer  $N_n$ , and  $A_{h_n,j} \cap S_n = \emptyset$  otherwise. It is not difficult to see that  $N_n \approx (r_n/h_n)^d$ .

For nonparametric regression estimation and for classification, Berrett and Butucea [4] and Berrett, Györfi and Walk [5] introduced a non-interactive privacy mechanism. In their setup locally privatised samples are created as follows: the statistician (data base provider) sets and announces some  $\sigma_W > 0$  and the *i*-th data holder with  $i \in \{1, \ldots, n\}$  generates and transmits to the statistician the data

$$Z_i := \{W_{n,i,j}, \quad j \le N_n\},$$
 (2)

where

$$W_{n,i,j} := \mathbb{I}_{\{X_i \in A_{h_n,j}\}} + \sigma_W \zeta_{i,j}. \tag{3}$$

Here the random variables  $\zeta_{i,j}$ , i = 1, ..., n, j = 1, 2, ... are i.i.d. according to a Laplace distribution with unit variance, which has the probability density

$$p(x) = \exp(-\sqrt{2}|x|)/\sqrt{2}.$$

Note that the *i*-th data holder transforms the *d*-dimensional vector  $X_i$  into randomised,  $N_n$ -dimensional data  $Z_i$ .

Let us briefly recall the definition of  $\alpha$ -local differential privacy. We refer the reader to the seminal paper by Duchi, Jordan and Wainwright [9] for a comprehensive introduction. A non-interactive privacy mechanism is a family of conditional distributions drawing the *i*-th privatised output  $Z_i$  from a measurable space  $(\mathcal{Z}, \mathcal{Z})$  given the corresponding raw datum  $X_i$ . Such a mechanism is said to satisfy the  $\alpha$ -LDP constraint if

$$\sup_{A \in \mathcal{Z}} \sup_{x, x' \in \mathbb{R}^d} \frac{\mathbb{P}\{Z_i \in A \mid X_i = x\}}{\mathbb{P}\{Z_i \in A \mid X_i = x'\}} \le e^{\alpha}, \quad i = 1, 2, \dots$$
 (4)

Here,  $\alpha$  is a non-negative privacy parameter with the interpretation that smaller values of  $\alpha$  lead to a stronger privacy guarantee.

Standard calculations show that the mechanism defined via (2) and (3) satisfies the LDP constraint provided that the standard deviation  $\sigma_W = \sigma_W(\alpha)$  is chosen sufficiently large, namely  $\sigma_W \geq 2^{3/2}\alpha^{-1}$ , see [4] and [5]. In the sequel, we assume equality such that

$$\sigma_W = 2^{3/2} \alpha^{-1} \tag{5}$$

holds.

Though the privacy mechanism defined through (2) and (3) has been suggested in others papers already for univariate density estimation, we propose a novel estimator based on multidimensional privatised data. When we discuss our results, we follow [9] and restrict ourselves to small values of  $\alpha \in [0, 1]$ , which represents the interesting regime with strong privacy guarantees. As the main result, the rate of convergence of our density estimate and the minimax lower bound are matching for such small privacy levels  $\alpha$ .

Using the privatised data in (3) one can compute the empirical distribution function

$$G_{n,j}(z) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{W_{n,i,j} \le z\}}$$

for all  $j \leq N_n$ . Our private density estimator will be defined in terms of the  $G_{n,j}(0), j \leq N_n$ , only. One has that

$$G_{n,j}(z) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{\mathbb{I}_{\{X_i \in A_{h_n,j}\}} + \sigma_W \zeta_{i,j} \leq z\}}$$

$$= \frac{1}{n} \sum_{i=1}^{n} (\mathbb{I}_{\{X_i \in A_{h_n,j}\}} \mathbb{I}_{\{1 + \sigma_W \zeta_{i,j} \leq z\}} + (1 - \mathbb{I}_{\{X_i \in A_{h_n,j}\}}) \mathbb{I}_{\{\sigma_W \zeta_{i,j} \leq z\}})$$

$$= \frac{1}{n} \sum_{i=1}^{n} (\mathbb{I}_{\{X_i \in A_{h_n,j}\}} (\mathbb{I}_{\{1 + \sigma_W \zeta_{i,j} \leq z\}} - \mathbb{I}_{\{\sigma_W \zeta_{i,j} \leq z\}}) + \mathbb{I}_{\{\sigma_W \zeta_{i,j} \leq z\}}).$$

Denoting with H the standard Laplace distribution function, conditioning on the raw data  $\mathcal{D}_n = \{X_1, \dots, X_n\}$  yields

$$\mathbb{E}\{G_{n,j}(0) \mid \mathcal{D}_n\} = \frac{1}{n} \sum_{i=1}^{n} (\mathbb{I}_{\{X_i \in A_{h_n,j}\}} (H(-1/\sigma_W) - H(0)) + H(0))$$
$$= H(0) + \mu_n(A_{h_n,j}) (H(-1/\sigma_W) - H(0)),$$

which can be rearranged to obtain

$$\mu_n(A_{h_n,j}) = \frac{1/2 - \mathbb{E}\{G_{n,j}(0) \mid \mathcal{D}_n\}}{1/2 - H(-1/\sigma_W)}.$$
 (6)

Based on (6), we introduce

$$\tilde{\mu}_n(A_{h_n,j}) = \frac{1/2 - G_{n,j}(0)}{1/2 - H(-1/\sigma_W)} \tag{7}$$

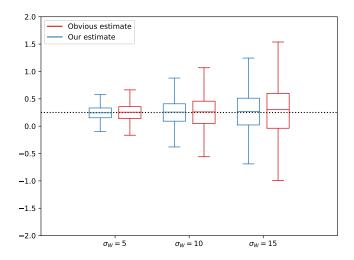


Figure 1: Finite sample comparison of the obvious estimator  $\hat{\mu}_n(A_{h_n,j})$  defined in (9) (red boxplots) and our novel estimator  $\tilde{\mu}_n(A_{h_n,j})$  defined in (7) (blue boxplots). Simulations were performed for n=1000 data holders in the local privacy framework, and the boxplots illustrate the behaviour of the estimators over  $n_{\text{sim}}=1000$  iterations. Different values of  $\sigma_W \in \{5,10,15\}$  were considered as indicated on the x-axis. In the considered example, the true value of  $\mu(A_{h_n,j})$  is 0.25 as indicated by the black dashed horizontal line.

as an estimator of  $\mu_n(A_{h_n,j})$ , and finally define the corresponding density estimator from the privatised data  $Z_1, \ldots, Z_n$ 

$$\tilde{f}_n(\boldsymbol{x}) = \frac{\tilde{\mu}_n(A_{h_n,j})}{h_n^d} \mathbb{I}_{\{j \le N_n\}} \quad \text{if } \boldsymbol{x} \in A_{h_n,j}.$$
 (8)

Note that the  $\tilde{\mu}_n(A_{h_n,j})$  in (7) are bounded but not necessarily non-negative.

An alternative estimator  $\hat{f}_n$ , which was proposed already in the seminal work [9], is obtained by replacing  $\tilde{\mu}_n(A_{h_n,j})$  in (8) with

$$\hat{\mu}_n(A_{h_n,j}) = \frac{1}{n} \sum_{i=1}^n W_{n,i,j}.$$
(9)

Because of

$$\mathbb{E}\{\tilde{\mu}_n(A_{h_n,j}) \mid \mathcal{D}_n\} = \mathbb{E}\{\hat{\mu}_n(A_{h_n,j}) \mid \mathcal{D}_n\} = \mu_n(A_{h_n,j}),$$

the bias of both estimators coincides with the one of the non-private histogram, and the results obtained in the sequel for the estimator  $\tilde{f}_n$  hold

equally true for the alternative estimator with only minor modifications necessary in the proofs but the same requirements concerning the choice of the bandwidth parameter  $h_n$ . However, for large values of  $\sigma_W$  (which are of particular interest from a privacy point of view) the less intuitive definition of  $\tilde{\mu}_n(A_{h_n,j})$  in (7) outperforms the obvious estimator  $\hat{\mu}_n(A_{h_n,j})$  in (9) in finite sample studies. A typical example of this phenomenon is illustrated in Figure 1. For this reason, we state and prove theoretical results in this paper exclusively for the estimator  $\tilde{f}_n$  as defined in (8). Further results from computer experiments, that investigate the performance of the privatised histogram estimator  $\tilde{f}_n$  (and natural modifications of this estimator), will be presented in Section 6 below.

# 3 Universal consistency results

Our first result establishes the strong universal  $L_1$  and pointwise consistency of the privatised histogram and can be seen as an LDP-analogue of Theorem 1.

**Theorem 3.** Assume that  $S_n \uparrow \mathbb{R}^d$ . If  $h_n \to 0$  and  $nh_n^{2d}/\log n \to \infty$ , then

$$\lim_{n\to\infty}\widetilde{f}_n(\boldsymbol{x}) = f(\boldsymbol{x}) \quad almost \ surely$$

for Lebesgue-almost every  $\mathbf{x} \in \mathbb{R}^d$ . If, in addition,  $nh_n^d/(r_n^d \log n) \to \infty$ , then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |f(\boldsymbol{x}) - \tilde{f}_n(\boldsymbol{x})| d\boldsymbol{x} = 0 \quad almost \ surely.$$

We remark that the proof of the pointwise consistency in Section 7.1 can equally be used to prove universal pointwise consistency of the non-private histogram  $f_n$ . In that case one can use Bennett's inequality (cf. [6], Theorem 2.9) instead of Hoeffding's inequality, and the (weaker) conditions to obtain universal pointwise consistency in that case turn out to be  $h_n \to 0$  and  $nh_n^d/\log n \to \infty$ . This result seems to be novel as well and might be of independent interest, since establishing universal pointwise consistency usually comes along with quite technical assumptions, see Vidal-Sanz [12].

As we mentioned in the Introduction, from an  $L_1$  consistent density estimate one can derive a distribution estimate consistent in total variation. Let  $\mu_n^*$  be the distribution estimate derived from the histogram estimate  $\tilde{f}_n$ :

$$\mu_n^*(A) = \int_A \tilde{f}_n(\boldsymbol{x}) d\boldsymbol{x} = \sum_{j=1}^{N_n} \int_{A \cap A_{h_n,j}} \tilde{f}_n(\boldsymbol{x}) d\boldsymbol{x} = \sum_{j=1}^{N_n} \frac{\lambda(A \cap A_{h_n,j})}{\lambda(A_{h_n,j})} \tilde{\mu}_n(A_{h_n,j}),$$

where  $\lambda$  is the Lebesgue measure. Thus, under the conditions of Theorem 3 one gets

$$\lim_{n \to \infty} \text{TV}(\mu, \mu_n^*) = 0 \quad \text{almost surely.}$$

If the distribution  $\mu$  is a mixture of absolutely continuous and discrete distributions, then from non-private data, Barron, Györfi and van der Meulen [2] introduced a distribution estimate that is consistent in total variation. Without knowing the support of the discrete component, we guess that from private data it is impossible to estimate such a distribution  $\mu$  consistently in total variation.

# 4 Rate of convergence

Our next theorem provides an LDP-analogue of Theorem 2 cited in the Introduction and states a weak consistency rate for Lipschitz functions with compact support.

**Theorem 4.** Assume that  $S_n \uparrow \mathbb{R}^d$ . Under the conditions of Theorem 2, one has that

$$\mathbb{E}\left\{\int_{\mathbb{R}^d} |f(\boldsymbol{x}) - \widetilde{f}_n(\boldsymbol{x})| d\boldsymbol{x}\right\} \le L\sqrt{d}h_n + \frac{\lambda(S_n)}{(1 - e^{-\sqrt{2}/\sigma_W})\sqrt{nh_n^{2d}}}.$$

If  $\sigma_W$  is sufficiently large ( $\sigma_W \geq 2$  works), then

$$\frac{1}{1 - e^{-\sqrt{2}/\sigma_W}} \le \sigma_W \tag{10}$$

holds, which together with the choice  $h_n = c \cdot (n/\sigma_W^2)^{-1/(2d+2)}$  implies

$$\mathbb{E}\left\{\int_{\mathbb{R}^d} |f(\boldsymbol{x}) - \tilde{f}_n(\boldsymbol{x})| \mathrm{d}\boldsymbol{x}\right\} \le c^* \cdot \lambda(S_n) \cdot (n/\sigma_W^2)^{-1/(2d+2)}.$$

Using the relation (5), it is easily checked that (10) holds in the usually considered privacy regime where  $\alpha \in [0, 1]$ . In this case, the convergence rate obtained in Theorem 4 can be rewritten as

$$\mathbb{E}\left\{\int_{\mathbb{R}^d} |f(\boldsymbol{x}) - \tilde{f}_n(\boldsymbol{x})| d\boldsymbol{x}\right\} \le c' \cdot \lambda(S_n) \cdot (n\alpha^2)^{-1/(2d+2)}.$$
 (11)

Note that the validity of relation (10) for sufficiently large values of  $\sigma_W$  is in coincidence with the message illustrated in Figure 1, namely that our estimator  $\tilde{f}_n$  outperforms the obvious estimator  $\hat{f}_n$ .

The factor  $\lambda(S_n)$  in the rate of convergence of Theorem 4 is caused by the fact that the support of the target density is unknown. If a priori knowledge on the support of the density was given, for instance by knowing that  $\operatorname{supp}(f) \subseteq \{x \in \mathbb{R}^d : ||x|| \le 1\}$ , one could choose  $\lambda(S_n)$  of order O(1)and this extra factor in the rate would disappear. For this scenario we will determine a matching lower bound in the Section 5. Moreover, we remark that Theorem 4 can easily be extended from Lipschitz to more general Hölder continuous functions: in the proof of such theorem, only the bound for the bias term would change.

In contrast to the classical histogram estimator  $f_n$  the privatised histogram estimator  $\tilde{f}_n$  is not necessarily a probability density function, neither positivity, nor the property that the estimator integrates to one are in general satisfied. For this reason we modify this estimator by taking its positive part and then normalise such that the modified estimator integrates to 1:

$$\tilde{f}_{n,\mathrm{mod}}(\boldsymbol{x}) = rac{ ilde{f}_n(\boldsymbol{x}) ee 0}{\int_{\mathbb{R}^d} ( ilde{f}_n(\boldsymbol{z}) ee 0) \mathrm{d} \boldsymbol{z}}.$$

If  $f_{n,\text{mod}}(\boldsymbol{x})$  takes only non-positive values, we put  $f_{n,\text{mod}}(\boldsymbol{x}) = 0$ . It is well-known (see [8], pp. 269-70) that this modification reduces the  $L^1$ -error of the estimator, and the error is certainly bounded by the constant 2. Hence, under the conditions of Theorem 4 we obtain the rate

$$\mathbb{E}\left\{\int_{\mathbb{R}^d} |f(\boldsymbol{x}) - \tilde{f}_{n,\text{mod}}(\boldsymbol{x})| d\boldsymbol{x}\right\} = O(r_n^d/(n\alpha^2)^{1/(2d+2)} \wedge 2).$$
 (12)

Let us mention that rates of convergence for nonparametric density estimation under LDP have already been derived in the literature. In their seminal work, Duchi, Jordan and Wainwright [9] study privatised orthogonal series estimators in terms of the standard trigonometric basis and measure performance by means of the mean integrated squared error. The authors consider the univariate case and mention that a Laplace perturbed version of the classical histogram (that is, the obvious estimator  $\hat{f}_n$  mentioned in Section 2) achieves the optimal private rate for Lipschitz continuous functions. Butucea, Dubois, Kroll and Saumard [7] study wavelet estimators and derive minimax optimal rates over Besov ellipsoids. Moreover, a datadriven estimator is proposed in that paper that adapts to an unknown level of Besov smoothness. Minimax rates for the  $L_1$ -error over Besov ellipsoids of are derived as a special case of their general results. However, their analysis is restricted to the case of univariate densities.

The rate derived in Theorem 4 indicates a worsening of the curse of dimensionality under LDP by a factor of 2 in the exponent. This phenomenon has already been established by Rohde and Steinberger [10] for the estimation of linear functionals under local differential privacy.

#### 5 Minimax lower bound

The purpose of this section is to prove that the rate from Theorem 4 is essentially optimal, that is, no other privacy mechanism together with an estimator respecting the  $\alpha$ -LDP constraint can attain a faster rate of convergence. Here, we will even allow for a richer class of admissible privacy mechanisms than only the non-interactive ones introduced in Section 2. These more general mechanisms allow for a certain amount of interaction, and we refer the reader to [9] for a precise definition of the privacy mechanism that are admissible in the framework of LDP.

For a constant L > 0, we now consider the Lipschitz class  $\mathcal{F}^d_{\text{Lip}}(L)$  defined as the class of all densities  $f: \mathbb{R}^d \to \mathbb{R}$  vanishing outside  $[0,1]^d$  and satisfying the Lipschitz condition

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \le L ||\boldsymbol{x} - \boldsymbol{y}||,$$

where  $\|\cdot\|$  denotes the Euclidean norm of  $\mathbb{R}^d$ .

The following theorem complements the convergence rate derived in Theorem 4 by a lower bound that matches the upper bound. Its proof essentially relies on an appropriate information theoretical inequality between Kullback-Leibler divergence of the privatised data on the one side and the squared total-variation distance of the raw data on the other side, which has been derived in [9] and is cited in Section 7. Duchi, Jordan and Wainwright [9] extended the theory of Le Cam, Fano and Assouad and developed a rich class of lower bound techniques for locally private data.

**Theorem 5** (Private lower bound for  $\mathcal{F}_{Lip}^d$ ). For all sufficiently large L, we have

$$\inf_{\substack{\widetilde{f} \\ Q \in \mathcal{Q}_{\alpha}}} \sup_{f \in \mathcal{F}_{\mathrm{Lip}}^d(L)} \mathbb{E}_f \left[ \int_{\mathbb{R}^d} |\widetilde{f}(\boldsymbol{x}) - f(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \right] \gtrsim (n(e^{\alpha} - 1)^2)^{-\frac{1}{2d + 2}},$$

where the infimum is taken over all, potentially interactive, privacy channels Q ensuring  $\alpha$ -differential privacy, and all estimators  $\tilde{f}$  based on the privatised observations  $Z_1, \ldots, Z_n$ .

As for Theorem 4, one can reformulate this result for the interesting privacy regime where  $\alpha \in [0, 1]$ . Then the lower bound in Theorem 5 can

be rewritten as

$$\tilde{c} \cdot (n\alpha^2)^{-1/(2d+2)}.\tag{13}$$

Thus, the upper bound (11) and the lower bound (13) are matching, i.e., the obtained rate (11) is optimal and cannot be improved by any, potentially interactive, privacy mechanism. We emphasise that the privacy mechanism defined by (2) and (3) is of the preferential, non-interactive, form.

By some appropriate adjustments this lower bound might be generalised to more general Lipschitz classes as considered in Devroye and Györfi [8]. However, these classes contain smoother functions which cannot be estimated with the optimal rate via the piecewise constant histogram estimator.

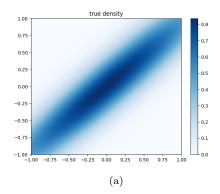
# 6 Simulation study

We tested the modified privatised histogram estimator  $\tilde{f}_{n,\text{mod}}$  defined in (12) in a small simulation study. Results of these experiments are reported in this section. We considered the two-dimensional case where the raw data were generated from a truncated two-dimensional Gaussian. More precisely,  $X_1, \ldots, X_n$  were drawn i.i.d.  $\sim \mathcal{N}(\mathbf{0}, \Sigma)$  restricted to the square  $[-1, 1]^2$  where the covariance matrix was chosen as

$$\Sigma = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 0.9 \end{pmatrix}.$$

The true density function of this data and a typical raw data sample of size n = 500 are plotted in Figure 2.

In light of the drastic sample size reduction under LDP, we used a large sample size of  $n=10^5$  for the simulations. We considered different values for the perturbation parameter  $\sigma_W$ , namely  $\sigma_W \in \{15, 10, 5, 0\}$ . Here, setting  $\sigma_W = 0$  corresponds to the case without privacy constraints, and in this case we consider the classical histogram estimator  $f_n$  defined in the Introduction. Recall that the privacy parameter  $\alpha$  is related to the variance  $\sigma_W^2$  appearing in the perturbation of the histogram via the identity  $\sigma_W^2 = 8/\alpha^2$  (see (5)): the non-zero values of  $\sigma_W$  we consider here correspond to values of  $\alpha$  approximately equal to 0.19, 0.28, and 0.56, respectively. The case  $\sigma_W = 0$  can be interpreted as formally setting  $\alpha = \infty$ . In this latter case, the condition (4) is always satisfied. Note that even in the case of the weak privacy guarantee with  $\alpha = 0.56$ , an extreme noise with standard deviation



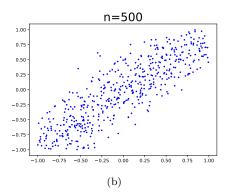


Figure 2: (a) True probability density function corresponding to a correlated truncated two-dimensional Gaussian distribution. (b) i.i.d. sample of size n = 500 from the density function in (a).

 $\sigma_W$  equal to 5 has to be added which shows the difficulty of inference under LDP.

In order to further illustrate this effect we compare the information stored in the classical histogram of any data holder with the one of the corresponding perturbed histogram. The non-privatized histogram generated locally by any data holder can be stored as a binary matrix with exactly one entry equal to 1 indicating the cell, where the respective data holder's observation is located. An example of such a matrix is given by

The strong effect of the anonymization procedure becomes apparent, when considering the perturbed version of this matrix, where we have take  $\sigma_W = 5$ . A realisation of such a perturbed matrix containing the  $W_{n,i,j}$  in (3) for a fixed i is given by

$$\begin{pmatrix} -0.18 & -1.99 & 2.64 & 0.83 & -8.85 \\ 4.36 & 0.58 & 2.59 & -1.66 & 2.17 \\ 2.01 & -2.49 & -0.83 & 8.20 & -0.50 \\ 0.12 & -0.84 & -13.20 & 2.15 & 6.01 \\ -3.97 & 4.39 & 2.45 & 4.28 & 1.51 \end{pmatrix}$$

where entries are rounded to the second decimal.

For our simulations, we split the domain  $[-1,1]^2$  into  $k \times k$ -grids of squares for k=3,4,5. This corresponds to choices of the bandwidth parameter equal to 2/3, 1/2, and 2/5, respectively. The resulting estimators are given in Figures 3–5. In all these figures the privatised histogram is plotted on the square  $[-1,1]^2$ . Already this limited number of simulations confirms the general message from the theoretical results, that a larger value of  $\sigma_W$ , which yields more privacy, must be paid for by a coarser resolution, i.e., a larger bandwidth in contrast to the non-privacy framework. Whereas for the  $3 \times 3$ -grids the 'trend' of the data concentrating close to the diagonal can be detected for all values of  $\sigma_W$ , this is only restrictively possible for  $\sigma_W = 15$  and the  $4 \times 4$  and  $5 \times 5$  grid. For the  $5 \times 5$  grid the choice  $\sigma_W = 5$  is still close to the non-privatized histogram, and the estimator for  $\sigma_W = 15$  now completely fails to detect the trend in the data.

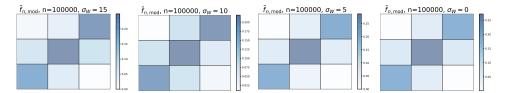


Figure 3:  $3 \times 3$  grids

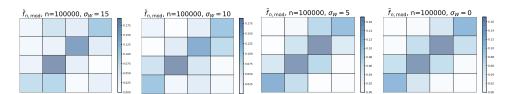


Figure 4:  $4 \times 4$  grids

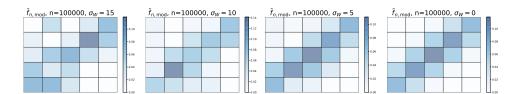


Figure 5:  $5 \times 5$  grids

#### 7 Proofs

#### 7.1 Proof of Theorem 3

First, we prove the pointwise consistency. The proof is based on the decomposition

$$|\widetilde{f}_n(\boldsymbol{x}) - f(\boldsymbol{x})| \le |\mathbb{E}\widetilde{f}_n(\boldsymbol{x}) - f(\boldsymbol{x})| + |\widetilde{f}_n(\boldsymbol{x}) - \mathbb{E}\widetilde{f}_n(\boldsymbol{x})|.$$
 (14)

First, for all  $n \geq n_0$  with  $n_0 = n_0(\boldsymbol{x})$  chosen sufficiently large we have  $\boldsymbol{x} \in S_n$ . For such n, the identity

$$\mathbb{E}\widetilde{f}_n(oldsymbol{x}) = \mathbb{E}f_n(oldsymbol{x}) = rac{\mu(A_n(oldsymbol{x}))}{h_n^d}$$

holds, where we denote with  $A_n(\mathbf{x})$  the unique cube from the collection of  $A_{h_n,j}$  that contains the considered  $\mathbf{x}$ . Then, the consistency of the bias term (the first term on the right-hand side of (14)) follows from the generalised Lebesgue density theorem (cf. [13], Theorem 7.16) as follows: let  $B_n(\mathbf{x})$  denote the smallest cube centred at  $\mathbf{x}$  containing  $A_n(\mathbf{x})$ . Again, let  $\lambda$  denote the Lebesgue measure. If there is a universal constant c > 0 such that

$$\lambda(B_n(\boldsymbol{x})) \le c\lambda(A_n(\boldsymbol{x})),\tag{15}$$

then  $h_n \to 0$  implies  $|\mathbb{E}\widetilde{f}_n(\boldsymbol{x}) - f(\boldsymbol{x})| \to 0$  for  $\lambda$ -almost every  $\boldsymbol{x}$ . Obviously, (15) is satisfied with  $c = 2^d$ .

In order to bound the second term on the right-hand side of (14), we use Hoeffding's inequality (in the formulation of [6], Theorem 2.8) which can be applied, since the random variables  $W_{n,i,j}$  for  $i=1,\ldots,n$  and fixed j are independent. Denoting with j=j(n) the index such that  $A_n(\boldsymbol{x})=A_{h_n,j}$  we can thus conclude

$$\begin{split} & \mathbb{P}(|\tilde{f}_{n}(\boldsymbol{x}) - \mathbb{E}\tilde{f}_{n}(\boldsymbol{x})| > \varepsilon) \\ & = \mathbb{P}(|\tilde{\mu}_{n}(A_{n}(\boldsymbol{x})) - \mu(A_{n}(\boldsymbol{x}))| > \varepsilon h_{n}^{d}) \\ & = \mathbb{P}(|G_{n,j}(0) - \mathbb{E}G_{n,j}(0)| > \varepsilon h_{n}^{d}(1/2 - H(-1/\sigma_{W}))) \\ & = \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}(\mathbb{I}_{\{W_{n,i,j\leq0}\}} - \mathbb{E}[\mathbb{I}_{\{W_{n,i,j\leq0}\}}]\right)\right| > \varepsilon h_{n}^{d}(1/2 - H(-1/\sigma_{W}))\right) \\ & \leq 2\exp(-\varepsilon^{2}nh_{n}^{2d}(1 - 2H(-1/\sigma_{W}))^{2}/2). \end{split}$$

The assumption of the theorem now guarantees that

$$\sum_{n=1}^{\infty} \mathbb{P}(|\widetilde{f}_n(\boldsymbol{x}) - \mathbb{E}\widetilde{f}_n(\boldsymbol{x})| > \varepsilon) < \infty.$$

Then, almost sure convergence follows using the Borel-Cantelli lemma.

We start the proof of  $L_1$  consistency by recalling the identity  $|a - b| = 2(a - b)_+ + b - a$  which is used to write

$$\int_{\mathbb{R}^d} |f(\boldsymbol{x}) - \tilde{f}_n(\boldsymbol{x})| d\boldsymbol{x}$$

$$= 2 \int_{\mathbb{R}^d} (f(\boldsymbol{x}) - \tilde{f}_n(\boldsymbol{x}))_+ d\boldsymbol{x} + \int_{\mathbb{R}^d} \tilde{f}_n(\boldsymbol{x}) d\boldsymbol{x} - \int_{\mathbb{R}^d} f(\boldsymbol{x}) d\boldsymbol{x}.$$

The pointwise consistency together with Lebesgue dominated convergence theorem implies that

$$\int_{\mathbb{R}^d} (f(\boldsymbol{x}) - \tilde{f}_n(\boldsymbol{x}))_+ d\boldsymbol{x} \to 0 \quad \text{almost surely.}$$

Thus, we have to show that

$$\int_{\mathbb{R}^d} \tilde{f}_n(\boldsymbol{x}) d\boldsymbol{x} = \sum_{j=1}^{N_n} \tilde{\mu}_n(A_{h_n,j}) \to 1 \quad \text{almost surely.}$$

By the triangle inequality, we have

$$\left| \sum_{j=1}^{N_n} \tilde{\mu}_n(A_{h_n,j}) - 1 \right| \le \left| \sum_{j=1}^{N_n} \mu_n(A_{h_n,j}) - 1 \right| + \left| \sum_{j=1}^{N_n} \tilde{\mu}_n(A_{h_n,j}) - \sum_{j=1}^{N_n} \mu_n(A_{h_n,j}) \right|.$$

The first term on the right hand side tends to 0 almost surely (using the strong universal consistency of the standard histogram from Theorem 1), and it remains to consider the second term. Put

$$\Delta_n := \left| \sum_{j=1}^{N_n} \tilde{\mu}_n(A_{h_n,j}) - \sum_{j=1}^{N_n} \mu_n(A_{h_n,j}) \right|.$$

For any  $\varepsilon > 0$ , consider the probability  $\mathbb{P}(\Delta_n > \varepsilon)$ . Conditioning on  $\mathcal{D}_n$  yields

$$\mathbb{P}(\Delta_n > \varepsilon) = \mathbb{E}[\mathbb{E}[\mathbb{I}_{\{\Delta_n > \varepsilon\}} | \mathcal{D}_n]].$$

Setting  $C(\sigma_W) = (1/2 - H(-1/\sigma_W))^{-1}$  implies

$$\Delta_n = C(\sigma_W) \left| \sum_{j=1}^{N_n} (G_{n,j}(0) - \mathbb{E}[G_{n,j}(0)|\mathcal{D}_n]) \right|$$

$$= C(\sigma_W) n^{-1} \left| \sum_{j=1}^{N_n} \sum_{i=1}^n (\mathbb{I}_{\{W_{n,i,j\leq 0}\}} - \mathbb{E}[\mathbb{I}_{\{W_{n,i,j\leq 0}\}} | \mathcal{D}_n]) \right|.$$

Applying Hoeffding's inequality to the conditional distribution of the random variables  $W_{n,i,j}$ ,  $i=1,\ldots,n,\ j=1,\ldots,N_n$  given  $\mathcal{D}_n$  (note that conditional on  $\mathcal{D}_n$  all these random variables are independent) yields

$$\mathbb{E}[\mathbb{I}_{\{\Delta_n > \varepsilon\}} | \mathcal{D}_n] \le 2 \exp\left(-\frac{2n\varepsilon^2}{C(\sigma_W)^2 N_n}\right)$$

Hence,

$$\sum_{n=1}^{\infty} \mathbb{P}(\Delta_n > \varepsilon) \lesssim \sum_{n=1}^{\infty} \exp\left(-\frac{2n\varepsilon^2}{C(\sigma_W)^2 N_n}\right),\,$$

and the sum on the right-hand side converges provided that  $\frac{n}{N_n \log n} \to \infty$  which is equivalent to the condition  $\frac{nh_n^d}{r_n^d \log n} \to \infty$  stated in the assumptions of the theorem (recall that  $N_n \asymp r_n^d/h_n^d$ ).

#### 7.2 Proof of Theorem 4

We bound the integrated error by the sum of bias and stochastic error, that is,

$$\mathbb{E}\left\{\int_{\mathbb{R}^d} |f(\boldsymbol{x}) - \widetilde{f}_n(\boldsymbol{x})| d\boldsymbol{x}\right\} \le b_n + s_n$$

where

$$b_n = \int_{\mathbb{R}^d} |f(\boldsymbol{x}) - \mathbb{E}\{\tilde{f}_n(\boldsymbol{x})\}| d\boldsymbol{x},$$
  $s_n = \mathbb{E}\left\{\int_{\mathbb{R}^d} |\mathbb{E}\{\tilde{f}_n(\boldsymbol{x})\} - \tilde{f}_n(\boldsymbol{x})| d\boldsymbol{x}\right\}.$ 

We mentioned earlier that the bias of the ordinary histogram  $f_n$  and the bias of  $\tilde{f}_n$  are identical. Using the fact that the support of f is contained in  $S_n$  for all sufficiently large n, Theorem 2 implies

$$b_n \le L\sqrt{d}h_n.$$

The bound for the stochastic error  $s_n$  is calculated as

$$\mathbb{E}\left\{\int_{\mathbb{R}^d} |\mathbb{E}\{\tilde{f}_n(\boldsymbol{x})\} - \tilde{f}_n(\boldsymbol{x})|\mathrm{d}\boldsymbol{x}\right\} = \sum_{j=1}^{N_n} \mathbb{E}\left\{|\mu(A_{h_n,j}) - \tilde{\mu}_n(A_{h_n,j})|\right\}$$

$$\leq \sum_{j=1}^{N_n} \sqrt{\text{Var}(\tilde{\mu}_n(A_{h_n,j}))}$$

$$\leq \frac{1}{\sqrt{n}} \sum_{j=1}^{N_n} \frac{1}{1 - 2H(-1/\sigma_W)}$$

$$= \frac{N_n}{(1 - e^{-\sqrt{2}/\sigma_W})\sqrt{n}}$$

$$= \frac{\lambda(S_n)}{(1 - e^{-\sqrt{2}/\sigma_W})\sqrt{nh_n^{2d}}}.$$

Combining the bounds for bias and stochastic error yields the claim of the theorem.

#### 7.3 Proof of Theorem 5

In the sequel, whenever  $\mu_{\theta}$  denotes the distribution of the original i.i.d. data  $X_1, \ldots, X_n$  with density  $f_{\theta}$ , we denote for a privacy mechanism Q with  $\mu_{\theta}^{\mathbf{Z}}$  the resulting distribution of the privatised data  $\mathbf{Z} = (Z_1, \ldots, Z_n)$  (the dependence of this distribution on the mechanism Q is suppressed for the sake of convenience).

Let  $\mathbb{P}$  and  $\mathbb{Q}$  be two probability measures on the same measurable space  $(\mathcal{X}, \mathcal{X})$ , and denote by p and q their densities with respect to some dominating measure  $\nu$ . Recall that the total variation distance, the Hellinger distance, and the Kullback-Leibler distance are defined as

$$\begin{aligned} & \text{TV}(\mathbb{P}, \mathbb{Q}) = \frac{1}{2} \int_{\mathcal{X}} |p(\boldsymbol{x}) - q(\boldsymbol{x})| \nu(\mathrm{d}\boldsymbol{x}), \\ & H(\mathbb{P}, \mathbb{Q}) = \left( \int_{\mathcal{X}} (\sqrt{p(\boldsymbol{x})} - \sqrt{q(\boldsymbol{x})})^2 \nu(\mathrm{d}\boldsymbol{x}) \right)^{1/2}, \\ & \text{KL}(\mathbb{P}, \mathbb{Q}) = \int_{\mathcal{X}} \log \left( \frac{p(\boldsymbol{x})}{q(\boldsymbol{x})} \right) p(\boldsymbol{x}) \nu(\mathrm{d}\boldsymbol{x}), \end{aligned}$$

respectively (the last definition being valid for  $\mathbb{P} \ll \mathbb{Q}$ ; otherwise one defines  $\mathrm{KL}(\mathbb{P},\mathbb{Q}) = +\infty$ ). For the proof we need the following result which links the Kullback-Leibler distance of the privatised observations  $Z_1, \ldots, Z_n$  (generated by a potentially interactive privacy mechanism) to the total-variation distance of the original random variables.

**Lemma 1** (Consequence of [9], Theorem 1). Let  $\alpha \geq 0$ . For any  $\alpha$ -local differentially private mechanism  $Q \in \mathcal{Q}_{\alpha}$ 

$$\mathrm{KL}(\mu_{\theta}^{\mathbf{Z}}, \mu_{\theta'}^{\mathbf{Z}}) \le 4n(e^{\alpha} - 1)^{2}\mathrm{TV}^{2}(\mu_{\theta}, \mu_{\theta'}).$$

For the proof of Theorem 5, we need to introduce some notation first. Define the function  $g_0: [0,1]^d \to \mathbb{R}$  by

$$g_0(\mathbf{x}) = L \min_{i=1,...,d} \min\{x_i, 1 - x_i\}.$$

Then, denoting with h the 1-Lipschitz function  $h: [0,1] \to \mathbb{R}$  with  $h(x) = x\mathbb{I}_{\{x \in [0,1/2]\}} + (1-x)\mathbb{I}_{\{x \in (1/2,1]\}}$ , we have

$$|g_{0}(\boldsymbol{x}) - g_{0}(\boldsymbol{x}')| = L \left| \min_{i=1,\dots,d} \min\{x_{i}, 1 - x_{i}\} - \min_{i=1,\dots,d} \min\{x'_{i}, 1 - x'_{i}\} \right|$$

$$\leq L \max_{i=1,\dots,d} \left| \min\{x_{i}, 1 - x_{i}\} - \min\{x'_{i}, 1 - x'_{i}\} \right|$$

$$= L \max_{i=1,\dots,d} \left| h(x_{i}) - h(x'_{i}) \right|$$

$$\leq L \max_{i=1,\dots,d} \left| x_{i} - x'_{i} \right|$$

$$\leq L \|\boldsymbol{x} - \boldsymbol{x}'\|,$$

showing that  $g_0$  belongs to  $\mathcal{F}_{Lip}^d(L)$ .

For a positive integer k (that will be specified below) set  $A = [0, 1/(2k))^d$ . Define  $\mathbf{y_j} = (y_{j_1}, \dots, y_{j_d}) \in [0, 1]^d$  for  $\mathbf{j} \in \{0, \dots, k-1\}^d$  by  $y_{j_i} = 1/4 + j_i/(2k)$ , and further put  $A_{\mathbf{j}} = \mathbf{y_j} + A$  for the same values of  $\mathbf{j}$ . Now, consider the function  $g \colon A \to \mathbb{R}$  defined as

$$g(\boldsymbol{x}) = \frac{(-1)^{\#\{i \in \{1,\dots,d\}: 1/(4k) \le x_i < 1/(2k)\}}}{4k} \cdot g_0(\tau(x_1),\dots,\tau(x_d))$$

where

$$\tau(x) = \begin{cases} 4kx, & \text{if } 0 \le x < \frac{1}{4k}, \\ 4k(x - 1/(4k)), & \text{if } \frac{1}{4k} \le x < \frac{1}{2k}. \end{cases}$$

Take  $\theta = (\theta_j) \in \{\pm 1\}^{k^d}$ . With such a  $\theta$  we associate the function defined by

$$f_{\theta}(x) = \begin{cases} f_0(x), & \text{if } x \notin \bigcup_j A_j, \\ f_0(x) + \theta_j g(x - y_j), & \text{if } x \in A_j. \end{cases}$$

Here, the function  $f_0$  is chosen such that it is constant on the block  $\bigcup_j A_j = [1/4, 3/4)^d$ , and outside this block it is defined in such a way that  $f_0$  is positive, integrates to 1 and satisfies the  $\mathcal{F}_{\text{Lip}}^d(L)$  condition. The existence of such a function  $f_0$  is guaranteed whenever L is sufficiently large. The definition of g (see Figure 6 for an illustration in the case d = k = 2) guarantees that the hypotheses  $f_\theta$  are density functions and belong to  $\mathcal{F}_{\text{Lip}}^d$ .

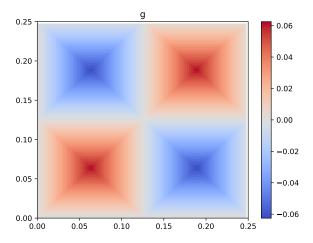


Figure 6: Heatmap of the function  $g: [0, 0.25]^2 \to \mathbb{R}$  for the case d = k = 2 with L = 1. This function is used to construct the hypotheses  $f_{\theta}$  in the proof of the lower bound. Note that positive and negative parts of the functions cancel each other when integrating over the whole area  $[0, 0.25)^2$ .

After these preparations, we now consider privatised data  $\mathbf{Z} = (Z_1, \dots, Z_n)$  generated by an arbitrary  $\alpha$ -LDP privacy mechanism  $Q \in \mathcal{Q}_{\alpha}$  (generating observations in an arbitrary measurable space  $(\mathcal{Z}, \mathcal{Z})$ ) and let us denote with  $\tilde{f}$  an arbitrary estimator based on these data. Then, denoting with  $\mathbb{E}_f$  and  $\mathbb{E}_{\theta}$  the expectation operator when the true density is f and  $f_{\theta}$ , respectively, we have

$$\sup_{f \in \mathcal{F}_{\text{Lip}}^{d}} \mathbb{E}_{f} \left[ \int_{[0,1]^{d}} |\widetilde{f}(\boldsymbol{x}) - f(\boldsymbol{x})| d\boldsymbol{x} \right] \ge \sup_{\theta \in \{\pm 1\}^{k^{d}}} \mathbb{E}_{\theta} \left[ \int_{[0,1]^{d}} |\widetilde{f}(\boldsymbol{x}) - f_{\theta}(\boldsymbol{x})| d\boldsymbol{x} \right]$$

$$\ge \frac{1}{2^{k^{d}}} \sum_{\theta \in \{\pm 1\}^{k^{d}}} \mathbb{E}_{\theta} \left[ \int_{[0,1]^{d}} |\widetilde{f}(\boldsymbol{x}) - f_{\theta}(\boldsymbol{x})| d\boldsymbol{x} \right]$$

$$\ge \frac{1}{2^{k^{d}}} \sum_{\theta \in \{\pm 1\}^{k^{d}}} \sum_{j} \mathbb{E}_{\theta} \left[ \int_{A_{j}} |\widetilde{f}(\boldsymbol{x}) - f_{\theta}(\boldsymbol{x})| d\boldsymbol{x} \right]$$

$$= \frac{1}{2^{k^{d}}} \sum_{j} \sum_{\theta \in \{\pm 1\}^{k^{d}}} \mathbb{E}_{\theta} \left[ \int_{A_{j}} |\widetilde{f}(\boldsymbol{x}) - f_{\theta}(\boldsymbol{x})| d\boldsymbol{x} \right]$$

$$= \frac{1}{2^{k^{d}+1}} \sum_{j} \sum_{\theta \in \{\pm 1\}^{k^{d}}} \left\{ \mathbb{E}_{\theta} \left[ \int_{A_{j}} |\widetilde{f}(\boldsymbol{x}) - f_{\theta}(\boldsymbol{x})| d\boldsymbol{x} \right] \right\}$$

$$+ \mathbb{E}_{\theta^{(j)}} \left[ \int_{A_j} |\widetilde{f}(\boldsymbol{x}) - f_{\theta^{(j)}}(\boldsymbol{x})| d\boldsymbol{x} \right] \right\}, \tag{16}$$

where we denote for some given  $\theta \in \{\pm 1\}^{k^d}$  with  $\theta^{(j)}$  the element of  $\{\pm 1\}^{k^d}$  satisfying  $\theta_i^{(j)} = \theta_i$  for  $i \neq j$  and  $\theta_j^{(j)} = -\theta_j$  (sign reversal at the coordinate with index j). Consider the Hellinger affinity

$$\rho(\mu_{\theta}^{\mathbf{Z}}, \mu_{\theta^{(j)}}^{\mathbf{Z}}) := \int_{\mathcal{Z}^n} \sqrt{\psi_{\theta}^{\mathbf{Z}}(\mathbf{z}) \psi_{\theta^{(j)}}^{\mathbf{Z}}(\mathbf{z})} \nu(\mathrm{d}\mathbf{z})$$

where  $\psi_{\theta}^{\mathbf{Z}}$  and  $\psi_{\theta^{(j)}}^{\mathbf{Z}}$  denote densities of the measure  $\mu_{\theta}^{\mathbf{Z}}$  and  $\mu_{\theta^{(j)}}^{\mathbf{Z}}$ , respectively, with respect to some dominating measure, say  $\nu$ . Using the elementary inequality  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$  for  $a,b \geq 0$  and the Cauchy-Schwarz inequality, we obtain

$$\rho(\mu_{\theta}^{\mathbf{Z}}, \mu_{\theta(j)}^{\mathbf{Z}}) = \int_{\mathcal{Z}^{n}} \frac{\left(\int_{A_{j}} |f_{\theta}(\mathbf{x}) - f_{\theta(j)}(\mathbf{x})| d\mathbf{x}\right)^{1/2}}{\left(\int_{A_{j}} |f_{\theta}(\mathbf{x}) - f_{\theta(j)}(\mathbf{x})| d\mathbf{x}\right)^{1/2}} \sqrt{\psi_{\theta}^{\mathbf{Z}}(\mathbf{z}) \psi_{\theta(j)}^{\mathbf{Z}}(\mathbf{z})} \nu(d\mathbf{z})$$

$$\leq \int_{\mathcal{Z}^{n}} \frac{\left(\int_{A_{j}} |f_{\theta}(\mathbf{x}) - \tilde{f}(\mathbf{x})| d\mathbf{x}\right)^{1/2}}{\left(\int_{A_{j}} |f_{\theta}(\mathbf{x}) - f_{\theta(j)}(\mathbf{x})| d\mathbf{x}\right)^{1/2}} \sqrt{\psi_{\theta}^{\mathbf{Z}}(\mathbf{z})} \sqrt{\psi_{\theta(j)}^{\mathbf{Z}}(\mathbf{z})} \nu(d\mathbf{z})$$

$$+ \int_{\mathcal{Z}^{n}} \frac{\left(\int_{A_{j}} |f_{\theta}(\mathbf{x}) - \tilde{f}(\mathbf{x})| d\mathbf{x}\right)^{1/2}}{\left(\int_{A_{j}} |f_{\theta}(\mathbf{x}) - f_{\theta(j)}(\mathbf{x})| d\mathbf{x}\right)^{1/2}} \sqrt{\psi_{\theta(j)}^{\mathbf{Z}}(\mathbf{z})} \sqrt{\psi_{\theta}^{\mathbf{Z}}(\mathbf{z})} \nu(d\mathbf{z})$$

$$\leq \left(\int_{\mathcal{Z}^{n}} \frac{\int_{A_{j}} |f_{\theta}(\mathbf{x}) - \tilde{f}(\mathbf{x})| d\mathbf{x}}{\int_{A_{j}} |f_{\theta}(\mathbf{x}) - f_{\theta(j)}(\mathbf{x})| d\mathbf{x}} \psi_{\theta}^{\mathbf{Z}}(\mathbf{z}) \nu(d\mathbf{z})\right)^{1/2}$$

$$+ \left(\int_{\mathcal{Z}^{n}} \frac{\int_{A_{j}} |\tilde{f}(\mathbf{x}) - f_{\theta(j)}(\mathbf{x})| d\mathbf{x}}{\int_{A_{j}} |f_{\theta}(\mathbf{x}) - f_{\theta(j)}(\mathbf{x})| d\mathbf{x}} \psi_{\theta(j)}^{\mathbf{Z}}(\mathbf{z}) \nu(d\mathbf{z})\right)^{1/2}.$$

Consequently, using  $(a+b)^2 \le 2a^2 + 2b^2$ ,

$$\frac{1}{2}\rho^{2}(\mu_{\theta}^{\mathbf{Z}}, \mu_{\theta^{(j)}}^{\mathbf{Z}}) \int_{A_{j}} |f_{\theta}(\mathbf{x}) - f_{\theta^{(j)}}(\mathbf{x})| d\mathbf{x}$$

$$\leq \mathbb{E}_{\theta} \left[ \int_{A_{j}} |f_{\theta}(\mathbf{x}) - \widetilde{f}(\mathbf{x})| d\mathbf{x} \right] + \mathbb{E}_{\theta^{(j)}} \left[ \int_{A_{j}} |f_{\theta^{(j)}}(\mathbf{x}) - \widetilde{f}(\mathbf{x})| d\mathbf{x} \right]. \quad (17)$$

Let us now determine a lower bound for the quantity  $\rho(\mu_{\theta}^{\mathbf{Z}}, \mu_{\theta(j)}^{\mathbf{Z}})$ . For this, we first derive an upper bound for the Kullback-Leibler divergence

 $\mathrm{KL}(\mu_{\theta}^{\mathbf{Z}}, \mu_{\theta(j)}^{\mathbf{Z}})$ . Lemma 1 directly provides the estimate

$$\mathrm{KL}(\mu_{\theta}^{\mathbf{Z}}, \mu_{\theta(j)}^{\mathbf{Z}}) \leq 4n(e^{\alpha} - 1)^{2}\mathrm{TV}^{2}(\mu_{\theta}, \mu_{\theta(j)}).$$

Further, by definition of  $f_{\theta}$  and  $f_{\theta(j)}$ ,

$$TV(\mu_{\theta}, \mu_{\theta(j)}) = \frac{1}{2} \int_{A_{j}} |f_{\theta}(\boldsymbol{x}) - f_{\theta(j)}(\boldsymbol{x})| d\boldsymbol{x}$$

$$= \int_{A} |g(\boldsymbol{x})| d\boldsymbol{x}$$

$$= \frac{2^{d}}{4k} \int_{[0,1/(4k))^{d}} g_{0}(4kx_{1}, \dots, 4kx_{d}) dx_{1} \dots dx_{d}$$

$$= \frac{2^{d}}{(4k)^{d+1}} \int_{[0,1)^{d}} g_{0}(\boldsymbol{x}) d\boldsymbol{x}.$$

Now choose k as the smallest integer  $\geq 1$  such that

$$(n(e^{\alpha}-1)^2 \vee 1) \frac{1}{(2k)^{2d+2}} \left( \int_{[0,1)^d} g_0(\boldsymbol{x}) d\boldsymbol{x} \right)^2 \le 1,$$

which yields  $\mathrm{KL}(\mu_{\theta}^{\mathbf{Z}}, \mu_{\theta^{(j)}}^{\mathbf{Z}}) \leq 1$ . Using the relation  $H^2 \leq \mathrm{KL}$  between squared Hellinger and Kullback-Leibler distance (cf. [11], Equation (2.19)) together with the identity  $\rho = 1 - \frac{1}{2}H^2$  for the Hellinger affinity (cf. [11], p. 83, property (iii) of the Hellinger distance), we obtain the estimate

$$\rho(\mu_{\theta}^{\mathbf{Z}}, \mu_{\theta(\mathbf{j})}^{\mathbf{Z}}) \ge \frac{1}{2},$$

which is independent of the index j. Combining this last estimate with (17) and putting the result into (16) yields

$$\sup_{f \in \mathcal{F}_{\text{Lip}}^d(L)} \mathbb{E}_f \left[ \int_{[0,1]^d} |\widetilde{f}(\boldsymbol{x}) - f(\boldsymbol{x})| d\boldsymbol{x} \right] \ge \frac{1}{8} \sum_{\boldsymbol{j}} \int_{A_{\boldsymbol{j}}} |f_{\boldsymbol{\theta}}(\boldsymbol{x}) - f_{\boldsymbol{\theta}^{(\boldsymbol{j})}}(\boldsymbol{x})| d\boldsymbol{x}$$

$$= \frac{k^d}{4} \int_{A} |g(\boldsymbol{x})| d\boldsymbol{x}$$

$$\approx k^{-1}$$

$$\approx (n(e^{\alpha} - 1)^2)^{-\frac{1}{2d+2}} \wedge 1,$$

where we used that  $k \simeq (n(e^{\alpha}-1)^2 \vee 1)^{\frac{1}{2d+2}}$ . This proves the desired lower bound.

# Acknowledgements

The research of Martin Kroll was supported by the German Research Foundation (DFG) under the grant DFG DE 502/27-1.

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