1) Z_t = état caché au temps

$(Z_t) \sim \mathcal{M}(\nu, \pi)$ $\qquad Z_{tq} = \mathbb{1}\{Z_t = q\}$

- $$\log p(Z) = \sum_q Z_{1q} \log \nu_q + \sum_{t \geq 2} \sum_{q,\ell} Z_{t-1,q} Z_{t,\ell} \log \pi_{q\ell}$$

- $\log p(Y|Z)$
$$= \sum_{t \geq 1} \sum_k Z_{tk} \left(-\frac{1}{2} \log |\Sigma_k| - \frac{1}{2} \| Y_t - \mu_k \|^2_{\Sigma_k^{-1}} \right) + \text{cst}$$

- $$\log p_\theta(Y, Z) = \log p_\theta(Z) + \log p_\theta(Y|Z)$$

2) $\tau_{tk} = E(Z_{tk} | Y)$ $\qquad \eta_{tq\ell} = E(Z_{t-1,q} Z_{t\ell} | Y)$

$$E[\log p(Y, Z) | Y] = Q(\theta)$$
$$= \sum_q \tau_{1q} \log \nu_q + \sum_{t \geq 2} \sum_{q,\ell} \eta_{tq\ell} \log \pi_{q\ell}$$
$$+ \sum_{t \geq 1} \sum_q \tau_{tq} \left(-\frac{1}{2} \log |\Sigma_q| - \frac{1}{2} \| Y_t - \mu_q \|^2_{\Sigma_q^{-1}} \right)$$

3) ν : $\quad Q(\theta) + \lambda \left(\sum_q \nu_q - 1 \right)$

$$\partial_{\nu_q}^u = \tau_{1q} / \nu_q + \lambda = 0$$
$$\iff \nu_q = -\lambda \tau_{1q}$$
$$\Rightarrow \hat{\nu}_q = \tau_{1q} \qquad \left(\text{car } \sum_q \nu_q = 1\right)$$

π : $\quad Q(\theta) + \sum_q \lambda_q \left(\sum_\ell \pi_{q\ell} - 1 \right)$

$$\partial_{\pi_{q\ell}}^u = \sum_{t \geq 2} \eta_{tq\ell} / \pi_{q\ell} + \lambda_q = 0$$

$$\Leftrightarrow \quad \pi_{q\ell} = -d \sum_{t\geq 2} \eta_{tq\ell}$$

$$\hat{\pi}_{q\ell} = \frac{\sum_{t\geq 2} \eta_{tq\ell}}{\sum_{h} \sum_{t\geq 2} \eta_{tqh}} \qquad \left(\sum_{\ell} \pi_{q\ell} = 1 \right)$$

$$\partial_{\mu_q} Q = \partial_{\mu_q} - \frac{1}{2} \sum_{t\geq 1} \sum_{q} \tau_{tq} \, \| Y_t - \mu_q \|^2_{\Sigma_q^{-1}}$$

$$= -\cancel{\frac{1}{2}} \sum_{t\geq 1} \cancel{\sum_{q}} \tau_{tq} \left(-\cancel{2} \, \Sigma_q^{-1} (Y_t - \mu_q) \right)$$

$$= \Sigma_q^{-1} \left(\sum_{t\geq 1} \tau_{tq} Y_t - N_q \mu_q \right)$$

$$N_q = \sum_{t\geq 1} \tau_{tq} \quad (\sim \text{effectif de l'état } q)$$

$$= 0 \quad \Leftrightarrow \quad \hat{\mu}_q = \frac{1}{N_q} \sum_{t\geq 1} \tau_{tq} Y_t$$

$$\Sigma_q : \quad Q(\theta) = -\frac{1}{2} \left(N_q \log |\Sigma_q| + \sum_{t\geq 1} \tau_{tq} \, \| Y_t - \mu_q \|^2_{\Sigma_q^{-1}} \right)$$

$$= -\frac{1}{2} \Big(-N_q \log |\Sigma_q^{-1}|$$

$$+ \sum_{t\geq 1} \tau_{tq} \, \mathrm{tr}\left((Y_t - \mu_q)(Y_t - \mu_q)^T \Sigma_q^{-1} \right) \Big)$$

$$\partial_{\Sigma_q^{-1}} Q = -\frac{1}{2} \left(-N_q (\Sigma_q^{-1})^{-1} + \sum_{t\geq 1} \tau_{tq} (Y_t - \mu_q)(Y_t - \mu_q)^T \right)$$

$$\rightarrow \hat{\Sigma}_q = \frac{1}{N_q} \sum_{t\geq 1} \tau_{tq} (Y_t - \hat{\mu}_q)(Y_t - \hat{\mu}_q)^T$$

$$= \frac{1}{N_q} \sum_{t\geq 1} \tau_{tq} Y_t Y_t^T - \hat{\mu}_q \hat{\mu}_q^T$$

4) $\theta = (\nu, \pi, (\mu_q), (\Sigma_q))$

$\#\text{param} = (K-1) + K(K-1) + 2K + 3K$

$= (K+1)(K-1) + 5K$

si σ est fixe : $\#parm = K(K-1) + 5K$

$$BIC(K) = \log p_{\hat{\theta}_K}(Y) - \frac{\log(n)}{2} \# parms$$