RECONSTRUCTION OF COMPACT SETS

Contents

Throughout this chapter, K will denote a compact set in the Euclidean d-space $\mathbb{R}^d$, usually with no additional regularity assumption.

1. Distance Function and Hausdorff Distance

Recall that the *distance function* to K, denoted by d_K, is defined by

$$\begin{aligned} \mathrm{d}_K \colon \mathbb{R}^d &\to \mathbb{R}_{\geqslant 0} \\ x &\mapsto \min_{p\in K} \|x-p\| . \end{aligned}$$

Definition 1.1 (Offset)**.** The *r-offset of K*, also called *tubular neighborhood* in geometry, is the set K^r of points at distance at most r of K, or equivalently the sublevel set $K^r := \{x \in \mathbb{R}^d ; \mathrm{d}_K(x) \leqslant r\}$.

Definition 1.2 (Hausdorff distance)**.** The *Hausdorff distance* between two compact subsets A and B of $\mathbb{R}^d$ can be defined in term of offsets:

$$\mathrm{d}_{\mathrm{H}}(A,B) := \min\{r \geqslant 0 \text{ s.t. } B \subset A^r \text{ and } A \subset B^r\}.$$

Loosely speaking, a finite set $\mathcal{P}$ is within Hausdorff distance r from a compact set K if it is sampled r-close to K ($\mathcal{P} \subset K^r$) and densely in K at scale r ($K \subset P^r$).

As seen in last lesson, alternative characterizations of the Hausdorff distance are given by

$$\mathrm{d}_{\mathrm{H}}(A,B) = \max\left\{\sup_{a\in A} \mathrm{d}_B(a), \sup_{b\in B} \mathrm{d}_A(b)\right\},$$

and

$$\mathrm{d}_{\mathrm{H}}(A,B) = \|\mathrm{d}_B - \mathrm{d}_A\|_\infty ,$$

where $\|f\|_\infty = \sup_{x\in\mathbb{R}^d} |f(x)|$.

2. Critical Point of Distance Functions

The objective of this section is to give a sufficient condition ensuring that the topology of the offsets K^s do not depend on $s \in [r, R]$.

Definition 2.1 (Homeomorphism)**.** Two topological spaces X, Y are called *homeomorphic* if there exists a continuous bijection $f : X \to Y$ whose inverse is also continuous. The function $f : X \to Y$ is called an *homeomorphism* between X and Y.

Definition 2.2 (Isotopy)**.** Two subsets $X, Y \subset \mathbb{R}^d$ are said to be *isotopic* if there exists a map $f : [0,1] \times X \to \mathbb{R}^d$ such that:
– $f(0,\cdot) = \mathrm{id}_X$;
– $f(1, X) = Y$;
– for all $t \in [0,1]$, $f(t,\cdot)$ is a homeomorphism onto its image.
In particular, two isotopic sets are homeomorphic.

To give a motivation, we recall the following celebrated result:

Lemma 2.3 (Isotopy Lemma). *Let $g : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^1$ proper function (meaning that the sublevel sets of g are compact). Assume that $g^{-1}([r,R])$ contains no critical point*[1]*. Then,*
– *There exists a homeomorphism $\Phi : [0, R-r] \times g^{-1}(\{r\}) \to g^{-1}([r,R])$.*
– *$g^{-1}(\{r\})$ and $g^{-1}(\{R\})$ are isotopic.*

Sketch of proof. Consider the vector field $V(x) := \nabla g(x)/\left\|\nabla g(x)\right\|^2$ (thus explaining the requirement $\nabla g(x) \neq 0$), and for $x \in g^{-1}(r)$, define $\Phi(x,t)$ by integrating the vector field, i.e.

$$\Phi(x,0) = x \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}\Phi(x,t) = V(\Phi(x,t)).$$

It is (rather) easy to check that $\Phi(\cdot,t) : g^{-1}(r) \to g^{-1}(r+t)$ is a homeomorphism, so that $g^{-1}(r)$ and $g^{-1}(R)$ are isotopic. □

The difficulty for applying this proposition to distance functions is that these functions are usually non-smooth (i.e. merely Lipschitz-regular). Grove introduced the following definition, which allows to define critical points without using the gradient of d_K.

Definition 2.4 (Projection Function)**.** A point p of K that realizes the minimum in the definition (1) of the distance function $\mathrm{d}_K(x)$ is called a *projection* of x on K. The set of such projections is denoted $\mathrm{proj}_K(x)$, and is always non-empty by compactness of K:

$$\mathrm{proj}_K(x) = \operatorname*{argmin}_{p \in K} \left\|x - p\right\|.$$

Definition 2.5 (Critical Point for Distance Function)**.** Let K be a compact subset of $\mathbb{R}^d$. A point $x \in \mathbb{R}^d \setminus K$ is called *critical* for d_K if it belongs to the convex hull of its projection set, i.e. $x \in \mathrm{conv}(\mathrm{proj}_K(x))$

Example 2.6. Let us check that this definition makes sense.
– If $K = \{p, q\} \subset \mathbb{R}^2$ with $p \neq q$, then the only critical point is $x = \frac{1}{2}(p+q)$.

[1] A point $x \in \mathbb{R}^d$ is called critical for $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ if $\nabla g(x) = 0$.

– If $K = \{p, q, r\} \subset \mathbb{R}^2$ consists of three distinct points, denote by c the circumcenter of the triangle $[p, q, r]$ (i.e. the center of its circumscribed circle, i.e. the intersection of the the line segment bisectors). The midpoints of the edges are always critical points. Moreover, if c belongs to $\mathrm{conv}(pqr)$, it is critical (and one can check that if c belongs to the interior of $\mathrm{conv}(pqr)$, the critical point induces a change in topology of the offsets).

Theorem 2.7 (Grove's Isotopy Lemma). *Let K be a compact set, and assume that there exists no critical point in $K^{[r,R]} := \mathrm{d}_K^{-1}([r, R])$ for $0 < r \leqslant R < +\infty$. Then,*

– *There exists a homeomorphism $\Phi : \partial(K^r) \times [0, R - r] \to K^{[r,R]}$ such that $\Phi(\cdot, 0) = \mathrm{id}_{\partial(K^r)}$.*
– *K^r and K^R are isotopic.*

Proof. Similar but much more technical. □

3. Homotopy Equivalence and Weak Feature Size

The rest of these notes aim at answering the following question:

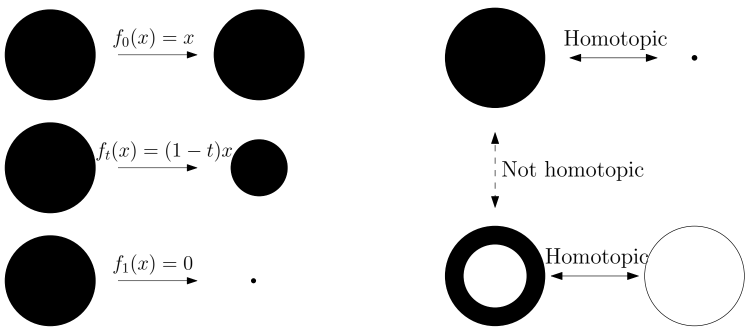
> Given a sample $\mathcal{P}$ sampled densely enough near a "regular" set $K \subset \mathbb{R}^d$, may we "recover the topology" of K^r with $\mathcal{P}^s$ for some wisely chosen $s \geqslant 0$?

To answer this question, we shall identify what a "regular" set may mean, and in what sense we can "recover topology".

Definition 3.1 (Weak Feature Size)**.** The *weak feature size* of a compact set $K \subset \mathbb{R}^d$ is defined by

$$\mathrm{wfs}(K) = \sup\{R > 0 \mid \mathrm{d}_K^{-1}((0, R)) \text{ contains no critical point}\}.$$

Definition 3.2 (Homotopy)**.** Two continuous functions $f_0, f_1 : X \to Y$ are called *homotopic* if there exists a continuous function $f : [0, 1] \times X \to Y$ such that $f_0 = f(0, \cdot)$ and $f_1 = f(1, \cdot)$. We denote the fact that f_0 and f_1 are homotopic by $f_0 \simeq f_1$.



(A) (B)

Figure 1. An example of two maps that are homotopic (a) and examples of spaces that are homotopy equivalent, but not homeomorphic (b).

Definition 3.3 (Homotopy Equivalence)**.** A function $f : X \to Y$ is a *homotopy equivalence* between X and Y if there exists a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. If this occurs, we say that the two spaces are called *homotopy equivalent* or that they have the same *homotopy type.*

Example 3.4 (See Figure 1)**.**

- The unit ball $X = \mathrm{B}(0, 1)$ is not homeomorphic to $Y = \{0\}$, but X and Y are nonetheless homotopy equivalent. Consider $f : X \to Y$ the projection map and $g : Y \hookrightarrow X$ the inclusion. Then, $g \circ f = \mathrm{id}_Y$ and $g \circ f = f$. Now, let $f_t : x \in X \mapsto (1 - t)x + tf(x)$. Then, $f_0 = \mathrm{id}_X$ and $f_1 = f$.
- A circle and an annulus are also homotopy equivalent.
- A circle and a point are not homotopy equivalent, as we will see in next chapter on homology.

Remark 3.5. The homotopy equivalences considered in the previous examples are called *rectractions.* Namely, $f : X \to Y$ is a retraction if the restriction of f to Y is the identity map. A *deformation retract* is a homotopy between a retraction $f : X \to Y$ and the identity map on X.

Example 3.6. Let K be a compact set, and assume that $K^{[r,R]} = \mathrm{d}_K^{-1}([r, R])$ does not contain any critical point. Consider $f : \partial(K^r) \to K^{[r,R]}$ the inclusion map, and let $\Phi : \partial(K^r) \times [r, R] \to K^{[r,R]}$ be the homeomorphism given by Grove's Isotopy Lemma (Theorem 2.7). Define $g : K^{[r,R]} \to \partial(K^r)$ as $g = \pi_1 \circ \Phi^{-1}$, where π_1 is the projection map on the first factor. Our goal is to show that g is the homotopic inverse of f, and therefore that f is a homotopy equivalence. First notice that $g \circ f = \mathrm{id}_{\partial(K^r)}$. For $t \in [0, 1]$, define

$$\begin{aligned} \phi_t : \partial(K^r) \times [0, R - r] &\to K^{[r,R]} \\ (x, s) &\mapsto \Phi(x, ts) \end{aligned}$$

and $g_t = \phi_t \circ \Phi^{-1} : K^{[r,R]} \to K^{[r,R]}$. Then, $g_0 = g$ is homotopic to $g_1 = \mathrm{id}_{K^{[r,R]}}$. Under the same assumptions, the inclusion $K^r \hookrightarrow K^R$ is also a homotopy equivalence.

Theorem 3.7 (Chazal, Lieutier). *Let $K, L \subset \mathbb{R}^d$ be two compact sets such that $\mathrm{d_H}(K, L) = \varepsilon < \varepsilon_0 := \frac{1}{2}\min(\mathrm{wfs}(K), \mathrm{wfs}(L))$.*

Then, for every $0 < r, s \leqslant \varepsilon_0$, the offsets K^r and L^s are homotopic.

Remark 3.8. – In general, r cannot be set to 0 in Theorem 3.7. Said otherwise, even if $\mathrm{wfs}(K) > 0$, the homotopy type of K and that of its small thickenings are not always the same An example is given by the so-called *Warsaw circle* that we now describe. Let $K \subset \mathbb{R}^2$ be the union of $K_1 = \{0\} \times [-2, 1]$, $K_2 = [0, 1] \times \{-2\}$, $K_3 = \{1\} \times [-2, 1]$ and $K_4 = \{(x, \sin(2\pi/x)), x \in [0, 1]\}$ (see Figure 2).

One can easily check that K is a simply connected compact set with positive $\mathrm{wfs}(K) > 0$, while the offsets of K are homeomorphic to annuli and that K is the boundary of a topological disk [26, 2.4.8].

consists of a planar closed curve containing oscillations similar to the ones of the graph of $x \mapsto \sin(1/x)$ near 0. Because of these oscillations, the curve is simply connected. However any sufficiently small offset of the curve has a nontrivial fundamental group and hence a different homotopy type as the curve itself.

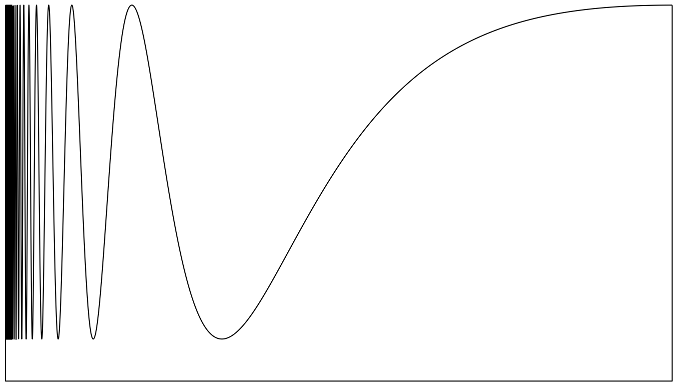


FIGURE 2. A compact set with positive weak feature size whose homotopy type differs from that of its offsets.

– Critical points are unstable with respect to the Hausdorff metric $\mathrm{d_H}$. Hence, Theorem 3.7 should not be considered as a reconstruction result from point clouds. Indeed, if $L = \mathcal{P}$ is a finite point cloud sampled on K, then $\mathrm{wfs}(\mathcal{P}) = \frac{1}{2}\min_{x\neq y\in\mathcal{P}}\|y-x\|$ is half the minimum interpoint distance in $\mathcal{P}$, while $\mathrm{d_H}(K,\mathcal{P})$ would typically be of the order of the maximum interpoint distance $\max_{x\in\mathcal{P}}\mathrm{d}_{\mathcal{P}\setminus\{x\}}(x)$ of $\mathcal{P}$. As a result, the condition "$\mathrm{d_H}(K,\mathcal{P}) < \frac{1}{2}\min(\mathrm{wfs}(K),\mathrm{wfs}(\mathcal{P}))$" may not be fulfilled.

Proof. Let $\delta > 0$ be such that $2\varepsilon_0 > 2\varepsilon + \delta$. Since $\mathrm{d_H}(K,L) \leqslant \varepsilon$, we have the following commutative diagram, where each map is an inclusion:

$$\begin{tikzcd}
K^\delta \arrow[r, "a_0"] \arrow[dr, "d_0" near start] & K^{\delta+\varepsilon} \arrow[r, "a_1"] \arrow[dr, "d_1" near start] & K^{\delta+2\varepsilon} \\
L^\delta \arrow[r, "b_0"] \arrow[ur, "c_0" near start] & L^{\delta+\varepsilon} \arrow[r, "b_1"] \arrow[ur, "c_1" near start] & L^{\delta+2\varepsilon}
\end{tikzcd}$$

By Grove's isotopy lemma (Theorem 2.7), we know that K^r and K^δ are isotopic, as well as L^s and L^δ. Also from Theorem 2.7, we know that the inclusion $a_0 : K^\delta \to K^{\delta+\varepsilon}$ is a homotopy equivalence and therefore that there exist a map $a_0^{-1} : K^{\delta+\varepsilon} \to K^\delta$ such that $a_0 a_0^{-1} \simeq \mathrm{id}_{K^{\delta+\varepsilon}}$ and $a_0^{-1}a_0 \simeq \mathrm{id}_{K^\delta}$. Similar statements hold for a_1, b_0 and b_1 as well. By transitivity of homotopy equivalence, it therefore suffices to prove that e.g. c_1 is a homotopy equivalence. More precisely, we will prove that $b_1^{-1}\circ d_1 \circ a_1^{-1}$ is a homotopic inverse for c_1. Indeed, since all maps are inclusion, we have

$$d_1 \circ a_0 = b_1 \circ d_0, \text{ i.e. } d_1 \simeq b_1 \circ d_0 \circ a_0^{-1}.$$

Then,

$$\begin{aligned} c_1 \circ b_1^{-1} \circ d_1 \circ a_1^{-1} &\simeq c_1 \circ b_1^{-1} \circ b_1 \circ d_0 \circ a_0^{-1} \circ a_1^{-1} \\ &\simeq c_1 \circ d_0 \circ a_0^{-1} \circ a_1^{-1} \\ &\simeq \mathrm{id}_{K^{\delta+2\varepsilon}}, \end{aligned}$$

where we used the equality $c_1 \circ d_0 = a_1 \circ a_0$ to get to the last line. □

4. Generalized Gradient of d_K and μ-Critical Points

Remark 4.1. The distance function to a compact set $K \subset \mathbb{R}^d$ has the following properties:

(i) d_K is 1-Lipschitz, and therefore differentiable almost everywhere.
(ii) d_K is differentiable at x iff $\mathrm{proj}_K(x)$ is a singleton. We denote

$$\mathrm{Med}(K) := \{x \in \mathbb{R}^d \mid \mathrm{card}(\mathrm{proj}_K(x)) > 1\},$$

the *medial axis of* K, which by the previous remark has zero Lebesgue measure, and $\pi_K : \mathbb{R}^d \setminus \mathrm{Med}(K) \to K$ the uniquely defined projection.
(iii) $\phi_K := \|\cdot\|^2 - \mathrm{d}_K^2$ is convex. For this reason, we say that d_K is *semi-concave*. In particular, d_K^2 is as regular as a concave function.

Definition 4.2 (Generalized Gradient)**.** Given $x \in \mathbb{R}^d \setminus K$, let $\Gamma_K(x)$ be the convex hull of $\mathrm{proj}_K(x)$ and define the *generalized gradient* of d_K at x as

$$\nabla^g \mathrm{d}_K(x) := \frac{x - \pi_{\Gamma_K(x)}(x)}{\mathrm{d}_K(x)}.$$

Example 4.3. – Given $x \notin K$, $\nabla^g \mathrm{d}_K(x) = 0$ iff $x \in \mathrm{conv}(\mathrm{proj}_K(x))$ iff x is critical;
– $\|\nabla^g \mathrm{d}_K(x)\| = 1 \implies \mathrm{d}_K$ is differentiable at x;
– If $K = \{p_1, \dots, p_n\}$, the generalized gradient can be evaluated easily, and its flow follows the boundary of the Voronoi cells.

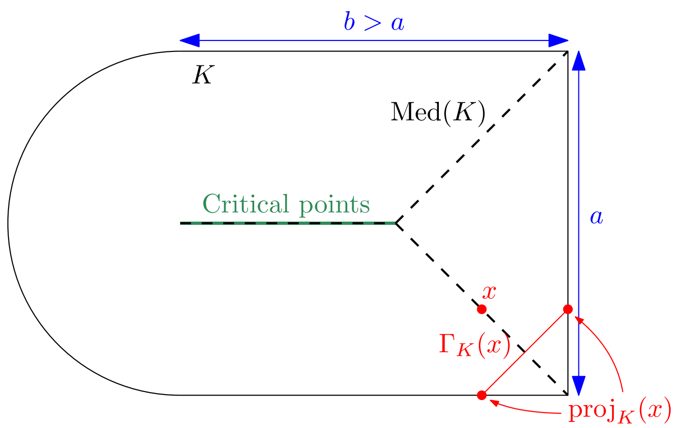


Figure 3. A set $K \subset \mathbb{R}^2$ and its associated medial axis and critical points. Notice that for the point $x \in \mathrm{Med}(K)$ displayed, $\|\nabla^g \mathrm{d}_K(x)\| = \mathrm{d}_{\Gamma_K(x)}(x)/\mathrm{d}_K(x) = 1/\sqrt{2}$.

Theorem 4.4 (Lieutier). *The vector field $\nabla^g \mathrm{d}_K$ is integrable, i.e.*

$$\forall x \in \mathbb{R}^d \setminus K, \exists \gamma : [0, +\infty] \to \mathbb{R}^d \text{ s.t. } \begin{cases} \gamma(0) = x \\ \gamma'(t) = \nabla^g \mathrm{d}_K(\gamma(t)) \end{cases}$$

Moreover, if $\sigma : [0, R] \to \mathbb{R}^d$ is the reparametrization of γ by arclength (assuming that γ does not encounter critical points in the corresponding *time interval), one has*

$$\mathrm{d}_K(\sigma(r)) = \mathrm{d}_K(x) + \int_0^r \|\nabla^g \mathrm{d}_K(\sigma(s))\| \,\mathrm{d}s.$$

Definition 4.5 (Subdifferential, Directional Derivatives)**.** Given a function $f : \mathbb{R}^d \to \mathbb{R}$, we denote

$$\partial f(x) = \{g \in \mathbb{R}^d \mid \forall y \in \mathbb{R}^d,\ f(y) \geqslant f(x) + \langle y - x, g\rangle\},$$
$$f'(x; v) := \limsup_{t\to 0^+} \frac{f(x+tv) - f(x)}{t}.$$

Lemma 4.6. *Given $x \in \mathbb{R}^d$, denote $\Gamma_K(x)$ the convex hull of* $\mathrm{proj}_K(x)$ *and* $\phi_K = \|\cdot\|^2 - \mathrm{d}_K^2$.

(i) $\partial\phi_K(x) = 2\Gamma_K(x)$
(ii) $\phi'_K(x; v) = \max_{p\in\Gamma_K(x)} 2\langle p, v\rangle$
(iii) $\left(\mathrm{d}_K^2\right)'(x; v) = \min_{p\in\Gamma_K(x)} 2\langle x - p, v\rangle$
(iv) $\mathrm{d}'_K(x; v) = \min_{p\in\Gamma_K(x)} \langle \frac{x-p}{\mathrm{d}_K(x)}, v\rangle$
(v) $\|\nabla^g \mathrm{d}_K(x)\| = \max_{\|v\|=1} \mathrm{d}'_K(x; v)$.

Partial proof. (i) Since $\mathrm{d}_K(x)^2 = \inf_{p\in K} \|x - p\|^2$,

$$\phi_K(x) = \max_{p\in K} \|x\|^2 - \|x - p\|^2 = \max_{p\in K} \|p\|^2 - 2\langle x, p\rangle,$$

for any projection $p \in \mathrm{proj}_K(x)$ and $y \in \mathbb{R}^d$ we get

$$\phi_K(y) \geqslant \|p\|^2 - 2\langle y, p\rangle = \phi_K(x) + \langle x - y, 2p\rangle.$$

This gives us the inclusion $2\,\mathrm{proj}_K(x) \subset \partial\phi_K(x)$, and taking convex combinations of elements in $\mathrm{proj}_K(x)$ we get $2\Gamma_K(x) \subset \partial\phi_K(x)$.

(ii) From convex analysis, we know that $\phi'_K(x; v) = \max_{g\in\partial\phi_K(x)}\langle g, v\rangle$. With (i), this gives us the conclusion.

(iii)–(v) Follow easily.

□

The following result provides quantitative stability of μ-critical points.

Theorem 4.7 (Chazal, Cohen-Steiner, Lieutier). *Let $K, L \subset \mathbb{R}^d$ be two compact sets with $\mathrm{d_H}(K, L) \leqslant \varepsilon$ and let $x \in \mathbb{R}^d$ such that $\|\nabla^g \mathrm{d}_K(x)\| \leqslant \mu$. Then, there exists a point y such that*

$$\|x - y\| \leqslant 2\sqrt{\varepsilon \mathrm{d}_K(x)}$$
$$\mu' := \|\nabla^g \mathrm{d}_L(y)\| \leqslant \mu + 2\sqrt{\varepsilon/\mathrm{d}_K(x)}.$$

Proof.

Step 1. Consider $p = \pi_{\Gamma_K(x)}(x)$, so that $2p$ belongs to $\partial\phi_K(x)$, i.e. for every point $y \in \mathbb{R}^d$,

$$\|y\|^2 - \mathrm{d}_K^2(y) \geqslant \|x\|^2 - \mathrm{d}_K^2(x) + 2\langle x - y, p\rangle$$

Since

$$\|x\|^2 - \|y\|^2 = \langle x - y, x + y\rangle = \langle x - y, y - x + 2x\rangle$$
$$= -\|x - y\|^2 + 2\langle x, x - y\rangle,$$

we have

$$\mathrm{d}_K^2(x) + \|x-y\|^2 + 2\langle x-y, x-p\rangle \geqslant \mathrm{d}_K^2(y),$$

so that by Cauchy-Schwarz,

$$\mathrm{d}_K^2(x) + \|x-y\|^2 + 2\mu\mathrm{d}_K(x)\,\|x-y\| \geqslant \mathrm{d}_K^2(y).$$

Step 2. Now, for $R > 0$ to be chosen later, take $\sigma : [0, R] \to \mathbb{R}^d$ the arclength parametrization of an integral curve of $\nabla^g \mathrm{d}_L$ starting from $\sigma(0) = x$. If it reaches a critical point before the desired length R, there is nothing to prove. If not, taking $y = \sigma(R)$ one has

$$\mathrm{d}_L(y) = \mathrm{d}_L(x) + \int_0^R \|\nabla^g \mathrm{d}_L(\sigma(r))\|\,\mathrm{d}r.$$

There must therefore exist $p \in \sigma([0, R])$ such that

$$\|\nabla^g \mathrm{d}_L(p)\| \leqslant \frac{\mathrm{d}_L(y) - \mathrm{d}_L(x)}{R}.$$

Our goal is now to upper bound the norm of this generalized gradient. Since we assumed that $\mathrm{d}_{\mathrm{H}}(K, L) \leqslant \varepsilon$ we have using (4) that

$$\begin{aligned}
\mathrm{d}_L(y) - \mathrm{d}_L(x) &\leqslant \mathrm{d}_K(y) - \mathrm{d}_K(x) + 2\varepsilon \\
&\leqslant \mathrm{d}_K(x)\left(\sqrt{1 + 2\frac{\mu\,\|x-y\|}{\mathrm{d}_K(x)} + \frac{\|x-y\|^2}{\mathrm{d}_K(x)^2}} - 1\right) + 2\varepsilon \\
&\leqslant \mu\,\|x-y\| + \frac{\|x-y\|^2}{2\mathrm{d}_K(x)} + 2\varepsilon,
\end{aligned}$$

where we used the concavity inequality $\sqrt{1+x} - 1 \leqslant \frac{x}{2}$. Dividing the inequality by R and recalling that $\|x-y\| \leqslant R$, we get

$$\mu' := \|\nabla^g \mathrm{d}_L(p)\| \leqslant \mu + \frac{R}{2\mathrm{d}_K(x)} + 2\frac{\varepsilon}{R}.$$

Now, take R so that the last two terms are equal, i.e. $R^2 = 4\mathrm{d}_K(x)\varepsilon$, we get the theorem.

□

Definition 4.8 (μ-Critical Point)**.** Let $K \subset \mathbb{R}^d$ be a compact set. A point $x \in \mathbb{R}^d$ is called *μ-critical for* d_K if $\left\|\nabla \mathrm{d}_K^g(x)\right\| \leqslant \mu$. The *critical function* of d_K is defined as

$$\chi_K(r) = \min_{x \in \partial(K^r)} \left\|\nabla \mathrm{d}_K^g(x)\right\|.$$

Remark 4.9. Thanks to the upper semi-continuity of the subdifferential of a convex function, the function $x \mapsto \left\|\nabla \mathrm{d}_K^g(x)\right\|$ is lower semi-continuous. This implies that the minimum in the definition of χ_K is indeed attained.

Corollary 4.10. *Let K, L be two compact sets with $\mathrm{d}_{\mathrm{H}}(K, L) \leqslant \varepsilon$. Assume that $\chi_K \geqslant \mu$ on $[r, R]$ (with $r > 0$). Then,*

$$\chi_L \geqslant \mu - 2\sqrt{\varepsilon/r} \text{ on } [r + 2\sqrt{\varepsilon R}, R - 2\sqrt{\varepsilon R}].$$

5. Reconstruction of Compact Sets

Definition 5.1 (μ-Reach)**.** The μ-reach of a compact set is defined as

$$\begin{aligned}\mathrm{reach}_\mu(K) &= \sup\{r \geqslant 0 \mid \chi_K \geqslant \mu \text{ on } [0,r]\}\\ &= \sup\{r \geqslant 0 \mid \forall x \in K^r, \|\nabla^g \mathrm{d}_K(x)\| \geqslant \mu\}.\end{aligned}$$

Example 5.2. Note that:

(i) $\mu \mapsto \mathrm{reach}_\mu(K)$ is nonincreasing.
(ii) For $\mu > 0$, $\mathrm{reach}_\mu(K) \leqslant \mathrm{wfs}(K)$.
(iii) For $\mu = 1$, denote $\mathrm{reach}(K) := \mathrm{reach}_1(K)$. Then,

$$\mathrm{reach}(K) = \sup\{r \geqslant 0 \mid K^r \cap \mathrm{Med}(K) = \emptyset\};$$

(iv) For compact smooth submanifolds, $0 < \mathrm{reach}(K) \leqslant$ minimum radius of curvature.
(v) For piecewise linear submanifolds, $\mathrm{reach}(K) = 0$ but there exists $\mu > 0$ such that $\mathrm{reach}_\mu(K) > 0$.
(vi) The set $K \subset \mathbb{R}^2$ of Figure 3 has $\mathrm{wfs}(K) = a$, and

$$\mathrm{reach}_\mu(K) = \begin{cases} a & \text{for } 0 < \mu \leqslant 1/\sqrt{2},\\ 0 & \text{for } 1/\sqrt{2} < \mu \leqslant 1\end{cases}$$

Exercise 5.3. Consider the set $K \subset \mathbb{R}^2$ as depicted in Figure 3, but for $b < a$. Draw its medial axis. What are its critical values? What are $\mathrm{wfs}(K)$, $\mathrm{reach}_\mu(K)$ $(0 < \mu < 1)$, and $\mathrm{reach}(K)$?

Theorem 5.4 (Chazal, Cohen-Steiner, Lieutier). *Let K be a compact set with positive μ-reach, and L be an approximation of K with $\mathrm{d_H}(K,L) \leqslant \varepsilon$. Then, d_L has no critical values in the interval*

$$I = (4\varepsilon/\mu^2, \mathrm{reach}_\mu(K) - 3\varepsilon).$$

In particular, if $\mathrm{d_H}(K,L) = \varepsilon < \kappa\,\mathrm{reach}_\mu(K)$ with $\kappa \leqslant \frac{\mu^2}{5\mu^2+12}$, then K^r and L^s are homotopy equivalent as soon as

$$0 < r < \mathrm{reach}_\mu(K) \text{ and } 4\frac{\varepsilon}{\mu^2} < s < \mathrm{reach}_\mu(K) - 3\varepsilon.$$

Proof.

Statement 1. We prove first the statement about critical values. Assume by contradiction that there exists $x \in \mathbb{R}^d$ such that $\mathrm{d}_L(x) \in I$ and x is critical for L, i.e. $\|\nabla \mathrm{d}_L(x)\| = \mu_x = 0$. Then, by the stability theorem, there exists y with

$$\|x-y\| \leqslant 2\sqrt{\varepsilon \mathrm{d}_L(x)}$$

$$\mu_y = \left\|\nabla \mathrm{d}_K^g(y)\right\| \leqslant \mu_x + 2\sqrt{\varepsilon/\mathrm{d}_L(x)} < \mu$$

To get the contradiction, it therefore suffices to establish that $\mathrm{d}_K(y) < \mathrm{reach}_\mu(K)$. To see this, recall that by semiconcavity one has (see (4)),

$$\begin{aligned}\mathrm{d}_L^2(y) &\leqslant \mathrm{d}_L^2(x) + \|x-y\|^2 + 2\mu_x\|x-y\|\\ &= \mathrm{d}_L^2(x) + \|x-y\|^2\\ &\leqslant \mathrm{d}_L^2(x) + 4\varepsilon \mathrm{d}_L(x).\end{aligned}$$

By concavity of the square root,

$$\begin{aligned} \mathrm{d}_L(y) &\leqslant \mathrm{d}_L(x)(1+2\varepsilon/\mathrm{d}_L(x)) = \mathrm{d}_L(x)+2\varepsilon \\ &< \mathrm{reach}_\mu(K) - 3\varepsilon + 2\varepsilon \end{aligned}$$

Finally, we use $\mathrm{d_H}(K,L) \leqslant \varepsilon$ to get $\mathrm{d}_K(x) < \mathrm{reach}_\mu(K)$, a contradiction.

Statement 2. Let us write $I = (s_-, s_+)$, i.e. $s_- = 4\varepsilon/\mu^2$ and $s_+ = \mathrm{reach}_\mu(K) - 3\varepsilon$. We want to apply Theorem 3.7 to K and $L' = L^{s_-}$. For this, note that

$$\mathrm{d_H}(K, L^{s_-}) \leqslant \mathrm{d_H}(K,L) + \mathrm{d_H}(L, L^{s_-}) \leqslant \varepsilon + 4\varepsilon/\mu^2,$$

and

$$\mathrm{wfs}(K) \geqslant \mathrm{reach}_\mu(K).$$

Furthermore, one can check that any point critical for L^{s_-} is also critical for L. As a result, denoting by $\mathrm{Crit}_0(A) \subset \mathbb{R}^d$ for the set of critical points of $A \subset \mathbb{R}^d$, we have

$$\mathrm{Crit}_0(L^{s_-}) \subset \mathrm{Crit}_0(L) \setminus L^{s_-} \subset (L^{s_+})^c,$$

where the second inclusion comes from the first statement. As $L^{s_+} = (L^{s_-})^{s_+ - s_-}$, we get

$$\begin{aligned} \mathrm{wfs}(L^{s_-}) &\geqslant s_+ - s_- \\ &= \mathrm{reach}_\mu(K) - 3\varepsilon - 4\varepsilon/\mu^2. \end{aligned}$$

As a result, Theorem 3.7 applies to K and $L' = L^{s_-}$ as soon as

$$\begin{aligned} &\mathrm{d_H}(K, L^s) < \frac{1}{2}\min(\mathrm{wfs}(K), \mathrm{wfs}(L^s)) \\ \Leftarrow\ &\varepsilon(2+8/\mu^2) < \mathrm{reach}_\mu(K) - \varepsilon(3+4/\mu^2) \\ \Leftrightarrow\ &\varepsilon(5+12/\mu^2) < \mathrm{reach}_\mu(K). \end{aligned}$$

□

Provided this condition is fulfilled, it states that K^r and $(L^{s_-})^{s'} = L^{s_- + s'}$ are homotopic for all $0 < r < \mathrm{reach}_\mu(K)$ and $0 < s' < s_+ - s_-$. The result follows by taking $s = s_- + s'$, which does range in $s_- + (0, s_+ - s_-) = (s_-, s_+) = I$.

6. Support Reconstruction from Random Point Clouds

6.1. Measure, Diameter, and Sampling.

Definition 6.1 ((a,b)-Standard Measure)**.** The distribution P is said to be (a,b)-*standard* at scale r_0 if for all $x \in \mathrm{supp}(P)$ and all $r \leqslant r_0$,

$$P\left(\mathrm{B}(x,r)\right) \geqslant a r^b.$$

Roughly speaking, a measure that is (a,b)-standard at scale r_0 behaves like the b-dimensional Lebesgue measure, though b needs not be an integer. This assumption is pretty popular in the literature on set estimation, and its properties will be used extensively in the results of this course. So far, we have considered the case $b = d$ in $\mathbb{R}^d$ for *Density Support Estimation.* As we will see shortly, such an assumption gives bounds on massiveness of the support $\mathrm{supp}(P) \subset \mathbb{R}^d$.

To measure massiveness of subsets $K \subset \mathbb{R}^d$, we will (again!) use packing and covering numbers. That is, numbers of balls optimally displayed at some scale r in K. A r-covering of $K \subset \mathbb{R}^d$ is a subset $\mathcal{X} = \{x_1, \ldots, x_i\} \subset K$ such that for all $x \in K$, $\mathrm{d}_{\mathcal{X}}(x) \leqslant r$. A r-packing of K is a subset $\mathcal{Y} = \{y_1, \ldots, y_i\} \subset K$ such that for all $y, y' \in \mathcal{Y}$, $\mathrm{B}(y, r) \cap \mathrm{B}(y', r) = \emptyset$ (or equivalently $\|y' - y\| > 2r$).

Definition 6.2 (Covering and Packing numbers)**.** For $K \subset \mathbb{R}^d$ and $r > 0$, the *covering number* $\mathrm{cv}(K, r)$ is the minimum number of balls of radius r that are necessary to cover K:

$$\mathrm{cv}(K, r) = \min \{k > 0 \mid \text{there exists a } r\text{-covering of cardinality } k\}.$$

The *packing number* $\mathrm{pk}(K, r)$ is the maximum number of disjoint balls of radius r that can be packed in K:

$$\mathrm{pk}(K, r) = \max \{k > 0 \mid \text{there exists a } r\text{-packing of cardinality } k\}.$$

For a given space K, covering and packing numbers usually have the same order of magnitude, as stated in the following result.

Proposition 6.3. *For all $K \subset \mathbb{R}^d$ and $r > 0$,*

$$\mathrm{pk}(K, 2r) \leqslant \mathrm{cv}(K, 2r) \leqslant \mathrm{pk}(K, r).$$

Proof. For the left-hand side inequality, notice that if K is covered by a family of balls of radius $2r$, each of these balls contains at most one point of a maximal packing $\mathcal{Y}$ at scale $2r$. Conversely, the right-hand side inequality follows from the fact that a maximal r-packing $\mathcal{Y}$ is always a $2r$-covering. If it was not the case, one could add a point x_0 such that $\mathrm{d}_{\mathcal{Y}}(x_0) > 2r$, which is impossible by maximality of $\mathcal{Y}$. □

It is crucial to note that any (a, b)-standard measure P has a controlled support massiveness, in the following sense.

Proposition 6.4. *Let P be a (a, b)-standard probability distribution at scale $r_0 > 0$. Then for $r \leqslant r_0$,*

$$\mathrm{pk}\left(\mathrm{supp}(P), r\right) \leqslant \frac{1}{ar^b}.$$

For $r \leqslant 2r_0$,

$$\mathrm{cv}\left(\mathrm{supp}(P), r\right) \leqslant \frac{2^b}{ar^b}.$$

Proof. Let $\mathcal{Y} = \{y_1, \ldots, y_N\}$, $N = \mathrm{pk}\left(\mathrm{supp}(P), r\right)$, be a maximal r-packing of $\mathrm{supp}(P)$. We have

$$\begin{aligned} 1 = P\big(\mathbb{R}^d\big) &\geqslant P\left(\cup_{i=1}^N \mathrm{B}(y_j, r)\right) \\ &\geqslant \sum_{i=1}^N P\left(\mathrm{B}(y_j, r)\right) \\ &\geqslant N a r^b = \mathrm{pk}\left(\mathrm{supp}(P), r\right) a r^b, \end{aligned}$$

hence the first result. The bound on $\mathrm{cv}\left(\mathrm{supp}(P), r\right)$ then follows from (6.3). □

If it is assumed to be path-connected, one can derive an upper bound on the diameter $\mathrm{diam}(K) = \sup_{x,y \in K} \|y - x\|$ of such a support. This is based on the following bound.

Lemma 6.5. *Let $K \subset \mathbb{R}^d$ be a bounded subset. If K is path-connected, then for all $\varepsilon > 0$,* $\mathrm{diam}(K) \leqslant 2\varepsilon\, \mathrm{cv}(K, \varepsilon)$.

Proof. Let $p, q \in K$ and $\gamma : [0,1] \to K$ be a continuous path joining $\gamma(0) = p$ and $\gamma(1) = q$. Writing $N = \mathrm{cv}(K, \varepsilon)$, let $x_1, \ldots, x_N \in \mathbb{R}^d$ be the centers of a minimal covering of K by open balls of radii ε. We let U_j denote $\{t \in [0,1] | \, \|\gamma(t) - x_j\| < \varepsilon\}$. By construction of the covering, there exists $x_{(1)} \in \{x_1, \ldots, x_N\}$ such that $\left\|p - x_{(1)}\right\| < \varepsilon$. Then $U_{(1)} \ni \gamma(0) = p$ is a non-empty open subset of $[0,1]$, so that $t_{(1)} = \sup U_{(1)}$ is positive. If $t_{(1)} = 1$, then $\left\|q - x_{(1)}\right\| \leqslant \varepsilon$, and in particular $\|q - p\| \leqslant 2\varepsilon$. If $t_{(1)} < 1$, since $U_{(1)}$ is an open subset of $[0,1]$, we see that $\gamma(t_{(1)}) \notin U_{(1)}$. But $\cup_{i=1}^N U_j$ is an open cover of $[0,1]$, which yields the existence $U_{(2)}$ such that $\gamma(t_{(1)}) \in U_{(2)}$, and for all $t < t_{(1)}$, $\gamma(t) \notin U_{(2)}$. Then consider $t_{(2)} = \sup U_{(2)}$, and so on. Doing so by induction, we build a sequence $0 < t_{(1)} < \ldots < t_{(k)} \leqslant 1$, for $k \leqslant N$, and distinct centers $x_{(1)}, \ldots, x_{(k)} \in \{x_1, \ldots, x_N\}$, such that $\left\|p - x_{(1)}\right\| < \varepsilon$, $\left\|q - x_{(k)}\right\| \leqslant \varepsilon$, with $\left\|\gamma(t_{(j)}) - x_{(j)}\right\| \leqslant \varepsilon$ for $1 \leqslant i \leqslant N$ and $\left\|\gamma(t_{(j)}) - x_{(j+1)}\right\| < \varepsilon$ for $1 \leqslant j \leqslant N-1$. In particular, $\left\|x_{(j)} - x_{(j+1)}\right\| \leqslant 2\varepsilon$ for all $1 \leqslant j \leqslant N-1$. To conclude, write

$$\begin{aligned}
\|p - q\| &\leqslant \left\|p - x_{(1)}\right\| + \left\|x_{(1)} - x_{(N)}\right\| + \left\|q - x_{(N)}\right\| \\
&\leqslant \varepsilon + \sum_{j=1}^{N-1} \left\|x_{(j)} - x_{(j+1)}\right\| + \varepsilon \\
&\leqslant 2N\varepsilon = 2\varepsilon\, \mathrm{cv}(K, \varepsilon).
\end{aligned}$$

Since this bound holds for all $p, q \in K$, we get the announced bound on the diameter of K. □

Thereby, the following Proposition 6.6 follows from Lemma 6.5 applied with $r = 2r_0$, together with Proposition 6.4.

Proposition 6.6. *If P is (a,b)-standard at scale r_0 and has a path-connected support* $\mathrm{supp}(P)$*, then*

$$\mathrm{diam}\left(\mathrm{supp}(P)\right) \leqslant 4 r_0^{1-b}/a.$$

Remark 6.7. Path-connectedness is crucial here. Indeed, consider $K_x = \mathrm{B}(-x, 1) \cup \mathrm{B}(x, 1) \subset \mathbb{R}^d$ for $\|x\|$ arbitrarily large. Then $\mathrm{diam}(K_x) = 2\|x\| + 1 \to \infty$ although the uniform distribution on K_x is (a, d)-standard at scale $r_0 = 1$ with fixed $a > 0$.

Further investigating the properties of (a,b)-standard measures at scale $r_0 > 0$, let us now give the convergence rate of a sample point cloud $\mathbb{X}_n = \{X_1, \ldots, X_n\}$ towards its underlying support $\mathrm{supp}(P)$.

Proposition 6.8. *Let P be an (a,b)-standard probability measure at scale $r_0 > 0$, and $\mathbb{X}_n = \{X_1, \ldots, X_n\}$ is an i.i.d. n-sample with common distribution P, then for all $r \leqslant 2r_0$,*

$$\mathbb{P}\left(\mathrm{d_H}\big(\mathrm{supp}(P), \mathbb{X}_n\big) > r\right) \leqslant \frac{4^b}{a r^b} \exp\left(-n \frac{a}{2^b} r^b\right).$$

In particular:

(i) For all $\alpha > 0$, there exists $C_{a,b,\alpha} > 0$ such that for n large enough so that $\left(C_{a,b,\alpha}\frac{\log n}{n}\right)^{1/b} \leqslant 2r_0$, with probability at least $1 - \left(\frac{1}{n}\right)^{\alpha}$,

$$\mathrm{d_H}\big(\mathrm{supp}(P), \mathbb{X}_n\big) \leqslant \left(C_{a,b,\alpha}\frac{\log n}{n}\right)^{1/b}.$$

(ii) For all $r \leqslant 2r_0$, $\mathrm{d_H}\big(\mathrm{supp}(P), \mathbb{X}_n\big) \leqslant r$ with probability at least $1-\delta$, as soon as

$$n \geqslant \frac{C'_{a,b}}{r^b}\left(\log\left(1/r\right) + \log\left(1/\delta\right)\right).$$

In other words, with n points, the typical density of sampling of a (a,b)-standard measure is of order $(\log n/n)^{1/b}$. Roughly speaking, it relies on the fact that standard measures have uniformly spread mass on their support. Hence, an n-sample would visit all the areas of its support with high probability. This comes from the fact that the massiveness of $\mathrm{supp}(P)$ (in terms of covering number) is controlled.

Proof. Since $\mathbb{X}_n \subset \mathrm{supp}(P)$ with probability one, the Hausdorff distance between $\mathbb{X}_n$ and $\mathrm{supp}(P)$ rewrites almost surely as

$$\mathrm{d_H}\big(\mathrm{supp}(P), \mathbb{X}_n\big) = \sup_{x\in\mathrm{supp}(P)} \min_{1\leqslant j\leqslant n} \left\|X_i - x\right\|.$$

For some $\delta > 0$ to be chosen later, consider a minimal δ-covering $\mathcal{X} = \{x_1, \ldots, x_N\}$ of $\mathrm{supp}(P)$, $N = \mathrm{cv}(\mathrm{supp}(P), \delta)$. By definition of a δ-covering, for all $x \in \mathrm{supp}(P)$ there exists some $x_{i_0} \in \mathcal{X}$ such that $\|x_{i_0} - x\| \leqslant \delta$. Hence,

$$\begin{aligned}\min_{1\leqslant i\leqslant n}\left\|X_i - x\right\| &\leqslant \min_{1\leqslant i\leqslant n}\left\|x_{i_0} - x\right\| + \left\|X_i - x_{i_0}\right\|\\ &\leqslant \delta + \min_{1\leqslant i\leqslant n}\left\|X_i - x_{i_0}\right\|\\ &\leqslant \delta + \max_{1\leqslant j\leqslant N}\min_{1\leqslant i\leqslant n}\left\|X_i - x_j\right\|.\end{aligned}$$

As a consequence,

$$\begin{aligned}\mathbb{P}\left(\mathrm{d_H}\big(\mathrm{supp}(P), \mathbb{X}_n\big) > r\right) &\leqslant \mathbb{P}\left(\max_{1\leqslant j\leqslant N}\min_{1\leqslant i\leqslant n}\left\|X_i - x_j\right\| > r - \delta\right)\\ &\leqslant \sum_{i=1}^{N}\mathbb{P}\left(\min_{1\leqslant i\leqslant n}\left\|X_i - x_j\right\| > r-\delta\right)\end{aligned}$$

But whenever $r - \delta \leqslant r_0$, for all $1 \leqslant j \leqslant N$,

$$\begin{aligned}\mathbb{P}\left(\min_{1\leqslant i\leqslant n}\left\|X_i - x_j\right\| > r-\delta\right) &= \prod_{1\leqslant i\leqslant n}\mathbb{P}\left(\left\|X_i - x_j\right\| > r - \delta\right)\\ &= \left(1 - P\left(\mathrm{B}(x_j, r-\delta)\right)\right)^n\\ &\leqslant \left(1 - a(r-\delta)^b\right)^n\\ &\leqslant \exp\left(-na(r-\delta)^b\right),\end{aligned}$$

where the last inequality follows from $1-t \leqslant e^{-t}$. Therefore, Proposition 6.4 yields for all $\delta \leqslant 2r_0$ such that $r-\delta \leqslant r_0$,

$$\begin{aligned}\mathbb{P}\left(\mathrm{d_H}\big(\mathrm{supp}(P), \mathbb{X}_n\big) > r\right) &\leqslant \mathrm{cv}(\mathrm{supp}(P), \delta) \exp\left(-na(r-\delta)^b\right) \\ &\leqslant \frac{2^b}{a\delta^b} \exp\left(-na(r-\delta)^b\right).\end{aligned}$$

Setting $\delta = r/2$ yields the announced result. □

6.2. **A Probabilistic Reconstruction Result.**

THEOREM 6.9. *Let P be a probability distribution and $K = \mathrm{supp}(P)$ be such that:*
– $\mathrm{reach}(K) > 0$*;*
– *P is (a,b)-standard at scale $r_0 > 0$.*
Let $\mathbb{X}_n = \{X_1, \ldots, X_n\}$ be a i.i.d. sample of P, and $\varepsilon < \frac{1}{17}\mathrm{reach}(K) \wedge 2r_0$. Then for all and $n \geqslant C_{a,b}\left(\log\left(1/\varepsilon\right) + \log\left(1/\delta\right)\right)/\varepsilon^b$,

$$\mathbb{P}\left(K^{\frac{1}{2}\mathrm{reach}(K)} \textit{ and } \mathbb{X}_n^{5\varepsilon} \textit{ are homotopy equivalent}\right) \geqslant 1-\delta.$$

Proof. From Proposition 6.8, $\mathbb{P}\left(\mathrm{d_H}\left(K, \mathrm{supp}(P)\right) \leqslant \varepsilon\right) \geqslant 1-\delta$. Hence, on this event, applying Theorem 5.4 with K, $L = \mathbb{X}_n$, $\mu = 1$, $\kappa = 1/17$, $r = \mathrm{reach}(K)/2$ and $s = 5\varepsilon$ yields the result. □

7. FURTHER SOURCES

These lecture notes are courtesy of Quentin Mérigot. Their structure mainly follows [CL05] and [CCSL09].

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