

Exercise Sheet 1

Exercise 1: Estimating the Bayes Error (10+10+10 P)

The Bayes decision rule for the two classes classification problem results in the Bayes error

$$P(\text{error}) = \int P(\text{error}|\mathbf{x}) p(\mathbf{x}) d\mathbf{x},$$

where

$$P(\text{error}|\mathbf{x}) = \min [P(\omega_1|\mathbf{x}), P(\omega_2|\mathbf{x})]$$

is the probability of error for a particular input \mathbf{x} . Interestingly, while class posteriors $P(\omega_1|\mathbf{x})$ and $P(\omega_2|\mathbf{x})$ can often be expressed analytically and are integrable, the error function has discontinuities that prevent its analytical integration, and therefore, direct computation of the Bayes error. Section 2.8 of Duda&Hart explains how considering an upper-bound to the Bayes error instead of the true error overcomes this difficulty.

- (a) *Show* that the full error can be upper-bounded as follows:

$$P(\text{error}) \leq \int \frac{2}{\frac{1}{P(\omega_1|\mathbf{x})} + \frac{1}{P(\omega_2|\mathbf{x})}} p(\mathbf{x}) d\mathbf{x}.$$

Note that the integrand is now continuous and corresponds to the harmonic mean of class posteriors weighted by $p(\mathbf{x})$.

- (b) *Show* using this result that for the univariate probability distributions

$$p(x|\omega_1) = \frac{\pi^{-1}}{1 + (x - \mu)^2} \quad \text{and} \quad p(x|\omega_2) = \frac{\pi^{-1}}{1 + (x + \mu)^2},$$

the Bayes error can be upper-bounded by:

$$P(\text{error}) \leq \frac{2 P(\omega_1) P(\omega_2)}{\sqrt{P(\omega_1)^2 + (4\mu^2 + 2) P(\omega_1) P(\omega_2) + P(\omega_2)^2}}$$

(Hint: you can use the identity $\int \frac{1}{ax^2+bx+c} dx = \frac{2\pi}{\sqrt{4ac-b^2}}$ for $b^2 < 4ac$.)

- (c) *Explain* how you would estimate the error if there was no upper-bounds that are both tight and analytically integrable. Discuss following two cases: (1) the data is low-dimensional and (2) the data is high-dimensional.

Exercise 2: Bayes Decision Boundaries (10+10+10 P)

One might speculate that, in some cases, the generated data $p(x|\omega_1)$ and $p(x|\omega_2)$ is of no use to improve the accuracy of a classifier, in which case one should only rely on prior class probabilities $P(\omega_1)$ and $P(\omega_2)$.

For the first part of this exercise, we assume that the data for each class is generated by the univariate Laplacian probability distributions:

$$p(x|\omega_1) = \frac{1}{2\sigma} \exp\left(-\frac{|x - \mu|}{\sigma}\right) \quad \text{and} \quad p(x|\omega_2) = \frac{1}{2\sigma} \exp\left(-\frac{|x + \mu|}{\sigma}\right).$$

where $\sigma > 0$.

- (a) *Compute* the Bayes optimal decision boundary (i.e. points $x \in \mathbb{R}$ where $P(\omega_1|x) = P(\omega_2|x)$).
- (b) *Determine* for which values of $P(\omega_1), P(\omega_2), \mu, \sigma$ the optimal decision is to always predict the first class (i.e. under which conditions $P(\text{error}|x) = P(\omega_2|x) \quad \forall x \in \mathbb{R}$).
- (c) *Repeat* the exercise for the case where the data for each class is generated by the univariate Gaussian probability distributions:

$$p(x|\omega_1) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) \quad \text{and} \quad p(x|\omega_2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x + \mu)^2}{2\sigma^2}\right).$$

where $\sigma > 0$.

Exercise 3: Programming (40 P)

Download the programming files on ISIS and follow the instructions.