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alternative definition of the natural logarithm

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The http://planetmath.org/NaturalLogarithm2natural logarithm function $\log x$ can be defined by an integral, as shown in the entry to which this entry is attached. However, it can also be defined as the inverse function of the exponential function $\exp x = e^x$.

In this entry, we show that this definition of $\log x$ yields a function that satisfies the logarithm laws $\log xy = \log x + \log y$ and $\log x^r = r \log x$ hold for any positive real numbers x and y and any real number r. We also show that $\log x$ is differentiable with respect to x on the interval $(1, \infty)$ with derivative $\frac{1}{x}$. Note that the logarithm laws imply that $\log 1 = 0$. The mean-value theorem implies that these properties characterize the logarithm function.

The proof of the first logarithm law is straightforward. Let x and y be positive real numbers. Then using the fact that e^x and $\log x$ are inverse functions, we find that

$$e^{\log xy} = xy = e^{\log x} \cdot e^{\log y} = e^{\log x + \log y}.$$

Since e^x is an injective function, the equation $e^{\log xy} = e^{\log x + \log y}$ implies the first logarithm law.

For the second logarithm law, observe that

$$e^{\log x^r} = x^r = (e^{\log x})^r = e^{r \log x}.$$

Since e^x and $\log x$ are inverse functions and e^x is differentiable, so is $\log x$. We can use the chain rule to find a formula for the derivative:

$$1 = \frac{dx}{dx} = \frac{d}{dx}[e^{\log x}] = e^{\log x} \frac{d}{dx}[\log x] = x \frac{d}{dx}[\log x].$$

Hence, $\frac{d}{dx}[\log x] = \frac{1}{x}$.