Parametic and Polar Review

Parametric Functions

We can represent x, y, and z in terms of other variables, typically t. We can represent this as

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

When graphing, the best approach is to plug in values for intermediate values and see how they behave.
Calculus:
If $x = f(t)$ and $y = g(t)$, then:

$$\frac{dx}{dt} = f'(t) \qquad \frac{dy}{dt} = g'(t) \tag{1}$$

If $f'(t) \neq 0$, then

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \tag{2}$$

If $g'(t)$ is 0, then the slope is vertical.

To find the length of a parametric curve, we use

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt \tag{3}$$

Where a and b are considered the endpoints in terms of time.

If we want to find an area, we use equations

$$A = \int_a^b y dx = \int_a^b y(t)x'(t) dt = -\int_a^b y'(t)x(t) dt \tag{4}$$

Polar Functions

Polar is a type of parametric function, where the cartesian plane is described in terms of r and θ. A good way to think about polar functions is θ determines how much you turn from the positive x-axis, and r determines how far a point is from the origin.

Basic conversion equations:

$$x = rcos\theta \qquad y = rsin\theta \qquad x^2 + y^2 = r^2 \tag{5}$$

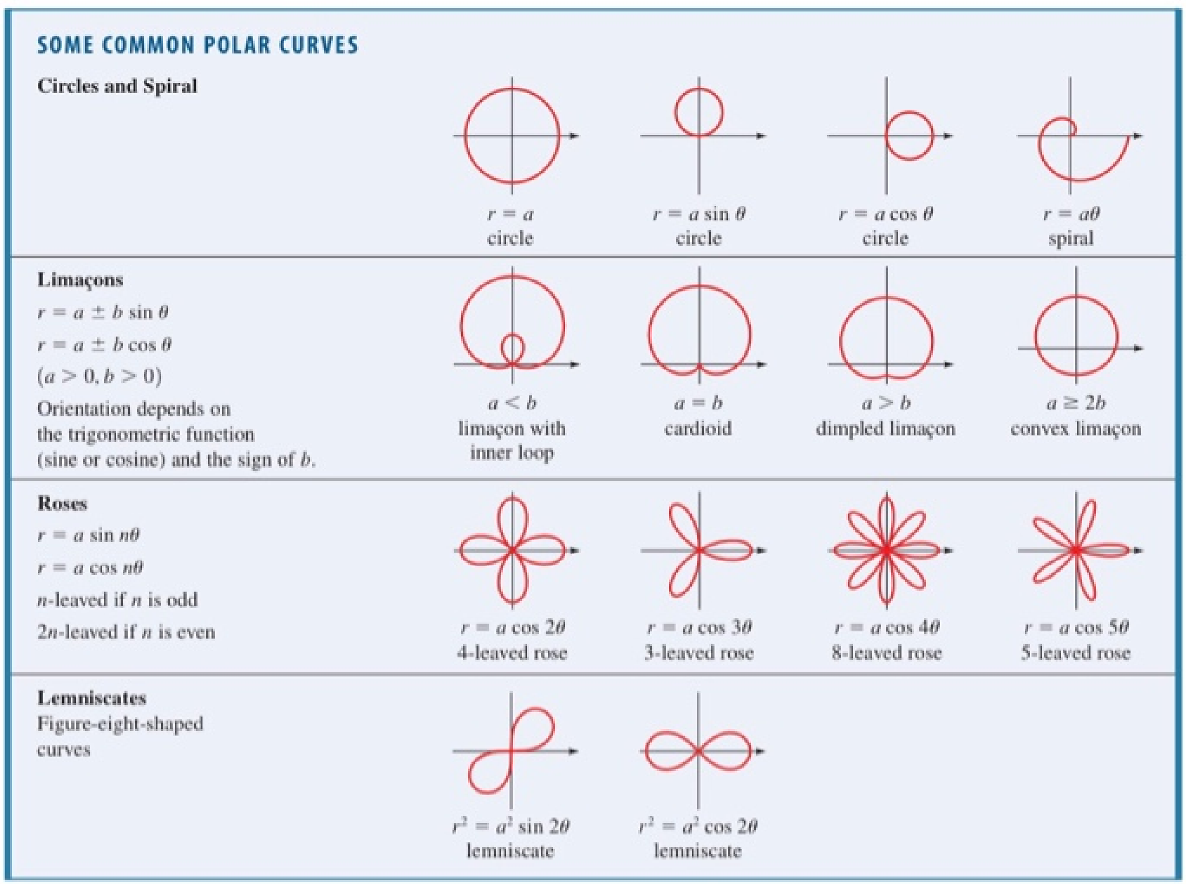
For polar area, we use the equation:

$$A = \int_a^b \frac{1}{2} r^2 d\theta \tag{6}$$

For length of a polar curve, we use:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \tag{7}$$

For graphing, go for it. Be aggressive. Here are some common ones to know



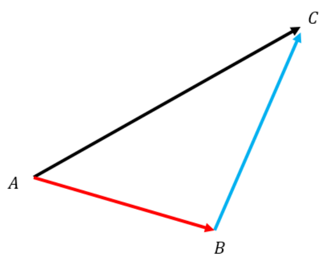
Vectors

Vectors Basics

Vectors is a quantity that has a direction and length. It is not associated with a point.
If we have a vectors $\vec{P} =< x', y', z' >$ and $\vec{Q} =< x, y, z >$, the vector from Q to P is described with $\vec{QP} =< x - x', y - y', z - z' >$

Vector Operations

We can do addition and subtractions. For addition, we can use head to tips method.



From this, we can get:

$$\vec{AB} + \vec{BC} = \vec{AC} \qquad \vec{BC} = \vec{AC} - \vec{AB} \tag{1}$$

When doing addition and subtraction, each component just gets added up

To find a magnitude, it goes as follows

$$|\vec{v}| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2} \tag{2}$$

There has been defined unit vectors, those who have the magnitude of 1. We can define vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ to be

$$\vec{i} =< 1, 0, 0 > \qquad \vec{j} =< 0, 1, 0 > \qquad \vec{k} =< 0, 0, 1 > \tag{3}$$

Multiplication using Vectors

Dot Product:
Given vector $\vec{P}$ and vector $\vec{Q}$, the dot product is

$$\vec{P} \cdot \vec{Q} = P_1Q_1 + P_2Q_2 + P_3Q_3 \tag{4}$$

It can also be computed as

$$\vec{P} \cdot \vec{Q} = |\vec{P}||\vec{Q}|cos(\theta) \tag{5}$$

Where θ is the angle between the two vectors

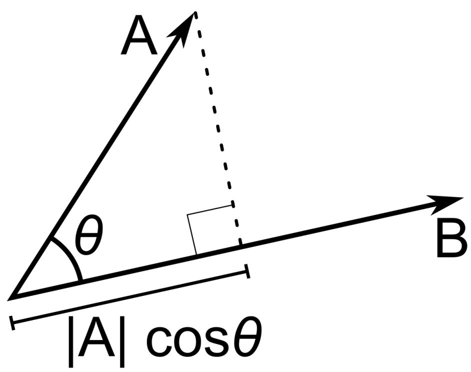
Cross Product:
The cross product takes in two vectors and represents a vector that is perpendicular to the two vectors. The direction is determined by right-hand rule (palm sweeps in the direction of θ, and the thumb is the direction of cross product). The formula is as follows:

$$\vec{P} \times \vec{Q} =< P_2Q_3 - P_3Q_2, P_3Q_1 - P_1Q_3, P_1Q_2 - P_2Q_1 > \tag{6}$$

It can also be computed as

$$|\vec{P} \times \vec{Q}| = |\vec{P}||\vec{Q}|sin(\theta) \tag{7}$$

Projections



We can consider projections to be the "shadow" of one vector on another. Here, we look at the projection of $\vec{A}$ onto $\vec{B}$. We can represent the projection to have the length of $|\vec{A}|cos(\theta)$. $cos(\theta)$ can be rewritten to be $\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$, hence we get

$$comp_b a = \frac{\vec{A}\cdot\vec{B}}{|\vec{B}|} \qquad\qquad proj_b a = comp_b a \left(\frac{|\vec{B}|}{\vec{B}}\right) \tag{8}$$

$comp_b a$ is the scalar projection, while $proj_b a$ is the vector projection.

Standard Basis Vectors

This is a side note, but going back to unit vectors, if we want to find a vector that is a unit vector, but is in the same direction of a given vector, then we simply divide by magnitude. This is called the standard basis vector

$$StandardBasisVector = \frac{\vec{P}}{|P|} \tag{9}$$

(Some) Applications of Vectors

Parallelagrams

If we have a parallelogram made up vectors $\vec{A}$ and $\vec{B}$, the the height can be represented as $|\vec{B}|sin(\theta)$, thus the area is

$$Area = |\vec{A}||\vec{B}|sin(\theta) = |\vec{A} \times \vec{B}| \tag{1}$$

This is a very important equation to build up to some things in multivariable calculus

Parallelapiped

If we have a Parallelapiped made up vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ the the volume is

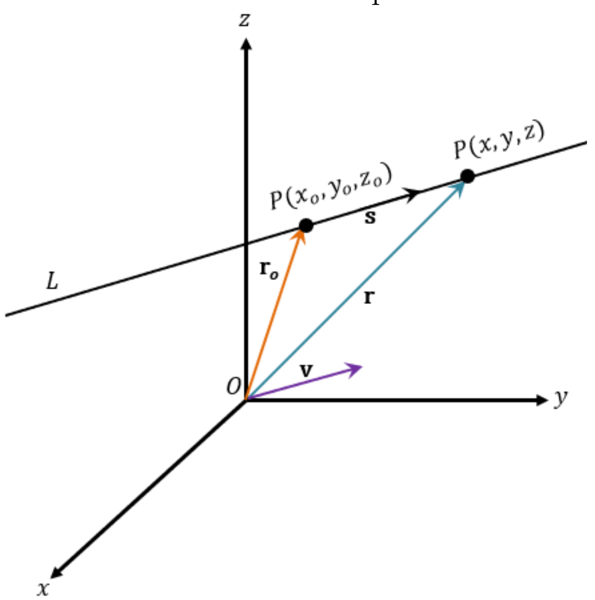
$$Volume = |\vec{C} \cdot (\vec{A} \times \vec{B})| \tag{2}$$

I will elaborate on this if I feel like it, but it is not in scope, so oh well.

Lines and Planes

Defining Lines by Vectors

Lines can be defined as a set of points in terms of vectors, as seen in the diagram below.



For a given point P, we can represent a vector from the origin to the point as $\vec{r}$. Given a starting point $\vec{r}_0$ and a directional unit vector $\vec{v}$, we can derive a general formula for a line to be

$$\vec{r} = \vec{r}_0 + t\vec{v} \tag{1}$$

where t is time. Intuition from the diagram is that we are just performing vector addition with $\vec{r}_0$ and an amount of change in the direction of $\vec{v}$.

From this we can get

$$\vec{r}_0 =< x_0, y_0, z_0 > \qquad \vec{r} =< x, y, z > \qquad \vec{v}_0 =< a, b, c > \tag{2}$$

and get the parametric definition through

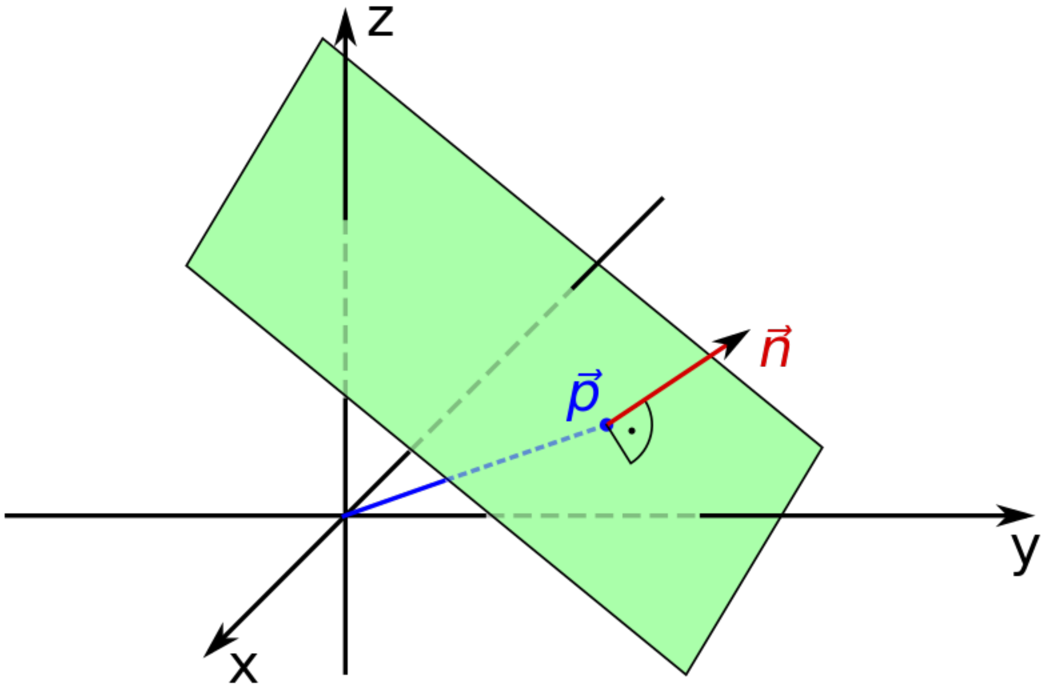
$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = z_0 + tc \tag{3}$$

Since for a given point, the time should be the same, if neither a, b, or c does not equal 0, then

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{4}$$

Defining Planes by Vectors

Just like lines, we can define a plane in regards to be a set of points in regard to vectors. We can define a plane to be a set of points perpendicular to point given a direction.



From the image above, we can infer that for some point on the plane $\vec{p}_0$, there is a perpendicular vector $\vec{n}$. This means if we have an arbitrary point $\vec{p}$ on the plane, we can use vector addition to deduce that

$$(\vec{p} - \vec{p}_0) \cdot \vec{n} = 0 \tag{5}$$

The reasoning is that we can have a vector from $\vec{p}_0$ to $\vec{p}$, and given the definition of a plane, it has to be perpendicular to $\vec{n}$. By definition of the dot product, the dot product should be 0.

This definition isn't particularly useful computationally, but we can say that

$$(x - x_0)a + (y - y_0)b + (z - z_0)c = 0 \tag{6}$$

given that $\vec{n} =< a, b, c >$. Given that $\vec{p}_0$, is a constant, we can further simplify towards

$$xa + yb + zc = d \tag{7}$$

This equation is pretty useful since we can still get a vector perpendicular to the plane from a, b, c.

Problem Solving Notes

When looking for intersections, take note of when the two "objects" are equal.
When constructing either a line or plane, try to look for either parallel lines (lines) or

perpendicular lines (planes).
When finding the shortest distance between the a plane and a point, use

$$Distance = \frac{xa + yb + zc + d}{\sqrt{a^2 + b^2 + c^2}} \tag{8}$$

Calculus with Vector-Valued Functions

Derivitives

Supposed we have $\vec{r}(t) =< f(t), g(t), h(t) >$ and $a \leq t \leq b$, then we can say

$$\vec{r}'(t) =< f'(t), g'(t), h'(t) > \tag{1}$$

geometrically, it is the tangent line to a given point on the curve.

If we wanted to get a tangent line at a given time t_0, then we define the line parametrically to be

$$x = f(t_0) + f'(t)t \qquad y = g(t_0) + g'(t)t \qquad z = h(t_0) + h'(t)t \tag{2}$$

Integration

Apparently all the notes I have for this is that it operationally gets distributed similarly to dervitives.

$$\int_a^b \vec{r}'(t)dt =< \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt > \tag{3}$$

Length

Similar to the previous definition, we are just throwing in one more term, so the broader equation is

$$Length = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}dt = \int_a^b |\vec{r}'(t)|dt \tag{4}$$

Functiosn of Two Variables

Domain and Range

In single variable, we used to say that $y = f(x)$, where y is a function of x. Here, we can say $z = f(x, y)$, where z is a function of x and y. Similar to single variable, the Domain D is defined to be the set of valid inputs of $(x,\ y)$, and the range is the values achieved from the domain.

Level Curves

We can think of this as 2-D slices of $f(x, y)$. We will be setting $f(x, y)$, or z, to a constant k.

Limits and Continuity

When finding the limit, we have to make sure the limit exists from all directions. It is very easy to find that a limit does not exist, since only one contradiction needs to exist. You could potentially do some delta/sigma nonsense, but it is not necessary

As a tip, rationals and polynomials' limits exist on the entire domain.

Partial Derivitives

We can measure rate of change when $z = f(x, b)$ when b is a constant and $z = f(a, y)$ when a is a constant. In regards to the limit definition, we can define the partial derivatives of $f(x, y)$ to be

$$\frac{\partial f}{\partial x}(a,b) = lim_{x\to 0}\frac{f(a+h,b)-f(a,b)}{h} \qquad \frac{\partial f}{\partial y}(a,b) = lim_{x\to 0}\frac{f(a,b+h)-f(a,b)}{h} \tag{1}$$

Operationally, it just means you treat variables that are not being differentiated to be constants. When taking second derivatives, the order does not matter.

Tangent Planes

If $f(x, y) = z$, then the plane tangent at the point (x_0, y_0, z_0) can be found with

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{2}$$

Linearization

We find a tangent plane at a point (a, b) for the function $f(x, y)$, replace z_0 to be $f(x_0, y_0)$. When close enough, it is a good approximation.

More Applications of Partial Derivitives

Chain Rule

Just like in single variable, there will be cases where our functions are defined in terms of intermediate variables. If we have a function $z = f(x, y)$, and $x = x(t)$ and $y = y(t)$, then we can say that

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{1}$$

When we get our final answer, we need to make sure that there are no intermediate variables present. In the case above, all x and y need to be converted in regards to time.

Implicit Differentiation

If we have a function $F(x, y) = 0$, then we can say that

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \tag{2}$$

We can write similarly for $\frac{dz}{dx}$ and etc.

Directional Derivatives and Gradients

We define a directional derivative to be a change in a function f when going in a direction $\vec{u}$. We can find this via.

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} \tag{3}$$

∇f is the gradient of f, where we define it to be

$$\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} > \tag{4}$$

We can maximize $D_{\vec{u}}f$ when the ∇f is in the direction of $\vec{u}$.

Normal Lines

Normal lines are lines that are perpendicular to a surface at a certain point. A unit vector $\vec{n}$ normal to a plane is shown to be

$$\vec{n} = \nabla f(x_0, y_0, z_0) \tag{5}$$

By vector definition of a line, we can deduce that a line normal to a plane at (x_0, y_0, z_0) would be

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)} \tag{6}$$

Absolute and Relative Extrema

Mins and Max in 3-D

Similarly to single var calculus, we first have to find critical points on the curve. We define critical points (a, b) to be when

$$f_x(a,b) = f_y(a,b) = 0 \tag{1}$$

Once we find critical points, we have to use the 2nd derivative test. we have a value D, which we define to be

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2 \tag{2}$$

If $D > 0$ it is an extrema. If $f_{xx} > 0$, it is a local min. If $f_{xx} < 0$, it is a local max. If $D < 0$, it is a saddle point (increase in one direction, decrease in the other direction). In every other case, we cannot say anything.

Optimization

These are basically word problems. You will know when it says "find the max or min or something". This is a clear sign that you will need to use optimization. The most common one is finding the maximum distance. Here, we need to remember that

$$Distance = \sqrt{(x-x_0)^2 + (y+y_0)^2 + (z-z+0)^2} \tag{3}$$

But when optimizing, we can delete the square root.

Lagrangian Multipliers

If we have a function $f(x, y, z)$ bounded by $g(x, y, z) = k$ and we are asked to find the max and min, we can use Lagrange Multipliers. We use that

$$\nabla f = \lambda \nabla g \tag{4}$$

where λ is a constant. The intuition is that at some point, there is a value of λ where $g(x, y, z)$ cleanly intersects $f(x, y, z)$ at one point, where the gradients of the two at the intersection are parallel.

Problem Solving

If we are looking for absolute extrema, we can proceed with (1) and (2). If we are bounded and are looking for extrema on and within the bound, we first need to find critical points within the bound, then find critical points on the bound. when finding critical points on the bound, we can do Lagrange bounds, or look for substitutions. If it devolves into single-variable calculus, make sure to not forget about the values at the edges.

Double Integrals

Rectangular Regions

In single-variable calculus, we define an integral to be the sum of infinitely small areas along an axis. We can use a similar intuition to understand double integrals. We can approximate an integral of function $f(x, y)$ to be,

$$Area \approx \Sigma_{i=1}^{\infty} \Sigma_{j=0}^{\infty} f(x_i, y_j) \Delta y \Delta x \tag{1}$$

We can apply limits to the approximation above similar to the way we have done in single variable, we get that

$$Area = \iint f(x, y) dA \tag{2}$$

The simplest case we can have is if we were to integrate over a rectangle defined to be $[a, b] \times [c, d]$, where the dimensions of x are from a to b, and y to be from c to d. Here, we can say that the area of a function $f(x, y)$ over the rectangle to be

$$Area = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx \tag{3}$$

In this case, the order of integration is not important, since the bounds are not dependent on a different variable.

Generalized Integrals

Most of the time, we are not integrating a function over a rectangle, Oftentimes, we are integrating over an area that is bounded by two other functions. We say an area is Type 1 if the area is bounded by y as a function of x. In this case, we would say that the area would be

$$\iint_D f(x, y) dA = \int_a^b \int_{f(x)}^{g(x)} f(x, y) dy dx \tag{4}$$

Where $g(x)$ is on top of $f(x)$, and x is bounded from a to b.

We say that a Type 2 area would be bounded by x as a function of y. The area would be

$$\iint_D f(x, y) dA = \int_c^d \int_{f(y)}^{g(y)} f(x, y) dx dy \tag{5}$$

Polar Double Integrals

We can use polar substitution to help simplify "roundish" functions. We can use the fact that $x = rcos(\theta)$, $y = rsin(\theta)$, and $x^2 + y^2 = r^2$ to say that

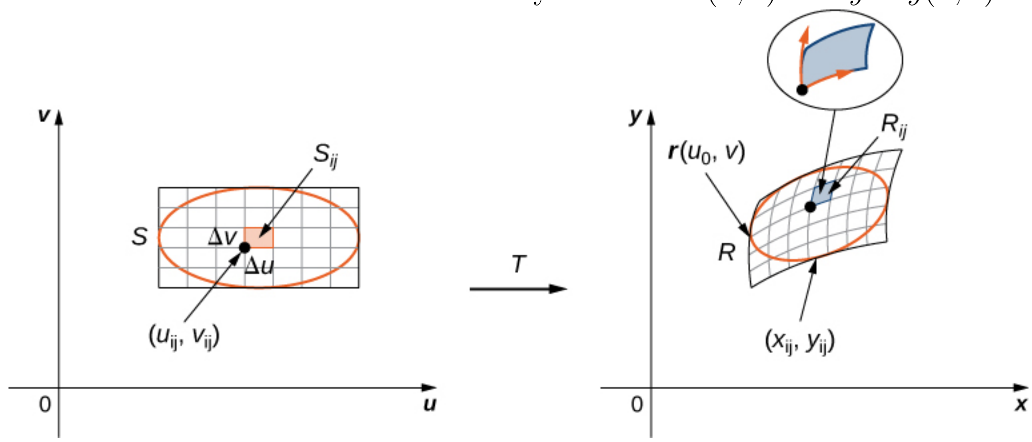
$$\iint_D f(x,y)dA = \int_a^b \int_\alpha^\beta f(rcos\theta, rsin\theta) r d\theta dr \qquad (6)$$

where a and b are the bounds for the radius r and α and β and the bounds for θ.

Change Of Variables and Triple Integrals

Changing Variables

In single variable calculus, we would perform a u-sub to represent a form a change of variables. In integrals of multiple variables, most surfaces will not be flat. However, we can change the variables so that it can be flat. We can say that $x = x(u, v)$ and $y = y(u, v)$.



As shown above, we often try to parameterize surfaces to become shapes that are easier to integrate, such as rectangles. Often times, when we graph the domain in terms of the new variables, it would be a rectangle.

If we want to integrate over a typical curved surface, we can still break up the surface into smaller areas. As shown above, we can represent the area to be parallelograms constructed by two vectors, with each vector moving in regards to one intermediate variable. By definition of area of a parallelogram in terms of vector, we can say that each "small" area would be

$$Area = |\Delta u\vec{r}_u \times \Delta v\vec{r}_v| = \Delta u\Delta v|\vec{r}_u \times \vec{r}_v| \tag{1}$$

$$dA = dudv|\vec{r}_u \times \vec{r}_v| \tag{2}$$

From this, we can finally say that if we are interesting $f(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$, then we can say

$$\iint_R f(x, y)dA = \iint_S f(x(u, v), y(u, v))Jdudv \tag{3}$$

We define J to be the Jacobian. We can compute it to be

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{4}$$

Triple Integrals

Double integrals are used to evaluate integrals of functions with two variables. We can use triple integrals to evaluate functions of three variables. A triple integral can be defined to be

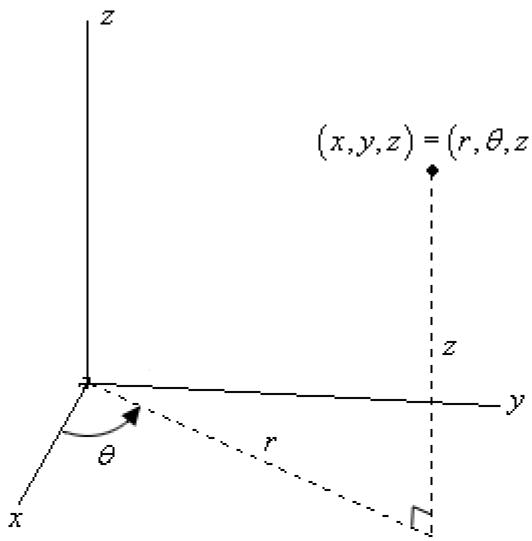
$$\iiint f(x, y, z)dV = \lim_{k->\infty} \lim_{n->\infty} \lim_{m->\infty} \Sigma_{i=1}^{m} \Sigma_{j=0}^{n} \Sigma_{h=0}^{n} f(x_i, y_j, z_h)dzdydx \tag{5}$$

Operationally, they are the same as double integrals. When doing problem solving, we often say that if $E = (x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)$, then

$$\iiint_E f dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz \right) dA \tag{6}$$

Cylindrical and Spherical Coordinates

Cylindircal Coordinates



Polar integrals are a good 2-D representation of round-ish things. Similarly in 3-D, we can use cylindrical coordinates to describe shapes where there is some sort of revolution in the shape. If we say that

$$x = rcos\theta \qquad y = rsin\theta \qquad z = z \tag{1}$$

then we can say that

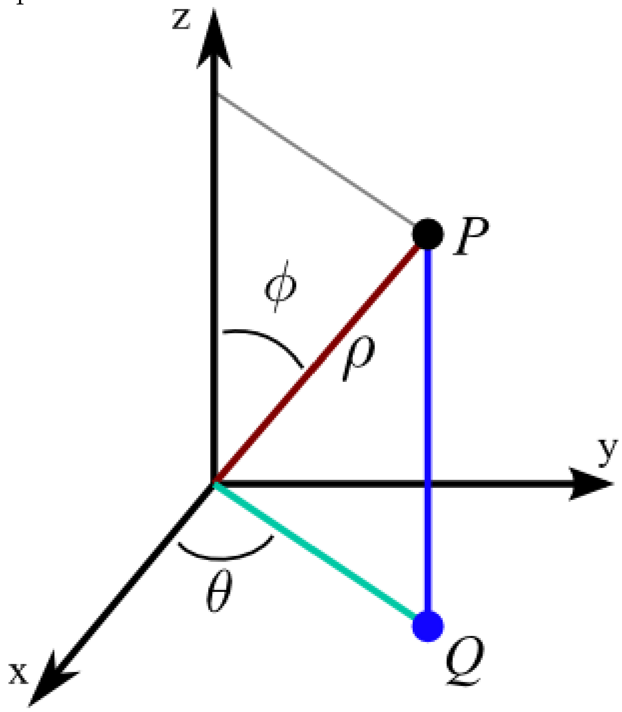
$$\iiint_E f(x, y, z)dV = \iiint f(rcos\theta, rsin\theta, z)rdrd\theta dz \tag{2}$$

Here, the Jacobian is computed to be r, hence why there is a r in the integral.

As mentioned above, we can use cylindrical coordinates if we are dealing with a shape that has some sort of axis or revolution. More specifically, if a shape is defined to have one of its variables be equal to itself, while two other variables are affected by it. At the most basic level, you can think of a cylinder where each cross-section is a unit circle defined by x and y, you can continue to extend that framework to more complex shapes.

Spherical Coordinates

We can also represent a shape in regard to spherical coordinates. We can define a point in regards to the variables ρ, θ, and ϕ. ρ is the distance between a point and the origin. θ is the angle between the point and the positive x-axis. ϕ is the angle between the point and the positive z-axis.



We can then say that

$$x = \rho sin\phi cos\theta \qquad y = \rho sin\phi sin\theta \qquad z = \rho cos\phi \qquad x^2 + y^2 + z^2 = \rho^2 \tag{3}$$

We can finally say that

$$\iiint_E f(x, y, z)dV = \iiint f(\rho sin\phi cos\theta, \rho sin\phi sin\theta, \rho cos\phi)\rho^2 sin\theta d\rho d\theta d\phi \tag{4}$$

It is good to remember that ϕ has the potential values between 0 to π and θ has the values between 0 to 2π

Vector Fields and Gradient

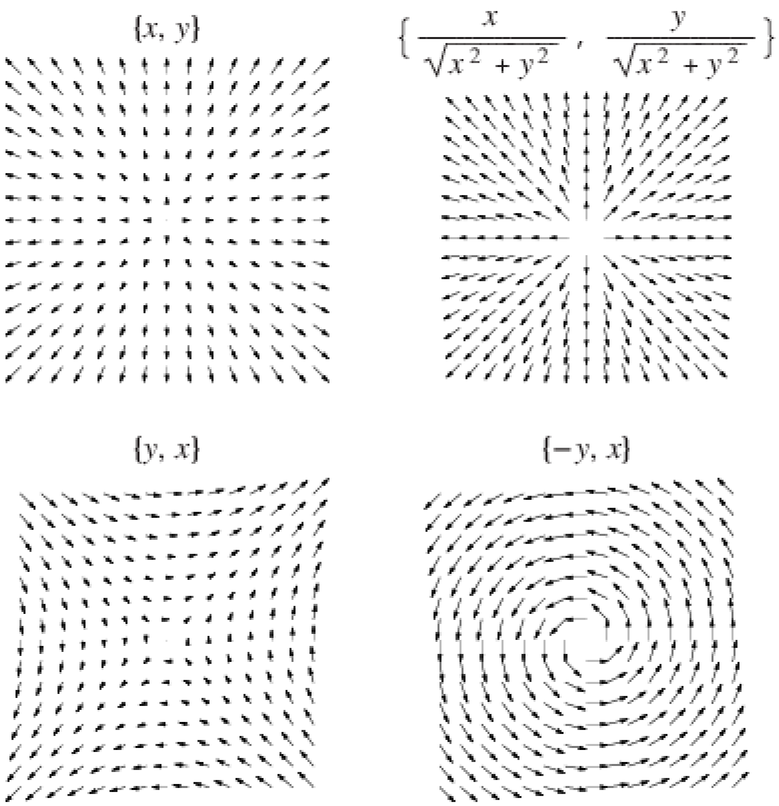
Vector Fields

If we have the x, y plane, then we can define a vector field

$$\vec{F}(x,y) =< P(x,y), Q(x,y) > \tag{1}$$

Similarly in 3-D, we can define a vector field

$$\vec{F}(x,y,z) =< P(x,y,z), Q(x,y,z), R(x,y,z) > \tag{2}$$



when matching vector fields, its often helpful to consider properties such as the behavior in each quadrant and sign analysis (i.e. if something is squared, it is always going to be positive)

Conservative Vector Fields

In single-variable calculus, we had slope field diagrams, where arrows represented tangent lines, or derivatives, of a function. It would be nice if a vector field also somehow represented a multivariable function. We can define a conservative vector field to be one where

$$\vec{F}(x,y,z) =< f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) > \tag{3}$$

for some function $f(x, y, z)$. By Clairaut's theorem, we can determine a vector field is conservative if for $\vec{F}(x, y, z) =< P, Q, R >$, then

$$P_y = Q_x \qquad P_z = R_x \qquad Q_z = R_y \tag{4}$$

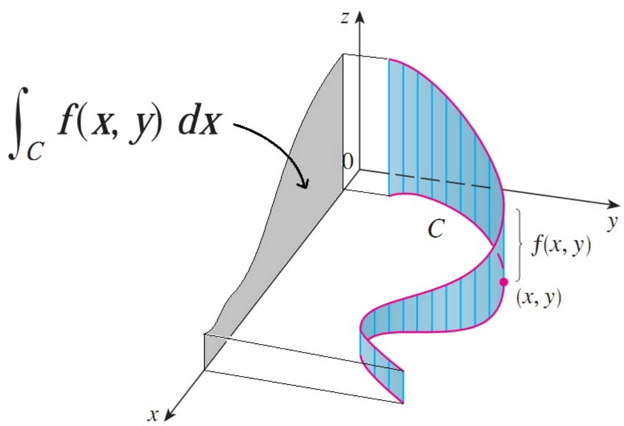
If we are asked to find the function of a conservative vector field, we can use integration techniques to sort of "reverse engineer" the function. We can use the fact $\int f_x dx = smth + f(y, z)$ for each variable, and deduce what f is from this.

Line Integrals and Fundamental Theorem of It

Line Integral

Supposed we have a curve C defined $\vec{r}(t) =< x(t), y(t), z(t) >$, where $a \leq t \leq b$ and a function f. We define the line integra to be

$$\int_C f dS = \int_a^b f(\vec{r}(t))|\vec{r}'(t)|dt \tag{1}$$



What the line integral generally shows geometrically is the area under a function if we follow the path of a line, similar to adding up the heights of a fence.

We can also take the line integral in regards to x or y. This is shown by $a \leq t \leq b$ and a function f. We define the line integra to be

$$\int_C f dx = \int_a^b f(\vec{r}(t))\vec{x}'(t)dt \qquad \int_C f dy = \int_a^b f(\vec{r}(t))\vec{y}'(t)dt \tag{2}$$

Fundamental Theorem of Line Integrals

Suppose we have a smooth curve C defined by $\vec{r}(t)$ and a function f with a trading ∇f that is continuous on C. Then,

$$\int_C \nabla f d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \tag{3}$$

We can prove this by expanding $\int_C \nabla f d\vec{r}$. We use the definition of ∇f and the fact that $d\vec{r} =< x'(t), y'(t), z'(t) > dt$ to get

$$\int_C \nabla f d\vec{r} = \int_a^b < f_x, f_y, f_z > \cdot < x'(t), y'(t), z'(t) > dt$$

Multiplied out we get

$$\int_a^b < f_x, f_y, f_z > \cdot < x'(t), y'(t), z'(t) > dt = \int_a^b f_x x'(t) + f_y y'(t) + f_z z'(t) dt$$

$$= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} dt = \int_a^b \frac{d}{dt} f(\vec{r}(t)) dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

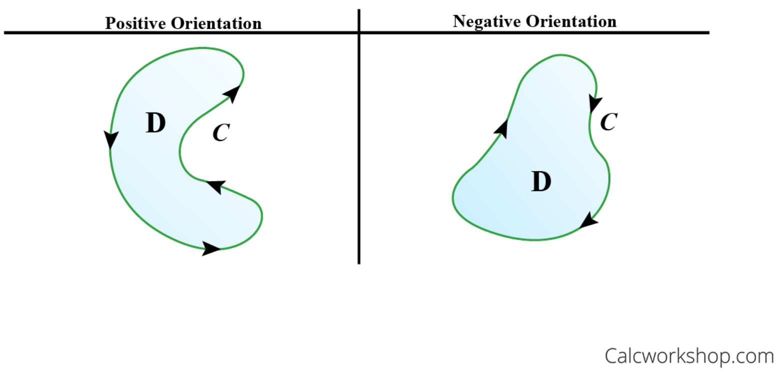
The 2nd to last step is because of the definition of chain rule.

A couple of important properties is that line integrals are path independent, meaning they only measure the change in f as it moves from one point to another. If and only if $\int_C F \cdot dr = 0$, then $\int_C F \cdot dr$ is independent of path. This is useful for loops over a conservative vector field, because a line integral over it will always be 0.

Green's Theorem

Orientation

We can say that if we travel along the curve in a counter-clockwise direction, the curve is positively oriented. An alternative definition is a positively oriented curve will have the surface on its left side. Likewise, if we are traveling along a curve clockwise, it is negatively oriented.



Green's Theorem

Suppose we have a positively oriented, smooth, simple closed curve C in a plane that bounds an area D. If P and Q have continuous partial derivatives the open region that contains D, then

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \tag{1}$$

The following proof is in the case of a rectangle defined by $[a, b] \times [c, d]$. C_1 is defined to be the line from (a, c) to (b, c), C_2 is defined to be the line from (b, c) to (b, d), C_3 is defined to be the line from (b, d) to (a, d), and C_4 is defined to be the line from (a, d) to (a, c). We can break up $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$ to $\iint_D \frac{\partial Q}{\partial x} dA - \iint_D \frac{\partial P}{\partial y} dA$. We can also break up $\int_C Pdx + Qdy$ to $\int_C Pdx + \int_C Qdy$. We can first focus on the Q component, starting with the double integral by saying

$$\iint_D \frac{\partial Q}{\partial x} dA = \int_c^d \int_a^b \frac{\partial Q}{\partial x} dxdy = \int_c^d Q(b, y) - Q(a, y)dy$$

We can expand $\int_C Qdy$ by

$$\int_C Qdy = \int_{C_1} Qdy + \int_{C_2} Qdy + \int_{C_3} Qdy + \int_{C_4} Qdy$$

Along C_1 and C_2, dy is 0, since y is constant while we are traveling along it. Its respective line integrals are 0 as a result. Focusing on C_2, we can say that $\int_{C_2} Qdy = \int_c^d Q(b,y)dy$. Likewise for C_4, we can say that $\int_{C_4} Qdy = \int_c^d Q(a,y)dy$. However, we can say that C_2 is moving positively oriented direction, C_4 is moving in a negatively oriented direction, thus we put a negative sign in front of the integral. The line integral becomes

$$\int_C Qdy = \int_{C_2} Qdy + \int_{C_4} Qdy = \int_c^d Q(b,y)dy - \int_c^d Q(a,y)dy$$

By equality, $\int_C Qdy = \iint_D \frac{\partial Q}{\partial x} dA$. Moving on to the other half of the theorem, we evaluate the double integral as follows

$$-\iint_D \frac{\partial P}{\partial y} dA = -\int_a^b \int_c^d \frac{\partial P}{\partial y} dydx = -\int_a^b Q(x,d) - Q(x,c)dy$$

We can evaluate the line integral as follows

$$\int_C Pdx = \int_{C_1} Pdx + \int_{C_2} Pdx + \int_{C_3} Pdx + \int_{C_4} Pdx$$

Since along C_2 and C_4, dx is 0, their respective integrals are 0. We can also parameterize the line integrals along C_1 and C_3 by saying $\int_{C_1} Pdx = \int_a^b P(x,c)dx$ and $\int_{C_1} Pdx = \int_a^b P(x,d)dx$. If we are saying C_1 is positively oriented, C_2 is negatively oriented, thus putting a negative sign in front of the integral

$$\int_C Pdx = \int_{C_2} Pdx + \int_{C_4} Pdx = \int_a^b P(x,c)dx - \int_a^b P(x,d)dx$$

This is equal to the double integral calculated above by the property of integrals, thus proving Green's theorem in this particular case.

Curl and Divergence

Curl

Curl can be represented as $\nabla \times f$ for some vector field $f =< P, Q, R >$. Computed out, curl is as follows

$$\nabla \times \vec{f} =< \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} > \quad (1)$$

Geometrically, we can think that curl measures how much the vector field "rotates" at a given point.

Divergence

We can define divergence to be equal to $\nabla \cdot \vec{f}$. Computed out, divergence is found as follows

$$\nabla \cdot \vec{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad (2)$$

Physically, divergence is defined to be how much the vector field behaves like a "source" in regards to flux. If divergence is greater than 0, it is like a "source" and if it is less than 0, then it is like a sink.

Properties

From this, we can derive a couple of neat properties. We first find that

$$\nabla \times (\nabla f) = 0 \quad (3)$$

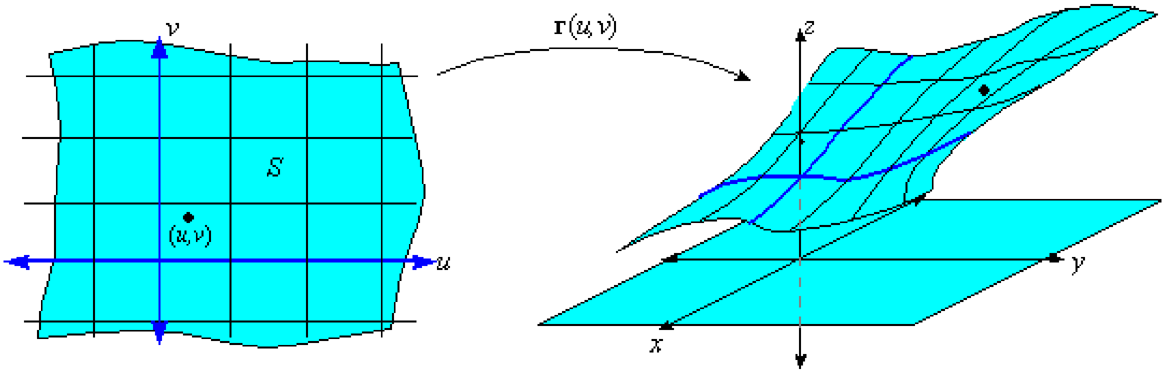
Go search up computationally how this works, but it is just algebra once you expand it. We can also find that

$$div(curl(\vec{F})) = 0 \quad (4)$$

Parametric Surfaces and Surface Integrals

Paramertizing Srfaces

Just like curves in 2-D can be parametrized in terms of one variable, surfaces of 3-D can be parametrized in terms of two variables.



The basic idea is that the (u, v) system is in 2-D, and we can map points from that system into a surface in the 3-D Cartesian plane. We use $\vec{r}(u, v)$ to represent a conversion from 2-D to 3-D, where

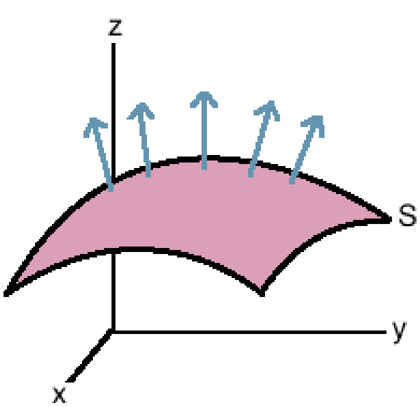
$$\vec{r}(u, v) =< x(u, v), y(u, v), z(u, v) > \tag{1}$$

In the special case of a graph, where $z = f(x, y)$, we can simply parameterize it by saying

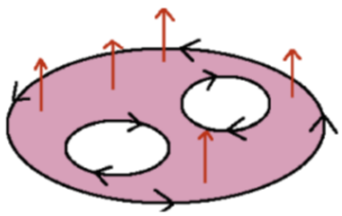
$$\vec{r}(x, y) =< x, y, f(x, y) > \tag{2}$$

Orientation of Surfaces

A surface is orientable if it has two sides you can walk on. This implies there is a continuous normal vector on each side. Usually, the problem will define which side or a normal vector of the orientable surface we will be integrating over.



When finding the orientation of the boundary of the surface, it is positively oriented when it is traveling with the chosen side on its left. More hands-on, we can use the right-hand rule to find the direction of the curve. We point our thumb in the direction of the chosen normal vector, and the direction our hand sweeps in is the positive orientation of the boundary curve.



Surface Integrals

The surface integral of function g over a parametrized surface S is

$$\iint_S g dS = \iint_D g(r(u,v))|r_u \times r_v| dA \tag{3}$$

In the case for a graph where $z = f(x,y)$, then the surface integral is

$$\iint_S g dS = \iint_D g(x,y,f(x,y))\sqrt{(f_x^2 + f_y^2 + 1)} dA \tag{4}$$

Flux

We define flux to be the surface integral of a vector field through a surface S.

$$\iint_S \vec{F} dS = \iint_S \vec{F} \cdot \vec{n} dS \tag{5}$$

If we say that $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v dA \tag{6}$$

When computing the normal vector, if we reverse the order of $\vec{r}_u$ and $\vec{r}_v$, then we will get the normal vector going in the other direction.

Stokes Theorem and Required Proofs

Stokes Theorem

If S is an oriented surface and C is a positively oriented boundary of the surface, then

$$\iint_S \nabla \times \vec{F} \cdot \vec{n} dS = \int_C \vec{F} \cdot d\vec{r} \tag{1}$$

We are asked to prove in the case where it is a graph and in cases where $\vec{F} =< 0, 0, R >$. This is for the case if it is a graph and $\vec{F} =< P, Q, R >$. We can expand the curl function in the integral as follows.

$$\iint_S \nabla \times \vec{F} \cdot dS =$$

$$\iint_D \left(-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right) dA$$

If we paramterize the curve such that $x = x(t)$,$y = y(t)$, and $z = g(x(t), y(t))$ such that $a \leq t \leq b$, then we can evaluate the line integral as such.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt$$

By chain rule, we can see that $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$, thus substituting that in gets

$$\int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \right) \right) dt$$

$$= \int_a^b \left(\left(P + R\frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right) dt$$

We define the 2-D curve C_1 that corresponds to the (x, y) components of the curve the 3-D curve C.

$$= \int_{C_1} \left(P + R\frac{\partial z}{\partial x} \right) dx + \left(Q + R\frac{\partial z}{\partial y} \right) dy$$

By Green's Theorem, we say that

$$= \iint_D \frac{\partial}{\partial x} \left(P + R\frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left(Q + R\frac{\partial z}{\partial x} \right) dA$$

Using Chain Rule again, we remember P and Q are defined in terms of x, y, and z, and z itself is in terms of x and y. After expanding out and simplifying, we get

$$= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y} \right) - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x} \right) dA$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} - \frac{\partial P}{\partial y} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} \right) dA$$

$$= \iint_D \left(-\frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D \left(-\left(\frac{\partial R}{\partial y}\frac{\partial z}{\partial x} - \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} \right) - \left(\frac{\partial P}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} \right) + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D \left(-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_S \nabla \times \vec{F} \cdot dS$$

Problem Solving Notes

Make sure your orientation is correct. Follow the right-hand rule and make sure your thumb is pointing in the right direction. The motivation of Stokes is to somehow relate flux across a surface to its boundary. Good indicators that we should be using Stoke's is either the problem is asking for the flux of a curl, or if we are asked for a line integral where the curve is the boundary for some shape.

Divergence Theorem and Proof

Divergence Theorem

If E is a 3-D solid region, and S is the oriented surface that bounds it by choosing the outward pointing normal vector, then for vector fields $\vec{F}$

$$\iiint_E \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot \vec{n} dS \tag{1}$$

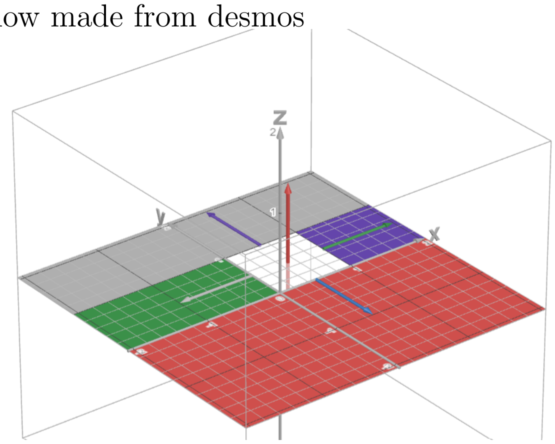
There are three cases we need to know for divergence theorem, but they are all similar enough. During the exam, they will always be where the vector field is only defined by one thing, and the volume will be a cube-ish. If we say that $\vec{F} =< P, Q, R >$ We expand out both sides of Divergence Theorem to see that

$$\iiint_E \frac{\partial P}{\partial x} dV + \iiint_E \frac{\partial Q}{\partial y} dV + \iiint_E \frac{\partial R}{\partial z} dV$$
$$= \iint_S P\vec{i} \cdot \vec{n} dS + \iint_S Q\vec{j} \cdot \vec{n} dS + \iint_S R\vec{k} \cdot \vec{n} dS$$

We will need to show the corresponding letters equal to each other (i.e. the P will match with itself, and so on). For each "subset" we can consider them to be in a cubical shape, where the base is a square defined by the "other-two" variables. For example, to prove that $\iiint_E \frac{\partial R}{\partial z} dV = \iint_S R\vec{k} \cdot \vec{n} dS$, we can define the region to be $E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$. For the triple integral, it can be simplified as follows

$$\iiint_E \frac{\partial R}{\partial z} dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x, y, z) dz \right) dA = \iint_D R(x, y, u_2(x, y)) - R(x, y, u_1(x, y)) dA$$

We now have to evaluate curl. We need to consider all 6 sides of the shape. We need to consider The normal vector of each side. Notice that the curl has the vector $\vec{k} =< 0, 0, 1 >$ in it, and it is dot multiplied with the normal vector of the surface. Consider the diagram below made from desmos



The white square could be D. The red vector is $\vec{k}$, and all other vectors are normal to their respective side. Notice that If we take the dot product of two perpendicular vectors, it is equal to 0. Therefore, we we are taking the curl, we only need to consider the side that is associated with $u_1(x,y)$, S_1, and the side associated with $u_2(x,y)$,S_2, since those have normal vectors that are either anti-parallel to $\vec{k}$, or not perpendicular to it. As a result, curl can be evaluated as follows

$$\begin{aligned}\iint_S R\vec{k}\cdot\vec{n}dS = \iint_{S_1} R\vec{k}\cdot\vec{n}dS + \iint_{S_2} R\vec{k}\cdot\vec{n}dS = \iint_D R(x,y,u_2(x,y))dA + \iint_D R(x,y,u_1(x,y))(-1)dA \\ = \iint_D R(x,y,u_2(x,y))dA - \iint_D R(x,y,u_1(x,y))dA = \iiint_E \frac{\partial R}{\partial z}dV\end{aligned}$$

The proof is done, and canbe replicated for all the other variables.