Section 8 Motion/Vibration Analysis

8.1 Recurring Abbreviations

$C_{1/x}$ number of cycles to achieve $1/x$ amplitude
D damping
D_1, D_2 peak-to-peak displacement (subsequent)
FV final value
F(t)   *forcing function*
f frequency, cycles/sec $= \omega/(2\pi)$
HCAR   half cycle amplitude ratio (i.e., x_2/x_1, x_3/x_2, etc.)
Im   imaginary axis
M mass
MP   peak overshoot
Re   real axis
rms   root mean square
s_1, s_2 equation roots of second order
T period $= 1/f = 2\pi/\omega_d$ (seconds)
T_d delay time (i.e., time to 50% of FV)
T_r rise time (i.e., time from 10% to 90% of FV)
T_p time to peak amplitude
TPR   transient peak ratio
T_s settling time (time to settle within x% of FV)
$T_{1/2}$ time to achieve $1/x$ amplitude
x displacement
x_1, x_2 peak displacements (subsequent)
v velocity
v_o peak velocity
ε $= \zeta\omega_n / \omega_d = \zeta/[1-\zeta^2]^{.5}$
ϕ phase lag (radians)
ζ damping coefficient (non-dimensional)
σ damping rate $= \zeta\omega_n = 1/\tau$
τ time constant $= 1/\zeta\omega_n$
ω frequency, radians/sec
ω_d damped natural frequency (rad/sec)
ω_n natural frequency (rad/sec)

8.2 First Order Motion

Found in classical aircraft roll and spiral modes. Named first-order because the motions are described by mathematics using the first derivative of a parameter.

8.2.1 Elements of First Order Motion

- Mechanical analogy contains elements of mass, damping and sometimes a forcing function.
- Example: Determine the vertical velocity of a diver as she hits the water at 10 ft/s (assume constant body position & neutral buoyancy)

Summing vertical forces
$$\sum F_{vert} : M\frac{dV}{dt} + DV = 0$$

$$\frac{dV}{V} + \frac{D}{M}dt = 0$$

$$\int \frac{dV}{V} = -\int \frac{D}{M}dt$$

Since D & M are constant
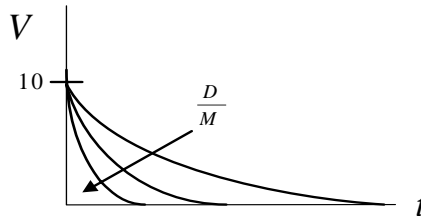$$nV = \frac{-D}{M}t + C = \frac{-D}{M}t + \ln c$$

$$V_C = ce^{\frac{-D}{M}t} \quad \text{(complementary solution)}$$

Apply initial condition
$$V(t=0) = 10 \; ft/s \Rightarrow$$
$$10 = C$$

$$V_T = 10e^{\frac{-D}{M}t}$$

Plot response over time

- Exponential rate of decay described by *D/M* ratio

- Example 2: Diver with 20 lb submerged weight releases from zero velocity at top of pool (quiescent condition).

Solve using Laplace analysis methods:

$$M\dot{V} + DV = 20 \text{ (step input)}$$

$$M[sV(s) - V(0)] + DV(s) = \frac{20}{s}$$

$$sV(s) - V(0) + \frac{D}{M}V(s) = \frac{20}{M}\frac{1}{s}$$

$$V(s)\left(s + \frac{D}{M}\right) = \frac{1}{s}\frac{20}{M} + V(0) \qquad V(s) = \frac{\frac{1}{s}\frac{20}{M} + V(0)}{s + \frac{D}{M}} = \left\{\frac{\frac{20}{M}}{s\left(s + \frac{D}{M}\right)}\right\} + \frac{V(0)}{s + \frac{D}{M}}$$

$$\text{use partial fraction} = \left\{\frac{\frac{20}{M}}{s\left(s + \frac{D}{M}\right)}\right\} = \frac{A}{s} + \frac{B}{s + \frac{D}{M}} \qquad \left\{\frac{\frac{20}{M}}{s\left(s + \frac{D}{M}\right)}\right\} = \frac{A\left(s + \frac{D}{M}\right)}{s\left(s + \frac{D}{M}\right)} + \frac{Bs}{s\left(s + \frac{D}{M}\right)} = \frac{A\left(s + \frac{D}{M}\right) + Bs}{s\left(s + \frac{D}{M}\right)}$$

$$\text{let } s = \frac{-D}{M}; \quad \frac{20}{M} = B\frac{-D}{M} \Rightarrow B = \frac{-20}{M}, \quad \text{let} \quad s = 0: \frac{20}{M} = A\frac{D}{M} \Rightarrow A = \frac{20}{D}$$
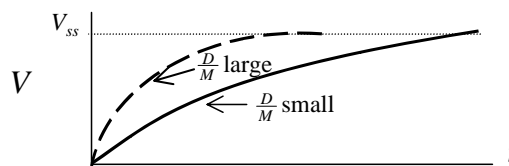
$$V(s) = \left\{\frac{\frac{20}{D}}{S} + \frac{\frac{-20}{D}}{s + \frac{D}{M}}\right\} + \frac{V_{(0)}}{s + \frac{D}{M}},$$

$$v(t) = \left\{\frac{20}{D} - \frac{20}{D}e^{-\left(\frac{D}{M}\right)t}\right\} + V_{(0)}e^{-\left(\frac{D}{M}\right)t} \quad \text{since } V(0) = 0 \text{ then } v(t) = \frac{20}{D}\left[1 - e^{-\left(\frac{D}{M}\right)t}\right]$$

Note that $\frac{20}{D}$ is the steady state value i.e. $v(t) = V_{ss}\left[1 - e^{\frac{-D}{M}t}\right]$

This "force/damping" ratio is merely a scaling factor for the steady state.

- Several methods can be used to describe the quickness of convergence toward steady state (i.e., time to 99.999 % of V_{ss}, time to 1/2 V_{ss}).
- By convention, we use a % that directly reflects the exponent.
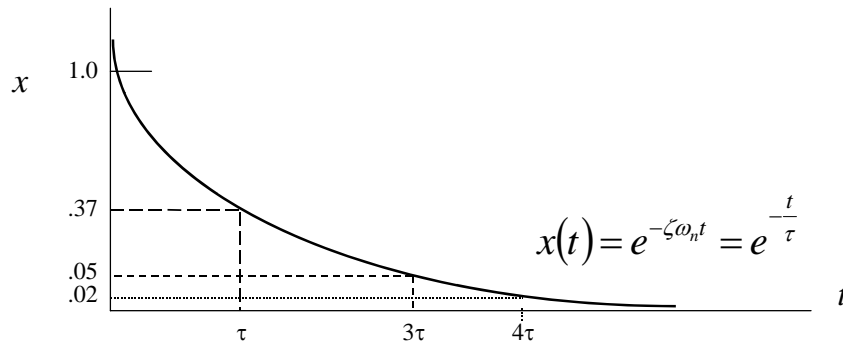- Establish a time constant τ based on D/M.

$$\frac{D}{M} \equiv \frac{1}{\tau} \qquad \text{so} \qquad e^{-\frac{D}{M}t} = e^{-\frac{t}{\tau}}$$

when $t = \tau$,   then $e^{-\frac{t}{\tau}} = e^{-\frac{\tau}{\tau}} = e^{-1} = 0.36788$

so, after $\tau\left[\left(\frac{D}{M}\right)^{-1}\right]$ seconds have elapsed, $V = V_{ss}\left[1 - 0.36788\right] = 63.212\% V_{ss}$
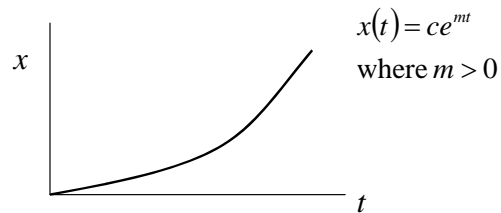
τ = time constant (time for parameter to reach 63% of its steady state value)

8.2.2 First Order Motion Descriptive Parameters

$$x(t) = e^{-\zeta\omega_n t} = e^{-\frac{t}{\tau}}$$

when $t = 0.6931\tau$: $x = e^{-.6931}$ $= 0.5$ (time to half amplitude)
when $t = \tau$: $x = e^{-1}$ $= 0.37$
when $t = 3\tau$ $x = e^{-3}$ $= 0.05$
when $t = 4\tau$: $x = e^{-4}$ $= 0.02$

- If exponent > 0, then motion is divergent.

$$x(t) = ce^{mt}$$
where $m > 0$

- τ again describes the exponential rate of divergence.
- By convention, the "time to double amplitude" (t_2) is usually applied as the evaluation metric.
- $x_{(t2)} = 2x_{(0)}$ where $x_{(0)} = ce^{m0}$

$$\text{Therefore} \quad x_2 = 2c$$
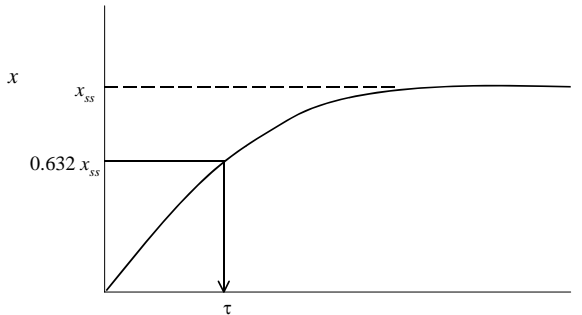$$2c = ce^{mt_2}$$
$$2 = e^{mt_2}$$
$$\ln 2 = mt_2$$
$$\therefore \text{Time to double amplitude} \quad \frac{0.6931}{m} = t_2 = 0.693\tau$$

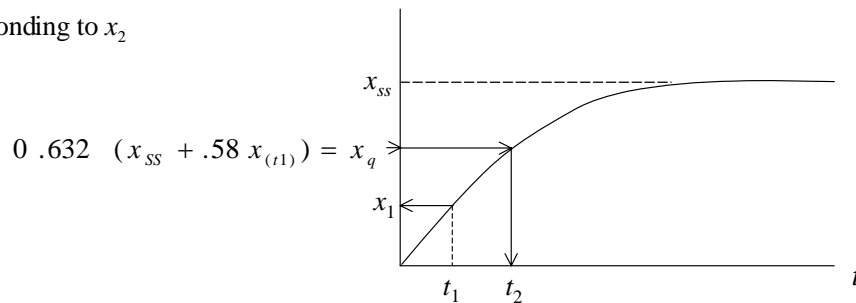## 8.2.3   Determining τ from Step Input Time History

Method #1

τ = time to reach $0.632\ x_{ss}$
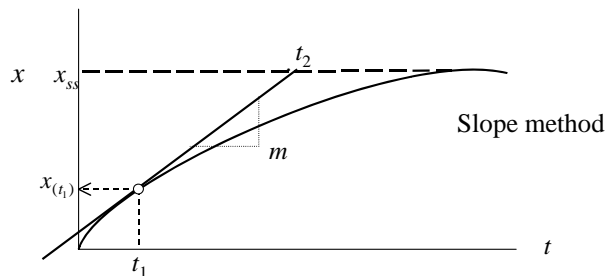
Method #2

1. Define x_{ss}
2. Measure x_1 at some time t_1
3. Calculate $x_2 = 0.632 \left(x_{ss} + .58 x_{(t_1)}\right)$
4. Find t_2 corresponding to x_2
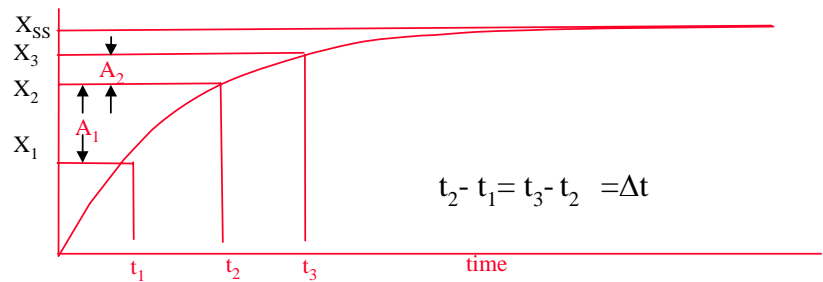5. $t_2 - t_1 = \tau$

$$0.632\ (x_{ss} + .58\ x_{(t1)}) = x_q$$

Method #3

1. Pick any time t_1.
2. Draw tangent at t_1.
3. Note t_2 where tangent intercepts x_{ss}.
4. $\tau = t_2 - t_1$

Slope method
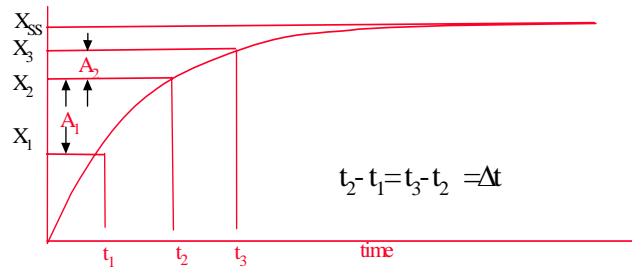
Method #4 When X_{SS} is unknown use

$$\tau = \frac{\Delta t}{\ln\left(\dfrac{A_1}{A_2}\right)}$$
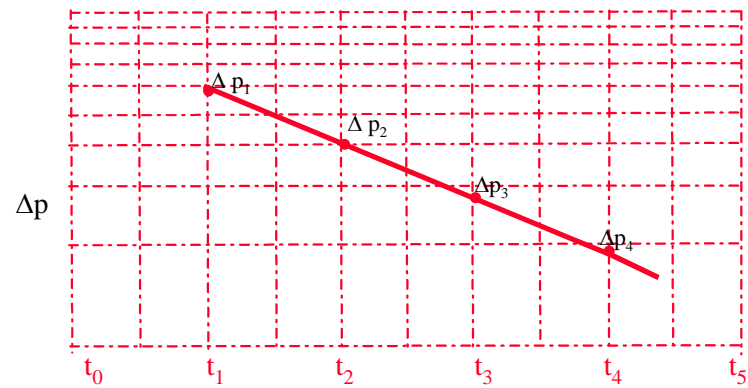

$$t_2 - t_1 = t_3 - t_2 = \Delta t$$

Method #5 When Xss is known, use $\tau = \dfrac{-\Delta t}{\ln\left[\dfrac{X_{SS} - X_1}{X_{SS} - X_2}\right]}$

Linearity check:
Note parameter change be-
tween even time increments.

Plot parameter changes vs
elapsed time on semi-log
scale

$$t_2 - t_1 = t_3 - t_2 = \Delta t$$

Slope of line equals τ

8.3 Second Order Motion

Found in classical aircraft phugoid, Dutch roll and short period modes as well as noise filter and vibration test-ing. Named second-order because the motions are described by mathematics using the second derivative of a parameter.

8.3.1 Elements of Second Order Motion

Mechanical systems have elements of spring, mass, and usually damping. Forcing functions can be included (see illustration).

K = spring stiffness (F/x)
x = displacement from equilibrium
M = mass
$F_{(t)}$ = forcing function
D = damping
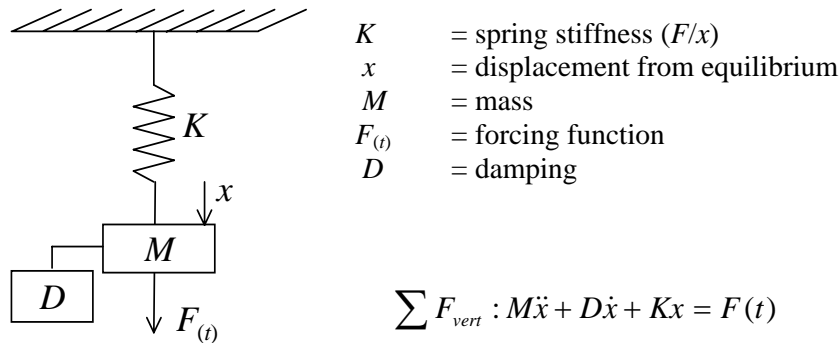
$$\sum F_{vert} : M\ddot{x} + D\dot{x} + Kx = F(t)$$

Natural character is observed when system is allowed to move with no external input [$F(t) = 0$]

$$\sum F_{vert} : M\ddot{x} + D\dot{x} + Kx = 0$$

Apply operator technique:

$$\text{let} \quad x = e^{st}, \dot{x} = se^{st}, \ddot{x} = s^2 e^{st}$$

$$\Rightarrow e^{st}\left(Ms^2 + Ds + K\right) = 0$$

Divide out e^{st}, since it never equals zero, the characteristic equation remains:

$$Ms^2 + Ds + K = 0 \quad \text{or} \quad s^2 + \frac{D}{M}s + \frac{K}{M} = 0$$

The values of s that satisfy this equation are called the roots

$$\left[x = c_1 e^{s_1 t} + c_2 e^{s_2 t}\right]$$

Solve for the roots using the quadratic equation

8.3.2 Second Order Motion Descriptive Parameters

Solution (x) calculated as

$$x = c_1 e^{s_1 t} + c_2 e^{s_2 t} \quad \text{where} \quad s_1, s_2 = \frac{-D}{2M} \pm j \sqrt{\frac{K}{M} - \left(\frac{D}{2M}\right)^2}$$

Apply Euler's identity for complex conjugate roots

$$x = A e^{\frac{-D}{2M} t} \sin\left(\sqrt{\frac{K}{M} - \left(\frac{D}{2M}\right)^2} \, t + \phi\right)$$

- ϕ defines the **phase shift**.
- *A* defines the **initial amplitude**.
- The real part of the root [$D/2M$] defines the **envelope** of the motion.
- The imaginary part of the root identifies the **damped frequency** of the oscillations, ω_d (rad/sec).

$$\omega_d = \sqrt{\frac{K}{M} - \left(\frac{D}{2M}\right)^2}$$

- If damping is reduced to $D = 0$ then only $[K/M]^{1/2}$ remains. This is the undamped or **"natural" frequency** (ω_n).

$$\omega_n \equiv \sqrt{\frac{K}{M}}$$

- If $\left(\frac{D}{2M}\right)^2 = \frac{K}{M}$ then D is conisidered to be critical [just enough to prevent oscillations]

$$D_{crit} = \sqrt{\frac{K}{M}} \, 2M = 2\sqrt{KM}$$

- For oscillatory motion, actual system damping is typically expressed as a fraction critical damping. Define **damping ratio** as

$$\zeta \equiv \frac{D}{D_{crit}} = \frac{D}{2\sqrt{KM}}$$

Combining $\omega_d \equiv \sqrt{\dfrac{K}{M} - \left(\dfrac{D}{2M}\right)^2}$

with $\omega_n \equiv \sqrt{\dfrac{K}{M}}$ and $\zeta = \dfrac{D}{2\sqrt{KM}}$
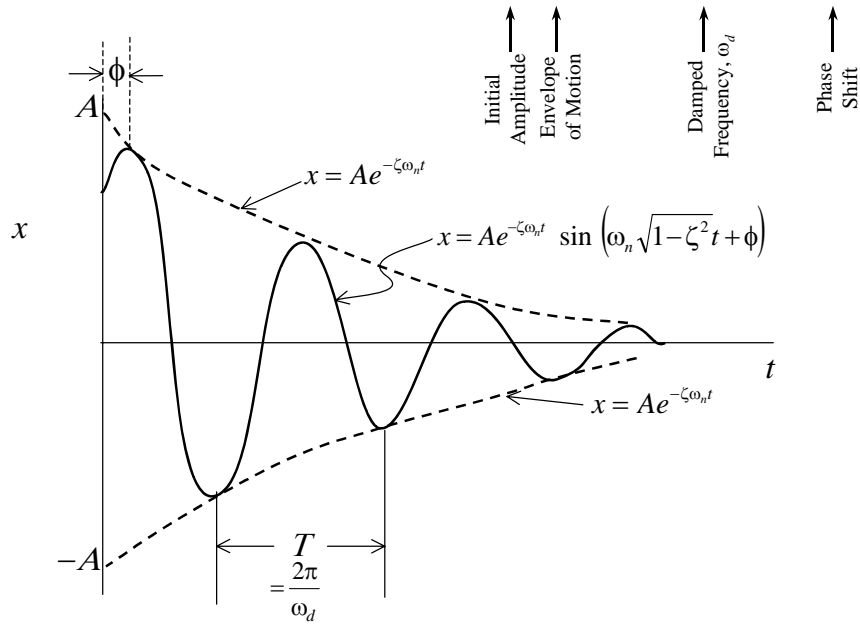
gives $\omega_d \equiv \omega_n \sqrt{1 - \zeta^2}$ and $\dfrac{D}{2M} = \zeta \omega_n$

The values can be substituted to give

$$x = Ae^{-\frac{D}{2M}t} \sin\left(\sqrt{\frac{K}{M} - \left(\frac{D}{2M}\right)^2} + \phi\right)$$

$$= Ae^{-\zeta\omega_n t} \sin\left(\omega_d + \phi\right)$$

$$= Ae^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2} + \phi\right)$$

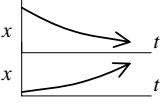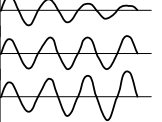where $\phi = \tan^{-1}\dfrac{\omega_d}{\zeta\omega_n} = \tan^{-1}\dfrac{\omega_n\sqrt{1-\zeta}}{\omega_n} = \tan^{-1}\dfrac{\sqrt{1-\zeta}}{\zeta}$

$$x = Ae^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \phi\right)$$

Initial Amplitude   Envelope of Motion   Damped Frequency, ω_d Phase Shift

$x = Ae^{-\zeta\omega_n t}$

$x = Ae^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \phi\right)$

$x = Ae^{-\zeta\omega_n t}$

$T = \dfrac{2\pi}{\omega_d}$

$$\text{Roots} \quad s_{1,2} = \frac{-D}{2M} \pm j\sqrt{\frac{K}{M} - \left(\frac{D}{2M}\right)^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Possible Solutions:

if $\left(\frac{D}{2M}\right)^2$	then roots are	and system response is	
$> \dfrac{k}{m}$	Real and unequal	Non-oscillatory ~ convergent if $D > 0$ ~ divergent if $D < 0$	
$= \dfrac{k}{m}$	Real and equal	Non-oscillatory ~ convergent if $D > 0$ ~ divergent if $D < 0$	
$< \dfrac{k}{m}$	Complex (purely imaginary if $D = 0$)	Non-oscillatory ~ convergent if $D > 0$ ~ neutral if $D = 0$ ~ divergent if $D < 0$	

- The various combination of K, M, and D and their effects on system response can be related to damping ratio ζ as follows:

$\zeta > 1$	Real & unequal roots exponential, convergent	$x(t) = c_1 e^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t} + c_2 e^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t}$
$\zeta = \pm 1$	Real & equal roots exponential, conv or div	$x(t) = c_1 e^{-\zeta\omega_n t} + c_2 t e^{-\zeta\omega_n t}$
$0 < \zeta < 1$	Complex pair roots sinusoidal, convergent	$x(t) = A e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$
$\zeta = 0$	Imaginary pair roots sinusoidal, neutral	$x(t) = A\sin(\omega_n t + \phi)$ $= A\cos\omega_n t$
$-1 < \zeta < 0$	Complex pair roots sinusoidal, divergent	$x(t) = A e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$
$\zeta < -1$	Real & unequal roots exponential, divergent	$x(t) = c_1 e^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t} + c_2 e^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t}$

Damping ratio effect on second order system

Convergent Sinusoid

Response of various second order systems to an impulse input.

$(x)t = e^{-0.3t} \sin (2.99t), \ \zeta = 0.1, \ \omega_n = 3$

Second-order systems are oscillatory if
$-1 > \zeta > 1$.

X

Time (sec)

Motion typically described by ω_n and ζ

Neutrally Damped Sinusoid

$(x)t = \cos 3t , \ \zeta = 0$

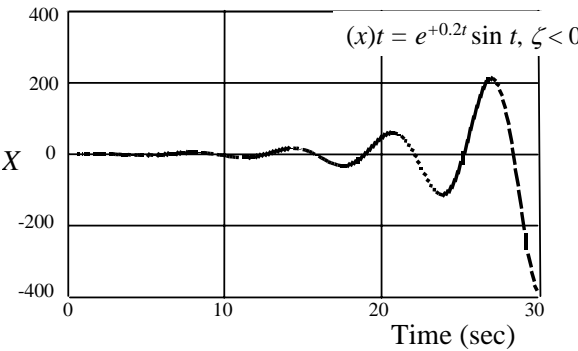T, ω_d, ω_n and ζ are linked such that knowledge of any two will yield the other two.

X

Time (sec)
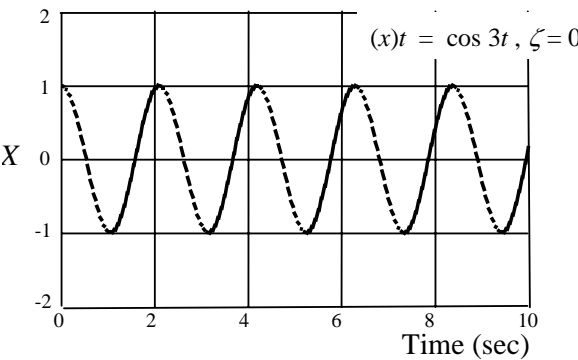
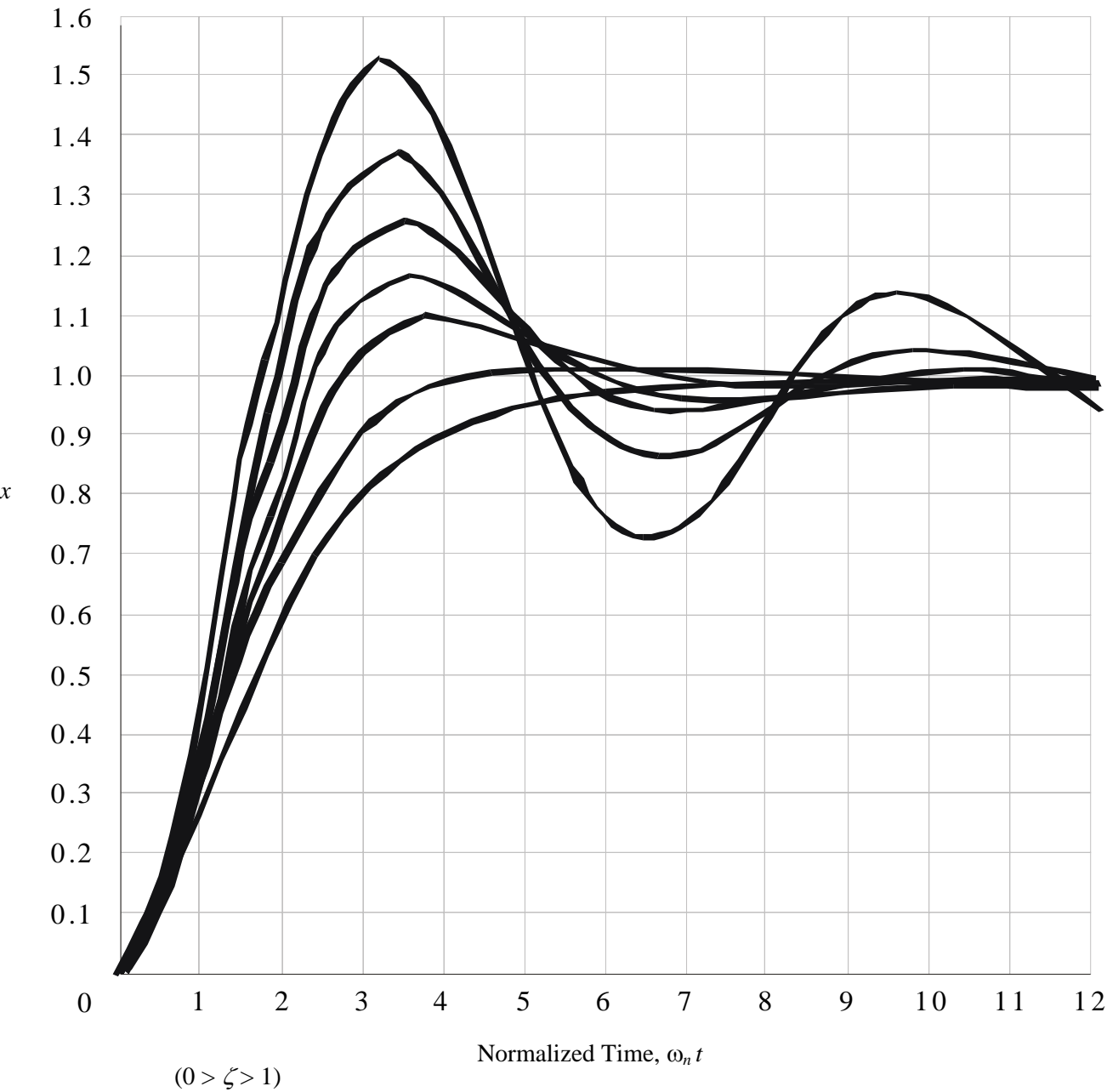### Divergent Sinusoid

$(x)t = e^{+0.2t} \sin t, \ \zeta < 0$

X

Time (sec)

2nd order system response to **unit step input** for **underdamped systems**

Normalized Time, $\omega_n t$

$(0 > \zeta > 1)$

Decay rates (for $0 < \zeta < 1$) and Useful Insights

- Time to decay
$$\zeta\omega_n = \frac{\ln\frac{x_1}{x_2}}{t_2 - t_1} \Rightarrow \Delta t = \frac{\ln\frac{x_1}{x_2}}{\zeta\omega_n}$$

  Time to $\frac{1}{2}$ amplitude
$$\equiv \Delta t_{\frac{1}{2}} = \frac{\ln\frac{2}{1}}{\zeta\omega_n} = \frac{0.6931}{\zeta\omega_n}$$

  Time to $\frac{1}{10}$ amplitude
$$\equiv \Delta t_{\frac{1}{10}} = \frac{\ln\frac{10}{1}}{\zeta\omega_n} = \frac{2.3026}{\zeta\omega_n}$$

  Time to $\frac{1}{x}$ amplitude
$$\equiv \Delta t_{\frac{1}{x}} = \frac{\ln x}{\zeta\omega_n}$$

- # cycles to $\frac{1}{2}$ amplitude $\text{# cycles} = \dfrac{\Delta t}{T} = \dfrac{\Delta t \omega_d}{2\pi}$

  # cycles to $\frac{1}{2}$ amplitude
$$\equiv C_{\frac{1}{2}} = \frac{0.6931}{\zeta\omega_n}\frac{\omega_d}{2\pi} = \frac{0.6931}{2\pi}\frac{\omega_d}{\zeta\omega_n} = \frac{0.1103}{\varepsilon}$$

  # cycles to $\frac{1}{10}$ amplitude
$$\equiv C_{\frac{1}{10}} = \frac{2.3026}{\zeta\omega_n}\frac{\omega_d}{2\pi} = \frac{2.3026}{2\pi}\frac{\omega_d}{\zeta\omega_n} = \frac{0.3665}{\varepsilon}$$

  # cycles to $\frac{1}{x}$ amplitude
$$\equiv C_{\frac{1}{x}} = \frac{\ln x}{2\pi}\frac{\omega_d}{\zeta\omega_n}$$

- Same analysis applies for exponential growth

  For same Δt $\dfrac{x_1}{x_2} = \dfrac{x_2}{x_3} = \dfrac{x_3}{x_4}$ etc

  Therefore $\dfrac{x_1}{x_3} = \dfrac{x_1}{x_2}\dfrac{x_2}{x_3} = \left(\dfrac{x_1}{x_2}\right)^2$ $\Rightarrow$ $HCAR = \sqrt{\text{full cycle amp ratio}}$

$$\text{HCAR} \equiv \frac{x_1}{x_2} = e^{\zeta\omega_n(t_2-t_1)} = e^{\zeta\omega_n(\Delta t)} = e^{\zeta\omega_n\left(\frac{T}{2}\right)} = e^{\zeta\omega_n\left(\frac{2\pi}{2\omega_d}\right)}$$
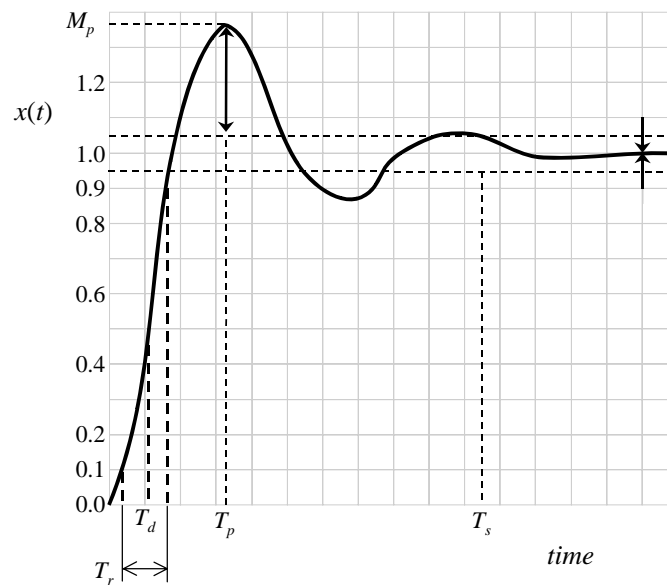
$$= e^{\zeta\omega_n\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right)} = e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \quad \text{Note } HCAR = f(\text{only})$$

$$\Rightarrow \ln\frac{x_1}{x_2} = \frac{\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \zeta = \sqrt{\frac{\left(\ln\frac{x_1}{x_2}\right)^2}{\pi^2 + \left(\ln\frac{x_1}{x_2}\right)^2}}$$

This can be applied in graphical form (transient peak ratio method)

8.3.3 Determining Descriptive Parameters

Time domain metrics

Peak Value, *MP*: largest value

Final Value, *FV*: steady state value

Delay Time, T_d: 50% of final value
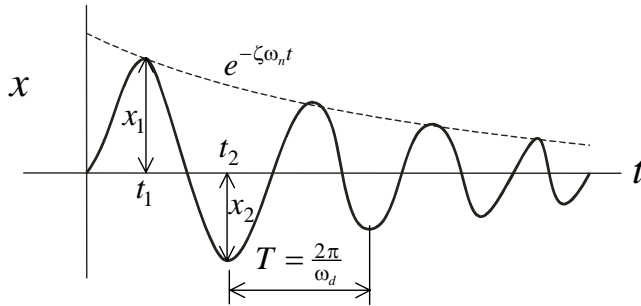
Rise Time, T_r: 10% - 90% of FV

Peak Time, T_p: time to MP

Settling Time, T_s: time to reach some defined % of final value

% Overshoot, *PO*: $\frac{MP-1}{1} \times 100\%$ *target value = unity*

Method #1 Basic Analysis

Note $x(t) = A e^{-\zeta\omega_n t} \sin(\omega_d t + 0)$

$[\zeta\omega_n]$ and $[\omega_d]$ describe a time history

$$f_d = \frac{1}{T} \qquad \omega_d = \frac{2\pi}{T} \qquad \boxed{\zeta\omega_n = \frac{\ln\frac{x_1}{x_2}}{(t_2 - t_1)}}$$

Easily measured values: can use any points on envelope

Define $\;\varepsilon \equiv \dfrac{\zeta\omega_n}{\omega_d} = \dfrac{\frac{\ln\frac{x_1}{x_2}}{(t_2 - t_1)}}{\frac{2\pi}{T}} = \dfrac{T \ln\frac{x_1}{x_2}}{2\pi(t_2 - t_1)}$

where $(t_2 - t_1) = \dfrac{T}{2} \Rightarrow \dfrac{\zeta\omega_n}{\omega_d} = \dfrac{\ln\frac{x_1}{x_2}}{\pi} = \varepsilon = \ln[x_1/x_2]/\pi$

recall $\omega_d = \omega_n\sqrt{1 - \zeta^2}$

$\varepsilon = \dfrac{\zeta\omega_n}{\omega_n\sqrt{1 - \zeta^2}} = \dfrac{\zeta}{\sqrt{1 - \zeta^2}} \Rightarrow \varepsilon^2 = \dfrac{\zeta^2}{1 - \zeta^2} \Rightarrow \varepsilon^2(1 - \zeta^2) = \zeta^2 \Rightarrow \varepsilon^2 = \zeta^2[1 + \varepsilon^2]$
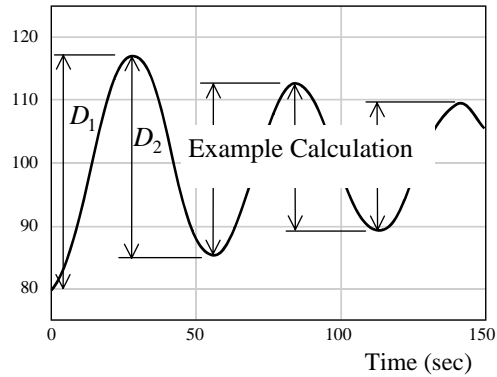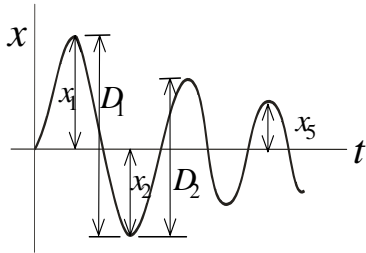
$$\zeta = \sqrt{\frac{\varepsilon^2}{1 + \varepsilon^2}} \qquad \omega_n = \frac{\zeta\omega_n}{\zeta} = \frac{\omega_d}{\sqrt{1 - \zeta^2}}$$

$\left|\dfrac{x_1}{x_2}\right| \equiv$ Half - Cycle Amplitude Ratio $(HCAR)$ $= e^{\zeta\omega_n(t_2 - t_1)}$

Method #2 Transient Peak Ratio Analysis

1) Measure either *D* or *x* distances as shown.

2) Note ratio of adjacent peak values (transient peak ratios).
3) Average several TPRs.
4) Use equation to find ζ:

$$\zeta = \sqrt{\frac{\left(\ln \frac{x_1}{x_2}\right)^2}{\pi^2 + \left(\ln \frac{x_1}{x_2}\right)^2}}$$

First TPR : $\dfrac{D_2}{D_1} = \dfrac{117-86}{117-80} = \dfrac{31}{37} = 0.8378$

Second TPR : $\dfrac{D_3}{D_2} = \dfrac{112-86}{117-86} = \dfrac{26}{31} = 0.8387$

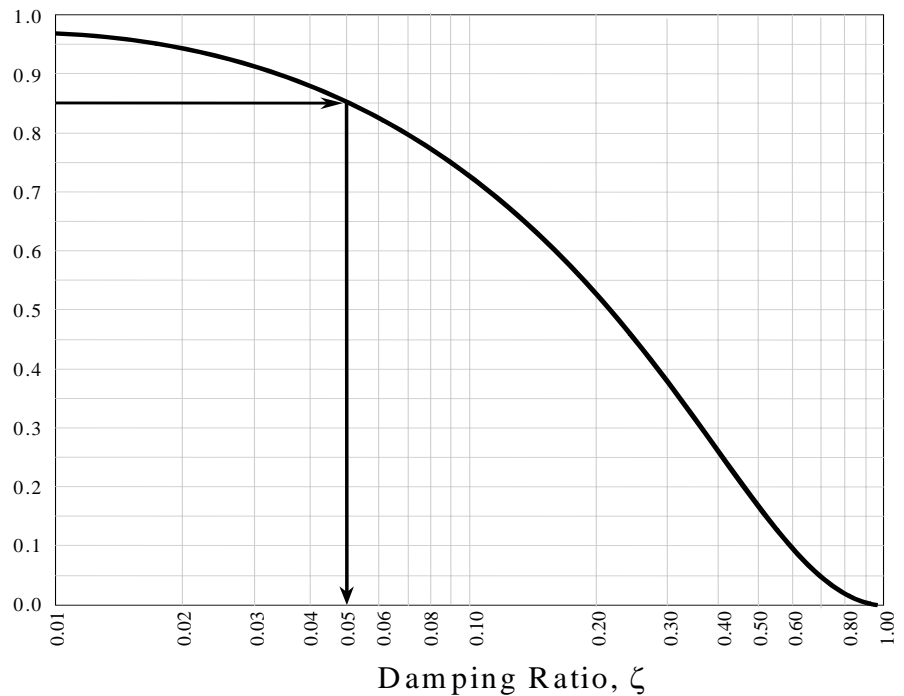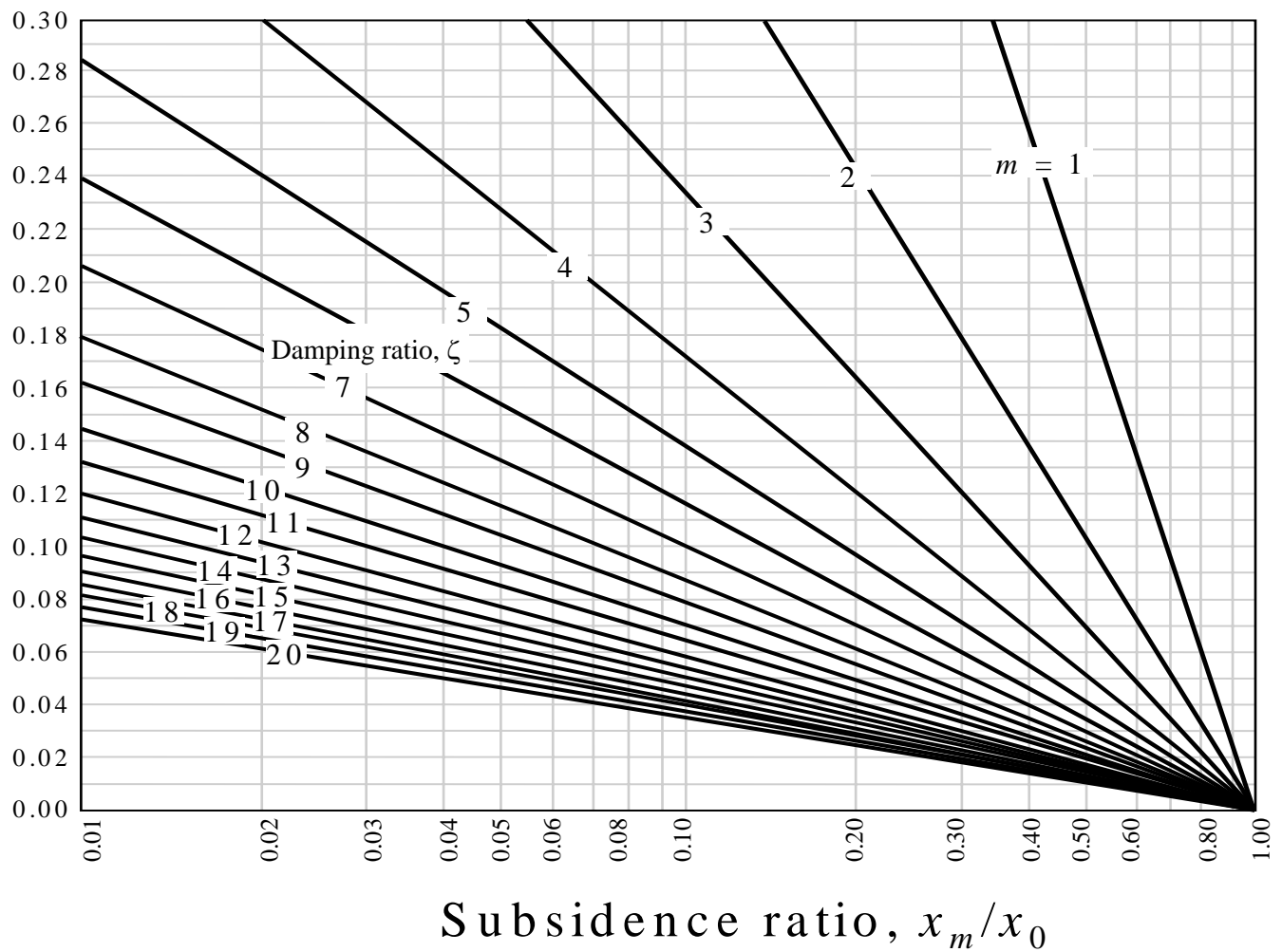Third TPR : $\dfrac{D_4}{D_3} = \dfrac{112-90}{112-86} = \dfrac{22}{26} = 0.8462$

Average TPR = 0.8409

4a) Can use
D_1/D_2 or x_1/x_2
ratios in above equation.

4b) In lieu of equation, use
adjacent look-up curve to find ζ.

4c) Time ratio method works better
with heavy damping.

Damping Ratio, ζ

Method #3   Multiple *TPR* Analysis

To determine damping ratio

      ~ Use the $m = 1$ line when comparing the next ratio. $\dfrac{\Delta x_1}{\Delta x_0} = \dfrac{2.5}{4.5} = 0.56$ $\zeta = 0.16$

                                                                    $\dfrac{\Delta x_2}{\Delta x_1} = \dfrac{1.5}{2.5} = 0.60$ $\zeta = 0.14$

      ~ Use the $m = 2$ line for comparing every other $\dfrac{\Delta x_2}{\Delta x_0} = \dfrac{1.5}{4.5} = 0.33$ $\zeta = 0.16$
        peak ratio.

                                                                    $\dfrac{\Delta x_3}{\Delta x_1} = \dfrac{1.0}{2.5} = 0.40$ $\zeta = 0.14$
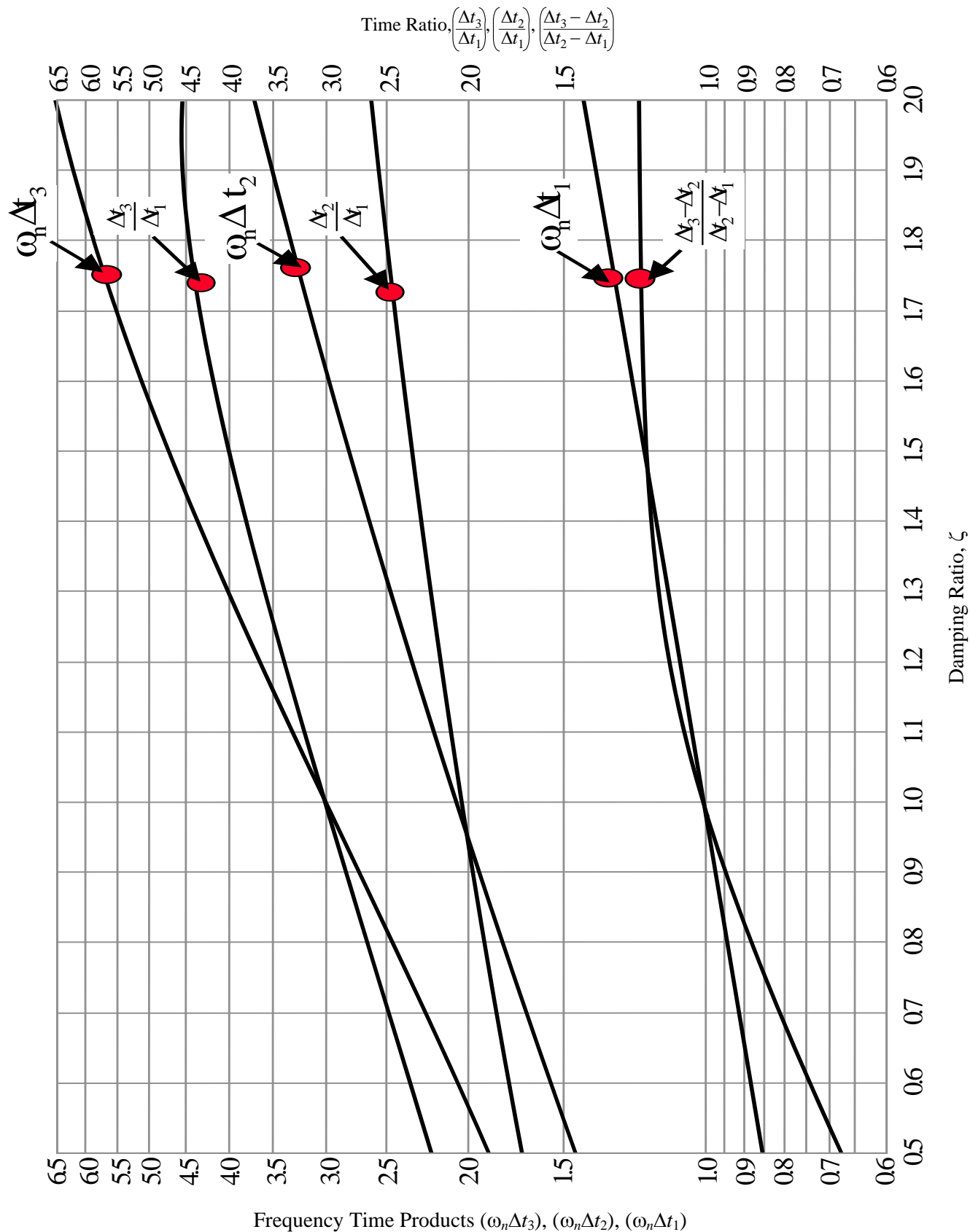
Method #4 Time Ratio Analysis

- If the damping ratio is between 0.5 and 1.0 (two or less overshoots), then the time ratio method can be used to determine frequency and damping ratio. Select a peak where the response if free.

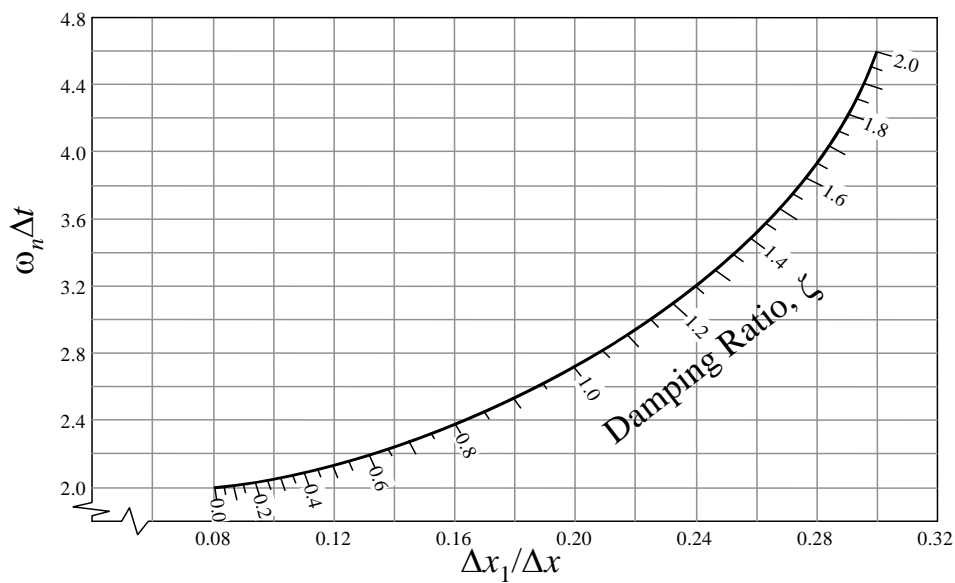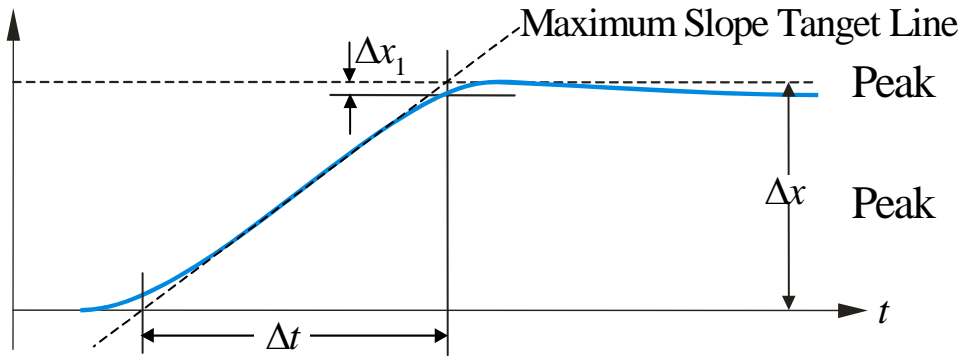- Note times for amplitude to reduce to 73.6%, 40.9%, and 19.9% of the peak value.
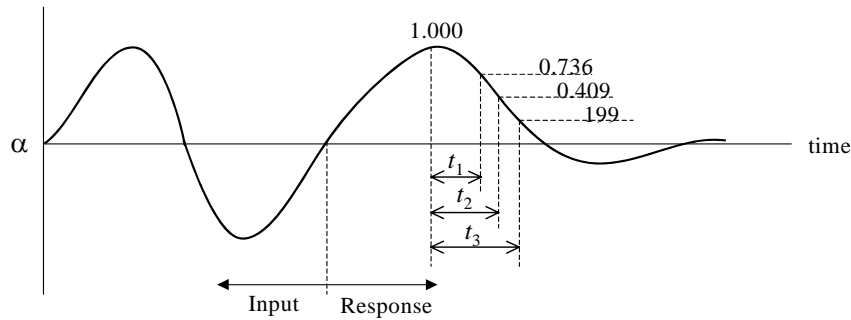
- Form the time ratios t_2/t_1, t_3/t_1, and $[t_3-t_2]/[t_2-t_1]$

- Enter the next figure at the time ratio side to find ζ for each time ratio.

- This technique is valid if the system transfer function has no zeros.

- If recorded measurements are not available and if the number of overshoots is between 2 and 6, then

$$\zeta \approx \frac{7-\#overshoots}{10}$$

Time Ratio, $\left(\dfrac{\Delta t_3}{\Delta t_1}\right), \left(\dfrac{\Delta t_2}{\Delta t_1}\right), \left(\dfrac{\Delta t_3 - \Delta t_2}{\Delta t_2 - \Delta t_1}\right)$

$\omega_n \Delta t_3$

$\dfrac{\Delta t_3}{\Delta t_1}$

$\omega_n \Delta t_2$

$\dfrac{\Delta t_2}{\Delta t_1}$

$\omega_n \Delta t_1$

$\dfrac{\Delta t_3 - \Delta t_2}{\Delta t_2 - \Delta t_1}$

Damping Ratio, ζ

Frequency Time Products $(\omega_n \Delta t_3), (\omega_n \Delta t_2), (\omega_n \Delta t_1)$

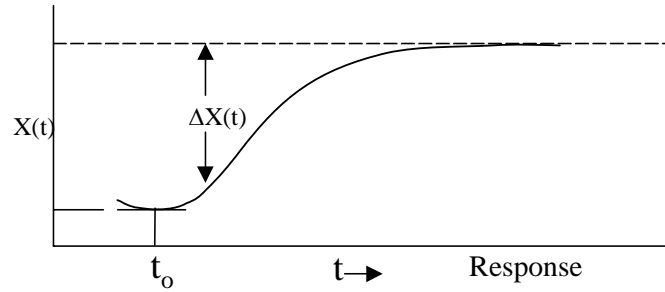**Method #5** Maximum Slope Analysis

- Calcu-   lation of ζ somewhat sensitive to Δx_1 measurement

- $\omega_n = \dfrac{\omega_n \Delta t}{\Delta t}$ not too sensitive to Δx_1

- Best if $0.5 \leq \zeta \leq 1.4$

- Initial overshoot approximation: let (step inputs only) $K \equiv \dfrac{X_{pK}}{X_{ss}} - 1$
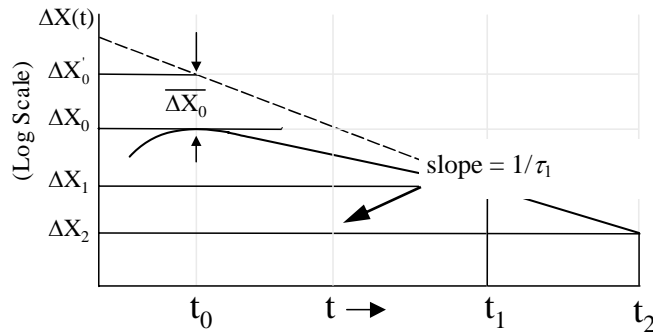
$$\zeta = \left[\frac{\left(\dfrac{-1}{\pi} \ln K \right)^2}{1 + \left(\dfrac{-1}{\pi} \ln K \right)^2} \right]^{\frac{1}{2}}$$

Method #6 Separated Real Root Analysis (when $\zeta > 1$)

1) Determine several steady state $\Delta X(t)$ values from time history

2) Plot ΔX vs t on semi-log scale

3) After the faster root has decayed, the semi-log plot will be a straight line whose slope determines the slower root $(1/\tau_1)$

$$\tau_1 = \frac{t_1 - t_2}{\ln\left(\dfrac{\Delta \chi_1}{\Delta \chi_2}\right)}$$

4) Determine by extrapolating the straight line portion of the response to establish the values

$$\left(\frac{1}{\tau_2}\right) \qquad\qquad\qquad \overline{\Delta \chi_0} \ \& \ \Delta \chi_0^{'}$$
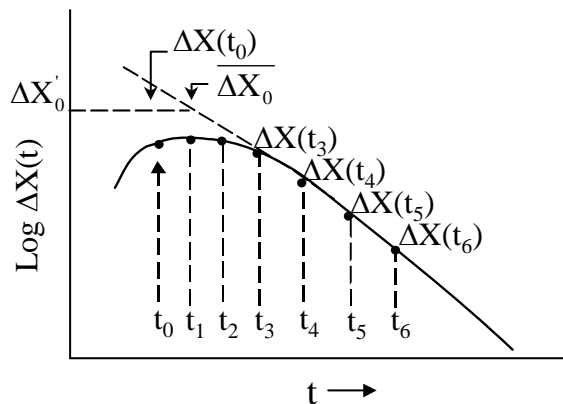
$$\tau_2 = \tau_1 \left(\frac{\overline{\Delta \chi_0}}{\Delta \chi_0^{'}}\right) \qquad \boxed{\omega_n = \sqrt{\frac{1}{\tau_1}\frac{1}{\tau_2}}} \ \text{and} \ \boxed{\zeta = \frac{-\left(\dfrac{1}{\tau_1} + \dfrac{1}{\tau_2}\right)}{2\omega_n} = \frac{-\left(\dfrac{1}{\tau_1} + \dfrac{1}{\tau_2}\right)}{2\sqrt{\dfrac{1}{\tau_1}\dfrac{1}{\tau_2}}}}$$

Method #7 Modified Separated Real Root Analysis

- Method #6 is sensitive to errors in determining steady state values

- Alternate method is to avoid need for steady state value

- Define $\Delta X(t) \equiv [\, x(t+\Delta T) - x(t)]$ where ΔT is a time increment

1) From time history, measure ΔX values according to definition

$\Delta T = (t_1 - t_o) = (t_2 - t_1)$, etc.
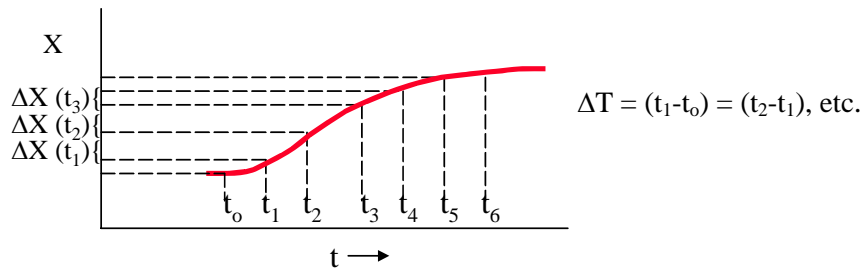
2) Plot $\Delta X(t)$ vs time on semi-log scale

3) Use previous method to determine roots and characteristics

- Gross error will result if ζ is actually <1

- If ζ is near 1, check results using time ratio or slope method

Method # 8   Frequency Sweep Analysis

Determine ω_n and ζ using sinusoidal inputs.

- This "forced response" method most useful when damping is heavy.
- For a second order system, output/input amplitude ratio and phase shift are a function of input frequency.
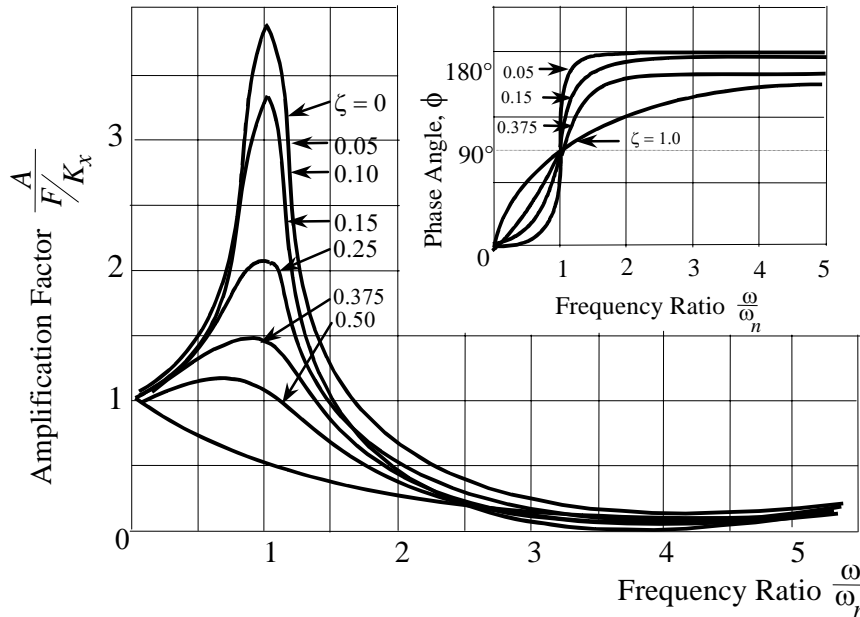
- Amplitude ratio peaks at "resonant" frequency, ω_r.

- Resonant peaks increase as ζ decreases below 0.707.

- Peak amplitude ratio "rolls off" as ζ increases above 0.707.

- Resonant frequency approaches natural frequency as damping decreases:

    $\omega_r = \omega_n[1 - 2\zeta^2]^{.5}$

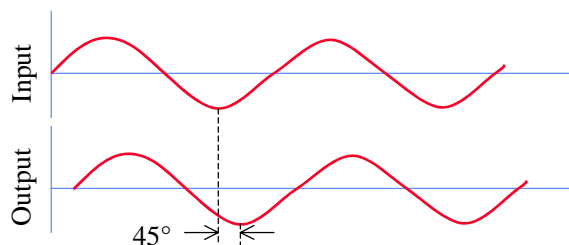- Phase shift = 90° if excited at ω_n, regardless of   damping ratio.
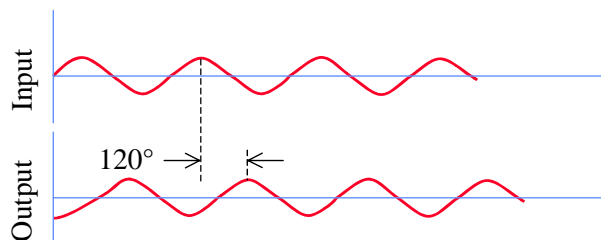
$$\zeta = 0.5(\omega_2 - \omega_1)/\omega_n$$
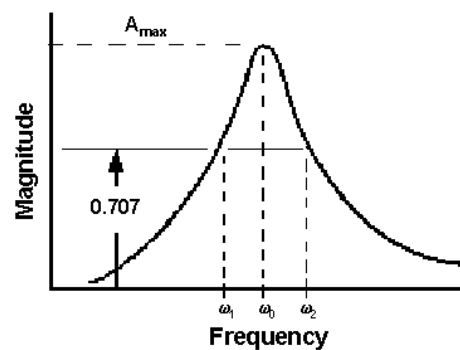
Frequency Sweep Analysis (continued)
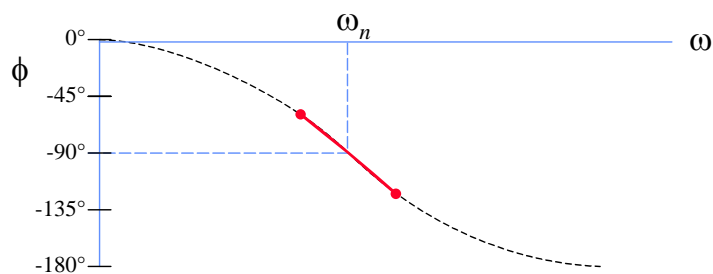
1.   Using sinusoidal inputs excite system @ ω near ω_n

2.   Measure phase lag (ϕ) of $\dfrac{output}{input}$

3.   Excite system @ another ω near ω_n
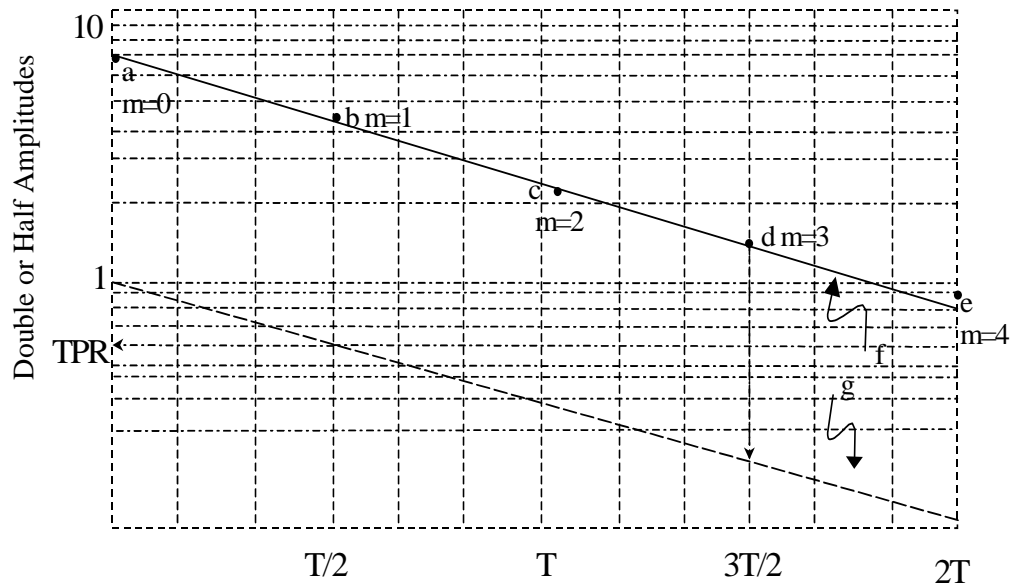
4.   Again Measure phase lag ϕ

5.   Plot ϕ vs input frequency

6. ω_n occurs at $\phi = 90°$

Linearity Check /Accuracy Improvement

1) On semi-log scale, plot ratio of initial amplitude to subsequent peak
   amplitudes at each half cycle (points *a-e*).

2) Fair straight line (*f*) through these points.

3) Draw line (*g*) parallel (*f*) intercepting the ordinate at *TPR*=1

4) Average *TPR* occurs at *T*/2 on line *g*

8.4 Complex Plane

Begin with sum of forces in spring-mass-damper example

$$M\ddot{x} + D\dot{x} + Kx = f(t)$$

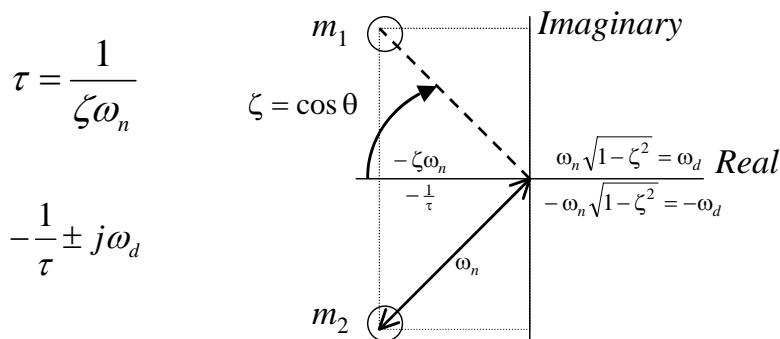$$\text{let} \quad x = e^{st}, \quad \text{find transient solution}$$

$$\left(Ms^2 + Ds + K\right)e^{st} = 0$$

Apply quadratic equation to solve for roots

$$s_{1,2} = \frac{-D}{2M} \pm \sqrt{\left(\frac{D}{2M}\right)^2 - \frac{K}{M}}$$
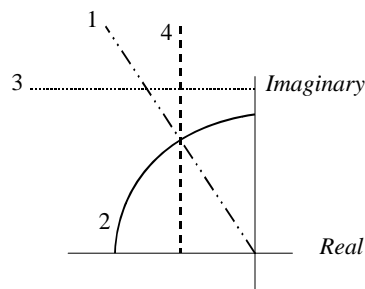
Recall previous analogy

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$\tau = \frac{1}{\zeta\omega_n}$$

$$\zeta = \cos\theta$$

$$-\frac{1}{\tau} \pm j\omega_d$$

Location of Roots on Complex Plane
1. Line of constant damping ratio ζ – varying $C_{1/n}$ and ω_n
2. Line of constant ω_n – varying ζ
3. Line of constant ω_d and period (T)
4. Line of constant real part ($\zeta\omega_n$) and time to damp ($T_{1/n}$)

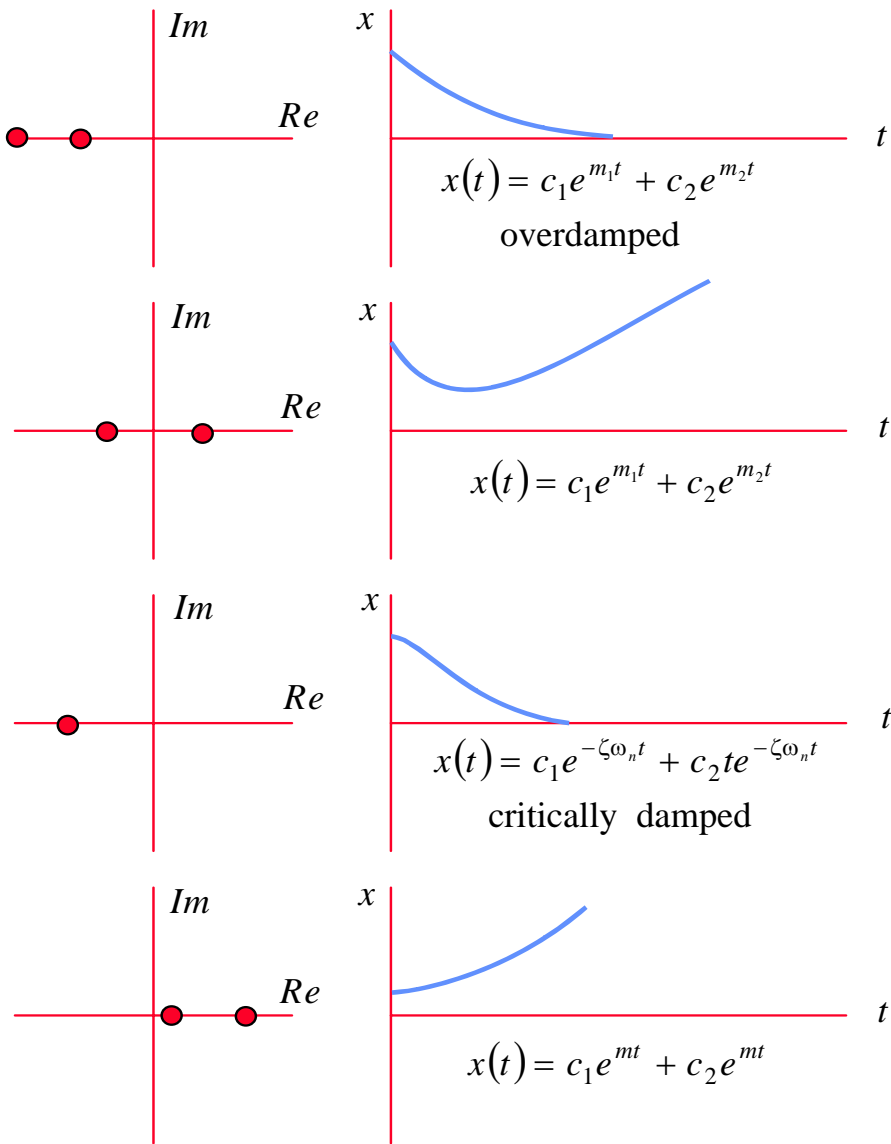$\sigma = \zeta\omega_n = 1/\tau = $ damping rate

Sample second order root plots and corresponding time histories

(time histories represent trends only)

Examples of "two real roots"

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$
overdamped

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$x(t) = c_1 e^{-\zeta \omega_n t} + c_2 t e^{-\zeta \omega_n t}$$
critically damped

$$x(t) = c_1 e^{mt} + c_2 e^{mt}$$

More sample second order root plots and corresponding time histories

Examples of "imaginary roots"

$$x(t) = A \sin\left(\omega_n t + \phi\right)$$
undamped

$$x(t) = A e^{-\zeta\omega_n t} \sin\left(\omega_d t + \phi\right)$$
lightly damped

$$x(t) = A e^{-\zeta\omega_n t} \sin\left(\omega_d t + \phi\right)$$
moderately damped

$$x(t) = A e^{\zeta\omega_n t} \sin\left(\omega_d t + \phi\right)$$
negatively damped
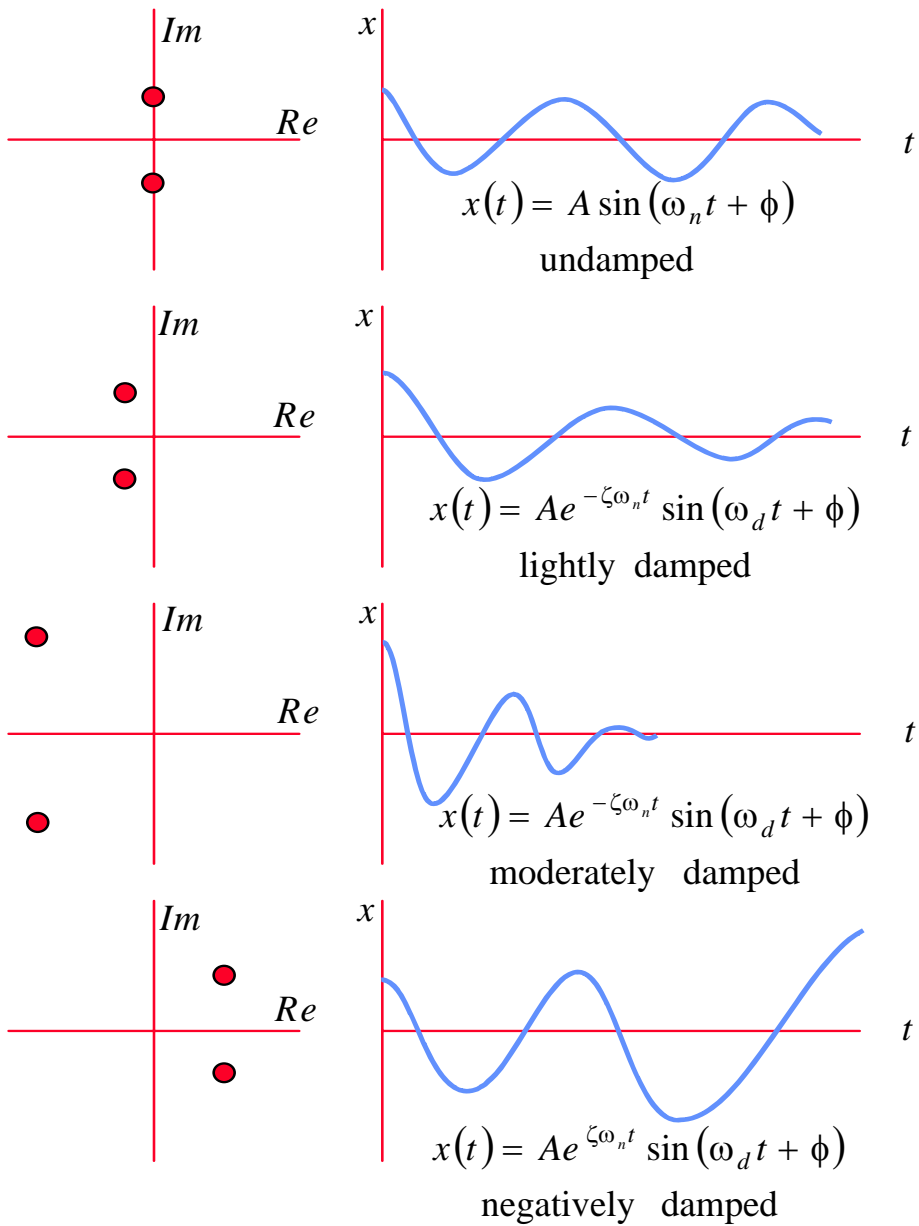
8.5 Parameter Conversions

For conversion of accelerometer measurements.
- For magnitude conversion substitute $2\pi f$ for $j\omega$.
- Assumes linear spectra.
- Conversion factor should be squared for power spectra.

Acceleration to velocity

to convert from	to	multiply by
ft/s² rms	*ft/s rms*	*1/jω*
ft/s² rms	*in/s rms*	*12/jω*
ft/s² rms	*in/s peak*	*16.97/jω*
g rms	*in/s rms*	*386/jω*
g rms	*in/s peak*	*545.8/jω*
m/s² rms	*mm/s rms*	*1000/jω*
m/s² rms	*mm/s peak*	*1414/jω*
g rms	*mm/s rms*	*9806/jω*
g rms	*mm/s peak*	*13865.7/jω*

Acceleration to Displacement

to convert from	to	multiply by
ft/s² rms	*in rms*	$12/(j\omega)^2$
ft/s² rms	*in p-p*	$33.9/(j\omega)^2$
ft/s² rms	*mil p-p*	$33.9\,E\,03/(j\omega)^2$
g rms	*in rms*	$386/(j\omega)^2$
g rms	*in p-p*	$1091.6\,E\,03/(j\omega)^2$
g rms	*mil p-p*	$1091.6\,E\,03/(j\omega)^2$
m/s² rms	*mm rms*	$1000/(j\omega)^2$
m/s² rms	*mm p-p*	$2828/(j\omega)^2$
m/s² rms	*micron p-p*	$2828\,E\,03/(j\omega)^2$

E= engineering exponent (x10 __)
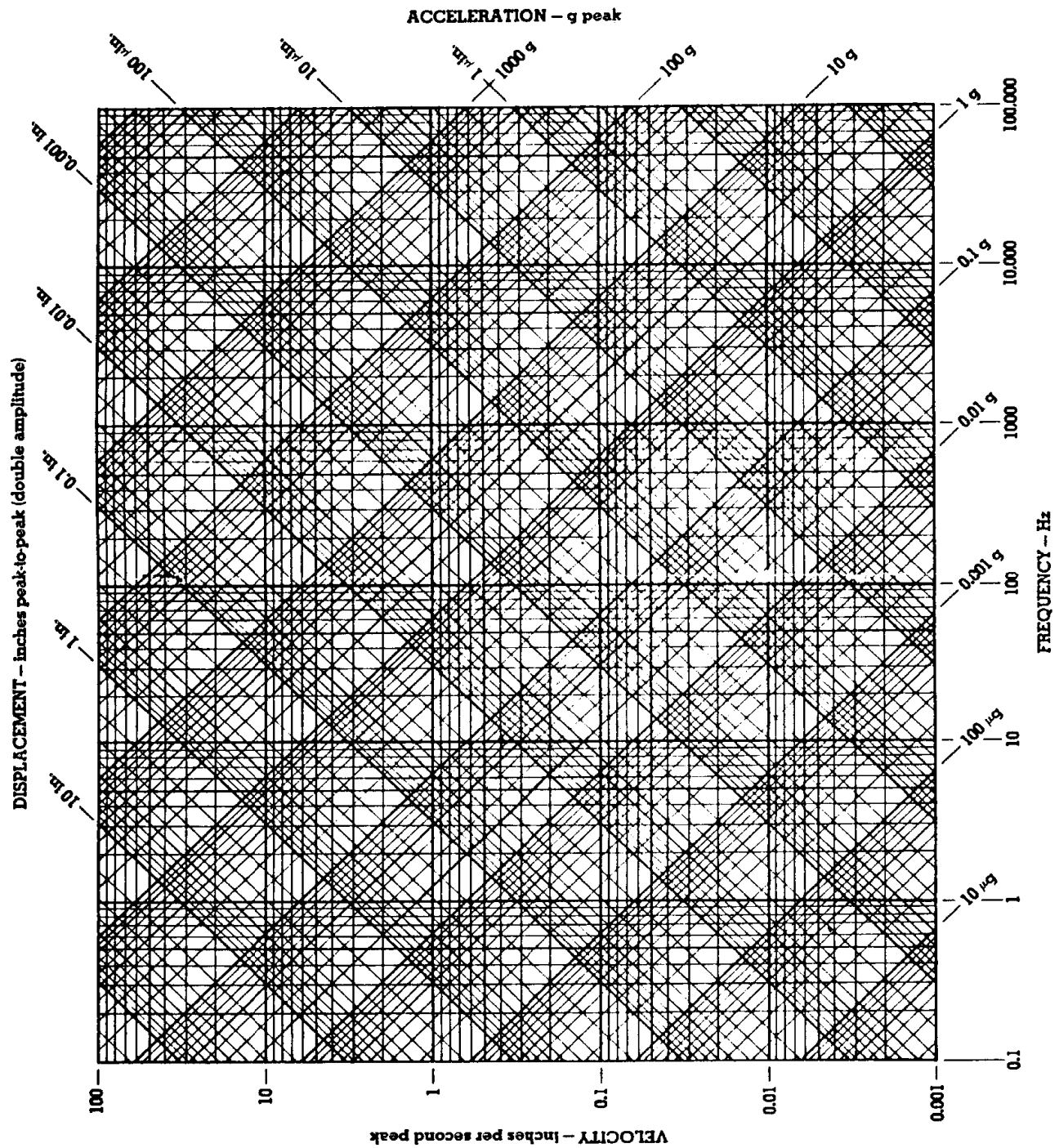*g* = 32.174 ft/sec2
in= inches
mil = thousandths of an inch
mm = milimeters
p-p = peak-to-peak
rms = root mean square

8.6 Vibration Nomograph

8.7 References

8.7.1   Lawless, Alan R., *Math and Physics for Flight Testers*, "Chapter 9, Motion Analysis," National Test Pilot School, Mojave CA, 1999.

8.7.2   Ward, Don, *Introduction to Flight Testing*, Texas A&M, Elsevier, 1993.

8.7.3   Lang, George F., *Understanding Vibration Measurements*, Application Note 9, Rockland Scientific Corporation, Rockleigh, New Jersey, December 1978.

8.7.4   *The Fundamentals of Modal Testing*, Application Note 243-3, Hewlett-Packard Company,

Additional Reading

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Jacobsen, Ludik S. and Ayre, Robert S., *Engineering Vibrations*, McGraw-Hill Book Company, New York, New York, 1958.

Meirovitch, Leonard, *Elements of Vibration Analysis*, McGraw-Hill Book Company, New York, New York, 1986.

Meirovitch, Leonard, *Analytical Methods in Vibrations*, Macmillan Publishing Company, New York, New York, 1967.

Myklestad, N.O., *Vibration Analysis*, McGraw-Hill Book Company, New York, New York, 1944.