Text as Data

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Three categories of documents

Hand labeled

- Training set (what we'll use to estimate model)
- Validation set (what we'll use to assess model)

Unlabeled

- Test set (what we'll use the model to categorize)

Label more documents than necessary to train model

Regression models

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- J will likely be large (perhaps $J > N$)

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Predictions will be variable

Mean Square Error

Suppose θ is some value of the true parameter

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$$\text{E}[(\hat{\theta} - \theta)^2] = E[\hat{\theta}^2] - 2\theta E[\hat{\theta}] + \theta^2$$

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To reduce MSE, we are willing to induce bias to decrease variance $\rightsquigarrow$ methods that shrink coefficeints toward zero

Ridge Regression

Penalty for model complexity

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$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y})$$

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$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \quad = \quad \sum_{i=1}^{N} \left(y_i - \ \beta_0 - \sum_{j=1}^{J} \beta_j x_{ij} \right)^2$$

Ridge Regression

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$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^{N} \left(y_i - \beta_0 - \sum_{j=1}^{J} \beta_j x_{ij} \right)^2 + \underbrace{\lambda \sum_{j=1}^{J} \beta_j^2}_{\text{Penalty}}$$

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where:

- $\beta_0 \rightsquigarrow$ intercept

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where:

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- $\lambda \rightsquigarrow$ penalty parameter

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where:

- $\beta_0 \rightsquigarrow$ intercept
- $\lambda \rightsquigarrow$ penalty parameter
- Standardized $\boldsymbol{X}$ (coefficients on same scale)

Ridge Regression$\rightsquigarrow$ Optimization

$$\boldsymbol{\beta}^{\text{Ridge}} = \text{arg min}_{\boldsymbol{\beta}} \left\{ f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \right\}$$

Ridge Regression $\rightsquigarrow$ Optimization

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Demean the data and set $\beta_0 = \bar{y} = \sum_{i=1}^{N} \frac{y_i}{N}$

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Ridge Regression $\rightsquigarrow$ Intuition (1)

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\widehat{\beta} &= \left(\boldsymbol{X}^{'}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} \\
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&= \left(\boldsymbol{I}_j + \lambda\boldsymbol{I}_j\right)^{-1}\widehat{\boldsymbol{\beta}} \\
\beta_j^{\text{Ridge}} &= \frac{\widehat{\beta}_j}{1+\lambda}
\end{aligned}
$$

Ridge Regression $\rightsquigarrow$ Intuition (2)

$$
\begin{aligned}
\boldsymbol{\beta}_j &\sim \text{Normal}(0, \tau^2) \\
y_i &\sim \text{Normal}(\beta_0 + \boldsymbol{x}_i^{'}\boldsymbol{\beta}, \sigma^2)
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Ridge Regression $\rightsquigarrow$ Intuition (2)

$$\begin{aligned} \boldsymbol{\beta}_j &\sim \text{Normal}(0, \tau^2) \\ y_i &\sim \text{Normal}(\beta_0 + \boldsymbol{x}_i^{'}\boldsymbol{\beta}, \sigma^2) \end{aligned}$$

$$p(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) \propto \prod_{j=1}^{J} p(\beta_j) \prod_{i=1}^{N} p(y_i|\boldsymbol{x}_i, \boldsymbol{\beta})$$

Ridge Regression$\leadsto$ Intuition (2)

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&\propto \prod_{j=1}^{J} \frac{1}{\sqrt{2\pi}\tau} \exp\left(-\frac{\beta_j^2}{2\tau^2}\right) \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_i - \beta_0 - \boldsymbol{x}_i^{'}\boldsymbol{\beta})^2}{2\sigma^2}\right)
\end{aligned}$$

Ridge Regression$\rightsquigarrow$ Intuition (2)

$$\log p(\beta|\boldsymbol{X}, \boldsymbol{Y}) \quad = \quad -\sum_{j=1}^{J} \frac{\beta_j^2}{2\tau^2} - \sum_{i=1}^{N} \frac{(y_i - \beta_0 - \boldsymbol{x}^{'}\boldsymbol{\beta})^2}{2\sigma^2}$$

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-2\sigma^2 \log p(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) &= \sum_{i=1}^{N}(y_i - \beta_0 - \boldsymbol{x}^{'}\boldsymbol{\beta})^2 + \sum_{j=1}^{J} \frac{\sigma^2}{\tau^2}\beta_j^2
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where:

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\end{aligned}
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where:

- $\lambda = \frac{\sigma^2}{\tau^2}$

Ridge Regression $\rightsquigarrow$ Intuition (3)

Definition

Suppose $\boldsymbol{X}$ *is an* $N \times J$ *matrix. Then* $\boldsymbol{X}$ *can be written as:*

$$\boldsymbol{X} = \underbrace{\boldsymbol{U}}_{N\times N}\underbrace{\boldsymbol{S}}_{N\times J}\underbrace{\boldsymbol{V}^{'}}_{J\times J}$$

Where:

$$\begin{aligned} \boldsymbol{U}^{'}\boldsymbol{U} &= \boldsymbol{I}_N \\ \boldsymbol{V}^{'}\boldsymbol{V} &= \boldsymbol{V}\boldsymbol{V}^{'} = \boldsymbol{I}_J \end{aligned}$$

$\boldsymbol{S}$ *contains* $\min(N, J)$ *singular values,* $\sqrt{\lambda_j} \geq 0$ *down the diagonal and then 0's for the remaining entries*

Ridge Regression $\rightsquigarrow$ Intuition (3)

Recall: PCA:

$$\frac{1}{N}\boldsymbol{X}^{'}\boldsymbol{X} = \underbrace{\boldsymbol{W}}_{\text{eigenvectors}} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_J \end{pmatrix} \underbrace{\boldsymbol{W}^{'}}_{\text{eigenvectors}}$$

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Using SVD:

$$\frac{1}{N}\boldsymbol{X}^{'}\boldsymbol{X} = \boldsymbol{V}\boldsymbol{S}^{'}\underbrace{\left(\boldsymbol{U}^{'}\boldsymbol{U}\right)}_{\boldsymbol{I}_J}\boldsymbol{S}\boldsymbol{V}^{'}$$

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&= \frac{1}{N}\boldsymbol{V}\boldsymbol{S}^{'}\boldsymbol{S}\boldsymbol{V}^{'} \\
&= \underbrace{\boldsymbol{V}}_{\text{eigenvectors}} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_J \end{pmatrix} \underbrace{\boldsymbol{V}^{'}}_{\text{eigenvectors}}
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Ridge Regression $\rightsquigarrow$ Intuition (3)

We can write the predicted values for a regular regression as

$$\begin{aligned}
\hat{Y} &= \boldsymbol{X}\hat{\boldsymbol{\beta}} \\
&= \boldsymbol{X}\left(\boldsymbol{X}^{'}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} \\
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$$

Where

$$
\tilde{\boldsymbol{S}} = \left[\boldsymbol{S}(\boldsymbol{S}^{'}\boldsymbol{S} + \lambda\boldsymbol{I}_J)^{-1}\boldsymbol{S}\right]
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$$\hat{Y}^{\text{ridge}} = \sum_{j=1}^{J}\boldsymbol{u}_j\frac{\lambda_j}{\lambda_j + \lambda}\boldsymbol{u}_j^{'}\boldsymbol{Y}$$

Degrees of Freedom for Ridge

We will say that the degrees of freedom for Ridge regression with penalty λ is

$$\text{dof}(\lambda) = \sum_{j=1}^{J} \frac{\lambda_j}{\lambda_j + \lambda}$$

Lasso Regression Objective Function

Different Penalty for Model Complexity

$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \;=\; \sum_{i=1}^{N} \left(y_i - \beta_0 - \sum_{j=1}^{J} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{J} \underbrace{|\beta_j|}_{\text{Penalty}}$$

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Lasso Regression Optimization

Definition

Coordinate Descent Algorithms:
Consider $g : \Re^J \rightarrow \Re$*. Our goal is to find* $\boldsymbol{x}^* \in \Re^J$ *such that* $g(\boldsymbol{x}^*) \leq g(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \Re$*.*
To find $\boldsymbol{x}^*$*:*
Until convergence: for each iteration t *and each coordinate* j

$$x_j^{t+1} \quad = \quad arg\ min_{x_j \in \Re} g(x_1^{t+1}, x_2^{t+1}, \ldots, x_{j-1}^{t+1}, x_j, x_{j+1}^t, \ldots, x_J^t)$$

Lasso Regression Optimization: Coordinate Descent

$$\tilde{f}(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{2N}\sum_{i=1}^{N}\left(y_i - \beta_0 - \sum_{j=1}^{J}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{J}|\beta_j|$$

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Lasso Regression $\leadsto$ Intuition 1, Soft Thresholding

Suppose again $\boldsymbol{X}^{'}\boldsymbol{X} = \boldsymbol{I}_J$

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- $\left(|\widehat{\beta}_j| - \lambda\right)_+ = \max(|\widehat{\beta}_j| - \lambda, 0)$

Lasso Regression$\rightsquigarrow$ Intuition 1, Soft Thresholding

Compare soft assignment

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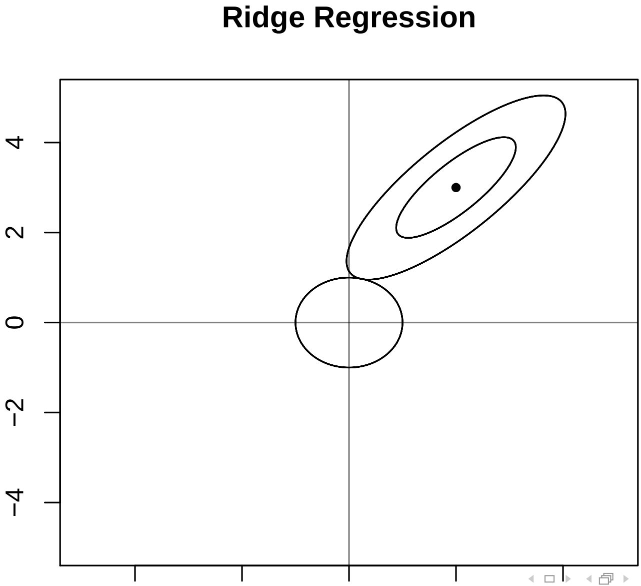
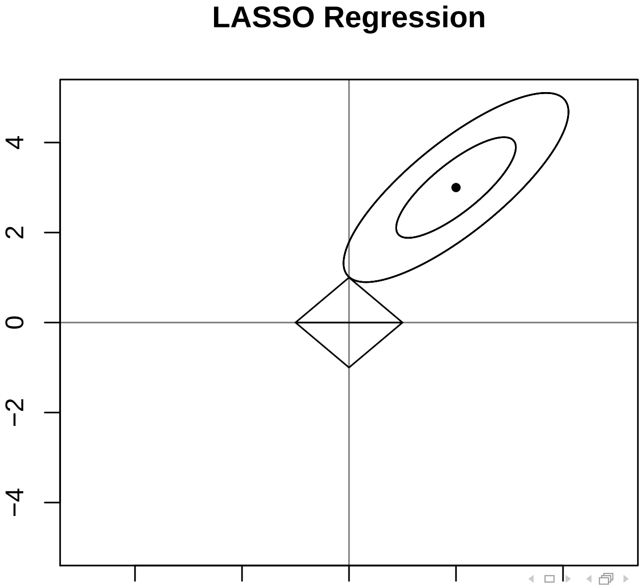
Why does LASSO induce sparsity?

Comparing Ridge and LASSO



Comparing Ridge and LASSO



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Ridge and LASSO: The Elastic-Net

Combining the two criteria $\rightsquigarrow$ Elastic-Net

$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \quad = \quad \frac{1}{2N}\sum_{i=1}^{N}\left(y_i - \beta_0 - \sum_{j=1}^{J}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{J}\left(\frac{1}{2}(1-\alpha)\beta_j^2 + \alpha|\beta_j|\right)$$

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Suppose observations i have dependent variables Y_i and covariates $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iP})$.

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Use MLE to obtain $\hat{\boldsymbol{\beta}}$.

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Training and Test Sets

The useful “fiction” of training and test sets:

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Even if no division, useful to think about systematic components of data.

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Suppose that we have:

$=$

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Training (in-sample) error is:

$$\text{Error}_{\text{in}} = \sum_{i \in \mathcal{T}} \frac{1}{N_{\text{train}}} L(Y_i, f(\hat{\boldsymbol{\beta}}, \boldsymbol{x}_i))$$

Loss Functions and Model Complexity

Suppose that we have:

- Training sets, $\mathcal{T}$, with $|\mathcal{T}| = N_{\text{train}}$
- Test sets, $\mathcal{O}$ with $|\mathcal{O}| = N_{\text{test}}$

Training (in-sample) error is:

$$\text{Error}_{\text{in}} = \sum_{i \in \mathcal{T}} \frac{1}{N_{\text{train}}} L(Y_i, f(\hat{\boldsymbol{\beta}}, \boldsymbol{x}_i))$$

We'd like to estimate out of sample performance with

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$$\text{Error} = \text{E}\left[\text{E}[L(\boldsymbol{Y}, f(\hat{\beta}, \boldsymbol{X}))|\mathcal{T}]\right]$$

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&= \text{E}[(f(\boldsymbol{x}_i) + \epsilon_i - \hat{f}(\boldsymbol{x}_i))^2 | \boldsymbol{x}_i = \boldsymbol{x}_0]
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&= \sigma^2_{\epsilon} + \left[f(\boldsymbol{x}_0) - \text{E}[\hat{f}(\boldsymbol{x}_0)]\right]^2 + E[\left(\hat{f}(\boldsymbol{x}_0) - E[\hat{f}(\boldsymbol{x}_0)]\right)^2]
\end{aligned}$$

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&= \text{Irreducible error} + \text{Bias}^2 + \text{Variance}
\end{aligned}
$$

Probit Regression (for motivational purposes)

Suppose:

$$\begin{aligned} Y_i &\sim \text{Bernoulli}(\pi_i) \\ \pi_i &= \Phi(\boldsymbol{\beta}^{'} \boldsymbol{x}_i) \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative normal distribution.
Implies log-likelihood

$$\log \text{L}(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^{N} \left[Y_i \log \Phi(\boldsymbol{\beta}^{'} \boldsymbol{x}_i) + (1 - Y_i) \log(1 - \Phi(\boldsymbol{\beta}^{'} \boldsymbol{x}_i)) \right]$$

Log-likelihood is a loss function $\rightsquigarrow$ overly optimistic: improves with more parameters

How Do We Build A Model?

There are many ways to fit models

And many choices made when performing model fit

How do we choose?

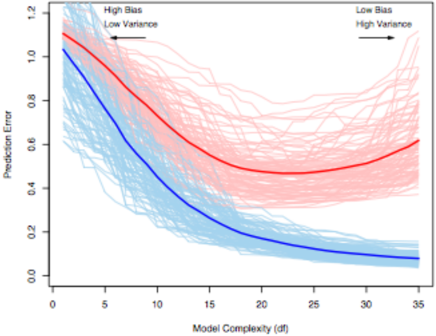

FIGURE 7.1. *Behavior of test sample and training sample error as the model complexity is varied. The light blue curves show the training error $\overline{\text{err}}$, while the light red curves show the conditional test error $\text{Err}_{\mathcal{T}}$ for 100 training sets of size 50 each, as the model complexity is increased. The solid curves show the expected test error Err and the expected training error $\text{E}[\overline{\text{err}}]$.*

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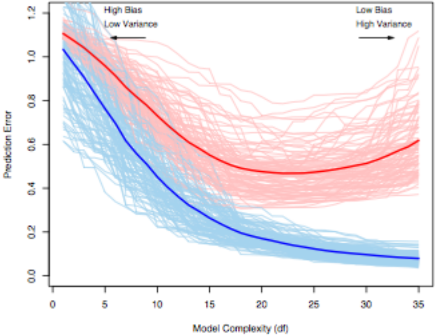

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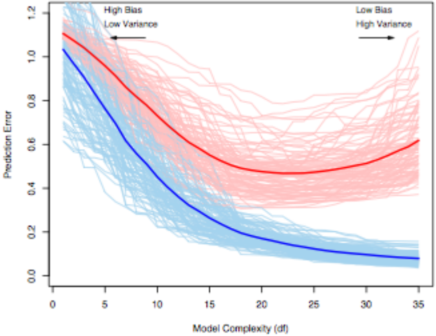
Bad way to choose: within sample model fit (HTF Figure 7.1)



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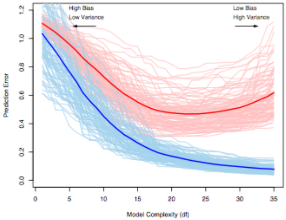

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Model overfit$\rightsquigarrow$ in sample error is optimistic:

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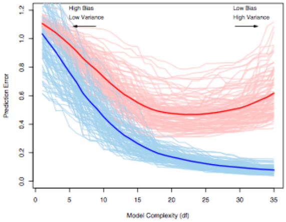

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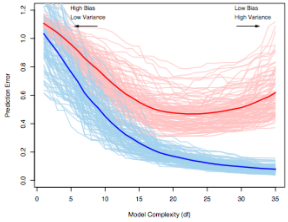

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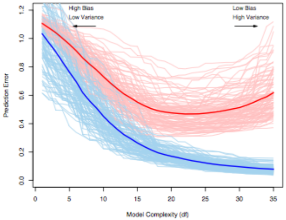

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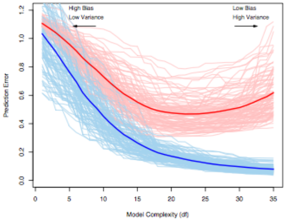

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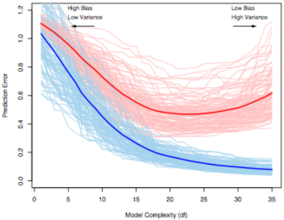

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- Characteristics found in both training and test set
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- Additional model complexity: idiosyncratic features of the training set
- Reduces error in training set, increases error in test set

How Do We Choose Covariates?

Best model depends on task

- Causal inference observational study: make treatment assignment ignorable
- Prediction: improve predictive performance

Stepwise Regression

Suppose we have P covariates.
2^P potential models

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Problematic:

1) Not optimal model selection (path dependent)
2) P-value $\neq$ objective of model

Analytic Solutions

Approximate optimism and compensate in loss function.

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Akaike Information Criterion (AIC) $\rightsquigarrow$ Minimize

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As $N \rightarrow \infty$

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Akaike Information Criterion (AIC) $\rightsquigarrow$ Minimize
As $N \rightarrow \infty$

$$- 2\text{E}[\log P_{\hat{\boldsymbol{\beta}}}(Y)] \quad = \quad -2 \left[\text{E}[\log \text{L}(\hat{\boldsymbol{\beta}} | \boldsymbol{X}, \boldsymbol{Y})] - d \right]$$

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- Intuition: balances model fit with penalty for complexity
- Derived from method to estimate optimism in likelihood based models
- Derived from a method to compute similarity between estimated model and true model (under assumptions of course)
- Can be extended to general models, though requires estimate of irresolvable error

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Bayesian Information Criterion (BIC) [Schwarz Criterion]

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$$\text{BIC} = -2 \log \text{L}(\widehat{\beta}|\boldsymbol{X}, \boldsymbol{Y}) + (\log N)d$$

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- Approximation to Bayes' factor

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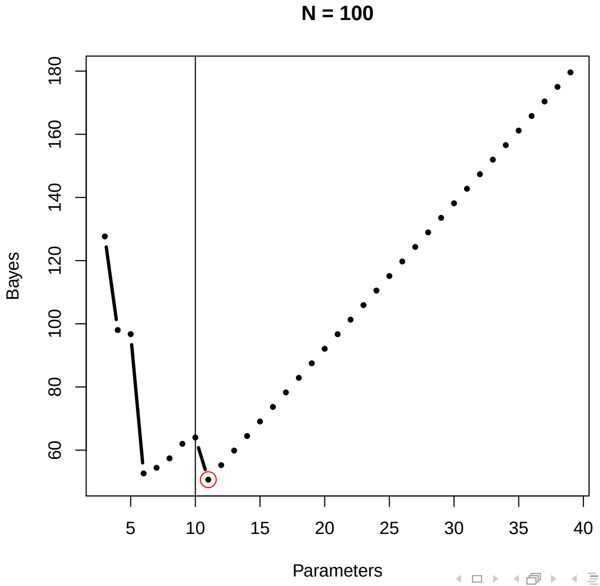
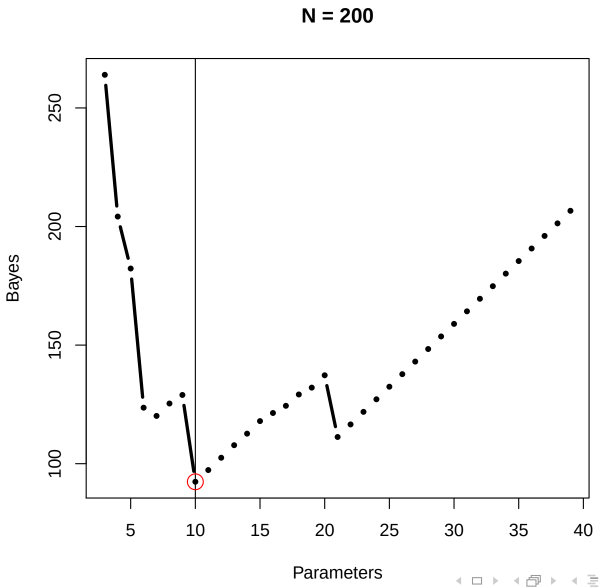
- Derived from Bayesian approach to model selection
- Approximation to Bayes' factor
- Penalizes more heavily than AIC

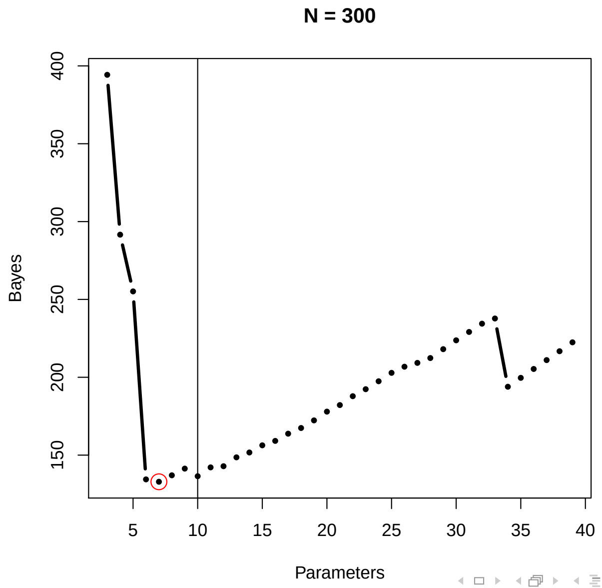
BIC or AIC?


N = 100
Bayes
Parameters
60
80
100
120
140
160
180
5
10
15
20
25
30
35
40

BIC or AIC?

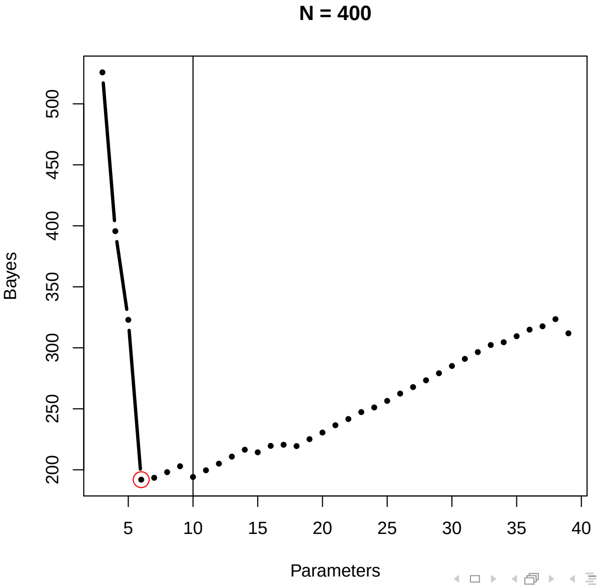


BIC or AIC?

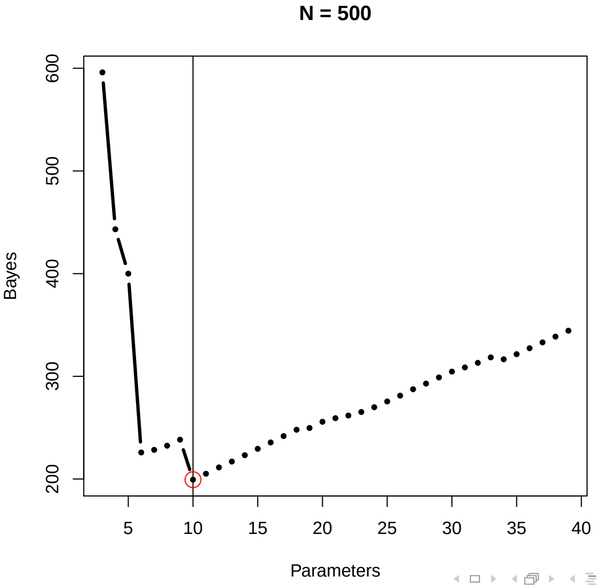


N = 300
Bayes
Parameters
400
350
300
250
200
150
5
10
15
20
25
30
35
40

BIC or AIC?


N = 400
Bayes
Parameters
500
450
400
350
300
250
200
5
10
15
20
25
30
35
40

BIC or AIC?



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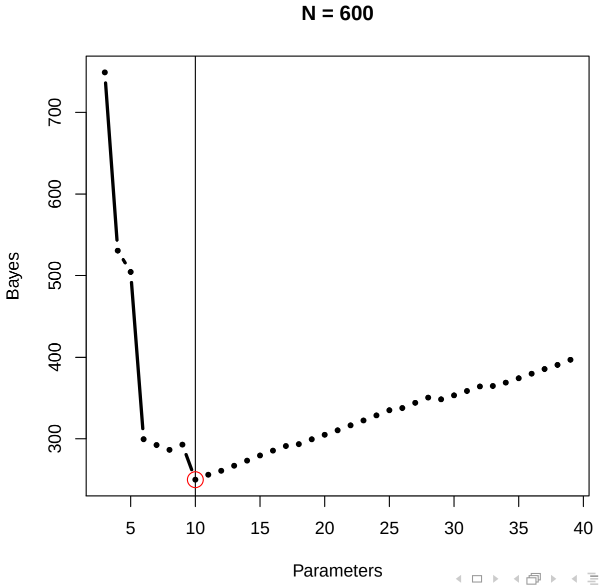

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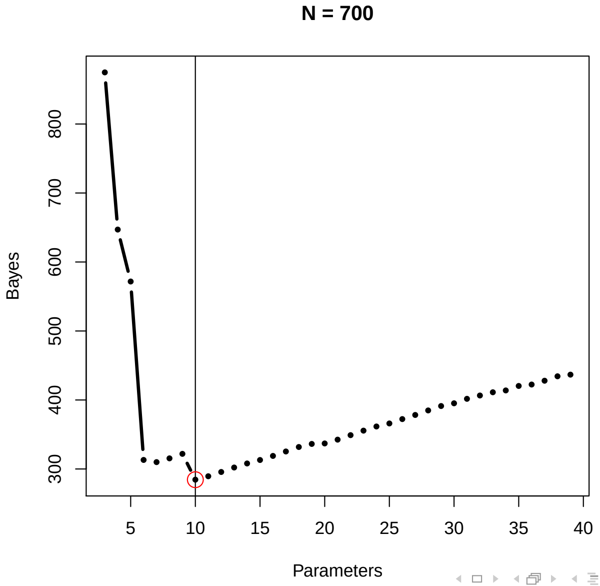
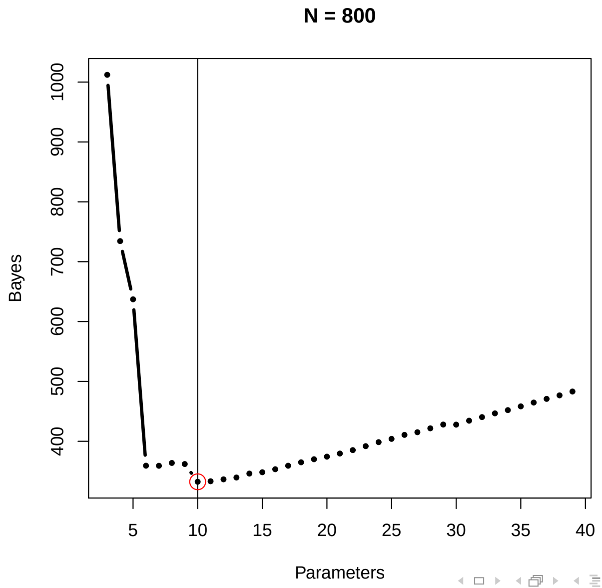
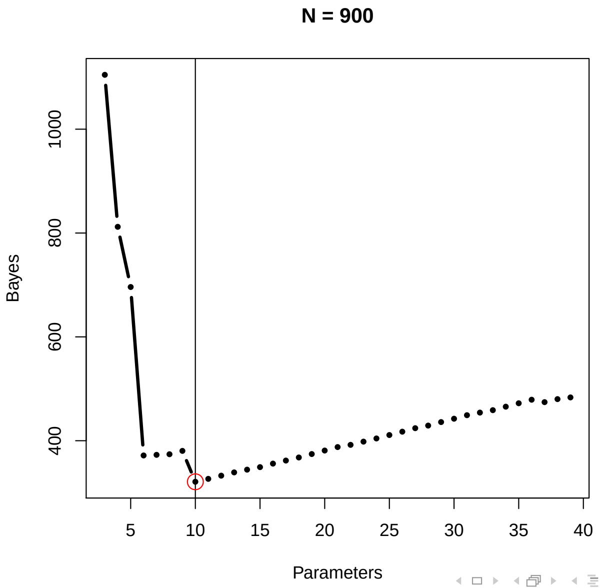

N = 700
Bayes
300
400
500
600
700
800
5
10
15
20
25
30
35
40
Parameters

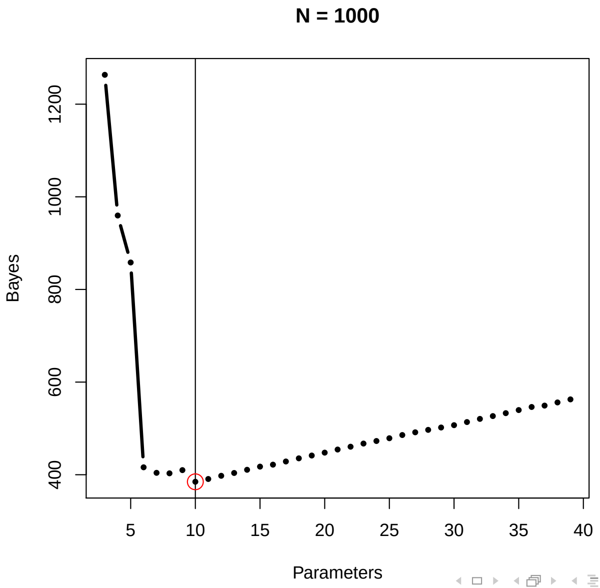
BIC or AIC?


N = 800
Bayes
1000
900
800
700
600
500
400
5
10
15
20
25
30
35
40
Parameters

BIC or AIC?

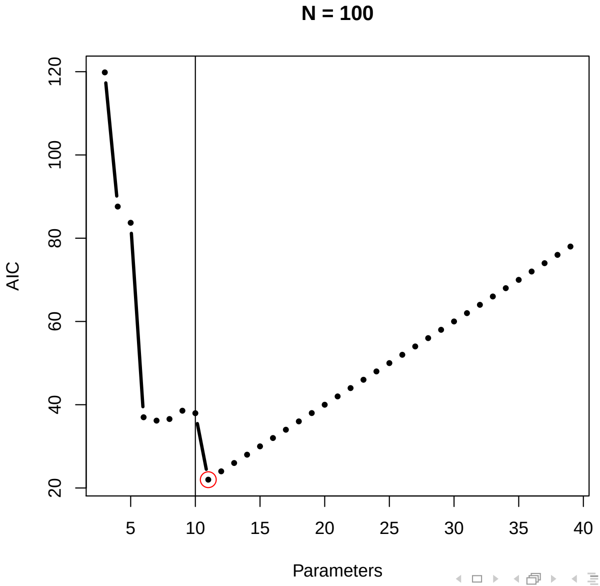


BIC or AIC?

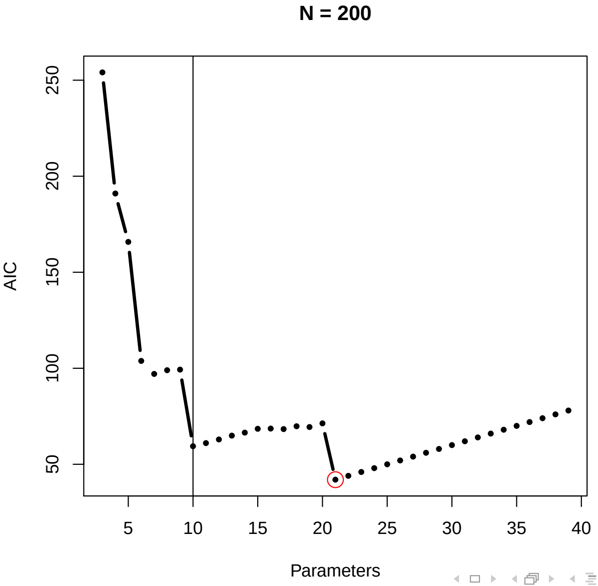


N = 1000
Bayes
1200
1000
800
600
400
5
10
15
20
25
30
35
40
Parameters

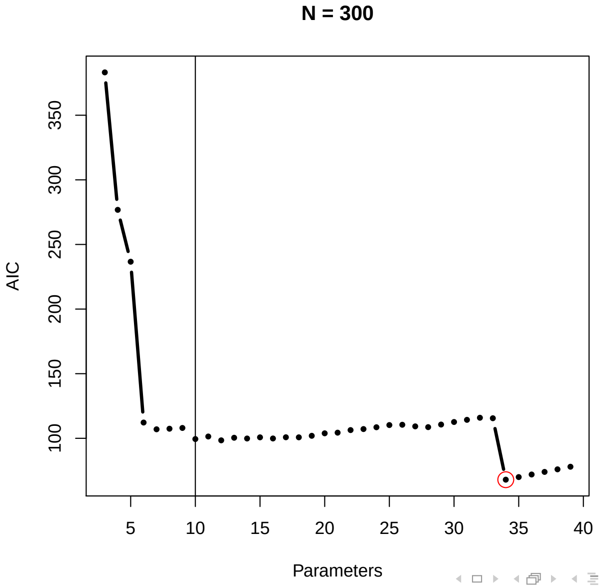
BIC or AIC?



BIC or AIC?

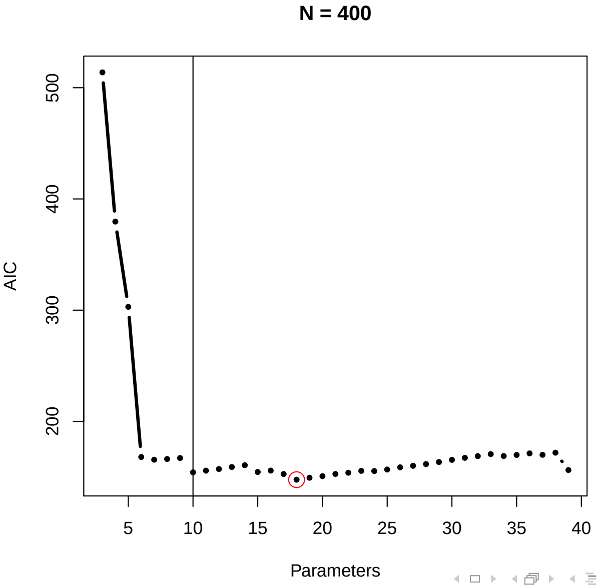


BIC or AIC?

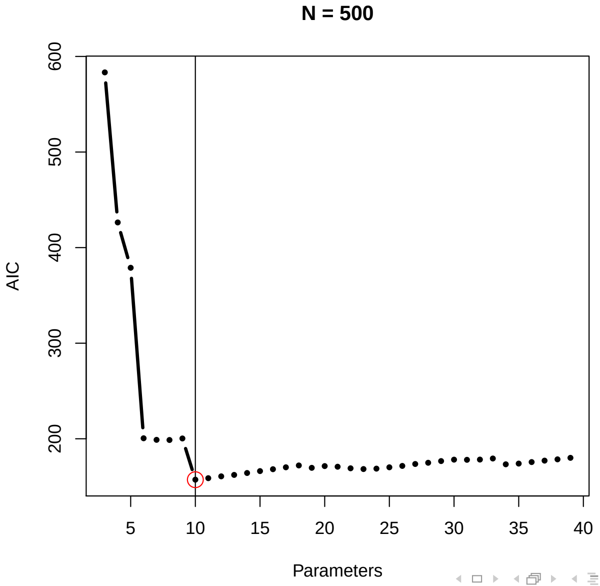


N = 300
AIC
350
300
250
200
150
100
5
10
15
20
25
30
35
40
Parameters

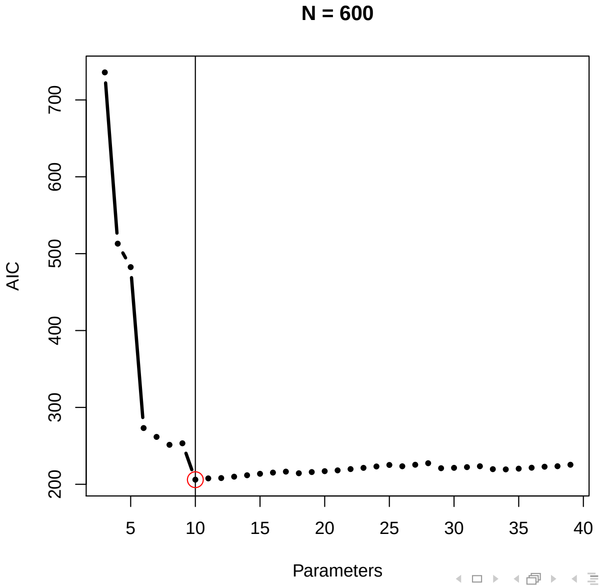
BIC or AIC?



BIC or AIC?

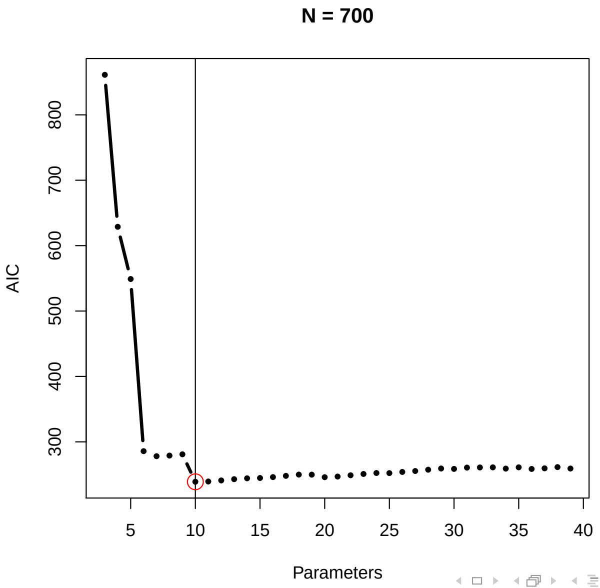


BIC or AIC?



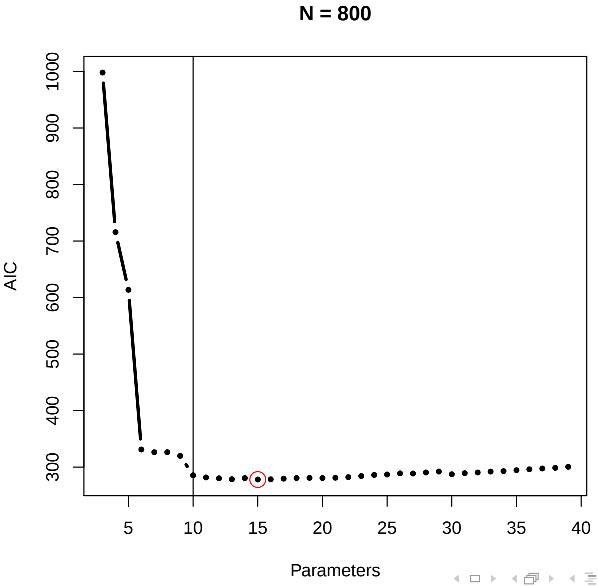
N = 600
AIC
200
300
400
500
600
700
5
10
15
20
25
30
35
40
Parameters

BIC or AIC?

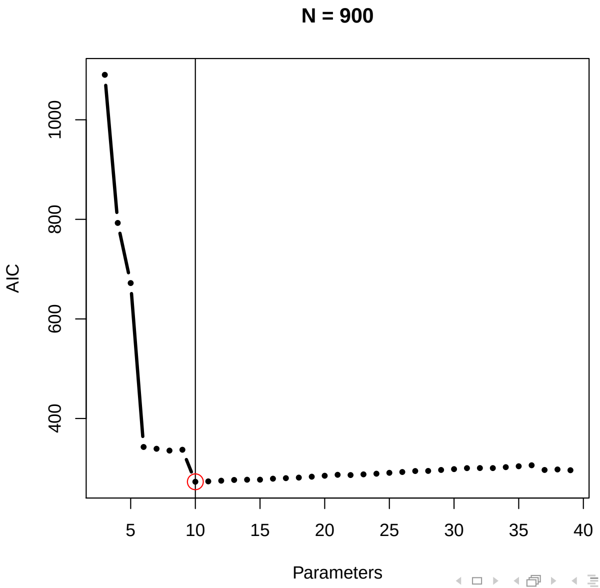


N = 700
AIC
800
700
600
500
400
300
5
10
15
20
25
30
35
40
Parameters

BIC or AIC?


N = 800
AIC
1000
900
800
700
600
500
400
300
5
10
15
20
25
30
35
40
Parameters

BIC or AIC?



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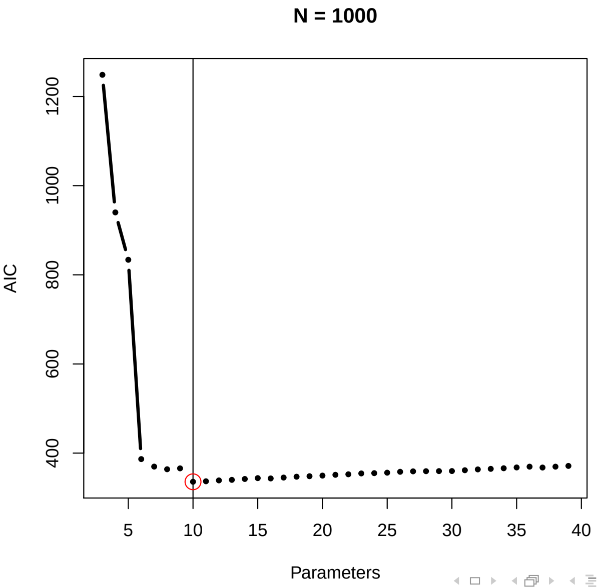

N = 1000
AIC
1200
1000
800
600
400
5
10
15
20
25
30
35
40
Parameters

BIC or AIC?

- BIC
  - Asymptotically consistent if true model is in choice set
  - As $N \rightarrow \infty$ will choose correct model with probability 1 (if available)
  - Small samples$\rightsquigarrow$ overpenalize
- AIC
  - No asymptotic guarantees $\rightsquigarrow$ derivation doesn't require truth in set. (KL-criteria)
  - In large samples$\rightsquigarrow$ favors complexity
  - Small samples$\rightsquigarrow$ avoids over penalization

How Do We Select A Model?

Analytic statistics for selection, include penalty for complexity

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- AIC : Akaka Information Criterion

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Analytic statistics for selection, include penalty for complexity

- AIC : Akaka Information Criterion
- BIC: Bayesian Information Criterion

How Do We Select A Model?

Analytic statistics for selection, include penalty for complexity

- AIC : Akaka Information Criterion
- BIC: Bayesian Information Criterion
- DIC: Deviance Information Criterion

How Do We Select A Model?

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Can work well, but...

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- Rely on specific loss function

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- Rely on asymptotic argument
- Rely on estimate of number of parameters

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- Rely on asymptotic argument
- Rely on estimate of number of parameters
- Extremely model dependent

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Can work well, but...

- Rely on specific loss function
- Rely on asymptotic argument
- Rely on estimate of number of parameters
- Extremely model dependent

Need: general tool for evaluating models, replicates decision problem

Cross-Validation: Some Intuition

Optimal division of data for prediction:

Cross-Validation: Some Intuition

Optimal division of data for prediction:

- Train: build model

Cross-Validation: Some Intuition

Optimal division of data for prediction:

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- Validation: assess model

Cross-Validation: Some Intuition

Optimal division of data for prediction:

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- Validation: assess model
- Test: predict remaining data

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K-fold Cross-validation idea: create many training and test sets.

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- Idea: use observations both in training and test sets

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K-fold Cross-validation idea: create many training and test sets.

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- Each step: use held out data to evaluate performance
- Avoid overfitting and have context specific penalty

Estimates:

$$\text{Error} \quad = \quad \mathrm{E}\left[\mathrm{E}[L(\boldsymbol{Y}, f(\hat{\beta}, \boldsymbol{X}))|\mathcal{T}]\right]$$

Cross-Validation: A How To Guide

Process:

Cross-Validation: A How To Guide

Process:

- Randomly partition data into K groups.

Cross-Validation: A How To Guide

Process:

- Randomly partition data into K groups.
  (Group 1, Group 2, Group3, . . ., Group K)

Cross-Validation: A How To Guide

Process:

- Randomly partition data into K groups.
  (Group 1, Group 2, Group3, . . ., Group K)
- Rotate through groups as follows

Cross-Validation: A How To Guide

Process:

- Randomly partition data into K groups.
  (Group 1, Group 2, Group3, ..., Group K)
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Step	Training	Validation ("Test")

Cross-Validation: A How To Guide

Process:

- Randomly partition data into K groups.
  (Group 1, Group 2, Group3, . . ., Group K)
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Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, . . ., Group K	Group 1

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- Randomly partition data into K groups.
  (Group 1, Group 2, Group3, ..., Group K)
- Rotate through groups as follows

Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, ..., Group K	Group 1
2	Group 1, Group3, Group 4, ..., Group K	Group 2

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Process:

- Randomly partition data into K groups.
  (Group 1, Group 2, Group3, . . ., Group K)
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Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, . . ., Group K	Group 1
2	Group 1, Group3, Group 4, . . ., Group K	Group 2
3	Group 1, Group 2, Group 4, . . ., Group K	Group 3

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Process:

- Randomly partition data into K groups.
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3	Group 1, Group 2, Group 4, ..., Group K	Group 3
⋮	⋮	⋮

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Step	Training	Validation ("Test")
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3	Group 1, Group 2, Group 4, ..., Group K	Group 3
⋮	⋮	⋮
K	Group 1, Group 2, Group 3, ..., Group K - 1	Group K

Cross-Validation: A How To Guide

Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, ..., Group K	Group 1
2	Group 1, Group3, Group 4, ..., Group K	Group 2
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Strategy:

Cross-Validation: A How To Guide

Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, ..., Group K	Group 1
2	Group 1, Group3, Group 4, ..., Group K	Group 2
3	Group 1, Group 2, Group 4, ..., Group K	Group 3
⋮	⋮	⋮
K	Group 1, Group 2, Group 3, ..., Group K - 1	Group K

Strategy:

- Divide data into K groups

Cross-Validation: A How To Guide

Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, ..., Group K	Group 1
2	Group 1, Group3, Group 4, ..., Group K	Group 2
3	Group 1, Group 2, Group 4, ..., Group K	Group 3
$\vdots$	$\vdots$	$\vdots$
K	Group 1, Group 2, Group 3, ..., Group K - 1	Group K

Strategy:

- Divide data into K groups
- Train data on $K-1$ groups. Estimate $\hat{f}^{-K}(\boldsymbol{\beta}, \boldsymbol{X})$

Cross-Validation: A How To Guide

Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, ..., Group K	Group 1
2	Group 1, Group3, Group 4, ..., Group K	Group 2
3	Group 1, Group 2, Group 4, ..., Group K	Group 3
⋮	⋮	⋮
K	Group 1, Group 2, Group 3, ..., Group K - 1	Group K

Strategy:

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- Predict values for K^{th}

Cross-Validation: A How To Guide

Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, . . ., Group K	Group 1
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⋮	⋮	⋮
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Strategy:

- Divide data into K groups
- Train data on $K-1$ groups. Estimate $\hat{f}^{-K}(\boldsymbol{\beta}, \boldsymbol{X})$
- Predict values for K^{th}
- Summarize performance with loss function: $L(\boldsymbol{Y}_i, \hat{f}^{-k}(\boldsymbol{\beta}, \boldsymbol{X}))$

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Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, ..., Group K	Group 1
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    - Mean square error, Absolute error, Prediction error, ...

Cross-Validation: A How To Guide

Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, ..., Group K	Group 1
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  - Mean square error, Absolute error, Prediction error, ...

CV(ind. classification) $= \frac{1}{N}\sum_{i=1}^{N} L(\boldsymbol{Y}_i, f^{-k}(\boldsymbol{\beta}, \boldsymbol{X}_i))$

Cross-Validation: A How To Guide

Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, ..., Group K	Group 1
2	Group 1, Group3, Group 4, ..., Group K	Group 2
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$\vdots$	$\vdots$	$\vdots$
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CV(ind. classification) $= \frac{1}{N}\sum_{i=1}^{N} L(\boldsymbol{Y}_i, f^{-k}(\boldsymbol{\beta}, \boldsymbol{X}_i))$

CV(proportions) $=$
$\frac{1}{K}\sum_{j=1}^{K}$ Mean Square Error Proportions from Group j

Cross-Validation: A How To Guide

Step	Training	Validation ("Test")
1	Group2, Group3, Group 4, ..., Group K	Group 1
2	Group 1, Group3, Group 4, ..., Group K	Group 2
3	Group 1, Group 2, Group 4, ..., Group K	Group 3
⋮	⋮	⋮
K	Group 1, Group 2, Group 3, ..., Group K - 1	Group K

Strategy:

- Divide data into K groups
- Train data on $K-1$ groups. Estimate $\hat{f}^{-K}(\boldsymbol{\beta}, \boldsymbol{X})$
- Predict values for K^{th}
- Summarize performance with loss function: $L(\boldsymbol{Y}_i, \hat{f}^{-k}(\boldsymbol{\beta}, \boldsymbol{X}))$
  - Mean square error, Absolute error, Prediction error, ...

  CV(ind. classification) $= \frac{1}{N}\sum_{i=1}^{N} L(\boldsymbol{Y}_i, f^{-k}(\boldsymbol{\beta}, \boldsymbol{X}_i))$

  CV(proportions) $=$
  $\frac{1}{K}\sum_{j=1}^{K}$ Mean Square Error Proportions from Group j
- Final choice: model with highest CV score

How Do We Select K? (HTF, Section 7.10)

Common values of K

- $K = 5$: Five fold cross validation
- $K = 10$: Ten fold cross validation
- $K = N$: Leave one out cross validation

Considerations:

- How sensitive are inferences to number of coded documents? (HTF, pg 243-244)
- 200 labeled documents
    - $K = N \rightarrow 199$ documents to train,
    - $K = 10 \rightarrow 180$ documents to train
    - $K = 5 \rightarrow 160$ documents to train
- 50 labeled documents
    - $K = N \rightarrow 49$ documents to train,
    - $K = 10 \rightarrow 45$ documents to train
    - $K = 5 \rightarrow 40$ documents to train
- How long will it take to run models?
    - $K-$fold cross validation requires $K\times$ One model run
- What is the correct loss function?

If you cross validate, you really need to cross validate (Section 7.10.2, ESL)

- Use CV to estimate prediction error
- All supervised steps performed in cross-validation
- Underestimate prediction error
- Could lead to selecting lower performing model

Example from Facebook Data

What do people say to legislators? (Franco, Grimmer, and Lee 2017)

1) Example: estimating classification error
   a) Accuracy in legislator posts: 75%
   b) Accuracy in public posts: 66.25%

Credit Claiming (Back to Ridge/Lasso, Grimmer, Westwood, and Messing 2014)

```
library(glmnet)
set.seed(8675309) ##setting seed
folds<- sample(1:10, nrow(dtm), replace=T) ##assigning to fold
out_of_samp<- c() ##collecting the predictions
```

Credit Claiming (Back to Ridge/Lasso, Grimmer, Westwood, and Messing 2014)

```
for(z in 1:10){
train<- which(folds!=z) ##the observations we will use to train the model

test<- which(folds==z) ##the observations we will use to test the model
part1<- cv.glmnet(x = dtm[train,], y = credit[train], alpha = 1, family =
'binomial') ##fitting the LASSO model on the data.
## alpha = 1 -> LASSO
## alpha = 0 -> RIDGE
## 0<alpha<1 -> Elastic-Net
out_of_samp[test]<- predict(part1, newx= dtm[test,], s = part1$lambda.min,
type ='class') ##predicting the labels
print(z) ##printing the labels
}
conf_table<- table(out_of_samp, credit) ##calculating the confusion table
> round(sum(diag(conf_table))/len(credit), 3)
[1] 0.844
```

Generalized Cross Validation and Ridge Regression

In some special cases there are analytic solutions:

Generalized Cross Validation and Ridge Regression

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$$\boldsymbol{\beta}^{\text{Ridge}} = \left(\boldsymbol{X}^{'}\boldsymbol{X} + \lambda \boldsymbol{I}_J\right)^{-1}\boldsymbol{X}^{'}\boldsymbol{Y}$$

Generalized Cross Validation and Ridge Regression

In some special cases there are analytic solutions:

$$
\begin{aligned}
\boldsymbol{\beta}^{\text{Ridge}} &= \left(\boldsymbol{X}^{'}\boldsymbol{X} + \lambda \boldsymbol{I}_J\right)^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} \\
\widehat{\boldsymbol{Y}} &= \boldsymbol{X}(\beta)^{\text{Ridge}}
\end{aligned}
$$

Generalized Cross Validation and Ridge Regression

In some special cases there are analytic solutions:

$$
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\widehat{\boldsymbol{Y}} &= \boldsymbol{X}(\boldsymbol{\beta})^{\text{Ridge}} \\
&= \underbrace{\boldsymbol{X}\left(\boldsymbol{X}^{'}\boldsymbol{X} + \lambda \boldsymbol{I}_J\right)^{-1}\boldsymbol{X}^{'}}_{\text{Hat Matrix}}\boldsymbol{Y}
\end{aligned}
$$

Generalized Cross Validation and Ridge Regression

In some special cases there are analytic solutions:

$$\begin{aligned}
\beta^{\text{Ridge}} &= \left(\boldsymbol{X}^{'}\boldsymbol{X} + \lambda \boldsymbol{I}_J\right)^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} \\
\widehat{\boldsymbol{Y}} &= \boldsymbol{X}(\beta)^{\text{Ridge}} \\
&= \underbrace{\boldsymbol{X}\left(\boldsymbol{X}^{'}\boldsymbol{X} + \lambda \boldsymbol{I}_J\right)^{-1}\boldsymbol{X}^{'}}_{\text{Hat Matrix}}\boldsymbol{Y} \\
\widehat{\boldsymbol{Y}} &= \underbrace{\boldsymbol{H}}_{\text{Smoother Matrix}}\boldsymbol{Y}
\end{aligned}$$

Generalized Cross Validation and Ridge Regression

In some special cases there are analytic solutions:

$$
\begin{aligned}
\beta^{\text{Ridge}} &= \left(\boldsymbol{X}^{'}\boldsymbol{X} + \lambda \boldsymbol{I}_J\right)^{-1} \boldsymbol{X}^{'}\boldsymbol{Y} \\
\widehat{\boldsymbol{Y}} &= \boldsymbol{X}(\beta)^{\text{Ridge}} \\
&= \underbrace{\boldsymbol{X}\left(\boldsymbol{X}^{'}\boldsymbol{X} + \lambda \boldsymbol{I}_J\right)^{-1} \boldsymbol{X}^{'}}_{\text{Hat Matrix}}\boldsymbol{Y} \\
\widehat{\boldsymbol{Y}} &= \underbrace{\boldsymbol{H}}_{\text{Smoother Matrix}} \quad \boldsymbol{Y}
\end{aligned}
$$

Generalized Cross Validation and Ridge Regression

Why do we care?

Generalized Cross Validation and Ridge Regression

Why do we care?
Leave one out cross validation

Generalized Cross Validation and Ridge Regression

Why do we care?
Leave one out cross validation

$$\text{Cross Validation}(1) = \frac{1}{N}\sum_{i=1}^{N}(Y_i - f(\boldsymbol{X}_{-i}, \boldsymbol{Y}_{-i}, \lambda, \hat{\beta}))^2$$

Generalized Cross Validation and Ridge Regression

Why do we care?
Leave one out cross validation

$$
\begin{aligned}
\text{Cross Validation}(1) &= \frac{1}{N}\sum_{i=1}^{N}(Y_i - f(\boldsymbol{X}_{-i}, \boldsymbol{Y}_{-i}, \lambda, \hat{\beta}))^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{Y_i - f(\boldsymbol{X}, \boldsymbol{Y}, \lambda, \hat{\beta})}{1 - H_{ii}}\right)^2
\end{aligned}
$$

Generalized Cross Validation and Ridge Regression

Calculating $\boldsymbol{H}$ can be computationally expensive

Generalized Cross Validation and Ridge Regression

Calculating $\boldsymbol{H}$ can be computationally expensive

- Trace($\boldsymbol{H}$) $\equiv$ Tr($\boldsymbol{H}$) $= \sum_{i=1}^{N} H_{ii}$

Generalized Cross Validation and Ridge Regression

Calling $\boldsymbol{H}$ can be computationally expensive

- Trace($\boldsymbol{H}$) $\equiv$ Tr($\boldsymbol{H}$) $= \sum_{i=1}^{N} H_{ii}$
- Tr($\boldsymbol{H}$) = Effective number of parameters (class regression = number of independent variables + 1)

Generalized Cross Validation and Ridge Regression

Calculating $\boldsymbol{H}$ can be computationally expensive

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- For Ridge regression:

Generalized Cross Validation and Ridge Regression

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- Tr($\boldsymbol{H}$) = Effective number of parameters (class regression = number of independent variables + 1)
- For Ridge regression:

$$\text{Tr}(\boldsymbol{H}) = \sum_{j=1}^{J} \frac{\lambda_j}{\lambda_j + \underbrace{\lambda}_{\text{Penalty}}}$$

Generalized Cross Validation and Ridge Regression

Calculating $\boldsymbol{H}$ can be computationally expensive

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