Standard Model Lagrangian

Including Neutrino Mass Terms

Corey Skinner

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$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}tr\left(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}\right) - \frac{1}{2}tr\left(\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}\right) && \text{(U(1), SU(2), and SU(3) gauge terms)}\\
&+ (\bar{\nu}_L, \bar{e}_L)\,\tilde{\sigma}^{\mu} i D_{\mu}\left(\begin{smallmatrix}\nu_L\\ e_L\end{smallmatrix}\right) + \bar{e}_R\sigma^{\mu} i D_{\mu} e_R + \bar{\nu}_R\sigma^{\mu} i D_{\mu}\nu_R + \text{(h.c.)} && \text{(lepton dynamical term)}\\
&- \frac{\sqrt{2}}{v}\left[(\bar{\nu}_L, \bar{e}_L)\,\phi M^e e_R + \bar{e}_R\bar{M}^e\bar{\phi}\left(\begin{smallmatrix}\nu_L\\ e_L\end{smallmatrix}\right)\right] && \text{(electron, muon, tauon mass term)}\\
&- \frac{\sqrt{2}}{v}\left[(-\bar{e}_L, \bar{\nu}_L)\,\phi^{*} M^{\nu}\nu_R + \bar{\nu}_R\bar{M}^{\nu}\phi^{T}\left(\begin{smallmatrix}-e_L\\ \nu_L\end{smallmatrix}\right)\right] && \text{(neutrino mass term)}\\
&+ \left(\bar{u}_L, \bar{d}_L\right)\tilde{\sigma}^{\mu} i D_{\mu}\left(\begin{smallmatrix}u_L\\ d_L\end{smallmatrix}\right) + \bar{u}_R\sigma^{\mu} i D_{\mu} u_R + \bar{d}_R\sigma^{\mu} i D_{\mu} d_R + \text{(h.c.)} && \text{(quark dynamical term)}\\
&- \frac{\sqrt{2}}{v}\left[\left(\bar{u}_L, \bar{d}_L\right)\phi M^d d_R + \bar{d}_R\bar{M}^d\bar{\phi}\left(\begin{smallmatrix}u_L\\ d_L\end{smallmatrix}\right)\right] && \text{(down, strange, bottom mass term)}\\
&- \frac{\sqrt{2}}{v}\left[\left(-\bar{d}_L, \bar{u}_L\right)\phi^{*} M^u u_R + \bar{u}_R\bar{M}^u\phi^{T}\left(\begin{smallmatrix}-d_L\\ u_L\end{smallmatrix}\right)\right] && \text{(up, charmed, top mass term)}\\
&+ \overline{\left(D_{\mu}\phi\right)}D^{\mu}\phi - \frac{m_h^2\left[\bar{\phi}\phi - \frac{v^2}{2}\right]^2}{2v^2} && \text{(Higgs dynamical and mass term)}
\end{aligned}
$$

Where (h.c.) means Hermitian conjugate of preceding terms, $\bar{\phi} = \text{(h.c.)}\phi = \phi^{\dagger} = \phi^{*T}$, and the derivative operators are:

$$D_{\mu}\left(\begin{smallmatrix}\nu_L\\ e_L\end{smallmatrix}\right) = \left[\partial_{\mu} - \frac{ig_1}{2}B_{\mu} + \frac{ig_2}{2}\mathbf{W}_{\mu}\right]\left(\begin{smallmatrix}\nu_L\\ e_L\end{smallmatrix}\right)$$

$$D_{\mu}\left(\begin{smallmatrix}u_L\\ d_L\end{smallmatrix}\right) = \left[\partial_{\mu} - \frac{ig_1}{6}B_{\mu} + \frac{ig_2}{2}\mathbf{W}_{\mu} + ig\mathbf{G}_{\mu}\right]\left(\begin{smallmatrix}u_L\\ d_L\end{smallmatrix}\right)$$

$$D_\mu \nu_R = \partial_\mu \nu_R$$

$$D_\mu e_R = \left[\partial_\mu - i g_1 B_\mu\right] e_R$$

$$D_\mu u_R = \left[\partial_\mu + \frac{i 2 g_1}{3} B_\mu + i g \mathbf{G}_\mu\right] u_r$$

$$D_\mu d_R = \left[\partial_\mu - \frac{i g_1}{3} B_\mu + i g \mathbf{G}_\mu\right] d_R$$

$$D_\mu \phi = \left[\partial_\mu + \frac{i g_1}{2} B_\mu + \frac{i g_2}{2} \mathbf{W}_\mu\right] \phi$$

ϕ is a 2-component comples Higgs field. Since $\mathcal{L}$ is SU(2) gauge invariant, a gauge can be chosen so ϕ has the form:

$$\phi^T = \frac{(0, \nu + h)}{\sqrt{2}}$$

$$< \phi >_0^T = (\text{expectation value of} \phi) = \frac{(0, \nu)}{\sqrt{2}}$$

Where ν is a real constant such that $\mathcal{L}_\phi = \overline{\left(\partial_\mu \phi\right)} \partial^\mu \phi - \frac{\mu\phi - m_h^2 \left[\bar{\phi}\phi - \frac{\nu^2}{2}\right]^2}{2\nu^2}$ is minimized, and h is a residual Higgs field. B_μ, $\mathbf{W}_\mu$, and $\mathbf{G}_\mu$ are the gauge boson vector potentials, and $\mathbf{W}_\mu$ and $\mathbf{G}_\mu$ are composed of 2×3 and 3×3 traceless Hermitian matrices. Their associated field tensors are:

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\mathbf{W}_{\mu\nu} = \partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + i g_2 \frac{\left(\mathbf{W}_\mu \mathbf{W}_\nu - \mathbf{W}_\nu \mathbf{W}_\mu\right)}{2}$$

$$\mathbf{G}_{\mu\nu} = \partial_\mu \mathbf{G}_\nu - \partial_\nu \mathbf{G}_\mu + \left(\mathbf{G}_\mu \mathbf{G}_\nu - \mathbf{G}_\nu \mathbf{G}_\mu\right)$$

The non-matrix A_μ, Z_μ, $W_\mu^\pm$ bosons are mixtures of $\mathbf{W}_\mu$ and B_μ components, according to the weak mixing angle θ_w:

$$A_\mu = W_{11\mu} \sin(\theta_w) + B_\mu \cos(\theta_w)$$

$$Z_\mu = W_{11\mu}\cos(\theta_w) - B_\mu \sin(\theta_w)$$

$$W_\mu^+ = W_\mu^{-*} = \frac{W_{12\mu}}{\sqrt{2}}$$

$$B_\mu = A_\mu \cos(\theta_w) - Z_\mu \sin(\theta_w)$$

$$W_{11\mu} = -W_{22\mu} = A_\mu \sin(\theta_w) + Z_\mu \cos(\theta_w)$$

$$W_{12\mu} = W_{21\mu}^* = \sqrt{2} W_\mu^+$$

$$\sin^2(\theta_w) = .2325(4)$$

The fermions include the leptons e_R, e_L, ν_R, ν_L and quarks u_R, u_L, d_R, d_L. They all have implicit 3-component generation indices, $e_i = (e, \mu, \tau)$, $\nu_i = \left(\nu_e, \nu_\mu, \nu_\tau\right)$, $u_i = (u, c, t)$, $d_i = (d, s, b)$, which contract into the fermion mass matrices $M_{ij}^e, M_{ij}^\nu, M_{ij}^u, M_{ij}^d$, and implicit 2-component indices which contract into the Pauli matrices:

$$\sigma^\mu = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$\tilde{\sigma}^\mu = \left[\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3\right]$$

$$tr\left(\sigma^i\right) = 0$$

$$\sigma^{\mu\dagger} = \sigma^\mu$$

$$tr(\sigma^\mu \sigma^\nu) = 2\delta^{\mu\nu}$$

The quarks also have implicit 3-component color indices which contract into $\mathbf{G}_\mu$. So $\mathcal{L}$ really has implicit sums over 3-component generation indices, 2-component Pauli indices, 3-component color indices in the quark terms, and 2-component SU(2) indices in $(\bar{\nu}_L, \bar{e}_L), \left(\bar{u}_L, \bar{d}_L\right), (-\bar{e}_L, \bar{\nu}_L), \left(-\bar{d}_L, \bar{u}_L\right), \bar{\phi}, \mathbf{W}_\mu, \left(\begin{smallmatrix} \nu_L \\ e_L \end{smallmatrix}\right), \left(\begin{smallmatrix} u_L \\ d_L \end{smallmatrix}\right), \left(\begin{smallmatrix} -e_L \\ \nu_L \end{smallmatrix}\right), \left(\begin{smallmatrix} -d_L \\ u_L \end{smallmatrix}\right), \phi$.

The electroweak and strong coupling constants, Higgs vacuum expectation value (VEV), and Higgs mass are:

$$g_1 = \frac{e}{\cos(\theta_w)}$$

$$g_2 = \frac{e}{\sin(\theta_w)}$$

$$g > 6.5e = g\left(m_\tau^2\right)$$

$$v = 246 \text{ GeV (PDG)} \approx \sqrt{2} \cdot 180 \text{ GeV (CG)}$$

$$m_h = 125.02(30) \text{ GeV}$$

Where $e = \sqrt{4\pi\alpha\hbar c} = \sqrt{\frac{4\pi}{137}}$ in natural units. Rewriting some things provides the mass of A_μ, Z_μ, $W_\mu^\pm$:

$$-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}tr\left(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}\right) = -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}\mathcal{W}^-_{\mu\nu}\mathcal{W}^{+\mu\nu} + \text{(higher order terms)}$$

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$W^\pm_{\mu\nu} = D_\mu W\nu^\pm - D_\nu W^\pm_\mu$$

$$D_\mu W^\pm_\nu = \left[\partial_\mu \pm ieA_\mu\right] W^\pm_\nu$$

$$D_\mu < \phi >_0 = \frac{iv}{\sqrt{2}} \begin{pmatrix} \frac{g_2 W_{12\mu}}{2} \\ \frac{g_1 B_\mu}{2} + \frac{g_2 W_{22\mu}}{2} \end{pmatrix} = \frac{ig_2 v}{2} \begin{pmatrix} \frac{W_{12\mu}}{\sqrt{2}} \\ \frac{\left(B_\mu \sin(\theta_w)/\cos(\theta_w) + W_{22\mu}\right)}{\sqrt{2}} \end{pmatrix} = \frac{ig_2 v}{2} \begin{pmatrix} W^+_\mu \\ \frac{-Z_\mu}{\sqrt{2}\cos(\theta_w)} \end{pmatrix}$$

$$m_A = 0$$

$$m_{W^\pm} = \frac{g_2 v}{2} = 80.425(38)\text{ GeV}$$

$$m_Z = \frac{g_2 v}{2\cos(\theta_w)} = 91.1876(21)\text{ GeV}$$

Ordinary 4-component Dirac fermions are composed of the left and right handed 2-component fields:

$$
\begin{aligned}
&e = \begin{pmatrix} e_{L1} \\ e_{R1} \end{pmatrix}, \nu_e = \begin{pmatrix} \nu_{L1} \\ \nu_{R1} \end{pmatrix}, u = \begin{pmatrix} u_{L1} \\ u_{R1} \end{pmatrix}, d = \begin{pmatrix} d_{L1} \\ d_{R1} \end{pmatrix}, && \text{(electron, electron neutrino, up and down quark)} \\
&\mu = \begin{pmatrix} e_{L2} \\ e_{R2} \end{pmatrix}, \nu_\mu = \begin{pmatrix} \nu_{L2} \\ \nu_{R2} \end{pmatrix}, c = \begin{pmatrix} u_{L2} \\ u_{R2} \end{pmatrix}, s = \begin{pmatrix} d_{L2} \\ d_{R2} \end{pmatrix}, && \text{(muon, muon neutrino, charm, and strange quark)} \\
&\tau = \begin{pmatrix} e_{L3} \\ e_{R3} \end{pmatrix}, \nu_\tau = \begin{pmatrix} \nu_{L3} \\ \nu_{R3} \end{pmatrix}, t = \begin{pmatrix} u_{L3} \\ u_{R3} \end{pmatrix}, b = \begin{pmatrix} d_{L3} \\ d_{R3} \end{pmatrix}, && \text{(tauon, tauon neutrino, top and bottom quark)} \\
&\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \text{ where } \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2Ig^{\mu\nu} && \text{(Dirac gamma matrices in chiral representation)}
\end{aligned}
$$

The corresponding antiparticles are related to the particles according to $\psi^e = -i\gamma^2\psi^*$ or $\psi_L^e = -i\sigma^2\psi_R^*$, $\psi_R^e = i\sigma^2\psi_L^*$. The fermion charges are the coefficients of A_μ when (8,10) are substituted into either the left or right handed derivative operators (2-4). The fermion masses are the singular values of the 3×3 fermion mass matrices M^ν, M^e, M^u, M^d:

$$M^e = \mathbf{U}_L^{e\dagger}\begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}\mathbf{U}_R^e$$

$$M^\nu = \mathbf{U}_L^{\nu\dagger}\begin{pmatrix} m_{\nu_e} & 0 & 0 \\ 0 & m_{\nu_\mu} & 0 \\ 0 & 0 & m_{\nu_\tau} \end{pmatrix}\mathbf{U}_R^\nu$$

$$M^u = \mathbf{U}_L^{u\dagger}\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}\mathbf{U}_R^u$$

$$M^d = \mathbf{U}_L^{d\dagger}\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}\mathbf{U}_R^d$$

$$
\begin{array}{llll}
m_e = .510998910(13)\text{ MeV}, & m_{\nu_e} \sim .001 - 2\text{ eV}, & m_u = 1.7 - 3.1\text{ MeV}, & m_d = 4.1 - 5.7\text{ MeV} \\
m_\mu = 105.658367(4)\text{ MeV}, & m_{\nu_\mu} \sim .001 - 2\text{ eV}, & m_c = 1.18 - 1.34\text{ GeV}, & m_s = 80 - 130\text{ MeV} \\
m_\tau = 1776.84(17)\text{ MeV}, & m_{\nu_\tau} \sim .001 - 2\text{ eV}, & m_t = 171.4 - 174.4\text{ GeV}, & m_b = 4.13 - 4.37\text{ GeV}
\end{array}
$$

Where the **U**s are 3×3 unitary matrices $\left(\mathbf{U}^{-1} = \mathbf{U}^\dagger\right)$. Consequently the "true fermions" with definite masses are actually linear combinations of those in $\mathcal{L}$, or conversely the fermions in $\mathcal{L}$ are linear combinations of the true fermions:

$$
e'_L = \mathbf{U}^e_L e_L\ ,\ e'_R = \mathbf{U}^e_R e_R\ ,\ \nu'_L = \mathbf{U}^\nu_L \nu_L\ ,\ \nu'_R = \mathbf{U}^\nu_R \nu_R\ ,\ u'_L = \mathbf{U}^u_L u_L\ ,\ u'_R = \mathbf{U}^u_R u_R\ ,\ d'_L = \mathbf{U}^d_L d_L\ ,\ d'_R = \mathbf{U}^d_R d_R
$$

$$
e_L = \mathbf{U}^{e\dagger}_L e'_L\ ,\ e_R = \mathbf{U}^{e\dagger}_R e'_R\ ,\ \nu_L = \mathbf{U}^{\nu\dagger}_L \nu'_L\ ,\ \nu_R = \mathbf{U}^{\nu\dagger}_R \nu'_R\ ,\ u_L = \mathbf{U}^{u\dagger}_L u'_L\ ,\ u_R = \mathbf{U}^{u\dagger}_R u'_R\ ,\ d_L = \mathbf{U}^{d\dagger}_L d'_L\ ,\ d_R = \mathbf{U}^{d\dagger}_R d'_R
$$

When $\mathcal{L}$ is written in terms of the true fermions, the **U**s fall out except in $\bar{u}'_L \mathbf{U}^u_L \tilde{\sigma}^\mu W^\pm_\mu \mathbf{U}^{d\dagger}_L d'_L$ and $\bar{\nu}'_L \mathbf{U}^\nu_L si\tilde{g}ma^\mu W^\pm_\mu \mathbf{U}^{e\dagger}_L e'_L$. Because of this, and some absorption of constants into the fermion fields, all the parameters in the **U**s are contained in only four components of the Cabibbo-Kobayashi-Maskawa matrix $\mathbf{V}^q = \mathbf{U}^u_L \mathbf{U}^{d\dagger}_L$ and four components of the Pontecorvo-Maki-Nakagawa-Sakata matrix $\mathbf{V}^l = \mathbf{U}^\nu_L \mathbf{U}^{e\dagger}_L$. The unitary matrices $\mathbf{V}^q$ and $\mathbf{V}^l$ are often parameterized as:

$$
\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} e^{\frac{-i\delta}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{i\delta}{2}} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} e^{\frac{i\delta}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{-i\delta}{2}} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
c_j = \sqrt{1 - s_j^2}
$$

$$
\begin{array}{llll}
\delta^q = 69(4)\text{ deg}, & s^q_{12} = 0.2253(7), & s^q_{23} = 0.041(1), & s^q_{13} = 0.0035(2), \\
\delta^l = ?, & s^l_{12} = 0.560(16), & s^l_{23} = 0.7(1), & s^l_{13} = 0.153(28)
\end{array}
$$

$\mathcal{L}$ is invariant under a U(1) ⊗ SU(2) gauge transformation with $U^{-1} = U^\dagger$, $detU = 1$, θ real:

$$\mathbf{W}_\mu \to U\mathbf{W}_\mu U^\dagger - \left(\frac{2i}{g_2}\right) U\partial_\mu U^\dagger$$

$$\mathbf{W}_{\mu\nu} \to U\mathbf{W}_{\mu\nu} U^\dagger$$

$$B_\mu \to B_\mu + \left(\frac{2}{g_1}\right)\partial_\mu \theta$$

$$B_{\mu\nu} \to B_{\mu\nu}$$

$$\phi \to e^{-i\theta} U\phi$$

$$\left(\begin{smallmatrix} \nu_L \\ e_L \end{smallmatrix}\right) \to e^{i\theta} U \left(\begin{smallmatrix} \nu_L \\ e_L \end{smallmatrix}\right)$$

$$\left(\begin{smallmatrix} u_L \\ d_L \end{smallmatrix}\right) \to e^{-\frac{i\theta}{3}} U \left(\begin{smallmatrix} u_L \\ d_L \end{smallmatrix}\right)$$

$$\nu_R \to \nu_R \qquad u_R \to e^{-\frac{4i\theta}{3}} u_R$$

$$e_R \to e^{2i\theta} e_R \qquad d_R \to e^{\frac{2i\theta}{3}} d_R$$

And under an SU(3) gauge transformation with $V^{-1} = V^\dagger$, $det V = 1$, and:

$$\mathbf{G}_\mu \to V\mathbf{G}_\mu V^\dagger - \left(\frac{i}{g}\right) V\partial_\mu V^\dagger$$

$$\mathbf{G}_{\mu\nu} \to V\mathbf{G}_{\mu\nu} V^\dagger$$

$$u_L \to V u_L \qquad d_L \to V d_L$$

$$u_R \to V u_R \qquad d_R \to V d_R$$