Standard Model Lagrangian

Including Neutrino Mass Terms

Corey Skinner

- From *An Introduction to the Standard Model of Particle Physics, 2nd Edition*, W.N. Cottingham and D.A. Greenwood, Cambridge University Press, Cambridge, 2007

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}tr\left(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}\right) - \frac{1}{2}tr\left(\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}\right) && \text{(U(1), SU(2), and SU(3) gauge terms)} \\
& + (\bar{\nu}_L, \bar{e}_L)\,\tilde{\sigma}^\mu i D_\mu \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \bar{e}_R \sigma^\mu i D_\mu e_R + \bar{\nu}_R \sigma^\mu i D_\mu \nu_R + \text{(h.c.)} && \text{(lepton dynamical term)} \\
& - \frac{\sqrt{2}}{v}\left[(\bar{\nu}_L, \bar{e}_L)\,\phi M^e e_R + \bar{e}_R \bar{M}^e \bar{\phi} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}\right] && \text{(electron, muon, tauon mass term)} \\
& - \frac{\sqrt{2}}{v}\left[(-\bar{e}_L, \bar{\nu}_L)\,\phi^* M^\nu \nu_R + \bar{\nu}_R \bar{M}^\nu \phi^T \begin{pmatrix} -e_L \\ \nu_L \end{pmatrix}\right] && \text{(neutrino mass term)} \\
& + \left(\bar{u}_L, \bar{d}_L\right)\tilde{\sigma}^\mu i D_\mu \begin{pmatrix} u_L \\ d_L \end{pmatrix} + \bar{u}_R \sigma^\mu i D_\mu u_R + \bar{d}_R \sigma^\mu i D_\mu d_R + \text{(h.c.)} && \text{(quark dynamical term)} \\
& - \frac{\sqrt{2}}{v}\left[\left(\bar{u}_L, \bar{d}_L\right)\phi M^d d_R + \bar{d}_R \bar{M}^d \bar{\phi} \begin{pmatrix} u_L \\ d_L \end{pmatrix}\right] && \text{(down, strange, bottom mass term)} \\
& - \frac{\sqrt{2}}{v}\left[\left(-\bar{d}_L, \bar{u}_L\right)\phi^* M^u u_R + \bar{u}_R \bar{M}^u \phi^T \begin{pmatrix} -d_L \\ u_L \end{pmatrix}\right] && \text{(up, charm, top mass term)} \\
& + \overline{\left(D_\mu \phi\right)} D^\mu \phi - \frac{m_h^2\left[\bar{\phi}\phi - \frac{v^2}{2}\right]^2}{2v^2} && \text{(Higgs dynamical and mass term)}
\end{aligned}
$$

Where (h.c.) means Hermitian conjugate of preceding terms, $\bar{\phi} = \text{(h.c.)}\phi = \phi^\dagger = \phi^{*T}$, and the derivative operators are:

$$
D_\mu \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \left[\partial_\mu - \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu\right]\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}
$$

$$
D_\mu \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \left[\partial_\mu - \frac{ig_1}{6}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu + ig\mathbf{G}_\mu\right]\begin{pmatrix} u_L \\ d_L \end{pmatrix}
$$

$$D_\mu \nu_R = \partial_\mu \nu_R$$

$$D_\mu e_R = \left[\partial_\mu - i g_1 B_\mu\right] e_R$$

$$D_\mu u_R = \left[\partial_\mu + \frac{i2g_1}{3} B_\mu + i g \mathbf{G}_\mu\right] u_r$$

$$D_\mu d_R = \left[\partial_\mu - \frac{i g_1}{3} B_\mu + i g \mathbf{G}_\mu\right] d_R$$

$$D_\mu \phi = \left[\partial_\mu + \frac{i g_1}{2} B_\mu + \frac{i g_2}{2} \mathbf{W}_\mu\right] \phi$$

ϕ is a 2-component complex Higgs field. Since $\mathcal{L}$ is SU(2) gauge invariant, a gauge can be chosen so ϕ has the form:

$$\phi^T = \frac{(0, \nu + h)}{\sqrt{2}}$$

$$< \phi >_0^T = (\text{expectation value of} \phi) = \frac{(0, \nu)}{\sqrt{2}}$$

Where ν is a real constant such that $\mathcal{L}_\phi = \overline{\left(\partial_\mu \phi\right)} \partial^\mu \phi - \frac{\mu\phi - m_h^2\left[\bar{\phi}\phi - \frac{\nu^2}{2}\right]^2}{2\nu^2}$ is minimized, and h is a residual Higgs field. B_μ, $\mathbf{W}_\mu$, and $\mathbf{G}_\mu$ are the gauge boson vector potentials, and $\mathbf{W}_\mu$ and $\mathbf{G}_\mu$ are composed of 2×3 and 3×3 traceless Hermitian matrices. Their associated field tensors are:

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\mathbf{W}_{\mu\nu} = \partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + i g_2 \frac{\left(\mathbf{W}_\mu \mathbf{W}_\nu - \mathbf{W}_\nu \mathbf{W}_\mu\right)}{2}$$

$$\mathbf{G}_{\mu\nu} = \partial_\mu \mathbf{G}_\nu - \partial_\nu \mathbf{G}_\mu + \left(\mathbf{G}_\mu \mathbf{G}_\nu - \mathbf{G}_\nu \mathbf{G}_\mu\right)$$

The non-matrix A_μ, Z_μ, $W_\mu^\pm$ bosons are mixtures of $\mathbf{W}_\mu$ and B_μ components, according to the weak mixing angle θ_w:

$$A_\mu = W_{11\mu} \sin(\theta_w) + B_\mu \cos(\theta_w)$$

$$Z_\mu = W_{11\mu}\cos(\theta_w) - B_\mu \sin(\theta_w)$$

$$W_\mu^+ = W_\mu^{-*} = \frac{W_{12\mu}}{\sqrt{2}}$$

$$B_\mu = A_\mu \cos(\theta_w) - Z_\mu \sin(\theta_w)$$

$$W_{11\mu} = -W_{22\mu} = A_\mu \sin(\theta_w) + Z_\mu \cos(\theta_w)$$

$$W_{12\mu} = W_{21\mu}^* = \sqrt{2} W_\mu^+$$

$$\sin^2(\theta_w) = .2325(4)$$

The fermions include the leptons e_R, e_L, ν_R, ν_L and quarks u_R, u_L, d_R, d_L. They all have implicit 3-component generation indices, $e_i = (e, \mu, \tau)$, $\nu_i = \left(\nu_e, \nu_\mu, \nu_\tau\right)$, $u_i = (u, c, t)$, $d_i = (d, s, b)$, which contract into the fermion mass matrices $M_{ij}^e, M_{ij}^\nu, M_{ij}^u, M_{ij}^d$, and implicit 2-component indices which contract into the Pauli matrices:

$$\sigma^\mu = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$\tilde{\sigma}^\mu = \left[\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3\right]$$

$$tr\left(\sigma^i\right) = 0$$

$$\sigma^{\mu\dagger} = \sigma^\mu$$

$$tr(\sigma^\mu \sigma^\nu) = 2\delta^{\mu\nu}$$

The quarks also have implicit 3-component color indices which contract into $\mathbf{G}_\mu$. So $\mathcal{L}$ really has implicit sums over 3-component generation indices, 2-component Pauli indices, 3-component color indices in the quark terms, and 2-component SU(2) indices in $(\bar{\nu}_L, \bar{e}_L), \left(\bar{u}_L, \bar{d}_L\right), (-\bar{e}_L, \bar{\nu}_L), \left(-\bar{d}_L, \bar{u}_L\right), \bar{\phi}, \mathbf{W}_\mu, \left(\begin{smallmatrix} \nu_L \\ e_L \end{smallmatrix}\right), \left(\begin{smallmatrix} u_L \\ d_L \end{smallmatrix}\right), \left(\begin{smallmatrix} -e_L \\ \nu_L \end{smallmatrix}\right), \left(\begin{smallmatrix} -d_L \\ u_L \end{smallmatrix}\right), \phi$.

The electroweak and strong coupling constants, Higgs vacuum expectation value (VEV), and Higgs mass are:

$$g_1 = \frac{e}{\cos(\theta_w)}$$

$$g_2 = \frac{e}{\sin(\theta_w)}$$

$$g > 6.5e = g\left(m_\tau^2\right)$$

$$v = 246 \text{ GeV (PDG)} \approx \sqrt{2} \cdot 180 \text{ GeV (CG)}$$

$$m_h = 125.02(30) \text{ GeV}$$

Where $e = \sqrt{4\pi\alpha\hbar c} = \sqrt{\frac{4\pi}{137}}$ in natural units. Rewriting some things provides the mass of A_μ, Z_μ, $W_\mu^\pm$:

$$-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}tr\left(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}\right) = -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}\mathcal{W}_{\mu\nu}^{-}\mathcal{W}^{+\mu\nu} + (\text{higher order terms})$$

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$W_{\mu\nu}^\pm = D_\mu W\nu^\pm - D_\nu W_\mu^\pm$$

$$D_\mu W_\nu^\pm = \left[\partial_\mu \pm ieA_\mu\right] W_\nu^\pm$$

$$D_\mu < \phi >_0 = \frac{iv}{\sqrt{2}} \begin{pmatrix} \frac{g_2 W_{12\mu}}{2} \\ \frac{g_1 B_\mu}{2} + \frac{g_2 W_{22\mu}}{2} \end{pmatrix} = \frac{ig_2 v}{2} \begin{pmatrix} \frac{W_{12\mu}}{\sqrt{2}} \\ \frac{\left(B_\mu \sin(\theta_w)/\cos(\theta_w) + W_{22\mu}\right)}{\sqrt{2}} \end{pmatrix} = \frac{ig_2 v}{2} \begin{pmatrix} W_\mu^+ \\ \frac{-Z_\mu}{\sqrt{2}\cos(\theta_w)} \end{pmatrix}$$

$$m_A = 0$$

$$m_{W^\pm} = \frac{g_2 v}{2} = 80.425(38)\text{ GeV}$$

$$m_Z = \frac{g_2 v}{2\cos(\theta_w)} = 91.1876(21)\text{ GeV}$$

Ordinary 4-component Dirac fermions are composed of the left and right handed 2-component fields:

$e = \binom{e_{L1}}{e_{R1}}, \nu_e = \binom{\nu_{L1}}{\nu_{R1}}, u = \binom{u_{L1}}{u_{R1}}, d = \binom{d_{L1}}{d_{R1}},$ (electron, electron neutrino, up and down quark)

$\mu = \binom{e_{L2}}{e_{R2}}, \nu_\mu = \binom{\nu_{L2}}{\nu_{R2}}, c = \binom{u_{L2}}{u_{R2}}, s = \binom{d_{L2}}{d_{R2}},$ (muon, muon neutrino, charm, and strange quark)

$\tau = \binom{e_{L3}}{e_{R3}}, \nu_\tau = \binom{\nu_{L3}}{\nu_{R3}}, t = \binom{u_{L3}}{u_{R3}}, b = \binom{d_{L3}}{d_{R3}},$ (tauon, tauon neutrino, top and bottom quark)

$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}$ where $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2I g^{\mu\nu}$ (Dirac gamma matrices in chiral representation)

The corresponding antiparticles are related to the particles according to $\psi^e = -i\gamma^2 \psi^*$ or $\psi^e_L = -i\sigma^2 \psi^*_R$, $\psi^e_R = i\sigma^2 \psi^*_L$. The fermion charges are the coefficients of A_μ when (8,10) are substituted into either the left or right handed derivative operators (2-4). The fermion masses are the singular values of the 3×3 fermion mass matrices M^ν, M^e, M^u, M^d:

$$M^e = \mathbf{U}_L^{e\dagger} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \mathbf{U}_R^e$$

$$M^\nu = \mathbf{U}_L^{\nu\dagger} \begin{pmatrix} m_{\nu_e} & 0 & 0 \\ 0 & m_{\nu_\mu} & 0 \\ 0 & 0 & m_{\nu_\tau} \end{pmatrix} \mathbf{U}_R^\nu$$

$$M^u = \mathbf{U}_L^{u\dagger} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \mathbf{U}_R^u$$

$$M^d = \mathbf{U}_L^{d\dagger} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \mathbf{U}_R^d$$

$m_e = .510998910(13)$ MeV, $m_{\nu_e} \sim .001 - 2$ eV, $m_u = 1.7 - 3.1$ MeV, $m_d = 4.1 - 5.7$ MeV
$m_\mu = 105.658367(4)$ MeV, $m_{\nu_\mu} \sim .001 - 2$ eV, $m_c = 1.18 - 1.34$ GeV, $m_s = 80 - 130$ MeV
$m_\tau = 1776.84(17)$ MeV, $m_{\nu_\tau} \sim .001 - 2$ eV, $m_t = 171.4 - 174.4$ GeV, $m_b = 4.13 - 4.37$ GeV

Where the **U**s are 3×3 unitary matrices $\left(\mathbf{U}^{-1} = \mathbf{U}^\dagger\right)$. Consequently the "true fermions" with definite masses are actually linear combinations of those in $\mathcal{L}$, or conversely the fermions in $\mathcal{L}$ are linear combinations of the true fermions:

$$e'_L = \mathbf{U}^e_L e_L \ , \ e'_R = \mathbf{U}^e_R e_R \ , \ \nu'_L = \mathbf{U}^\nu_L \nu_L \ , \ \nu'_R = \mathbf{U}^\nu_R \nu_R \ , \ u'_L = \mathbf{U}^u_L u_L \ , \ u'_R = \mathbf{U}^u_R u_R \ , \ d'_L = \mathbf{U}^d_L d_L \ , \ d'_R = \mathbf{U}^d_R d_R$$

$$e_L = \mathbf{U}^{e\dagger}_L e'_L \ , \ e_R = \mathbf{U}^{e\dagger}_R e'_R \ , \ \nu_L = \mathbf{U}^{\nu\dagger}_L \nu'_L \ , \ \nu_R = \mathbf{U}^{\nu\dagger}_R \nu'_R \ , \ u_L = \mathbf{U}^{u\dagger}_L u'_L \ , \ u_R = \mathbf{U}^{u\dagger}_R u'_R \ , \ d_L = \mathbf{U}^{d\dagger}_L d'_L \ , \ d_R = \mathbf{U}^{d\dagger}_R d'_R$$

When $\mathcal{L}$ is written in terms of the true fermions, the **U**s fall out except in $\bar{u}'_L \mathbf{U}^u_L \tilde{\sigma}^\mu W^\pm_\mu \mathbf{U}^{d\dagger}_L d'_L$ and $\bar{\nu}'_L \mathbf{U}^\nu_L si\tilde{g}ma^\mu W^\pm_\mu \mathbf{U}^{e\dagger}_L e'_L$. Because of this, and some absorption of constants into the fermion fields, all the parameters in the **U**s are contained in only four components of the Cabibbo-Kobayashi-Maskawa matrix $\mathbf{V}^q = \mathbf{U}^u_L \mathbf{U}^{d\dagger}_L$ and four components of the Pontecorvo-Maki-Nakagawa-Sakata matrix $\mathbf{V}^l = \mathbf{U}^\nu_L \mathbf{U}^{e\dagger}_L$. The unitary matrices $\mathbf{V}^q$ and $\mathbf{V}^l$ are often parameterized as:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} e^{\frac{-i\delta}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{i\delta}{2}} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} e^{\frac{i\delta}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{-i\delta}{2}} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$c_j = \sqrt{1 - s_j^2}$$

$\delta^q = 69(4)$ deg, $s^q_{12} = 0.2253(7)$, $s^q_{23} = 0.041(1)$, $s^q_{13} = 0.0035(2)$,
$\delta^l = ?$, $s^l_{12} = 0.560(16)$, $s^l_{23} = 0.7(1)$, $s^l_{13} = 0.153(28)$

$\mathcal{L}$ is invariant under a U(1) ⊗ SU(2) gauge transformation with $U^{-1} = U^\dagger$, $detU = 1$, θ real:

$$\mathbf{W}_\mu \rightarrow U\mathbf{W}_\mu U^\dagger - \left(\frac{2i}{g_2}\right) U\partial_\mu U^\dagger$$

$$\mathbf{W}_{\mu\nu} \rightarrow U\mathbf{W}_{\mu\nu} U^\dagger$$

$$B_\mu \rightarrow B_\mu + \left(\frac{2}{g_1}\right)\partial_\mu \theta$$

$$B_{\mu\nu} \rightarrow B_{\mu\nu}$$

$$\phi \rightarrow e^{-i\theta} U\phi$$

$$\left(\begin{smallmatrix} \nu_L \\ e_L \end{smallmatrix}\right) \rightarrow e^{i\theta} U \left(\begin{smallmatrix} \nu_L \\ e_L \end{smallmatrix}\right)$$

$$\left(\begin{smallmatrix} u_L \\ d_L \end{smallmatrix}\right) \rightarrow e^{-\frac{i\theta}{3}} U \left(\begin{smallmatrix} u_L \\ d_L \end{smallmatrix}\right)$$

$$\begin{aligned} &\nu_R \rightarrow \nu_R && u_R \rightarrow e^{-\frac{4i\theta}{3}} u_R \\ &e_R \rightarrow e^{2i\theta} e_R && d_R \rightarrow e^{\frac{2i\theta}{3}} d_R \end{aligned}$$

And under an SU(3) gauge transformation with $V^{-1} = V^\dagger$, $det V = 1$, and:

$$\mathbf{G}_\mu \rightarrow V\mathbf{G}_\mu V^\dagger - \left(\frac{i}{g}\right) V\partial_\mu V^\dagger$$

$$\mathbf{G}_{\mu\nu} \rightarrow V\mathbf{G}_{\mu\nu} V^\dagger$$

$$\begin{aligned} &u_L \rightarrow V u_L && d_L \rightarrow V d_L \\ &u_R \rightarrow V u_R && d_R \rightarrow V d_R \end{aligned}$$