Numerical Integration

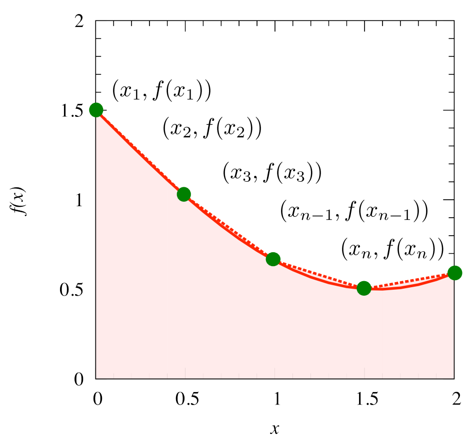
In an earlier lecture on the normal distribution, it was shown that the probability of obtaining a value within one standard deviation of the mean is

$$P(\text{within } \sigma) = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

In fact, it is impossible to evaluate this definite integral in an analytical manner, but it can be evaluated using numerical integration.

In general, numerical integration methods approximate the function of interest, between the integration limits, with a simpler function that can be integrated. This could be a straight line (trapezoidal rule) or quadratic (Simpson's 1/3 rule). For more accurate estimates the integration range is divided up into a number of intervals of equal width, the trapezoidal rule or Simpson's 1/3 rule is applied to each and the individual contributions summed (composite trapezoidal rule and composite Simpson's 1/3 rule).

The Composite Trapezoidal Rule



The composite trapezoidal rule is expressed

$$I = \int_{x_1}^{x_n} f(x)dx$$

$$I = \frac{h}{2}\left[f(x_1) + 2\left(\sum_{i=2}^{n-1} f(x_i)\right) + f(x_n)\right]$$

$$h = \frac{x_n - x_1}{n - 1}$$

If several composite trapezoidal rule calculations are performed using different interval widths, the results can be combined to determine an improved estimate.

A Richardson Extrapolation

Richardson extrapolation is a method of combining two estimates of an integral, determined using the composite trapezoidal rule, using different interval widths, to obtain a more accurate estimate. It has the form

$$I = I(h_2) + \frac{I(h_1) - I(h_2)}{1 - (h_1/h_2)^2}$$

$$\begin{aligned} I &= \text{improved estimate of integral} \\ I(h_1) &= \text{estimate of the intergral calculated using an interval width } h_1 \\ I(h_2) &= \text{estimate of the intergral calculated using an interval width } h_2 \end{aligned}$$

Your Task

Your task is to write a MATLAB function to perform numerical integration, to estimate the integral of a pre-defined function between two integration limits. In particular, it should perform a composite trapezoidal rule calculation for two interval widths and use Richardson extrapolation to combine the results to obtain a more accurate estimate.

More Details

The function should be defined at the command-line

```
>> f=@(x) (1/sqrt(2*pi))*exp(-(x^2)/2)
```

To ensure that the intervals are of an appropriate width, the user should define the number of intervals to use, rather than the actual interval width. The interval widths can then be computed by dividing the integration range by this number. The function being integrated, lower and upper integration limits and number of intervals should be passed to the function at the command-line

```
>> ll = -1; ul = 1; ni1 = 10; ni2 = 20;
>> [integral] = richardson(f,ll,ul,ni1,ni2)
```

where `ll` and `ul` are the lower and upper integration limits and `ni1` and `ni2` are the number of intervals to use.

Since two composite trapezoidal rule calculations have to be performed, it is efficient to write another function to perform this part of the computation. This can be called within your main function

```
[integral,iwidth] = ctrap(f,ll,ul,ni1)
```

and should return both the integral estimate and interval width.

B Romberg Integration

Romberg integration allows you to combine multiple estimates of an integral that were calculated using the composite trapezoidal rule and different interval widths.

$$I_{j,k} = \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{(k-1)} - 1}$$

$$\begin{aligned}
I_{j,k} &= \text{higher level estimate of the integral} \\
I_{j+1,k-1} &= \text{lower level estimate of the integral (more accurate)} \\
I_{j,k-1} &= \text{lower level estimate of the integral (less accurate)} \\
j &= \text{used to distinguish between the more and less accurate estimates} \\
&\quad (i.e.\ I_{j+1,k-1} \text{ is more accurate than } I_{j,k-1}, \text{ and } h_{j+1} = h_j/2) \\
k &= \text{used to distinguish level of integration} \\
&\quad (i.e.\ k = 1\text{:Composite Trapezoidal Rule; } k = 2\text{:}O(h^4);\ k = 3\text{:}O(h^6))
\end{aligned}$$

Your Task

Your task is to write a MATLAB function to perform numerical integration, to estimate the integral of a pre-defined function between two integration limits. In particular, it should perform a composite trapezoidal rule calculation for multiple interval widths and use Romberg integration to combine the results to obtain a more accurate estimate.

More Details

The function should be defined at the command-line

```
>> f=@(x) (1/sqrt(2*pi))*exp(-(x^2)/2)
```

To ensure that the intervals are of an appropriate width, the user should define the number of intervals to use, rather than the actual interval width. The interval widths can then be computed by dividing the integration range by this number. The function being integrated, lower and upper integration limits and number of intervals should be passed to your function at the command-line

```
>> ll = -1; ul = 1; ni = 8; nl = 4;
>> [integral] = romberg(f,ll,ul,ni,nl)
```

where `ll` and `ul` are the lower and upper integration limits, and `ni` the minimum number of intervals to use for the composite trapezoidal rule calculations and `nl` the number of integration levels to use. You should use your function for performing the composite trapezoidal rule from Part A, to calculate the initial estimates of the integral.

Testing Your MATLAB Script

Use your script(s) to determine the probability of obtaining a value within one standard deviation of the mean i.e. compute the definite integral at the top of the first page of the handout.

Try using larger and larger number of intervals and seeing how the estimated value changes. You should find it equal to about 0.682689492.