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Motto: Creative , Innovative , Mechatronics

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Chapter 1

Elementary Signals

This chapter begins with a discussion of elementary signals that may be applied to electric networks. The unit step, unit ramp, and delta functions are then introduced. The sampling and sifting properties of the delta function are defined and derived. Several examples for expressing a variety of waveforms in terms of these elementary signals are provided. Throughout this text, a left justified horizontal bar will denote the beginning of an example, and a right justified horizontal bar will denote the end of the example. These bars will not be shown whenever an example begins at the top of a page or at the bottom of a page. Also, when one example follows immediately after a previous example, the right justified bar will be omitted.

1.1 Signals Described in Math Form

Consider the network of Figure 1.1 where the switch is closed at time $t = 0$.

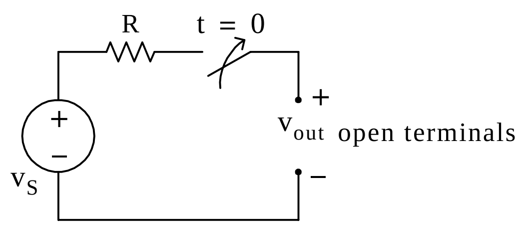


Figure 1.1. A switched network with open terminals

We wish to describe v_{out} in a math form for the time interval $-\infty < t < +\infty$. To do this, it is convenient to divide the time interval into two parts, $-\infty < t < 0$, and $0 < t < \infty$.

For the time interval $-\infty < t < 0$, the switch is open and therefore, the output voltage v_{out} is zero. In other words,

$$v_{out} = 0 \quad \text{for} \quad -\infty < t < 0 \tag{1.1}$$

For the time interval $0 < t < \infty$, the switch is closed. Then, the input voltage v_S appears at the output, i.e.,

$$v_{out} = v_S \quad \text{for} \quad 0 < t < \infty \tag{1.2}$$

Combining (1.1) and (1.2) into a single relationship, we obtain

$$\boxed{v_{out} = \begin{cases} 0 & -\infty < t < 0 \\ v_S & 0 < t < \infty \end{cases}} \tag{1.3}$$

We can express (1.3) by the waveform shown in Figure 1.2.

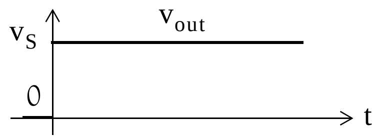


Figure 1.2. Waveform for v_{out} as defined in relation (1.3)

The waveform of Figure 1.2 is an example of a discontinuous function. A function is said to be *discontinuous* if it exhibits points of discontinuity, that is, the function jumps from one value to another without taking on any intermediate values.

1.2 The Unit Step Function $u_0(t)$

A well known discontinuous function is the *unit step function* $u_0(t)$ * which is defined as

$$\boxed{u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}} \tag{1.4}$$

It is also represented by the waveform of Figure 1.3.

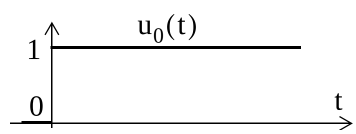


Figure 1.3. Waveform for $u_0(t)$

In the waveform of Figure 1.3, the unit step function $u_0(t)$ changes abruptly from 0 to 1 at $t = 0$. But if it changes at $t = t_0$ instead, it is denoted as $u_0(t - t_0)$. In this case, its waveform and definition are as shown in Figure 1.4 and relation (1.5) respectively.

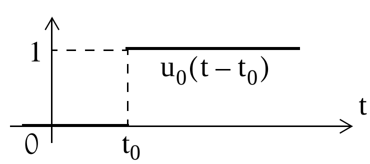


Figure 1.4. Waveform for $u_0(t - t_0)$

* *In some books, the unit step function is denoted as* $u(t)$, *that is, without the subscript 0. In this text, however, we will reserve the* $u(t)$ *designation for any input when we will discuss state variables in Chapter 5.*

$$u_0(t-t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases} \tag{1.5}$$

If the unit step function changes abruptly from 0 to 1 at $t = -t_0$, it is denoted as $u_0(t+t_0)$. In this case, its waveform and definition are as shown in Figure 1.5 and relation (1.6) respectively.

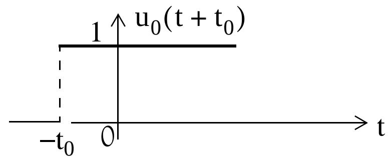


Figure 1.5. Waveform for $u_0(t+t_0)$

$$u_0(t+t_0) = \begin{cases} 0 & t < -t_0 \\ 1 & t > -t_0 \end{cases} \tag{1.6}$$

Example 1.1

Consider the network of Figure 1.6, where the switch is closed at time $t = T$.

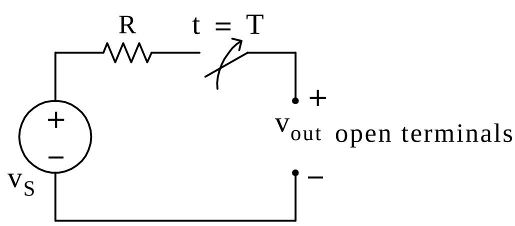


Figure 1.6. Network for Example 1.1

Express the output voltage v_{out} as a function of the unit step function, and sketch the appropriate waveform.

Solution:

For this example, the output voltage $v_{out} = 0$ for $t < T$, and $v_{out} = v_S$ for $t > T$. Therefore,

$$v_{out} = v_S u_0(t-T) \tag{1.7}$$

and the waveform is shown in Figure 1.7.

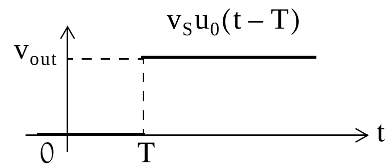


Figure 1.7. Waveform for Example 1.1

Other forms of the unit step function are shown in Figure 1.8.

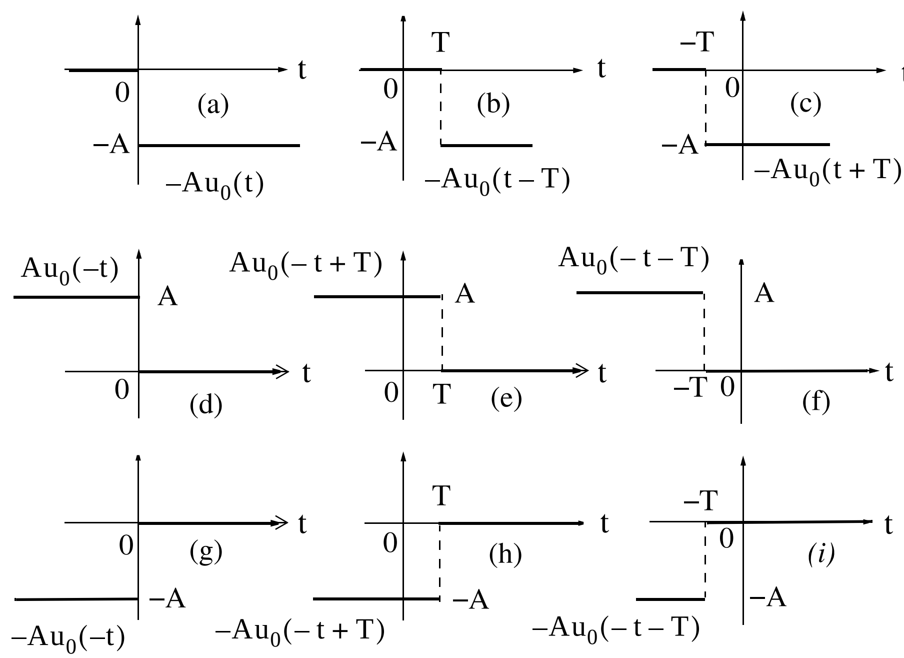


Figure 1.8. Other forms of the unit step function

Unit step functions can be used to represent other time–varying functions such as the rectangular pulse shown in Figure 1.9.

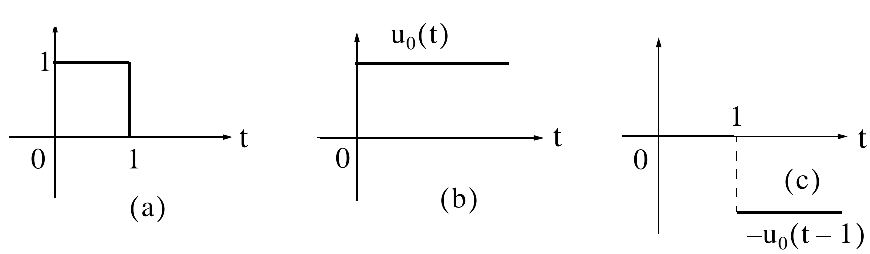


Figure 1.9. A rectangular pulse expressed as the sum of two unit step functions

Thus, the pulse of Figure 1.9(a) is the sum of the unit step functions of Figures 1.9(b) and 1.9(c) and it is represented as $u_0(t) - u_0(t-1)$.

The unit step function offers a convenient method of describing the sudden application of a voltage or current source. For example, a constant voltage source of $24\ V$ applied at $t = 0$, can be denoted as $24u_0(t)\ V$. Likewise, a sinusoidal voltage source $v(t) = V_m \cos\omega t\ V$ that is applied to a circuit at $t = t_0$, can be described as $v(t) = (V_m \cos\omega t) u_0(t - t_0)\ V$. Also, if the excitation in a circuit is a rectangular, or triangular, or sawtooth, or any other recurring pulse, it can be represented as a sum (difference) of unit step functions.

Example 1.2

Express the square waveform of Figure 1.10 as a sum of unit step functions. The vertical dotted lines indicate the discontinuities at $T, 2T, 3T$, and so on.

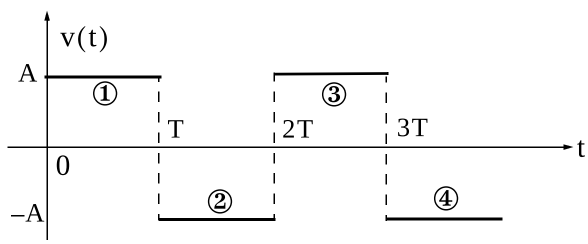


Figure 1.10. Square waveform for Example 1.2

Solution:

Line segment ① has height A, starts at $t = 0$, and terminates at $t = T$. Then, as in Example 1.1, this segment is expressed as

$$v_1(t) = A[u_0(t) - u_0(t-T)] \tag{1.8}$$

Line segment ② has height $-A$, starts at $t = T$ and terminates at $t = 2T$. This segment is expressed as

$$v_2(t) = -A[u_0(t-T) - u_0(t-2T)] \tag{1.9}$$

Line segment ③ has height A, starts at $t = 2T$ and terminates at $t = 3T$. This segment is expressed as

$$v_3(t) = A[u_0(t-2T) - u_0(t-3T)] \tag{1.10}$$

Line segment ④ has height $-A$, starts at $t = 3T$, and terminates at $t = 4T$. It is expressed as

$$v_4(t) = -A[u_0(t-3T) - u_0(t-4T)] \tag{1.11}$$

Thus, the square waveform of Figure 1.10 can be expressed as the summation of (1.8) through (1.11), that is,

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + v_3(t) + v_4(t) \\ &= A[u_0(t) - u_0(t-T)] - A[u_0(t-T) - u_0(t-2T)] \\ &\quad + A[u_0(t-2T) - u_0(t-3T)] - A[u_0(t-3T) - u_0(t-4T)] \end{aligned} \tag{1.12}$$

Combining like terms, we obtain

$$v(t) = A[u_0(t) - 2u_0(t-T) + 2u_0(t-2T) - 2u_0(t-3T) + \ldots] \tag{1.13}$$

Example 1.3

Express the symmetric rectangular pulse of Figure 1.11 as a sum of unit step functions.

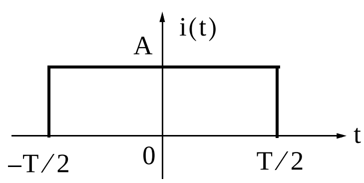


Figure 1.11. Symmetric rectangular pulse for Example 1.3

Solution:

This pulse has height A, starts at $t = -T/2$, and terminates at $t = T/2$. Therefore, with reference to Figures 1.5 and 1.8 (b), we obtain

$$i(t) = Au_0\left(t + \frac{T}{2}\right) - Au_0\left(t - \frac{T}{2}\right) = A\left[u_0\left(t + \frac{T}{2}\right) - u_0\left(t - \frac{T}{2}\right)\right] \tag{1.14}$$

Example 1.4

Express the symmetric triangular waveform of Figure 1.12 as a sum of unit step functions.

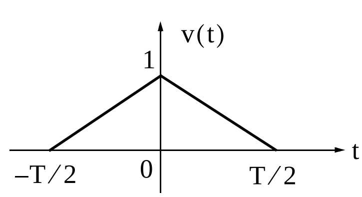


Figure 1.12. Symmetric triangular waveform for Example 1.4

Solution:

We first derive the equations for the linear segments ① and ② shown in Figure 1.13.

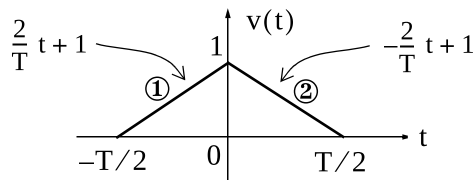


Figure 1.13. Equations for the linear segments of Figure 1.12

For line segment ①,

$$v_1(t) = \left(\frac{2}{T}t+1\right)\left[u_0\left(t+\frac{T}{2}\right)-u_0(t)\right] \tag{1.15}$$

and for line segment ②,

$$v_2(t) = \left(-\frac{2}{T}t+1\right)\left[u_0(t)-u_0\left(t-\frac{T}{2}\right)\right] \tag{1.16}$$

Combining (1.15) and (1.16), we obtain

$$\begin{aligned} v(t) &= v_1(t)+v_2(t) \\ &= \left(\frac{2}{T}t+1\right)\left[u_0\left(t+\frac{T}{2}\right)-u_0(t)\right]+\left(-\frac{2}{T}t+1\right)\left[u_0(t)-u_0\left(t-\frac{T}{2}\right)\right] \end{aligned} \tag{1.17}$$

Example 1.5

Express the waveform of Figure 1.14 as a sum of unit step functions.

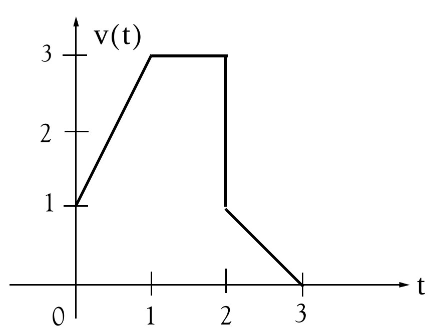


Figure 1.14. Waveform for Example 1.5

Solution:

As in the previous example, we first find the equations of the linear segments linear segments ① and ② shown in Figure 1.15.

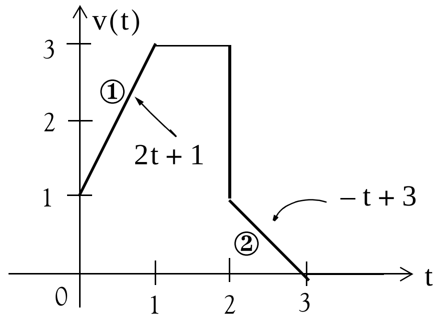


Figure 1.15. Equations for the linear segments of Figure 1.14

Following the same procedure as in the previous examples, we obtain

$$\begin{aligned} v(t) &= (2t+1)[u_0(t) - u_0(t-1)] + 3[u_0(t-1) - u_0(t-2)] \\ &\quad + (-t+3)[u_0(t-2) - u_0(t-3)] \end{aligned}$$

Multiplying the values in parentheses by the values in the brackets, we obtain

$$\begin{aligned} v(t) &= (2t+1)u_0(t) - (2t+1)u_0(t-1) + 3u_0(t-1) \\ &\quad - 3u_0(t-2) + (-t+3)u_0(t-2) - (-t+3)u_0(t-3) \end{aligned}$$

$$\begin{aligned} v(t) &= (2t+1)u_0(t) + [-(2t+1)+3]u_0(t-1) \\ &\quad + [-3+(-t+3)]u_0(t-2) - (-t+3)u_0(t-3) \end{aligned}$$

and combining terms inside the brackets, we obtain

$$v(t) = (2t+1)u_0(t) - 2(t-1)u_0(t-1) - tu_0(t-2) + (t-3)u_0(t-3) \tag{1.18}$$

Two other functions of interest are the *unit ramp function*, and the *unit impulse* or *delta function*. We will introduce them with the examples that follow.

Example 1.6

In the network of Figure 1.16 i_S is a constant current source and the switch is closed at time $t = 0$. Express the capacitor voltage $v_C(t)$ as a function of the unit step.

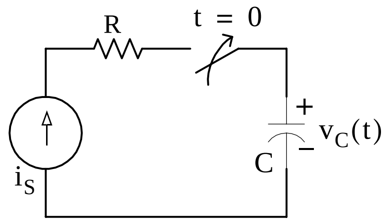


Figure 1.16. Network for Example 1.6

Solution:

The current through the capacitor is $i_C(t) = i_S = constant$, and the capacitor voltage $v_C(t)$ is

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau^{*} \tag{1.19}$$

where τ is a dummy variable.

Since the switch closes at $t = 0$, we can express the current $i_C(t)$ as

$$i_C(t) = i_S\, u_0(t) \tag{1.20}$$

and assuming that $v_C(t) = 0$ for $t < 0$, we can write (1.19) as

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_S\, u_0(\tau)d\tau = \underbrace{\frac{i_S}{C}\int_{-\infty}^{0} u_0(\tau)d\tau}_{0} + \frac{i_S}{C}\int_{0}^{t} u_0(\tau)d\tau \tag{1.21}$$

or

$$\boxed{v_C(t) = \frac{i_S}{C}\, t u_0(t)} \tag{1.22}$$

Therefore, we see that when a capacitor is charged with a constant current, the voltage across it is a linear function and forms a *ramp* with slope i_S / C as shown in Figure 1.17.

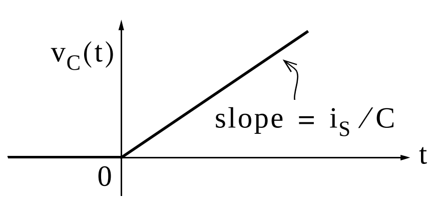


Figure 1.17. Voltage across a capacitor when charged with a constant current source

* *Since the initial condition for the capacitor voltage was not specified, we express this integral with* $-\infty$ *at the lower limit of integration so that any non-zero value prior to* $t < 0$ *would be included in the integration.*

1.3 The Unit Ramp Function $u_1(t)$

The *unit ramp function*, denoted as $u_1(t)$, is defined as

$$\boxed{u_1(t) = \int_{-\infty}^{t} u_0(\tau)d\tau} \tag{1.23}$$

where τ is a dummy variable.

We can evaluate the integral of (1.23) by considering the area under the unit step function $u_0(t)$ from $-\infty$ to t as shown in Figure 1.18.

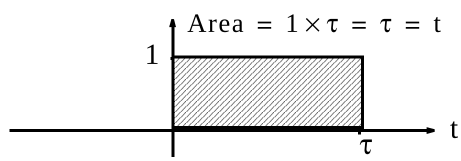


Figure 1.18. Area under the unit step function from $-\infty$ to t

Therefore, we define $u_1(t)$ as

$$\boxed{u_1(t) = \begin{cases} 0 & t < 0 \\ t & t \ge 0 \end{cases}} \tag{1.24}$$

Since $u_1(t)$ is the integral of $u_0(t)$, then $u_0(t)$ must be the derivative of $u_1(t)$, i.e.,

$$\boxed{\frac{d}{dt}u_1(t) = u_0(t)} \tag{1.25}$$

Higher order functions of t can be generated by repeated integration of the unit step function. For example, integrating $u_0(t)$ twice and multiplying by 2, we define $u_2(t)$ as

$$u_2(t) = \begin{cases} 0 & t < 0 \\ t^2 & t \ge 0 \end{cases} \quad \text{or} \quad u_2(t) = 2\int_{-\infty}^{t} u_1(\tau)d\tau \tag{1.26}$$

Similarly,

$$u_3(t) = \begin{cases} 0 & t < 0 \\ t^3 & t \ge 0 \end{cases} \quad \text{or} \quad u_3(t) = 3\int_{-\infty}^{t} u_2(\tau)d\tau \tag{1.27}$$

and in general,

$$u_n(t) = \begin{cases} 0 & t < 0 \\ t^n & t \ge 0 \end{cases} \quad \text{or} \quad u_n(t) = 3\int_{-\infty}^{t} u_{n-1}(\tau)d\tau \tag{1.28}$$

Also,

$$u_{n-1}(t) = \frac{1}{n}\frac{d}{dt}u_n(t) \tag{1.29}$$

Example 1.7

In the network of Figure 1.19, the switch is closed at time $t = 0$ and $i_L(t) = 0$ for $t < 0$. Express the inductor current $i_L(t)$ in terms of the unit step function.

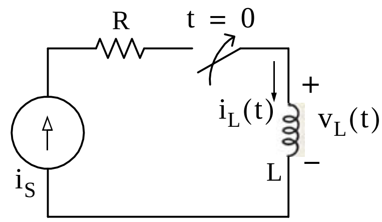


Figure 1.19. Network for Example 1.7

Solution:

The voltage across the inductor is

$$v_L(t) = L\frac{di_L}{dt} \tag{1.30}$$

and since the switch closes at $t = 0$,

$$i_L(t) = i_S\, u_0(t) \tag{1.31}$$

Therefore, we can write (1.30) as

$$\boxed{v_L(t) = Li_S\frac{d}{dt}u_0(t)} \tag{1.32}$$

But, as we know, $u_0(t)$ is constant (0 or 1) for all time except at $t = 0$ where it is discontinuous. Since the derivative of any constant is zero, the derivative of the unit step $u_0(t)$ has a non–zero value only at $t = 0$. The derivative of the unit step function is defined in the next section.

1.4 The Delta Function $\delta(t)$

The *unit impulse* or *delta function*, denoted as $\delta(t)$, is the derivative of the unit step $u_0(t)$. It is also defined as

$$\boxed{\int_{-\infty}^{t}\delta(\tau)d\tau = u_0(t)} \tag{1.33}$$

and

$$\boxed{\delta(t) = 0 \ \text{ for all } \ t \neq 0} \tag{1.34}$$

To better understand the delta function $\delta(t)$, let us represent the unit step $u_0(t)$ as shown in Figure 1.20 (a).

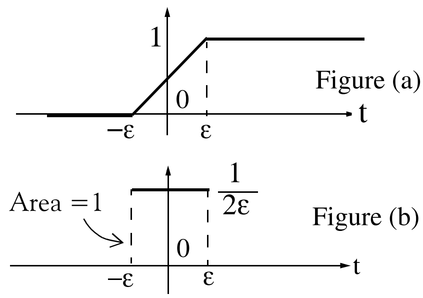


Figure 1.20. Representation of the unit step as a limit

The function of Figure 1.20 (a) becomes the unit step as $\varepsilon \to 0$. Figure 1.20 (b) is the derivative of Figure 1.20 (a), where we see that as $\varepsilon \to 0$, $1/2\varepsilon$ becomes unbounded, but the area of the rectangle remains 1. Therefore, in the limit, we can think of $\delta(t)$ as approaching a very large spike or impulse at the origin, with unbounded amplitude, zero width, and area equal to 1.

Two useful properties of the delta function are the sampling property and the sifting property.

1.4.1 The Sampling Property of the Delta Function $\delta(t)$

The *sampling property* of the delta function states that

$$\boxed{f(t)\delta(t-a) = f(a)\delta(t)} \tag{1.35}$$

or, when $a = 0$,

$$\boxed{f(t)\delta(t) = f(0)\delta(t)} \tag{1.36}$$

that is, multiplication of any function $f(t)$ by the delta function $\delta(t)$ results in sampling the function at the time instants where the delta function is not zero. The study of discrete–time systems is based on this property.

Proof:

Since $\delta(t) = 0$ for $t<0$ and $t>0$ then,

$$f(t)\delta(t) = 0 \text{ for } t<0 \text{ and } t>0 \tag{1.37}$$

We rewrite $f(t)$ as

$$f(t) = f(0) + [f(t) - f(0)] \tag{1.38}$$

Integrating (1.37) over the interval $-\infty$ to t and using (1.38), we obtain

$$\int_{-\infty}^{t} f(\tau)\delta(\tau)d\tau = \int_{-\infty}^{t} f(0)\delta(\tau)d\tau + \int_{-\infty}^{t} [f(\tau) - f(0)]\delta(\tau)d\tau \tag{1.39}$$

The first integral on the right side of (1.39) contains the constant term $f(0)$; this can be written outside the integral, that is,

$$\int_{-\infty}^{t} f(0)\delta(\tau)d\tau = f(0)\int_{-\infty}^{t} \delta(\tau)d\tau \tag{1.40}$$

The second integral of the right side of (1.39) is always zero because

$$\delta(t) = 0 \;\text{ for }\; t < 0 \;\text{ and }\; t > 0$$

and

$$[f(\tau) - f(0)]\big|_{\tau = 0} = f(0) - f(0) = 0$$

Therefore, (1.39) reduces to

$$\int_{-\infty}^{t} f(\tau)\delta(\tau)d\tau = f(0)\int_{-\infty}^{t} \delta(\tau)d\tau \tag{1.41}$$

Differentiating both sides of (1.41), and replacing τ with t, we obtain

$$\boxed{\begin{array}{c} f(t)\delta(t) = f(0)\delta(t) \\ \text{Sampling Property of } \delta(t) \end{array}} \tag{1.42}$$

1.4.2 The Sifting Property of the Delta Function $\delta(t)$

The *sifting property* of the delta function states that

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)} \tag{1.43}$$

that is, if we multiply any function $f(t)$ by $\delta(t-\alpha)$, and integrate from $-\infty$ to $+\infty$, we will obtain the value of $f(t)$ evaluated at $t = \alpha$.

Proof:

Let us consider the integral

$$\int_{a}^{b} f(t)\delta(t-\alpha)dt \quad \text{where } a < \alpha < b \tag{1.44}$$

We will use integration by parts to evaluate this integral. We recall from the derivative of products that

$$d(xy) = xdy + ydx \quad \text{or} \quad xdy = d(xy) - ydx \tag{1.45}$$

and integrating both sides we obtain

$$\int x dy = xy - \int y dx \tag{1.46}$$

Now, we let $x = f(t)$; then, $dx = f'(t)$. We also let $dy = \delta(t-\alpha)$; then, $y = u_0(t-\alpha)$. By substitution into (1.44), we obtain

$$\int_a^b f(t)\delta(t-\alpha)dt = f(t)u_0(t-\alpha)\Big|_a^b - \int_a^b u_0(t-\alpha)f'(t)dt \tag{1.47}$$

We have assumed that $a < \alpha < b$; therefore, $u_0(t-\alpha) = 0$ for $\alpha < a$, and thus the first term of the right side of (1.47) reduces to $f(b)$. Also, the integral on the right side is zero for $\alpha < a$, and therefore, we can replace the lower limit of integration a by α. We can now rewrite (1.47) as

$$\int_a^b f(t)\delta(t-\alpha)dt = f(b) - \int_\alpha^b f'(t)dt = f(b) - f(b) + f(\alpha)$$

and letting $a \to -\infty$ and $b \to \infty$ for any $|\alpha| < \infty$, we obtain

$$\boxed{\begin{array}{c}\displaystyle\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha) \\ \text{Sifting Property of } \delta(t)\end{array}} \tag{1.48}$$

1.5 Higher Order Delta Functions

An *nth-order delta function* is defined as the nth derivative of $u_0(t)$, that is,

$$\boxed{\delta^n(t) = \frac{\delta^n}{dt}[u_0(t)]} \tag{1.49}$$

The function $\delta'(t)$ is called *doublet*, $\delta''(t)$ is called *triplet*, and so on. By a procedure similar to the derivation of the sampling property of the delta function, we can show that

$$\boxed{f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(a)\delta(t-a)} \tag{1.50}$$

Also, the derivation of the sifting property of the delta function can be extended to show that

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta^n(t-\alpha)dt = (-1)^n \frac{d^n}{dt^n}[f(t)]\Big|_{t=\alpha}} \tag{1.51}$$

Example 1.8

Evaluate the following expressions:

a. $3t^4\delta(t-1)$ b. $\int_{-\infty}^{\infty} t\delta(t-2)dt$ c. $t^2\delta'(t-3)$

Solution:

a. The sampling property states that $f(t)\delta(t-a) = f(a)\delta(t-a)$ For this example, $f(t) = 3t^4$ and $a = 1$. Then,

$$3t^4\delta(t-1) = \{3t^4\big|_{t=1}\}\delta(t-1) = 3\delta(t-1)$$

b. The sifting property states that $\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$. For this example, $f(t) = t$ and $\alpha = 2$. Then,

$$\int_{-\infty}^{\infty} t\delta(t-2)dt = f(2) = t\big|_{t=2} = 2$$

c. The given expression contains the doublet; therefore, we use the relation

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(a)\delta(t-a)$$

Then, for this example,

$$t^2\delta'(t-3) = t^2\big|_{t=3}\delta'(t-3) - \frac{d}{dt}t^2\big|_{t=3}\delta(t-3) = 9\delta'(t-3) - 6\delta(t-3)$$

Example 1.9

a. Express the voltage waveform $v(t)$ shown in Figure 1.21 as a sum of unit step functions for the time interval $-1 < t < 7$ s.

b. Using the result of part (a), compute the derivative of $v(t)$ and sketch its waveform.

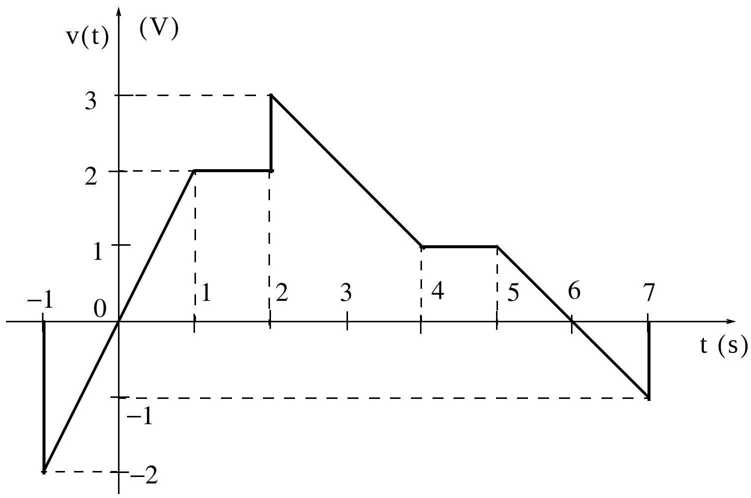


Figure 1.21. Waveform for Example 1.9

Solution:

a. We begin with the derivation of the equations for the linear segments of the given waveform as shown in Figure 1.22.

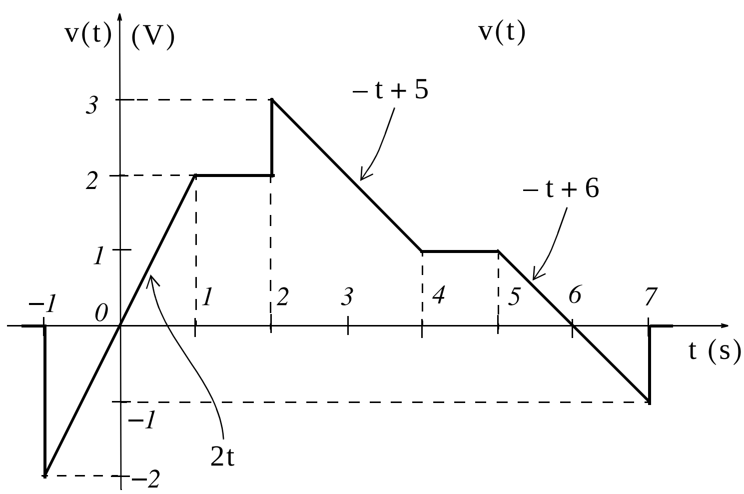


Figure 1.22. Equations for the linear segments of Figure 1.21

Next, we express $v(t)$ in terms of the unit step function $u_0(t)$, and we obtain

$$\begin{aligned} v(t) = {} & 2t[u_0(t+1) - u_0(t-1)] + 2[u_0(t-1) - u_0(t-2)] \\ & + (-t+5)[u_0(t-2) - u_0(t-4)] + [u_0(t-4) - u_0(t-5)] \\ & + (-t+6)[u_0(t-5) - u_0(t-7)] \end{aligned} \tag{1.52}$$

Multiplying and collecting like terms in (1.52), we obtain

$$v(t) = 2tu_0(t+1) - 2tu_0(t-1) - 2u_0(t-1) - 2u_0(t-2) - tu_0(t-2)$$
$$+ 5u_0(t-2) + tu_0(t-4) - 5u_0(t-4) + u_0(t-4) - u_0(t-5)$$
$$-tu_0(t-5) + 6u_0(t-5) + tu_0(t-7) - 6u_0(t-7)$$

or

$$v(t) = 2tu_0(t+1) + (-2t+2)u_0(t-1) + (-t+3)u_0(t-2)$$
$$+ (t-4)u_0(t-4) + (-t+5)u_0(t-5) + (t-6)u_0(t-7)$$

b. The derivative of $v(t)$ is

$$\begin{aligned}\frac{dv}{dt} = {} & 2u_0(t+1) + 2t\delta(t+1) - 2u_0(t-1) + (-2t+2)\delta(t-1) \\ & - u_0(t-2) + (-t+3)\delta(t-2) + u_0(t-4) + (t-4)\delta(t-4) \\ & - u_0(t-5) + (-t+5)\delta(t-5) + u_0(t-7) + (t-6)\delta(t-7)\end{aligned} \tag{1.53}$$

From the given waveform, we observe that discontinuities occur only at $t = -1$, $t = 2$, and $t = 7$. Therefore, $\delta(t-1) = 0$, $\delta(t-4) = 0$, and $\delta(t-5) = 0$, and the terms that contain these delta functions vanish. Also, by application of the sampling property,

$$2t\delta(t+1) = \{2t|_{t=-1}\}\delta(t+1) = -2\delta(t+1)$$
$$(-t+3)\delta(t-2) = \{(-t+3)|_{t=2}\}\delta(t-2) = \delta(t-2)$$
$$(t-6)\delta(t-7) = \{(t-6)|_{t=7}\}\delta(t-7) = \delta(t-7)$$

and by substitution into (1.53), we obtain

$$\begin{aligned}\frac{dv}{dt} = {} & 2u_0(t+1) - 2\delta(t+1) - 2u_0(t-1) - u_0(t-2) \\ & + \delta(t-2) + u_0(t-4) - u_0(t-5) + u_0(t-7) + \delta(t-7)\end{aligned} \tag{1.54}$$

The plot of dv/dt is shown in Figure 1.23.

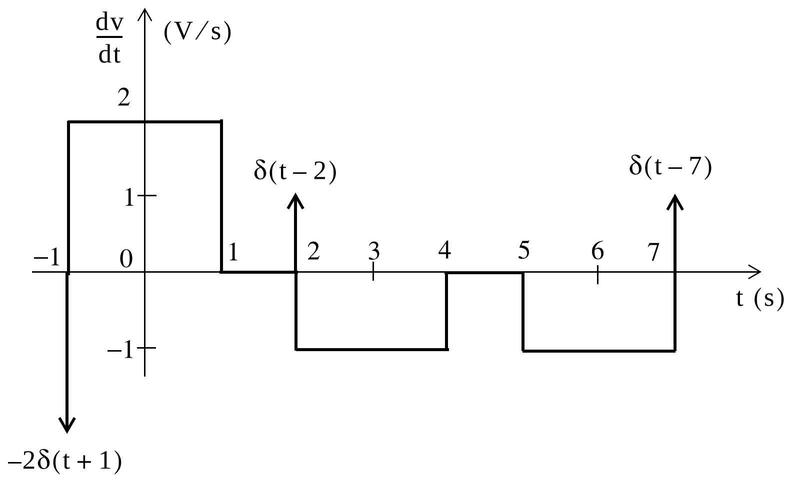


Figure 1.23. Plot of the derivative of the waveform of Figure 1.21

We observe that a negative spike of magnitude 2 occurs at $t = -1$, and two positive spikes of magnitude 1 occur at $t = 2$, and $t = 7$. These spikes occur because of the discontinuities at these points.

It would be interesting to observe the given signal and its derivative on the Scope block of the Simulink®* model of Figure 1.24. They are shown in Figure 1.25.

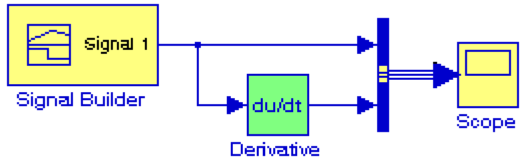


Figure 1.24. Simulink model for Example 1.9

The waveform created by the Signal Builder block is shown in Figure 1.25.

* *A brief introduction to Simulink is presented in Appendix B. For a detailed procedure for generating piece-wise linear functions with Simulink's Signal Builder block, please refer to Introduction to Simulink with Engineering Applications, ISBN 0−9744239−7−1*

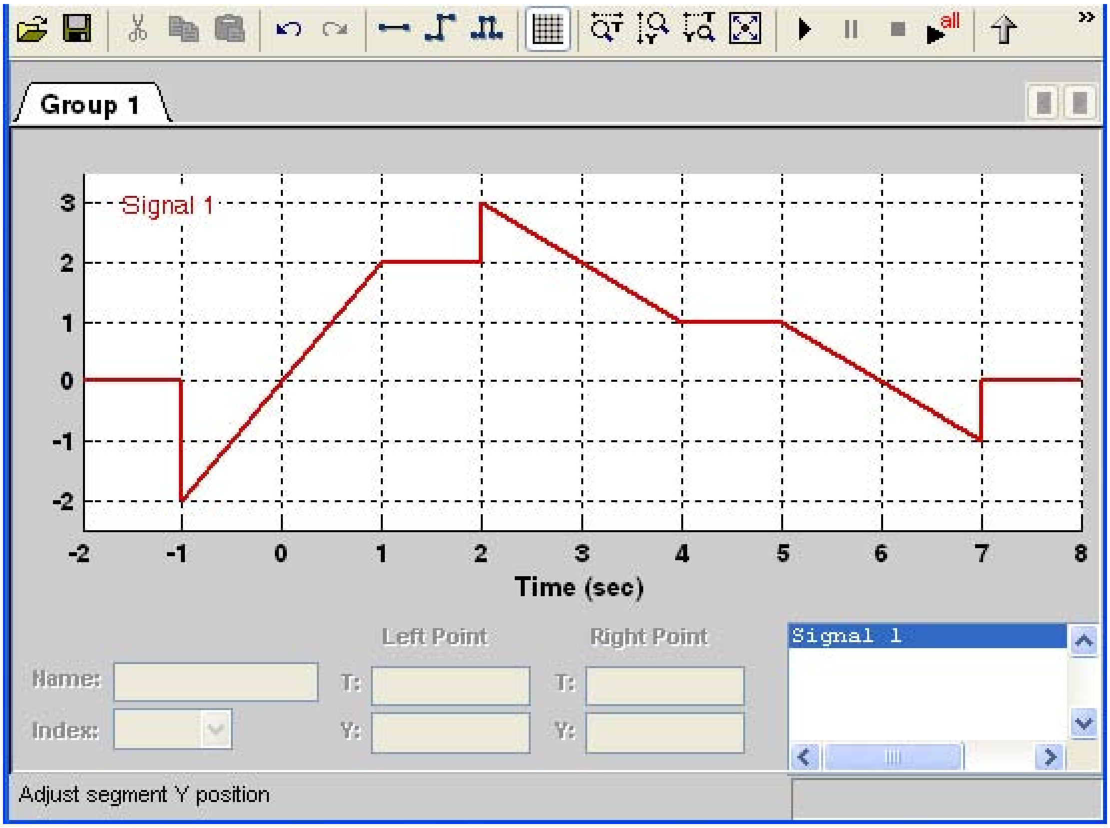


Figure 1.25. Piece−wise linear waveform for the Signal Builder block in Figure 1.24

The waveform in Figure 1.25 is created with the following procedure:

1. We open a new model by clicking on the new model icon shown as a blank page on the left corner of the top menu bar. Initially, the name **Untitled** appears on the top of this new model. We save it with the name Figure_1.25 and Simulink appends the .mdl extension to it.

2. From the **Sources** library, we drag the **Signal Builder** block into this new model. We also drag the **Derivative** block from the **Continuous** library, the **Bus Creator** block from the **Commonly Used Blocks** library, and the **Scope** block into this model, and we interconnect these blocks as shown in Figure 1.24.

3. We double−click on the Signal Builder block in Figure 1.24, and on the plot which appears as a square pulse, we click on the y−axis and we enter Minimum: −2.5, and Maximum: 3.5. Likewise we right−click anywhere on the plot and we specify the Change Time Range at Min time: −2, and Max time: 8.

4. To select a particular point, we position the mouse cursor over that point and we left−click. A circle is drawn around that point to indicate that it is selected.

5. To select a line segment, we left−click on that segment. That line segment is now shown as a thick line indicating that it is selected. To deselect it, we press the Esc key.

6. To drag a line segment to a new position, we place the mouse cursor over that line segment and the cursor shape shows the position in which we can drag the segment.
7. To drag a point along the y–axis, we move the mouse cursor over that point, and the cursor changes to a circle indicating that we can drag that point. Then, we can move that point in a direction parallel to the x–axis.
8. To drag a point along the x–axis, we select that point, and we hold down the Shift key while dragging that point.
9. When we select a line segment on the time axis (x–axis) we observe that at the lower end of the waveform display window the **Left Point** and **Right Point** fields become visible. We can then reshape the given waveform by specifying the Time (**T**) and Amplitude (**Y**) points.

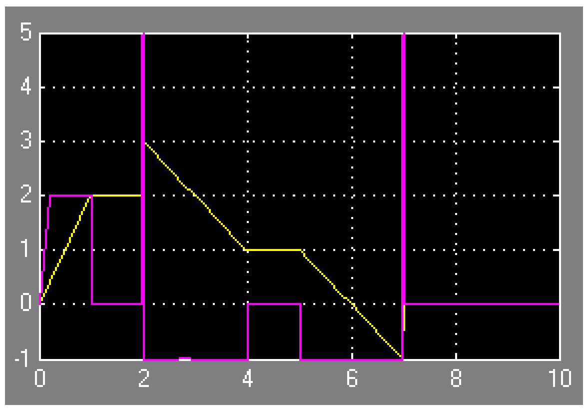


Figure 1.26. Waveforms for the Simulink model of Figure 1.24

The two positive spikes that occur at $t = 2$, and $t = 7$, are clearly shown in Figure 1.26.

MATLAB* has built-in functions for the unit step, and the delta functions. These are denoted by the names of the mathematicians who used them in their work. The unit step function $u_0(t)$ is referred to as **Heaviside(t)**, and the delta function $\delta(t)$ is referred to as **Dirac(t)**. Their use is illustrated with the examples below.

```
syms k a t;                      % Define symbolic variables
u=k*sym('Heaviside(t–a)')        % Create unit step function at t = a

u =
k*Heaviside(t-a)

d=diff(u)                        % Compute the derivative of the unit step function

d =
k*Dirac(t-a)
```

* *An introduction to MATLAB® is given in Appendix A.*

```
int(d)                    % Integrate the delta function

ans =
Heaviside(t-a)*k
```

1.6 Summary

- The unit step function $u_0(t)$ is defined as

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

- The unit step function offers a convenient method of describing the sudden application of a voltage or current source.
- The unit ramp function, denoted as $u_1(t)$, is defined as

$$u_1(t) = \int_{-\infty}^{t} u_0(\tau)d\tau$$

- The unit impulse or delta function, denoted as $\delta(t)$, is the derivative of the unit step $u_0(t)$. It is also defined as

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u_0(t)$$

and

$$\delta(t) = 0 \text{ for all } t \neq 0$$

- The sampling property of the delta function states that

$$f(t)\delta(t-a) = f(a)\delta(t)$$

or, when $a = 0$,

$$f(t)\delta(t) = f(0)\delta(t)$$

- The sifting property of the delta function states that

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$$

- The sampling property of the doublet function $\delta'(t)$ states that

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(a)\delta(t-a)$$

1.7 Exercises

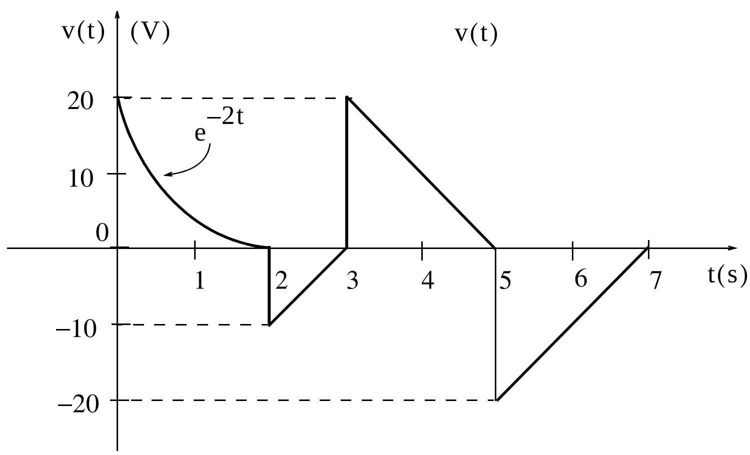
1. Evaluate the following functions:

a. $\sin t\delta\left(t-\frac{\pi}{6}\right)$ b. $\cos 2t\delta\left(t-\frac{\pi}{4}\right)$ c. $\cos^2 t\delta\left(t-\frac{\pi}{2}\right)$

d. $\tan 2t\delta\left(t-\frac{\pi}{8}\right)$ e. $\int_{-\infty}^{\infty} t^2 e^{-t}\delta(t-2)dt$ f. $\sin^2 t\delta^1\left(t-\frac{\pi}{2}\right)$

2.

a. Express the voltage waveform $v(t)$ shown below as a sum of unit step functions for the time interval $0 < t < 7$ s.



b. Using the result of part (a), compute the derivative of $v(t)$, and sketch its waveform. This waveform cannot be used with Sinulink's Function Builder block because it contains the decaying exponential segment which is a non–linear function.

Chapter 2

The Laplace Transformation

This chapter begins with an introduction to the Laplace transformation, definitions, and properties of the Laplace transformation. The initial value and final value theorems are also discussed and proved. It continues with the derivation of the Laplace transform of common functions of time, and concludes with the derivation of the Laplace transforms of common waveforms.

2.1 Definition of the Laplace Transformation

The *two–sided* or *bilateral* Laplace Transform pair is defined as

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt} \tag{2.1}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds} \tag{2.2}$$

where $\mathcal{L}\{f(t)\}$ denotes the Laplace transform of the time function $f(t)$, $\mathcal{L}^{-1}\{F(s)\}$ denotes the Inverse Laplace transform, and s is a complex variable whose real part is σ, and imaginary part ω, that is, $s = \sigma + j\omega$.

In most problems, we are concerned with values of time t greater than some reference time, say $t = t_0 = 0$, and since the initial conditions are generally known, the two–sided Laplace transform pair of (2.1) and (2.2) simplifies to the *unilateral* or *one–sided Laplace transform* defined as

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_{t_0}^{\infty} f(t)e^{-st}dt = \int_{0}^{\infty} f(t)e^{-st}dt} \tag{2.3}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds} \tag{2.4}$$

The Laplace Transform of (2.3) has meaning only if the integral converges (reaches a limit), that is, if

$$\left|\int_0^{\infty} f(t)e^{-st}dt\right| < \infty \tag{2.5}$$

To determine the conditions that will ensure us that the integral of (2.3) converges, we rewrite (2.5) as

$$\left|\int_0^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt\right| < \infty \tag{2.6}$$

The term $e^{-j\omega t}$ in the integral of (2.6) has magnitude of unity, i.e., $|e^{-j\omega t}| = 1$, and thus the condition for convergence becomes

$$\left|\int_0^{\infty} f(t)e^{-\sigma t}dt\right| < \infty \tag{2.7}$$

Fortunately, in most engineering applications the functions $f(t)$ are of *exponential order**. Then, we can express (2.7) as,

$$\left|\int_0^{\infty} f(t)e^{-\sigma t}dt\right| < \left|\int_0^{\infty} ke^{\sigma_0 t}e^{-\sigma t}dt\right| \tag{2.8}$$

and we see that the integral on the right side of the inequality sign in (2.8), converges if $\sigma > \sigma_0$. Therefore, we conclude that if $f(t)$ is of exponential order, $\mathcal{L}\{f(t)\}$ exists if

$$Re\{s\} = \sigma > \sigma_0 \tag{2.9}$$

where $Re\{s\}$ denotes the real part of the complex variable s.

Evaluation of the integral of (2.4) involves contour integration in the complex plane, and thus, it will not be attempted in this chapter. We will see in the next chapter that many Laplace transforms can be inverted with the use of a few standard pairs, and thus there is no need to use (2.4) to obtain the Inverse Laplace transform.

In our subsequent discussion, we will denote transformation from the time domain to the complex frequency domain, and vice versa, as

$$f(t) \Leftrightarrow F(s) \tag{2.10}$$

2.2 Properties and Theorems of the Laplace Transform

The most common properties and theorems of the Laplace transform are presented in Subsections 2.2.1 through 2.2.13 below.

* *A function* $f(t)$ *is said to be of exponential order if* $|f(t)| < ke^{\sigma_0 t}$ for all $t \ge 0$.

2.2.1 Linearity Property

The *linearity property* states that if

$$f_1(t), f_2(t), \ldots, f_n(t)$$

have Laplace transforms

$$F_1(s), F_2(s), \ldots, F_n(s)$$

respectively, and

$$c_1, c_2, \ldots, c_n$$

are arbitrary constants, then,

$$\boxed{c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t) \Leftrightarrow c_1 F_1(s) + c_2 F_2(s) + \ldots + c_n F_n(s)} \qquad (2.11)$$

Proof:

$$\begin{aligned}\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t)\} &= \int_{t_0}^{\infty} [c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t)]dt \\ &= c_1\int_{t_0}^{\infty} f_1(t)e^{-st}dt + c_2\int_{t_0}^{\infty} f_2(t)e^{-st}dt + \ldots + c_n\int_{t_0}^{\infty} f_n(t)e^{-st}dt \\ &= c_1 F_1(s) + c_2 F_2(s) + \ldots + c_n F_n(s)\end{aligned}$$

Note 1:

It is desirable to multiply $f(t)$ by the unit step function $u_0(t)$ to eliminate any unwanted non–zero values of $f(t)$ for $t < 0$.

2.2.2 Time Shifting Property

The *time shifting property* states that a right shift in the time domain by a units, corresponds to multiplication by e^{-as} in the complex frequency domain. Thus,

$$\boxed{f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)} \qquad (2.12)$$

Proof:

$$\mathcal{L}\{f(t-a)u_0(t-a)\} = \int_0^a 0e^{-st}dt + \int_a^{\infty} f(t-a)e^{-st}dt \qquad (2.13)$$

Now, we let $t - a = \tau$; then, $t = \tau + a$ and $dt = d\tau$. With these substitutions and with $a \to 0$, the second integral on the right side of (2.13) is expressed as

$$\int_0^{\infty} f(\tau)e^{-s(\tau + a)}d\tau = e^{-as}\int_0^{\infty} f(\tau)e^{-s\tau}d\tau = e^{-as}F(s)$$

2.2.3 Frequency Shifting Property

The *frequency shifting property* states that if we multiply a time domain function $f(t)$ by an exponential function e^{-at} where a is an arbitrary positive constant, this multiplication will produce a shift of the *s* variable in the complex frequency domain by a units. Thus,

$$\boxed{e^{-at}f(t) \Leftrightarrow F(s+a)} \tag{2.14}$$

Proof:

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^{\infty} e^{-at}f(t)e^{-st}dt = \int_0^{\infty} f(t)e^{-(s+a)t}dt = F(s+a)$$

Note 2:

A change of scale is represented by multiplication of the time variable t by a positive scaling factor a. Thus, the function $f(t)$ after scaling the time axis, becomes $f(at)$.

2.2.4 Scaling Property

Let a be an arbitrary positive constant; then, the *scaling property* states that

$$\boxed{f(at) \Leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right)} \tag{2.15}$$

Proof:

$$\mathcal{L}\{f(at)\} = \int_0^{\infty} f(at)e^{-st}dt$$

and letting $t = \tau/a$, we obtain

$$\mathcal{L}\{f(at)\} = \int_0^{\infty} f(\tau)e^{-s(\tau/a)}d\left(\frac{\tau}{a}\right) = \frac{1}{a}\int_0^{\infty} f(\tau)e^{-(s/a)\tau}d(\tau) = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Note 3:

Generally, the initial value of $f(t)$ is taken at $t = 0^-$ to include any discontinuity that may be present at $t = 0$. If it is known that no such discontinuity exists at $t = 0^-$, we simply interpret $f(0^-)$ as $f(0)$.

2.2.5 Differentiation in Time Domain Property

The *differentiation in time domain* property states that differentiation in the time domain corresponds to multiplication by s in the complex frequency domain, minus the initial value of $f(t)$ at $t = 0^-$. Thus,

$$\boxed{f'(t) = \frac{d}{dt} f(t) \Leftrightarrow sF(s) - f(0^-)} \tag{2.16}$$

Proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt$$

Using integration by parts where

$$\int v du = uv - \int u dv \tag{2.17}$$

we let $du = f'(t)$ and $v = e^{-st}$. Then, $u = f(t)$, $dv = -se^{-st}$, and thus

$$\mathcal{L}\{f'(t)\} = f(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}dt = \lim_{a \to \infty}\left[f(t)e^{-st}\Big|_{0^-}^{a}\right] + sF(s)$$

$$= \lim_{a \to \infty}[e^{-sa}f(a) - f(0^-)] + sF(s) = 0 - f(0^-) + sF(s)$$

The time differentiation property can be extended to show that

$$\boxed{\frac{d^2}{dt^2} f(t) \Leftrightarrow s^2F(s) - sf(0^-) - f'(0^-)} \tag{2.18}$$

$$\boxed{\frac{d^3}{dt^3} f(t) \Leftrightarrow s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)} \tag{2.19}$$

and in general

$$\boxed{\frac{d^n}{dt^n} f(t) \Leftrightarrow s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \ldots - f^{n-1}(0^-)} \tag{2.20}$$

To prove (2.18), we let

$$g(t) = f'(t) = \frac{d}{dt} f(t)$$

and as we found above,

$$\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0^-)$$

Then,

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0^-) = s[s\mathcal{L}[f(t)] - f(0^-)] - f'(0^-)$$

$$= s^2F(s) - sf(0^-) - f'(0^-)$$

Relations (2.19) and (2.20) can be proved by similar procedures.

We must remember that the terms $f(0^-)$, $f'(0^-)$, $f''(0^-)$, and so on, represent the initial conditions. Therefore, when all initial conditions are zero, and we differentiate a time function $f(t)$ n times, this corresponds to $F(s)$ multiplied by s to the nth power.

2.2.6 Differentiation in Complex Frequency Domain Property

This property states that *differentiation in complex frequency domain* and multiplication by minus one, corresponds to multiplication of $f(t)$ by t in the time domain. In other words,

$$\boxed{tf(t) \Leftrightarrow -\frac{d}{ds}F(s)} \tag{2.21}$$

Proof:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Differentiating with respect to s and applying *Leibnitz's rule** *for differentiation under the integral,* we obtain

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty}\frac{\partial}{\partial s}e^{-st}f(t)dt = \int_0^{\infty} -te^{-st}f(t)dt = -\int_0^{\infty}[tf(t)]e^{-st}dt = -\mathcal{L}[tf(t)]$$

In general,

$$\boxed{t^n f(t) \Leftrightarrow (-1)^n\frac{d^n}{ds^n}F(s)} \tag{2.22}$$

The proof for $n \geq 2$ follows by taking the second and higher–order derivatives of $F(s)$ with respect to s.

2.2.7 Integration in Time Domain Property

This property states that *integration in time domain* corresponds to $F(s)$ divided by s plus the initial value of $f(t)$ at $t = 0^-$, also divided by s. That is,

* *This rule states that if a function of a parameter* α *is defined by the equation* $F(\alpha) = \int_a^b f(x, \alpha)dx$ *where f is some known function of integration x and the parameter* α, *a and b are constants independent of x and* α, *and the partial derivative* $\partial f/\partial\alpha$ *exists and it is continuous, then* $\frac{dF}{d\alpha} = \int_a^b \frac{\partial(x,\alpha)}{\partial(\alpha)}dx$.

$$\boxed{\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(s)}{s} + \frac{f(0^-)}{s}} \tag{2.23}$$

Proof:

We begin by expressing the integral on the left side of (2.23) as two integrals, that is,

$$\int_{-\infty}^{t} f(\tau)d\tau = \int_{-\infty}^{0} f(\tau)d\tau + \int_{0}^{t} f(\tau)d\tau \tag{2.24}$$

The first integral on the right side of (2.24), represents a constant value since neither the upper, nor the lower limits of integration are functions of time, and this constant is an initial condition denoted as $f(0^-)$. We will find the Laplace transform of this constant, the transform of the second integral on the right side of (2.24), and will prove (2.23) by the linearity property. Thus,

$$\begin{aligned}\mathcal{L}\{f(0^-)\} &= \int_{0}^{\infty} f(0^-)e^{-st}dt = f(0^-)\int_{0}^{\infty} e^{-st}dt = f(0^-)\frac{e^{-st}}{-s}\bigg|_{0}^{\infty} \\ &= f(0^-)\times 0 - \left(-\frac{f(0^-)}{s}\right) = \frac{f(0^-)}{s}\end{aligned} \tag{2.25}$$

This is the value of the first integral in (2.24). Next, we will show that

$$\int_{0}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(s)}{s}$$

We let

$$g(t) = \int_{0}^{t} f(\tau)d\tau$$

then,

$$g'(t) = f(\tau)$$

and

$$g(0) = \int_{0}^{0} f(\tau)d\tau = 0$$

Now,

$$\mathcal{L}\{g'(t)\} = G(s) = s\mathcal{L}\{g(t)\} - g(0^-) = G(s) - 0$$

$$s\mathcal{L}\{g(t)\} = G(s)$$

$$\mathcal{L}\{g(t)\} = \frac{G(s)}{s}$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \tag{2.26}$$

and the proof of (2.23) follows from (2.25) and (2.26).

2.2.8 Integration in Complex Frequency Domain Property

This property states that *integration in complex frequency domain* with respect to s corresponds to division of a time function $f(t)$ by the variable t, provided that the limit $\lim_{t\to 0}\frac{f(t)}{t}$ exists. Thus,

$$\boxed{\frac{f(t)}{t} \Leftrightarrow \int_s^\infty F(s)ds} \tag{2.27}$$

Proof:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Integrating both sides from s to ∞, we obtain

$$\int_s^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty f(t)e^{-st}dt\right]ds$$

Next, we interchange the order of integration, i.e.,

$$\int_s^\infty F(s)ds = \int_0^\infty \left[\int_s^\infty e^{-st}ds\right] f(t)dt$$

and performing the inner integration on the right side integral with respect to s, we obtain

$$\int_s^\infty F(s)ds = \int_0^\infty \left[-\frac{1}{t}e^{-st}\Big|_s^\infty\right] f(t)dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

2.2.9 Time Periodicity Property

The *time periodicity* property states that a periodic function of time with period T corresponds to the integral $\int_0^T f(t)e^{-st}dt$ divided by $(1-e^{-sT})$ in the complex frequency domain. Thus, if we let $f(t)$ be a periodic function with period T, that is, $f(t) = f(t+nT)$, for $n = 1, 2, 3, \ldots$ we obtain the transform pair

$$\boxed{f(t+nT) \Leftrightarrow \frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}} \tag{2.28}$$

Proof:

The Laplace transform of a periodic function can be expressed as

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \int_{2T}^{3T} f(t)e^{-st}dt + \dots$$

In the first integral of the right side, we let $t = \tau$, in the second $t = \tau + T$, in the third $t = \tau + 2T$, and so on. The areas under each period of $f(t)$ are equal, and thus the upper and lower limits of integration are the same for each integral. Then,

$$\mathcal{L}\{f(t)\} = \int_0^T f(\tau)e^{-s\tau}d\tau + \int_0^T f(\tau+T)e^{-s(\tau+T)}d\tau + \int_0^T f(\tau+2T)e^{-s(\tau+2T)}d\tau + \dots \tag{2.29}$$

Since the function is periodic, i.e., $f(\tau) = f(\tau+T) = f(\tau+2T) = \dots = f(\tau+nT)$, we can write (2.29) as

$$\mathcal{L}\{f(\tau)\} = (1 + e^{-sT} + e^{-2sT} + \dots)\int_0^T f(\tau)e^{-s\tau}d\tau \tag{2.30}$$

By application of the binomial theorem, that is,

$$1 + a + a^2 + a^3 + \dots = \frac{1}{1-a} \tag{2.31}$$

we find that expression (2.30) reduces to

$$\mathcal{L}\{f(\tau)\} = \frac{\int_0^T f(\tau)e^{-s\tau}d\tau}{1-e^{-sT}}$$

2.2.10 Initial Value Theorem

The *initial value theorem* states that the initial value $f(0^-)$ of the time function $f(t)$ can be found from its Laplace transform multiplied by s and letting $s \to \infty$. That is,

$$\boxed{\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = f(0^-)} \tag{2.32}$$

Proof:

From the time domain differentiation property,

$$\frac{d}{dt} f(t) \Leftrightarrow sF(s) - f(0^-)$$

or

$$\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^-) = \int_0^\infty \frac{d}{dt} f(t) e^{-st} dt$$

Taking the limit of both sides by letting $s \to \infty$, we obtain

$$\lim_{s \to \infty} [sF(s) - f(0^-)] = \lim_{s \to \infty} \left[\lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^T \frac{d}{dt} f(t) e^{-st} dt \right]$$

Interchanging the limiting process, we obtain

$$\lim_{s \to \infty} [sF(s) - f(0^-)] = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^T \frac{d}{dt} f(t) \left[\lim_{s \to \infty} e^{-st} \right] dt$$

and since

$$\lim_{s \to \infty} e^{-st} = 0$$

the above expression reduces to

$$\lim_{s \to \infty} [sF(s) - f(0^-)] = 0$$

or

$$\lim_{s \to \infty} sF(s) = f(0^-)$$

2.2.11 Final Value Theorem

The *final value theorem* states that the final value $f(\infty)$ of the time function $f(t)$ can be found from its Laplace transform multiplied by s, then, letting $s \to 0$. That is,

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = f(\infty)} \qquad (2.33)$$

Proof:

From the time domain differentiation property,

$$\frac{d}{dt} f(t) \Leftrightarrow sF(s) - f(0^-)$$

or

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = sF(s) - f(0^-) = \int_0^\infty \frac{d}{dt}f(t)e^{-st}dt$$

Taking the limit of both sides by letting $s \to 0$, we obtain

$$\lim_{s \to 0}[sF(s)-f(0^-)] = \lim_{s \to 0}\left[\lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^T \frac{d}{dt}f(t)e^{-st}dt\right]$$

and by interchanging the limiting process, the expression above is written as

$$\lim_{s \to 0}[sF(s)-f(0^-)] = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^T \frac{d}{dt}f(t)\left[\lim_{s \to 0} e^{-st}\right]dt$$

Also, since

$$\lim_{s \to 0} e^{-st} = 1$$

it reduces to

$$\lim_{s \to 0}[sF(s)-f(0^-)] = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^T \frac{d}{dt}f(t)dt = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^T f(t) = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}}[f(T) - f(\varepsilon)] = f(\infty) - f(0^-)$$

Therefore,

$$\lim_{s \to 0} sF(s) = f(\infty)$$

2.2.12 Convolution in Time Domain Property

*Convolution** *in the time domain* corresponds to multiplication in the complex frequency domain, that is,

$$f_1(t)*f_2(t) \Leftrightarrow F_1(s)F_2(s) \tag{2.34}$$

* *Convolution is the process of overlapping two time functions* $f_1(t)$ *and* $f_2(t)$. *The convolution integral indicates the amount of overlap of one function as it is shifted over another function The convolution of two time functions* $f_1(t)$ *and* $f_2(t)$ *is denoted as* $f_1(t)*f_2(t)$, *and by definition,* $f_1(t)*f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau$ *where* τ *is a dummy variable. Convolution is discussed in detail in Chapter 6.*

Proof:

$$\mathcal{L}\{f_1(t)*f_2(t)\} = \mathcal{L}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau\right] = \int_0^{\infty}\left[\int_0^{\infty} f_1(\tau)f_2(t-\tau)d\tau\right]e^{-st}dt$$
$$= \int_0^{\infty} f_1(\tau)\left[\int_0^{\infty} f_2(t-\tau)e^{-st}dt\right]d\tau \tag{2.35}$$

We let $t-\tau = \lambda$; then, $t = \lambda+\tau$, and $dt = d\lambda$. Then, by substitution into (2.35),

$$\mathcal{L}\{f_1(t)*f_2(t)\} = \int_0^{\infty} f_1(\tau)\left[\int_0^{\infty} f_2(\lambda)e^{-s(\lambda+\tau)}d\lambda\right]d\tau = \int_0^{\infty} f_1(\tau)e^{-s\tau}d\tau\int_0^{\infty} f_2(\lambda)e^{-s\lambda}d\lambda$$
$$= F_1(s)F_2(s)$$

2.2.13 Convolution in Complex Frequency Domain Property

Convolution in the complex frequency domain divided by $1/2\pi j$, corresponds to multiplication in the time domain. That is,

$$f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi j}F_1(s)*F_2(s) \tag{2.36}$$

Proof:

$$\mathcal{L}\{f_1(t)f_2(t)\} = \int_0^{\infty} f_1(t)f_2(t)e^{-st}dt \tag{2.37}$$

and recalling that the Inverse Laplace transform from (2.2) is

$$f_1(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F_1(\mu)e^{\mu t}d\mu$$

by substitution into (2.37), we obtain

$$\mathcal{L}\{f_1(t)f_2(t)\} = \int_0^{\infty}\left[\frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F_1(\mu)e^{\mu t}d\mu\right]f_2(t)e^{-st}dt = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F_1(\mu)\left[\int_0^{\infty} f_2(t)e^{-(s-\mu)t}dt\right]d\mu$$

We observe that the bracketed integral is $F_2(s-\mu)$; therefore,

$$\mathcal{L}\{f_1(t)f_2(t)\} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F_1(\mu)F_2(s-\mu)d\mu = \frac{1}{2\pi j}F_1(s)*F_2(s)$$

For easy reference, the Laplace transform pairs and theorems are summarized in Table 2.1.

TABLE 2.1 Summary of Laplace Transform Properties and Theorems

	Property/Theorem	***Time Domain***	***Complex Frequency Domain***
1	Linearity	$c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t)$	$c_1 F_1(s) + c_2 F_2(s) + \ldots + c_n F_n(s)$
2	Time Shifting	$f(t-a)u_0(t-a)$	$e^{-as}F(s)$
3	Frequency Shifting	$e^{-as}f(t)$	$F(s+a)$
4	Time Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
5	Time Differentiation See also (2.18) through (2.20)	$\frac{d}{dt}f(t)$	$sF(s) - f(0^-)$
6	Frequency Differentiation See also (2.22)	$tf(t)$	$-\frac{d}{ds}F(s)$
7	Time Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{F(s)}{s} + \frac{f(0^-)}{s}$
8	Frequency Integration	$\frac{f(t)}{t}$	$\int_{s}^{\infty} F(s)ds$
9	Time Periodicity	$f(t+nT)$	$\frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}$
10	Initial Value Theorem	$\lim_{t \to 0} f(t)$	$\lim_{s \to \infty} sF(s) = f(0^-)$
11	Final Value Theorem	$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} sF(s) = f(\infty)$
12	Time Convolution	$f_1(t)*f_2(t)$	$F_1(s)F_2(s)$
13	Frequency Convolution	$f_1(t)f_2(t)$	$\frac{1}{2\pi j}F_1(s)*F_2(s)$

2.3 The Laplace Transform of Common Functions of Time

In this section, we will derive the Laplace transform of common functions of time. They are presented in Subsections 2.3.1 through 2.3.11 below.

2.3.1 The Laplace Transform of the Unit Step Function $u_0(t)$

We begin with the definition of the Laplace transform, that is,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

or

$$\mathcal{L}\{u_0(t)\} = \int_0^\infty 1e^{-st}dt = \left.\frac{-e^{st}}{s}\right|_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

Thus, we have obtained the transform pair

$$\boxed{u_0(t) \Leftrightarrow \frac{1}{s}} \qquad (2.38)$$

for $\mathrm{Re}\{s\} = \sigma > 0$.*

2.3.2 The Laplace Transform of the Ramp Function $u_1(t)$

We apply the definition

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

or

$$\mathcal{L}\{u_1(t)\} = \mathcal{L}\{t\} = \int_0^\infty te^{-st}dt$$

We will perform integration by parts by recalling that

$$\int udv = uv - \int vdu \qquad (2.39)$$

We let

$$u = t \text{ and } dv = e^{-st}$$

then,

$$du = 1 \text{ and } v = \frac{-e^{-st}}{s}$$

* *This condition was established in relation (2.9), Page 2−2.*

By substitution into (2.39),

$$\mathcal{L}\{t\} = \left.\frac{-te^{-st}}{s}\right|_0^{\infty} - \int_0^{\infty} \frac{-e^{-st}}{s}dt = \left.\left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]\right|_0^{\infty} \tag{2.40}$$

Since the upper limit of integration in (2.40) produces an indeterminate form, we apply *L' Hôpital's rule**, that is,

$$\lim_{t \to \infty} te^{-st} = \lim_{t \to \infty} \frac{t}{e^{st}} = \lim_{t \to \infty} \frac{\frac{d}{dt}(t)}{\frac{d}{dt}(e^{st})} = \lim_{t \to \infty} \frac{1}{se^{st}} = 0$$

Evaluating the second term of (2.40), we obtain $\mathcal{L}\{t\} = \frac{1}{s^2}$

Thus, we have obtained the transform pair

$$\boxed{t \Leftrightarrow \frac{1}{s^2}} \tag{2.41}$$

for $\sigma > 0$.

2.3.3 The Laplace Transform of $t^n u_0(t)$

Before deriving the Laplace transform of this function, we digress to review the *gamma* or *generalized factorial function* $\Gamma(n)$ which is an *improper integral*† but converges (approaches a limit) for all $n > 0$. It is defined as

* *Often, the ratio of two functions, such as* $\frac{f(x)}{g(x)}$, *for some value of x, say a, results in an indeterminate form. To work around this problem, we consider the limit* $\lim_{x \to a} \frac{f(x)}{g(x)}$, *and we wish to find this limit, if it exists. To find this limit, we use L'Hôpital's rule which states that if* $f(a) = g(a) = 0$, *and if the limit* $\frac{d}{dx}f(x)/\frac{d}{dx}g(x)$ *as x approaches a exists, then,* $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a}\left(\frac{d}{dx}f(x)/\frac{d}{dx}g(x)\right)$

† *Improper integrals are two types and these are:*

a. $\int_a^b f(x)dx$ *where the limits of integration a or b or both are infinite*

b. $\int_a^b f(x)dx$ *where f(x) becomes infinite at a value x between the lower and upper limits of integration inclusive.*

$$\boxed{\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx} \tag{2.42}$$

We will now derive the basic properties of the gamma function, and its relation to the well known factorial function

$$n! = n(n-1)(n-2) \cdot \quad \cdot 3 \cdot 2 \cdot 1$$

The integral of (2.42) can be evaluated by performing integration by parts. Thus, in (2.42) we let

$$u = e^{-x} \quad \text{and} \quad dv = x^{n-1}$$

Then,

$$du = -e^{-x} dx \quad \text{and} \quad v = \frac{x^n}{n}$$

and (2.42) is written as

$$\Gamma(n) = \left. \frac{x^n e^{-x}}{n} \right|_{x=0}^{\infty} + \frac{1}{n}\int_0^{\infty} x^n e^{-x} dx \tag{2.43}$$

With the condition that $n > 0$, the first term on the right side of (2.43) vanishes at the lower limit $x = 0$. It also vanishes at the upper limit as $x \to \infty$. This can be proved with L' Hôpital's rule by differentiating both numerator and denominator m times, where $m \ge n$. Then,

$$\lim_{x \to \infty} \frac{x^n e^{-x}}{n} = \lim_{x \to \infty} \frac{x^n}{ne^x} = \lim_{x \to \infty} \frac{\frac{d^m}{dx^m} x^n}{\frac{d^m}{dx^m} ne^x} = \lim_{x \to \infty} \frac{\frac{d^{m-1}}{dx^{m-1}} nx^{n-1}}{\frac{d^{m-1}}{dx^{m-1}} ne^x} = \dots$$

$$= \lim_{x \to \infty} \frac{n(n-1)(n-2)\dots(n-m+1)x^{n-m}}{ne^x} = \lim_{x \to \infty} \frac{(n-1)(n-2)\dots(n-m+1)}{x^{m-n} e^x} = 0$$

Therefore, (2.43) reduces to

$$\Gamma(n) = \frac{1}{n}\int_0^{\infty} x^n e^{-x} dx$$

and with (2.42), we have

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx = \frac{1}{n}\int_0^{\infty} x^n e^{-x} dx \tag{2.44}$$

By comparing the integrals in (2.44), we observe that

$$\boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}} \tag{2.45}$$

or

$$\boxed{n\Gamma(n) = \Gamma(n+1)} \tag{2.46}$$

It is convenient to use (2.45) for $n < 0$, and (2.46) for $n > 0$. From (2.45), we see that $\Gamma(n)$ becomes infinite as $n \to 0$.

For $n = 1$, (2.42) yields

$$\Gamma(1) = \int_0^{\infty} e^{-x} dx = -e^{-x}\Big|_0^{\infty} = 1 \tag{2.47}$$

and thus we have obtained the important relation,

$$\Gamma(1) = 1 \tag{2.48}$$

From the recurring relation of (2.46), we obtain

$$\begin{aligned} \Gamma(2) &= 1 \cdot \Gamma(1) = 1 \\ \Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1 = 2! \\ \Gamma(4) &= 3 \cdot \Gamma(3) = 3 \cdot 2 = 3! \end{aligned} \tag{2.49}$$

and in general

$$\boxed{\Gamma(n+1) = n!} \tag{2.50}$$

for $n = 1, 2, 3, \ldots$

The formula of (2.50) is a noteworthy relation; it establishes the relationship between the $\Gamma(n)$ function and the factorial $n!$

We now return to the problem of finding the Laplace transform pair for $t^n u_0 t$, that is,

$$\mathcal{L}\{t^n u_0 t\} = \int_0^{\infty} t^n e^{-st} dt \tag{2.51}$$

To make this integral resemble the integral of the gamma function, we let $st = y$, or $t = y/s$, and thus $dt = dy/s$. Now, we rewrite (2.51) as

$$\mathcal{L}\{t^n u_0 t\} = \int_0^{\infty} \left(\frac{y}{s}\right)^n e^{-y} d\left(\frac{y}{s}\right) = \frac{1}{s^{n+1}} \int_0^{\infty} y^n e^{-y} dy = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

Therefore, we have obtained the transform pair

$$\boxed{t^n u_0(t) \Leftrightarrow \frac{n!}{s^{n+1}}} \tag{2.52}$$

for positive integers of n and $\sigma > 0$.

2.3.4 The Laplace Transform of the Delta Function $\delta(t)$

We apply the definition

$$\mathcal{L}\{\delta(t)\} = \int_0^{\infty} \delta(t)e^{-st}dt$$

and using the sifting property of the delta function,* we obtain

$$\mathcal{L}\{\delta(t)\} = \int_0^{\infty} \delta(t)e^{-st}dt = e^{-s(0)} = 1$$

Thus, we have the transform pair

$$\boxed{\delta(t) \Leftrightarrow 1} \tag{2.53}$$

for all σ.

2.3.5 The Laplace Transform of the Delayed Delta Function $\delta(t-a)$

We apply the definition

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} \delta(t-a)e^{-st}dt$$

and again, using the sifting property of the delta function, we obtain

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} \delta(t-a)e^{-st}dt = e^{-as}$$

Thus, we have the transform pair

$$\boxed{\delta(t-a) \Leftrightarrow e^{-as}} \tag{2.54}$$

for $\sigma > 0$.

* *The sifting property of the* $\delta(t)$ *is described in Subsection 1.4.2, Chapter 1, Page 1−13.*

2.3.6 The Laplace Transform of $e^{-at}u_0(t)$

We apply the definition

$$\mathcal{L}\{e^{-at}u_0(t)\} = \int_0^{\infty} e^{-at}e^{-st}dt = \int_0^{\infty} e^{-(s+a)t}dt = \left(-\frac{1}{s+a}\right)e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$

Thus, we have the transform pair

$$\boxed{e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a}} \tag{2.55}$$

for $\sigma > -a$.

2.3.7 The Laplace Transform of $t^n e^{-at}u_0(t)$

For this derivation, we will use the transform pair of (2.52), i.e.,

$$t^n u_0(t) \Leftrightarrow \frac{n!}{s^{n+1}} \tag{2.56}$$

and the frequency shifting property of (2.14), that is,

$$e^{-at}f(t) \Leftrightarrow F(s+a) \tag{2.57}$$

Then, replacing s with $s+a$ in (2.56), we obtain the transform pair

$$\boxed{t^n e^{-at}u_0(t) \Leftrightarrow \frac{n!}{(s+a)^{n+1}}} \tag{2.58}$$

where n is a positive integer, and $\sigma > -a$. Thus, for $n = 1$, we obtain the transform pair

$$\boxed{te^{-at}u_0(t) \Leftrightarrow \frac{1}{(s+a)^2}} \tag{2.59}$$

for $\sigma > -a$.

For $n = 2$, we obtain the transform

$$\boxed{t^2 e^{-at}u_0(t) \Leftrightarrow \frac{2!}{(s+a)^3}} \tag{2.60}$$

and in general,

$$\boxed{t^n e^{-at} u_0(t) \Leftrightarrow \frac{n!}{(s+a)^{n+1}}} \tag{2.61}$$

for $\sigma > -a$.

2.3.8 The Laplace Transform of $\sin\omega t\, u_0(t)$

We apply the definition

$$\mathcal{L}\{\sin\omega t\, u_0(t)\} = \int_0^{\infty} (\sin\omega t)e^{-st}dt = \lim_{a\to\infty}\int_0^a (\sin\omega t)e^{-st}dt$$

and from tables of integrals*

$$\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

Then,

$$\mathcal{L}\{\sin\omega t\, u_0(t)\} = \lim_{a\to\infty} \left.\frac{e^{-st}(-s\sin\omega t - \omega\cos\omega t)}{s^2+\omega^2}\right|_0^a$$

$$= \lim_{a\to\infty}\left[\frac{e^{-as}(-s\sin\omega a - \omega\cos\omega a)}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right] = \frac{\omega}{s^2+\omega^2}$$

Thus, we have obtained the transform pair

$$\boxed{\sin\omega t\, u_0 t \Leftrightarrow \frac{\omega}{s^2+\omega^2}} \tag{2.62}$$

for $\sigma > 0$.

2.3.9 The Laplace Transform of $\cos\omega t\, u_0(t)$

We apply the definition

$$\mathcal{L}\{\cos\omega t\, u_0(t)\} = \int_0^{\infty} (\cos\omega t)e^{-st}dt = \lim_{a\to\infty}\int_0^a (\cos\omega t)e^{-st}dt$$

* *This can also be derived from* $\sin\omega t = \frac{1}{j2}(e^{j\omega t} - e^{-j\omega t})$, *and the use of (2.55) where* $e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a}$. *By the linearity property, the sum of these terms corresponds to the sum of their Laplace transforms. Therefore,* $\mathcal{L}[\sin\omega t u_0(t)] = \frac{1}{j2}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$

and from tables of integrals*

$$\int e^{ax} \cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

Then,

$$\mathcal{L}\{\cos\omega t\, u_0(t)\} = \lim_{a \to \infty} \left. \frac{e^{-st}(-s\cos\omega t + \omega\sin\omega t)}{s^2 + \omega^2} \right|_0^a$$

$$= \lim_{a \to \infty} \left[\frac{e^{-as}(-s\cos\omega a + \omega\sin\omega a)}{s^2 + \omega^2} + \frac{s}{s^2 + \omega^2} \right] = \frac{s}{s^2 + \omega^2}$$

Thus, we have the fransform pair

$$\boxed{\cos\omega t\; u_0 t \Leftrightarrow \frac{s}{s^2 + \omega^2}} \tag{2.63}$$

for $\sigma > 0$.

2.3.10 The Laplace Transform of $e^{-at}\sin\omega t\; u_0(t)$

From (2.62),

$$\sin\omega t u_0 t \Leftrightarrow \frac{\omega}{s^2 + \omega^2}$$

Using the frequency shifting property of (2.14), that is,

$$e^{-at} f(t) \Leftrightarrow F(s + a) \tag{2.64}$$

we replace s with $s + a$, and we obtain

$$\boxed{e^{-at}\sin\omega t\; u_0(t) \Leftrightarrow \frac{\omega}{(s + a)^2 + \omega^2}} \tag{2.65}$$

for $\sigma > 0$ and $a > 0$.

* *We can use the relation* $\cos\omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$ *and the linearity property, as in the derivation of the transform of* $\sin\omega t$ *on the footnote of the previous page. We can also use the transform pair* $\frac{d}{dt} f(t) \Leftrightarrow sF(s) - f(0^-)$; *this is the time differentiation property of* (2.16). *Applying this transform pair for this derivation, we obtain* $\mathcal{L}[\cos\omega t u_0(t)] = L\left[\frac{1}{\omega}\frac{d}{dt}\sin\omega t u_0(t)\right] = \frac{1}{\omega}\mathcal{L}\left[\frac{d}{dt}\sin\omega t u_0(t)\right] = \frac{1}{\omega} s \frac{\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2}$

2.3.11 The Laplace Transform of $e^{-at}\cos\omega t\, u_0(t)$

From (2.63),

$$\cos\omega t\, u_0(t) \Leftrightarrow \frac{s}{s^2+\omega^2}$$

and using the frequency shifting property of (2.14), we replace s with $s+a$, and we obtain

$$\boxed{e^{-at}\cos\omega t\, u_0(t) \Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2}} \qquad (2.66)$$

for $\sigma > 0$ and $a > 0$.

For easy reference, we have summarized the above derivations in Table 2.2.

TABLE 2.2 Laplace Transform Pairs for Common Functions

	$\mathbf{f(t)}$	$\mathbf{F(s)}$
1	$u_0(t)$	$1/s$
2	$tu_0(t)$	$1/s^2$
3	$t^n u_0(t)$	$\dfrac{n!}{s^{n+1}}$
4	$\delta(t)$	1
5	$\delta(t-a)$	e^{-as}
6	$e^{-at}u_0(t)$	$\dfrac{1}{s+a}$
7	$t^n e^{-at}u_0(t)$	$\dfrac{n!}{(s+a)^{n+1}}$
8	$\sin\omega t\, u_0(t)$	$\dfrac{\omega}{s^2+\omega^2}$
9	$\cos\omega t\, u_0(t)$	$\dfrac{s}{s^2+\omega^2}$
10	$e^{-at}\sin\omega t\, u_0(t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
11	$e^{-at}\cos\omega t\, u_0(t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

2.4 The Laplace Transform of Common Waveforms

In this section, we will present procedures for deriving the Laplace transform of common waveforms using the transform pairs of Tables 1 and 2. The derivations are described in Subsections 2.4.1 through 2.4.5 below.

2.4.1 The Laplace Transform of a Pulse

The waveform of a pulse, denoted as $f_P(t)$, is shown in Figure 2.1.

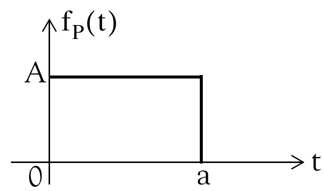


Figure 2.1. Waveform for a pulse

We first express the given waveform as a sum of unit step functions as we've learned in Chapter 1. Then,

$$f_P(t) = A[u_0(t) - u_0(t-a)] \tag{2.67}$$

From Table 2.1, Page 2–13,

$$f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)$$

and from Table 2.2, Page 2–22

$$u_0(t) \Leftrightarrow 1/s$$

Thus,

$$Au_0(t) \Leftrightarrow A/s$$

and

$$Au_0(t-a) \Leftrightarrow e^{-as}\frac{A}{s}$$

Then, in accordance with the linearity property, the Laplace transform of the pulse of Figure 2.1 is

$$\boxed{A[u_0(t) - u_0(t-a)] \Leftrightarrow \frac{A}{s} - e^{-as}\frac{A}{s} = \frac{A}{s}(1 - e^{-as})}$$

2.4.2 The Laplace Transform of a Linear Segment

The waveform of a linear segment, denoted as $f_L(t)$, is shown in Figure 2.2.

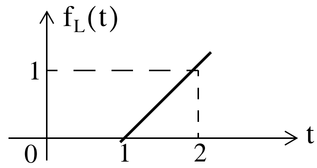


Figure 2.2. Waveform for a linear segment

We must first derive the equation of the linear segment. This is shown in Figure 2.3.

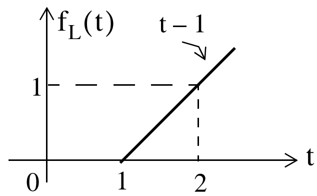


Figure 2.3. Waveform for a linear segment with the equation that describes it

Next, we express the given waveform in terms of the unit step function as follows:

$$f_L(t) = (t-1)u_0(t-1)$$

From Table 2.1, Page 2–13,

$$f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)$$

and from Table 2.2, Page 2–22,

$$tu_0(t) \Leftrightarrow \frac{1}{s^2}$$

Therefore, the Laplace transform of the linear segment of Figure 2.2 is

$$\boxed{(t-1)u_0(t-1) \Leftrightarrow e^{-s}\frac{1}{s^2}} \tag{2.68}$$

2.4.3 The Laplace Transform of a Triangular Waveform

The waveform of a triangular waveform, denoted as $f_T(t)$, is shown in Figure 2.4.

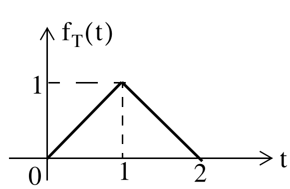


Figure 2.4. Triangular waveform

The equations of the linear segments are shown in Figure 2.5.

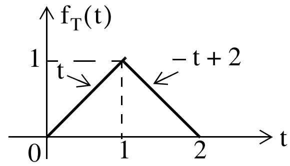


Figure 2.5. Triangular waveform with the equations of the linear segments

Next, we express the given waveform in terms of the unit step function.

$$\begin{aligned} f_T(t) &= t[u_0(t)-u_0(t-1)]+(-t+2)[u_0(t-1)-u_0(t-2)] \\ &= tu_0(t)-tu_0(t-1)-tu_0(t-1)+2u_0(t-1)+tu_0(t-2)-2u_0(t-2) \end{aligned}$$

Collecting like terms, we obtain

$$f_T(t) = tu_0(t)-2(t-1)u_0(t-1)+(t-2)u_0(t-2)$$

From Table 2.1, Page 2–13,

$$f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)$$

and from Table 2.2, Page 2–22,

$$tu_0(t) \Leftrightarrow \frac{1}{s^2}$$

Then,

$$tu_0(t)-2(t-1)u_0(t-1)+(t-2)u_0(t-2) \Leftrightarrow \frac{1}{s^2}-2e^{-s}\frac{1}{s^2}+e^{-2s}\frac{1}{s^2}$$

or

$$tu_0(t)-2(t-1)u_0(t-1)+(t-2)u_0(t-2) \Leftrightarrow \frac{1}{s^2}(1-2e^{-s}+e^{-2s})$$

Therefore, the Laplace transform of the triangular waveform of Figure 2.4 is

$$\boxed{f_T(t) \Leftrightarrow \frac{1}{s^2}(1-e^{-s})^2} \tag{2.69}$$

2.4.4 The Laplace Transform of a Rectangular Periodic Waveform

The waveform of a rectangular periodic waveform, denoted as $f_R(t)$, is shown in Figure 2.6. This is a periodic waveform with period $T = 2a$, and we can apply the time periodicity property

$$\mathcal{L}\{f(\tau)\} = \frac{\int_0^T f(\tau)e^{-s\tau}d\tau}{1-e^{-sT}}$$

where the denominator represents the periodicity of $f(t)$.

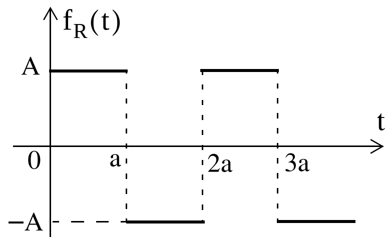


Figure 2.6. Rectangular periodic waveform

For this waveform,

$$\mathcal{L}\{f_R(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} f_R(t)e^{-st}dt = \frac{1}{1-e^{-2as}}\left[\int_0^a Ae^{-st}dt + \int_a^{2a}(-A)e^{-st}dt\right]$$

$$= \frac{A}{1-e^{-2as}}\left[\left.\frac{-e^{-st}}{s}\right|_0^a + \left.\frac{e^{-st}}{s}\right|_a^{2a}\right]$$

$$\mathcal{L}\{f_R(t)\} = \frac{A}{s(1-e^{-2as})}(-e^{-as}+1+e^{-2as}-e^{-as})$$

$$= \frac{A}{s(1-e^{-2as})}(1-2e^{-as}+e^{-2as}) = \frac{A(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})}$$

$$= \frac{A}{s}\frac{(1-e^{-as})}{(1+e^{-as})} = \frac{A}{s}\left(\frac{e^{as/2}e^{-as/2}-e^{-as/2}e^{-as/2}}{e^{as/2}e^{-as/2}+e^{-as/2}e^{-as/2}}\right)$$

$$= \frac{A}{s}\frac{e^{-as/2}}{e^{-as/2}}\left(\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right) = \frac{A}{s}\frac{\sinh(as/2)}{\cosh(as/2)}$$

$$\boxed{f_R(t) \Leftrightarrow \frac{A}{s}\tanh\left(\frac{as}{2}\right)} \qquad (2.70)$$

2.4.5 The Laplace Transform of a Half-Rectified Sine Waveform

The waveform of a half-rectified sine waveform, denoted as $f_{HW}(t)$, is shown in Figure 2.7. This is a periodic waveform with period $T = 2a$, and we can apply the time periodicity property

$$\mathcal{L}\{f(\tau)\} = \frac{\int_0^T f(\tau)e^{-s\tau}d\tau}{1-e^{-sT}}$$

where the denominator represents the periodicity of $f(t)$.

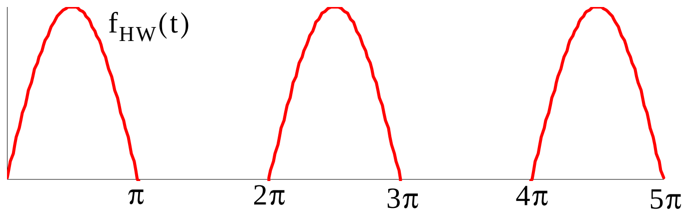


Figure 2.7. Half-rectified sine waveform*

For this waveform,

$$\mathcal{L}\{f_{HW}(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} f(t)e^{-st}dt = \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} \sin t e^{-st}dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-st}(s\sin t - \cos t)}{s^2+1}\right]\Bigg|_0^{\pi} = \frac{1}{(s^2+1)}\frac{(1+e^{-\pi s})}{(1-e^{-2\pi s})}$$

$$\mathcal{L}\{f_{HW}(t)\} = \frac{1}{(s^2+1)}\frac{(1+e^{-\pi s})}{(1+e^{-\pi s})(1-e^{-\pi s})}$$

$$\boxed{f_{HW}(t) \Leftrightarrow \frac{1}{(s^2+1)(1-e^{-\pi s})}} \qquad (2.71)$$

2.5 Using MATLAB for Finding the Laplace Transforms of Time Functions

We can use the MATLAB function laplace to find the Laplace transform of a time function. For examples, please type

```
help laplace
```

in MATLAB's Command prompt.

We will be using this function extensively in the subsequent chapters of this book.

* *This waveform was produced with the following MATLAB script:*

```
t=0:pi/64:5*pi; x=sin(t); y=sin(t–2*pi); z=sin(t–4*pi); plot(t,x,t,y,t,z); axis([0 5*pi 0 1])
```

2.6 Summary

- The two–sided or bilateral Laplace Transform pair is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds$$

where $\mathcal{L}\{f(t)\}$ denotes the Laplace transform of the time function $f(t)$, $\mathcal{L}^{-1}\{F(s)\}$ denotes the Inverse Laplace transform, and s is a complex variable whose real part is σ, and imaginary part ω, that is, $s = \sigma + j\omega$.

- The unilateral or one–sided Laplace transform defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{t_0}^{\infty} f(t)e^{-st}dt = \int_{0}^{\infty} f(t)e^{-st}dt$$

- We denote transformation from the time domain to the complex frequency domain, and vice versa, as

$$f(t) \Leftrightarrow F(s)$$

- The linearity property states that

$$c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t) \Leftrightarrow c_1 F_1(s) + c_2 F_2(s) + \ldots + c_n F_n(s)$$

- The time shifting property states that

$$f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)$$

- The frequency shifting property states that

$$e^{-at}f(t) \Leftrightarrow F(s+a)$$

- The scaling property states that

$$f(at) \Leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right)$$

- The differentiation in time domain property states that

$$f'(t) = \frac{d}{dt}f(t) \Leftrightarrow sF(s) - f(0^-)$$

Also,

$$\frac{d^2}{dt^2}f(t) \Leftrightarrow s^2F(s) - sf(0^-) - f'(0^-)$$

$$\frac{d^3}{dt^3}f(t) \Leftrightarrow s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$$

and in general

$$\frac{d^n}{dt^n}f(t) \Leftrightarrow s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \ldots - f^{n-1}(0^-)$$

where the terms $f(0^-)$, $f'(0^-)$, $f''(0^-)$, and so on, represent the initial conditions.

- The differentiation in complex frequency domain property states that

$$tf(t) \Leftrightarrow -\frac{d}{ds}F(s)$$

and in general,

$$t^nf(t) \Leftrightarrow (-1)^n\frac{d^n}{ds^n}F(s)$$

- The integration in time domain property states that

$$\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(s)}{s} + \frac{f(0^-)}{s}$$

- The integration in complex frequency domain property states that

$$\frac{f(t)}{t} \Leftrightarrow \int_{s}^{\infty} F(s)ds$$

provided that the limit $\lim_{t \to 0} \frac{f(t)}{t}$ exists.

- The time periodicity property states that

$$f(t + nT) \Leftrightarrow \frac{\int_0^T f(t)e^{-st}dt}{1 - e^{-sT}}$$

- The initial value theorem states that

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) = f(0^-)$$

- The final value theorem states that

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = f(\infty)$$

- Convolution in the time domain corresponds to multiplication in the complex frequency domain, that is,

$$f_1(t)*f_2(t) \Leftrightarrow F_1(s)F_2(s)$$

- Convolution in the complex frequency domain divided by $1/2\pi j$, corresponds to multiplication in the time domain. That is,

$$f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi j} F_1(s)*F_2(s)$$

- The Laplace transforms of some common functions of time are shown in Table 2.1, Page 2−13
- The Laplace transforms of some common waveforms are shown in Table 2.2, Page2−22
- We can use the MATLAB function **laplace** to find the Laplace transform of a time function

2.7 Exercises

1. Derive the Laplace transform of the following time domain functions:

   a. 12 b. $6u_0(t)$ c. $24u_0(t-12)$ d. $5tu_0(t)$ e. $4t^5u_0(t)$

2. Derive the Laplace transform of the following time domain functions:

   a. $j8$ b. $j5\angle{-90°}$ c. $5e^{-5t}u_0(t)$ d. $8t^7e^{-5t}u_0(t)$ e. $15\delta(t-4)$

3. Derive the Laplace transform of the following time domain functions:

   a. $(t^3+3t^2+4t+3)u_0(t)$ b. $3(2t-3)\delta(t-3)$

   c. $(3\sin 5t)u_0(t)$ d. $(5\cos 3t)u_0(t)$

   e. $(2\tan 4t)u_0(t)$ Be careful with this! Comment and you may skip derivation.

4. Derive the Laplace transform of the following time domain functions:

   a. $3t(\sin 5t)u_0(t)$ b. $2t^2(\cos 3t)u_0(t)$ c. $2e^{-5t}\sin 5t$

   d. $8e^{-3t}\cos 4t$ e. $(\cos t)\delta(t-\pi/4)$

5. Derive the Laplace transform of the following time domain functions:

   a. $5tu_0(t-3)$ b. $(2t^2-5t+4)u_0(t-3)$ c. $(t-3)e^{-2t}u_0(t-2)$

   d. $(2t-4)e^{2(t-2)}u_0(t-3)$ e. $4te^{-3t}(\cos 2t)u_0(t)$

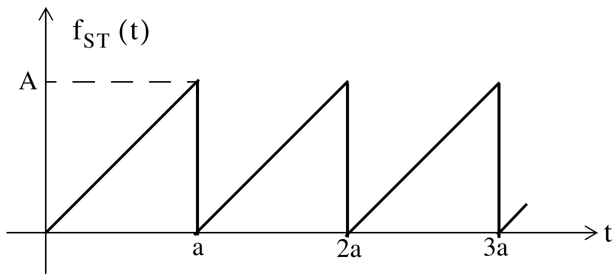
6. Derive the Laplace transform of the following time domain functions:

   a. $\dfrac{d}{dt}(\sin 3t)$ b. $\dfrac{d}{dt}(3e^{-4t})$ c. $\dfrac{d}{dt}(t^2\cos 2t)$ d. $\dfrac{d}{dt}(e^{-2t}\sin 2t)$ e. $\dfrac{d}{dt}(t^2e^{-2t})$

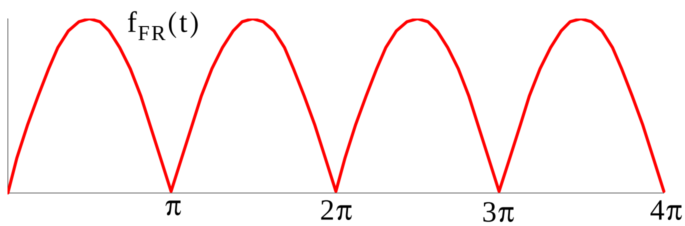
7. Derive the Laplace transform of the following time domain functions:

   a. $\dfrac{\sin t}{t}$ b. $\displaystyle\int_0^t \frac{\sin\tau}{\tau}d\tau$ c. $\dfrac{\sin at}{t}$ d. $\displaystyle\int_t^\infty \frac{\cos\tau}{\tau}d\tau$ e. $\displaystyle\int_t^\infty \frac{e^{-\tau}}{\tau}d\tau$

8. Derive the Laplace transform for the sawtooth waveform $f_{ST}(t)$ below.



9. Derive the Laplace transform for the full–rectified waveform $f_{FR}(t)$ below.



Write a simple MATLAB script that will produce the waveform above.

Chapter 3

The Inverse Laplace Transformation

This chapter is a continuation to the Laplace transformation topic of the previous chapter and presents several methods of finding the Inverse Laplace Transformation. The partial fraction expansion method is explained thoroughly and it is illustrated with several examples.

3.1 The Inverse Laplace Transform Integral

The Inverse Laplace Transform Integral was stated in the previous chapter; it is repeated here for convenience.

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds} \tag{3.1}$$

This integral is difficult to evaluate because it requires contour integration using complex variables theory. Fortunately, for most engineering problems we can refer to Tables of Properties, and Common Laplace transform pairs to lookup the Inverse Laplace transform.

3.2 Partial Fraction Expansion

Quite often the Laplace transform expressions are not in recognizable form, but in most cases appear in a rational form of s, that is,

$$F(s) = \frac{N(s)}{D(s)} \tag{3.2}$$

where $N(s)$ and $D(s)$ are polynomials, and thus (3.2) can be expressed as

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0} \tag{3.3}$$

The coefficients a_k and b_k are real numbers for $k = 1, 2, \ldots, n$, and if the highest power m of $N(s)$ is less than the highest power n of $D(s)$, i.e., $m < n$, $F(s)$ is said to be expressed as a *proper rational function*. If $m \ge n$, $F(s)$ is an *improper rational function.*

In a proper rational function, the roots of $N(s)$ in (3.3) are found by setting $N(s) = 0$; these are called the *zeros* of $F(s)$. The roots of $D(s)$, found by setting $D(s) = 0$, are called the *poles* of $F(s)$. We assume that $F(s)$ in (3.3) is a proper rational function. Then, it is customary and very convenient to make the coefficient of s^n unity; thus, we rewrite $F(s)$ as

$$F(s) = \frac{N(s)}{D(s)} = \frac{\frac{1}{a_n}(b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_1 s + b_0)}{s^n + \frac{a_{n-1}}{a_n}s^{n-1} + \frac{a_{n-2}}{a_n}s^{n-2} + \ldots + \frac{a_1}{a_n}s + \frac{a_0}{a_n}} \tag{3.4}$$

The zeros and poles of (3.4) can be real and distinct, repeated, complex conjugates, or combinations of real and complex conjugates. However, we are mostly interested in the nature of the poles, so we will consider each case separately, as indicated in Subsections 3.2.1 through 3.2.3 below.

3.2.1 Distinct Poles

If all the poles $p_1, p_2, p_3, \ldots, p_n$ of $F(s)$ are *distinct* (different from each another), we can factor the denominator of $F(s)$ in the form

$$F(s) = \frac{N(s)}{(s-p_1)\cdot(s-p_2)\cdot(s-p_3)\cdot\ldots\cdot(s-p_n)} \tag{3.5}$$

where p_k is distinct from all other poles. Next, using the *partial fraction expansion method,** we can express (3.5) as

$$F(s) = \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)} \tag{3.6}$$

where $r_1, r_2, r_3, \ldots, r_n$ are the *residues*, and $p_1, p_2, p_3, \ldots, p_n$ are the *poles* of $F(s)$.

To evaluate the residue r_k, we multiply both sides of (3.6) by $(s-p_k)$; then, we let $s \rightarrow p_k$, that is,

$$r_k = \lim_{s \rightarrow p_k}(s-p_k)F(s) = (s-p_k)F(s)\big|_{s = p_k} \tag{3.7}$$

Example 3.1

Use the partial fraction expansion method to simplify $F_1(s)$ of (3.8) below, and find the time domain function $f_1(t)$ corresponding to $F_1(s)$.

* *The partial fraction expansion method applies only to proper rational functions. It is used extensively in integration, and in finding the inverses of the Laplace transform, the Fourier transform, and the z-transform. This method allows us to decompose a rational polynomial into smaller rational polynomials with simpler denominators from which we can easily recognize their integrals and inverse transformations. This method is also being taught in intermediate algebra and introductory calculus courses.*

$$F_1(s) = \frac{3s+2}{s^2+3s+2} \tag{3.8}$$

Solution:

Using (3.6), we obtain

$$F_1(s) = \frac{3s+2}{s^2+3s+2} = \frac{3s+2}{(s+1)(s+2)} = \frac{r_1}{(s+1)} + \frac{r_2}{(s+2)} \tag{3.9}$$

The residues are

$$r_1 = \lim_{s \to -1}(s+1)F(s) = \left.\frac{3s+2}{(s+2)}\right|_{s=-1} = -1 \tag{3.10}$$

and

$$r_2 = \lim_{s \to -2}(s+2)F(s) = \left.\frac{3s+2}{(s+1)}\right|_{s=-2} = 4 \tag{3.11}$$

Therefore, we express (3.9) as

$$F_1(s) = \frac{3s+2}{s^2+3s+2} = \frac{-1}{(s+1)} + \frac{4}{(s+2)} \tag{3.12}$$

and from Table 2.2, Chapter 2, Page 2–22, we find that

$$e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a} \tag{3.13}$$

Therefore,

$$F_1(s) = \frac{-1}{(s+1)} + \frac{4}{(s+2)} \Leftrightarrow (-e^{-t} + 4e^{-2t})\,u_0(t) = f_1(t) \tag{3.14}$$

The residues and poles of a rational function of polynomials such as (3.8), can be found easily using the MATLAB **residue(a,b)** function. For this example, we use the script

```
Ns = [3, 2]; Ds = [1, 3, 2]; [r, p, k] = residue(Ns, Ds)
```

and MATLAB returns the values

```
r =
     4
    -1
p =
    -2
    -1
k =
     []
```

For the MATLAB script above, we defined Ns and Ds as two vectors that contain the numerator and denominator coefficients of $F(s)$. When this script is executed, MATLAB displays the r and p vectors that represent the residues and poles respectively. The first value of the vector r is associated with the first value of the vector p, the second value of r is associated with the second value of p, and so on.

The vector k is referred to as the *direct term* and it is always empty (has no value) whenever $F(s)$ is a proper rational function, that is, when the highest degree of the denominator is larger than that of the numerator. For this example, we observe that the highest power of the denominator is s^2, whereas the highest power of the numerator is s and therefore the direct term is empty.

We can also use the MATLAB **ilaplace(f)** function to obtain the time domain function directly from $F(s)$. This is done with the script that follows.

```
syms s t; Fs=(3*s+2)/(s^2+3*s+2); ft=ilaplace(Fs); pretty(ft)
```

When this script is executed, MATLAB displays the expression

```
4 exp(-2 t)- exp(-t)
```

Example 3.2

Use the partial fraction expansion method to simplify $F_2(s)$ of (3.15) below, and find the time domain function $f_2(t)$ corresponding to $F_2(s)$.

$$F_2(s) = \frac{3s^2 + 2s + 5}{s^3 + 12s^2 + 44s + 48} \tag{3.15}$$

Solution:

First, we use the MATLAB **factor(s)** symbolic function to express the denominator polynomial of $F_2(s)$ in factored form. For this example,

```
syms s; factor(s^3 + 12*s^2 + 44*s + 48)
```

```
ans =
(s+2)*(s+4)*(s+6)
```

Then,

$$F_2(s) = \frac{3s^2 + 2s + 5}{s^3 + 12s^2 + 44s + 48} = \frac{3s^2 + 2s + 5}{(s+2)(s+4)(s+6)} = \frac{r_1}{(s+2)} + \frac{r_2}{(s+4)} + \frac{r_3}{(s+6)} \tag{3.16}$$

The residues are

$$r_1 = \left.\frac{3s^2+2s+5}{(s+4)(s+6)}\right|_{s=-2} = \frac{9}{8} \tag{3.17}$$

$$r_2 = \left.\frac{3s^2+2s+5}{(s+2)(s+6)}\right|_{s=-4} = -\frac{37}{4} \tag{3.18}$$

$$r_3 = \left.\frac{3s^2+2s+5}{(s+2)(s+4)}\right|_{s=-6} = \frac{89}{8} \tag{3.19}$$

Then, by substitution into (3.16) we obtain

$$F_2(s) = \frac{3s^2+2s+5}{s^3+12s^2+44s+48} = \frac{9/8}{(s+2)} + \frac{-37/4}{(s+4)} + \frac{89/8}{(s+6)} \tag{3.20}$$

From Table 2.2, Chapter 2, Page 2–22,

$$e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a} \tag{3.21}$$

Therefore,

$$F_2(s) = \frac{9/8}{(s+2)} + \frac{-37/4}{(s+4)} + \frac{89/8}{(s+6)} \Leftrightarrow \left(\frac{9}{8}e^{-2t} - \frac{37}{4}e^{-4t} + \frac{89}{8}e^{-6t}\right)u_0(t) = f_2(t) \tag{3.22}$$

Check with MATLAB:

```
syms s t; Fs = (3*s^2 + 4*s + 5) / (s^3 + 12*s^2 + 44*s + 48); ft = ilaplace(Fs)
```

```
ft =
-37/4*exp(-4*t)+9/8*exp(-2*t)+89/8*exp(-6*t)
```

3.2.2 Complex Poles

Quite often, the poles of $F(s)$ are complex,* and since complex poles occur in complex conjugate pairs, the number of complex poles is even. Thus, if p_k is a complex root of $D(s)$, then, its complex conjugate pole, denoted as p_k^*, is also a root of $D(s)$. The partial fraction expansion method can also be used in this case, but it may be necessary to manipulate the terms of the expansion in order to express them in a recognizable form. The procedure is illustrated with the following example.

* *A review of complex numbers is presented in Appendix C*

Example 3.3

Use the partial fraction expansion method to simplify $F_3(s)$ of (3.23) below, and find the time domain function $f_3(t)$ corresponding to $F_3(s)$.

$$F_3(s) = \frac{s+3}{s^3+5s^2+12s+8} \tag{3.23}$$

Solution:

Let us first express the denominator in factored form to identify the poles of $F_3(s)$ using the MATLAB **factor(s)** symbolic function. Then,

```
syms s; factor(s^3 + 5*s^2 + 12*s + 8)
```

```
ans =
(s+1)*(s^2+4*s+8)
```

The **factor(s)** function did not factor the quadratic term. We will use the **roots(p)** function.

```
p=[1  4  8]; roots_p=roots(p)
```

```
roots_p =
  -2.0000 + 2.0000i
  -2.0000 - 2.0000i
```

Then,

$$F_3(s) = \frac{s+3}{s^3+5s^2+12s+8} = \frac{s+3}{(s+1)(s+2+j2)(s+2-j2)}$$

or

$$F_3(s) = \frac{s+3}{s^3+5s^2+12s+8} = \frac{r_1}{(s+1)} + \frac{r_2}{(s+2+j2)} + \frac{r_2{}^*}{(s+2-j2)} \tag{3.24}$$

The residues are

$$r_1 = \left.\frac{s+3}{s^2+4s+8}\right|_{s=-1} = \frac{2}{5} \tag{3.25}$$

$$\begin{aligned} r_2 &= \left.\frac{s+3}{(s+1)(s+2-j2)}\right|_{s=-2-j2} = \frac{1-j2}{(-1-j2)(-j4)} = \frac{1-j2}{-8+j4} \\ &= \frac{(1-j2)}{(-8+j4)}\frac{(-8-j4)}{(-8-j4)} = \frac{-16+j12}{80} = -\frac{1}{5}+\frac{j3}{20} \end{aligned} \tag{3.26}$$

$$r_2{}^* = \left(-\frac{1}{5}+\frac{j3}{20}\right)^* = -\frac{1}{5}-\frac{j3}{20} \tag{3.27}$$

By substitution into (3.24),

$$F_3(s) = \frac{2/5}{(s+1)} + \frac{-1/5 + j3/20}{(s+2+j2)} + \frac{-1/5 - j3/20}{(s+2-j2)} \tag{3.28}$$

The last two terms on the right side of (3.28), do not resemble any Laplace transform pair that we derived in Chapter 2. Therefore, we will express them in a different form. We combine them into a single term*, and now (3.28) is written as

$$F_3(s) = \frac{2/5}{(s+1)} - \frac{1}{5} \cdot \frac{(2s+1)}{(s^2+4s+8)} \tag{3.29}$$

For convenience, we denote the first term on the right side of (3.29) as $F_{31}(s)$, and the second as $F_{32}(s)$. Then,

$$F_{31}(s) = \frac{2/5}{(s+1)} \Leftrightarrow \frac{2}{5}e^{-t} = f_{31}(t) \tag{3.30}$$

Next, for $F_{32}(s)$

$$F_{32}(s) = -\frac{1}{5} \cdot \frac{(2s+1)}{(s^2+4s+8)} \tag{3.31}$$

From Table 2.2, Chapter 2, Page 2–22,

$$\begin{aligned} e^{-at} \sin\omega t u_0 t &\Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2} \\ e^{-at} \cos\omega t u_0 t &\Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2} \end{aligned} \tag{3.32}$$

Accordingly, we express $F_{32}(s)$ as

$$\begin{aligned} F_{32}(s) &= -\frac{2}{5}\left(\frac{s+\frac{1}{2}+\frac{3}{2}-\frac{3}{2}}{(s+2)^2+2^2)}\right) = -\frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)} + \frac{-3/2}{(s+2)^2+2^2)}\right) \\ &= -\frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)}\right) + \frac{6/10}{2}\left(\frac{2}{(s+2)^2+2^2)}\right) \\ &= -\frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)}\right) + \frac{3}{10}\left(\frac{2}{(s+2)^2+2^2)}\right) \end{aligned} \tag{3.33}$$

Addition of (3.30) with (3.33) yields

* *Here, we used MATLAB function* simple((–1/5 +3j/20)/(s+2+2j)+(–1/5 –3j/20)/(s+2–2j)). *The* **simple** *function, after several simplification tools that were displayed on the screen, returned* `(-2*s-1)/(5*s^2+20*s+40).`

$$F_3(s) = F_{31}(s) + F_{32}(s) = \frac{2/5}{(s+1)} - \frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2}\right) + \frac{3}{10}\left(\frac{2}{(s+2)^2+2^2}\right)$$

$$\Leftrightarrow \frac{2}{5}e^{-t} - \frac{2}{5}e^{-2t}\cos 2t + \frac{3}{10}e^{-2t}\sin 2t = f_3(t)$$

Check with MATLAB:

```
syms a s t w;           % Define several symbolic variables
Fs=(s + 3)/(s^3 + 5*s^2 + 12*s + 8); ft=ilaplace(Fs)
```

```
ft =
2/5*exp(-t)-2/5*exp(-2*t)*cos(2*t)
+3/10*exp(-2*t)*sin(2*t)
```

3.2.3 Multiple (Repeated) Poles

In this case, $F(s)$ has simple poles, but one of the poles, say p_1, has a multiplicity m. For this condition, we express it as

$$F(s) = \frac{N(s)}{(s-p_1)^m(s-p_2)\dots(s-p_{n-1})(s-p_n)} \tag{3.34}$$

Denoting the m residues corresponding to multiple pole p_1 as $r_{11}, r_{12}, r_{13}, \dots, r_{1m}$, the partial fraction expansion of (3.34) is expressed as

$$\begin{aligned} F(s) = {} & \frac{r_{11}}{(s-p_1)^m} + \frac{r_{12}}{(s-p_1)^{m-1}} + \frac{r_{13}}{(s-p_1)^{m-2}} + \dots + \frac{r_{1m}}{(s-p_1)} \\ & + \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \dots + \frac{r_n}{(s-p_n)} \end{aligned} \tag{3.35}$$

For the simple poles $p_1, p_2, \dots, p_n$, we proceed as before, that is, we find the residues from

$$r_k = \lim_{s \to p_k}(s-p_k)F(s) = (s-p_k)F(s)\big|_{s=p_k} \tag{3.36}$$

The residues $r_{11}, r_{12}, r_{13}, \dots, r_{1m}$ corresponding to the repeated poles, are found by multiplication of both sides of (3.35) by $(s-p)^m$. Then,

$$\begin{aligned} (s-p_1)^m F(s) = {} & r_{11} + (s-p_1)r_{12} + (s-p_1)^2 r_{13} + \dots + (s-p_1)^{m-1} r_{1m} \\ & + (s-p_1)^m\left(\frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \dots + \frac{r_n}{(s-p_n)}\right) \end{aligned} \tag{3.37}$$

Next, taking the limit as $s \to p_1$ on both sides of (3.37), we obtain

$$\lim_{s \to p_1} (s-p_1)^m F(s) = r_{11} + \lim_{s \to p_1} [(s-p_1)r_{12} + (s-p_1)^2 r_{13} + \ldots + (s-p_1)^{m-1} r_{1m}]$$
$$+ \lim_{s \to p_1} \left[(s-p_1)^m \left(\frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)}\right)\right]$$

or

$$r_{11} = \lim_{s \to p_1} (s-p_1)^m F(s) \tag{3.38}$$

and thus (3.38) yields the residue of the first repeated pole.

The residue r_{12} for the second repeated pole p_1, is found by differentiating (3.37) with respect to s and again, we let $s \to p_1$, that is,

$$r_{12} = \lim_{s \to p_1} \frac{d}{ds}[(s-p_1)^m F(s)] \tag{3.39}$$

In general, the residue r_{1k} can be found from

$$(s-p_1)^m F(s) = r_{11} + r_{12}(s-p_1) + r_{13}(s-p_1)^2 + \ldots \tag{3.40}$$

whose $(m-1)$th derivative of both sides is

$$(k-1)! r_{1k} = \lim_{s \to p_1} \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}}[(s-p_1)^m F(s)] \tag{3.41}$$

or

$$r_{1k} = \lim_{s \to p_1} \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}}[(s-p_1)^m F(s)] \tag{3.42}$$

Example 3.4

Use the partial fraction expansion method to simplify $F_4(s)$ of (3.43) below, and find the time domain function $f_4(t)$ corresponding to $F_4(s)$.

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2} \tag{3.43}$$

Solution:

We observe that there is a pole of multiplicity 2 at $s = -1$, and thus in partial fraction expansion form, $F_4(s)$ is written as

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2} = \frac{r_1}{(s+2)} + \frac{r_{21}}{(s+1)^2} + \frac{r_{22}}{(s+1)} \tag{3.44}$$

The residues are

$$r_1 = \left.\frac{s+3}{(s+1)^2}\right|_{s=-2} = 1$$

$$r_{21} = \left.\frac{s+3}{s+2}\right|_{s=-1} = 2$$

$$r_{22} = \left.\frac{d}{ds}\left(\frac{s+3}{s+2}\right)\right|_{s=-1} = \left.\frac{(s+2)-(s+3)}{(s+2)^2}\right|_{s=-1} = -1$$

The value of the residue r_{22} can also be found without differentiation as follows:

Substitution of the already known values of r_1 and r_{21} into (3.44), and letting $s = 0$*, we obtain

$$\left.\frac{s+3}{(s+1)^2(s+2)}\right|_{s=0} = \left.\frac{1}{(s+2)}\right|_{s=0} + \left.\frac{2}{(s+1)^2}\right|_{s=0} + \left.\frac{r_{22}}{(s+1)}\right|_{s=0}$$

or

$$\frac{3}{2} = \frac{1}{2} + 2 + r_{22}$$

from which $r_{22} = -1$ as before. Finally,

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2} = \frac{1}{(s+2)} + \frac{2}{(s+1)^2} + \frac{-1}{(s+1)} \Leftrightarrow e^{-2t} + 2te^{-t} - e^{-t} = f_4(t) \tag{3.45}$$

Check with MATLAB:

```
syms s t; Fs=(s+3)/((s+2)*(s+1)^2); ft=ilaplace(Fs)
```

```
ft = exp(-2*t)+2*t*exp(-t)-exp(-t)
```

We can use the following script to check the partial fraction expansion.

```
syms s
Ns = [1  3];            % Coefficients of the numerator N(s) of F(s)
expand((s + 1)^2);      % Expands (s + 1)^2 to s^2 + 2*s + 1;
d1 = [1  2  1];         % Coefficients of (s + 1)^2 = s^2 + 2*s + 1 term in D(s)
d2 = [0  1  2];         % Coefficients of (s + 2) term in D(s)
Ds=conv(d1,d2);         % Multiplies polynomials d1 and d2 to express the
                        % denominator D(s) of F(s) as a polynomial
[r,p,k]=residue(Ns,Ds)
```

* *This is permissible since (3.44) is an identity.*

```
r =
    1.0000
   -1.0000
    2.0000
p =
   -2.0000
   -1.0000
   -1.0000
k =
     []
```

Example 3.5

Use the partial fraction expansion method to simplify $F_5(s)$ of (3.46) below, and find the time domain function $f_5(t)$ corresponding to the given $F_5(s)$.

$$F_5(s) = \frac{s^2 + 3s + 1}{(s+1)^3(s+2)^2} \tag{3.46}$$

Solution:

We observe that there is a pole of multiplicity 3 at $s = -1$, and a pole of multiplicity 2 at $s = -2$. Then, in partial fraction expansion form, $F_5(s)$ is written as

$$F_5(s) = \frac{r_{11}}{(s+1)^3} + \frac{r_{12}}{(s+1)^2} + \frac{r_{13}}{(s+1)} + \frac{r_{21}}{(s+2)^2} + \frac{r_{22}}{(s+2)} \tag{3.47}$$

The residues are

$$r_{11} = \left.\frac{s^2 + 3s + 1}{(s+2)^2}\right|_{s=-1} = -1$$

$$r_{12} = \left.\frac{d}{ds}\left(\frac{s^2 + 3s + 1}{(s+2)^2}\right)\right|_{s=-1}$$

$$= \left.\frac{(s+2)^2(2s+3) - 2(s+2)(s^2+3s+1)}{(s+2)^4}\right|_{s=-1} = \left.\frac{s+4}{(s+2)^3}\right|_{s=-1} = 3$$

$$r_{13} = \frac{1}{2!}\frac{d^2}{ds^2}\left(\frac{s^2+3s+1}{(s+2)^2}\right)\Bigg|_{s=-1} = \frac{1}{2}\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s^2+3s+1}{(s+2)^2}\right)\right]\Bigg|_{s=-1}$$

$$= \frac{1}{2}\frac{d}{ds}\left(\frac{s+4}{(s+2)^3}\right)\Bigg|_{s=-1} = \frac{1}{2}\left[\frac{(s+2)^3-3(s+2)^2(s+4)}{(s+2)^6}\right]\Bigg|_{s=-1}$$

$$= \frac{1}{2}\left(\frac{s+2-3s-12}{(s+2)^4}\right)\Bigg|_{s=-1} = \frac{-s-5}{(s+2)^4}\Bigg|_{s=-1} = -4$$

Next, for the pole at $s = -2$,

$$r_{21} = \frac{s^2+3s+1}{(s+1)^3}\Bigg|_{s=-2} = 1$$

and

$$r_{22} = \frac{d}{ds}\left(\frac{s^2+3s+1}{(s+1)^3}\right)\Bigg|_{s=-2} = \frac{(s+1)^3(2s+3)-3(s+1)^2(s^2+3s+1)}{(s+1)^6}\Bigg|_{s=-2}$$

$$= \frac{(s+1)(2s+3)-3(s^2+3s+1)}{(s+1)^4}\Bigg|_{s=-2} = \frac{-s^2-4s}{(s+1)^4}\Bigg|_{s=-2} = 4$$

By substitution of the residues into (3.47), we obtain

$$F_5(s) = \frac{-1}{(s+1)^3} + \frac{3}{(s+1)^2} + \frac{-4}{(s+1)} + \frac{1}{(s+2)^2} + \frac{4}{(s+2)} \qquad (3.48)$$

We will check the values of these residues with the MATLAB script below.

```
syms s;     % The function collect(s) below multiplies (s+1)^3 by (s+2)^2
            % and we use it to express the denominator D(s) as a polynomial so that we can
            % use the coefficients of the resulting polynomial with the residue function
Ds=collect(((s+1)^3)*((s+2)^2))

Ds =
s^5+7*s^4+19*s^3+25*s^2+16*s+4

Ns=[1 3 1];  Ds=[1 7 19 25 16 4];  [r,p,k]=residue(Ns,Ds)
r =
     4.0000
     1.0000
    -4.0000
     3.0000
    -1.0000
```

```
p =
   -2.0000
   -2.0000
   -1.0000
   -1.0000
   -1.0000
k =
     []
```

From Table 2.2, Chapter 2, Page 2–22,

$$e^{-at} \Leftrightarrow \frac{1}{s+a} \qquad te^{-at} \Leftrightarrow \frac{1}{(s+a)^2} \qquad t^{n-1}e^{-at} \Leftrightarrow \frac{(n-1)!}{(s+a)^n}$$

and with these, we derive $f_5(t)$ from (3.48) as

$$f_5(t) = -\frac{1}{2}t^2e^{-t} + 3te^{-t} - 4e^{-t} + te^{-2t} + 4e^{-2t} \tag{3.49}$$

We can verify (3.49) with MATLAB as follows:

```
syms s t; Fs=-1/((s+1)^3) + 3/((s+1)^2) - 4/(s+1) + 1/((s+2)^2) + 4/(s+2); ft=ilaplace(Fs)
```

```
ft = -1/2*t^2*exp(-t)+3*t*exp(-t)-4*exp(-t)
          +t*exp(-2*t)+4*exp(-2*t)
```

3.3 Case where F(s) is Improper Rational Function

Our discussion thus far, was based on the condition that $F(s)$ is a proper rational function. However, if $F(s)$ is an improper rational function, that is, if $m \ge n$, we must first divide the numerator $N(s)$ by the denominator $D(s)$ to obtain an expression of the form

$$F(s) = k_0 + k_1s + k_2s^2 + \ldots + k_{m-n}s^{m-n} + \frac{N(s)}{D(s)} \tag{3.50}$$

where $N(s)/D(s)$ is a proper rational function.

Example 3.6

Derive the Inverse Laplace transform $f_6(t)$ of

$$F_6(s) = \frac{s^2 + 2s + 2}{s+1} \tag{3.51}$$

Solution:

For this example, $F_6(s)$ is an improper rational function. Therefore, we must express it in the form of (3.50) before we use the partial fraction expansion method.

By long division, we obtain

$$F_6(s) = \frac{s^2+2s+2}{s+1} = \frac{1}{s+1}+1+s$$

Now, we recognize that

$$\frac{1}{s+1} \Leftrightarrow e^{-t}$$

and

$$1 \Leftrightarrow \delta(t)$$

but

$$s \Leftrightarrow ?$$

To answer that question, we recall that

$$u_0'(t) = \delta(t)$$

and

$$u_0''(t) = \delta'(t)$$

where $\delta'(t)$ is the doublet of the delta function. Also, by the time differentiation property

$$u_0''(t) = \delta'(t) \Leftrightarrow s^2F(s) - sf(0) - f'(0) = s^2F(s) = s^2 \cdot \frac{1}{s} = s$$

Therefore, we have the new transform pair

$$s \Leftrightarrow \delta'(t) \tag{3.52}$$

and thus,

$$F_6(s) = \frac{s^2+2s+2}{s+1} = \frac{1}{s+1}+1+s \Leftrightarrow e^{-t}+\delta(t)+\delta'(t) = f_6(t) \tag{3.53}$$

In general,

$$\frac{d^n}{dt^n}\delta(t) \Leftrightarrow s^n \tag{3.54}$$

We verify (3.53) with MATLAB as follows:

```
Ns = [1  2  2]; Ds = [1  1]; [r, p, k] = residue(Ns, Ds)
r =
     1
p =
    -1
```

```
k =
     1     1
```

The direct terms `k= [1 1]` above are the coefficients of $\delta(t)$ and $\delta'(t)$ respectively.

3.4 Alternate Method of Partial Fraction Expansion

Partial fraction expansion can also be performed with the *method of clearing the fractions*, that is, making the denominators of both sides the same, then equating the numerators. As before, we assume that $F(s)$ is a proper rational function. If not, we first perform a long division, and then work with the quotient and the remainder as we did in Example 3.6. We also assume that the denominator $D(s)$ can be expressed as a product of real linear and quadratic factors. If these assumptions prevail, we let $(s-a)$ be a linear factor of $D(s)$, and we assume that $(s-a)^m$ is the highest power of $(s-a)$ that divides $D(s)$. Then, we can express $F(s)$ as

$$F(s) = \frac{N(s)}{D(s)} = \frac{r_1}{s-a} + \frac{r_2}{(s-a)^2} + \ldots \frac{r_m}{(s-a)^m} \tag{3.55}$$

Let $s^2 + \alpha s + \beta$ be a quadratic factor of $D(s)$, and suppose that $(s^2 + \alpha s + \beta)^n$ is the highest power of this factor that divides $D(s)$. Now, we perform the following steps:

1. To this factor, we assign the sum of *n* partial fractions, that is,

$$\frac{r_1 s + k_1}{s^2 + \alpha s + \beta} + \frac{r_2 s + k_2}{(s^2 + \alpha s + \beta)^2} + \ldots + \frac{r_n s + k_n}{(s^2 + \alpha s + \beta)^n}$$

2. We repeat step 1 for each of the distinct linear and quadratic factors of $D(s)$
3. We set the given $F(s)$ equal to the sum of these partial fractions
4. We clear the resulting expression of fractions and arrange the terms in decreasing powers of s
5. We equate the coefficients of corresponding powers of s
6. We solve the resulting equations for the residues

Example 3.7

Express $F_7(s)$ of (3.56) below as a sum of partial fractions using the method of clearing the fractions.

$$F_7(s) = \frac{-2s+4}{(s^2+1)(s-1)^2} \tag{3.56}$$

Solution:

Using Steps 1 through 3 above, we obtain

$$F_7(s) = \frac{-2s+4}{(s^2+1)(s-1)^2} = \frac{r_1s+A}{(s^2+1)} + \frac{r_{21}}{(s-1)^2} + \frac{r_{22}}{(s-1)} \tag{3.57}$$

With Step 4,

$$-2s+4 = (r_1s+A)(s-1)^2 + r_{21}(s^2+1) + r_{22}(s-1)(s^2+1) \tag{3.58}$$

and with Step 5,

$$\begin{aligned} -2s+4 &= (r_1+r_{22})s^3 + (-2r_1+A-r_{22}+r_{21})s^2 \\ &\quad + (r_1-2A+r_{22})\,s + (A-r_{22}+r_{21}) \end{aligned} \tag{3.59}$$

Relation (3.59) will be an identity is s if each power of s is the same on both sides of this relation. Therefore, we equate like powers of s and we obtain

$$\begin{aligned} 0 &= r_1+r_{22} \\ 0 &= -2r_1+A-r_{22}+r_{21} \\ -2 &= r_1-2A+r_{22} \\ 4 &= A-r_{22}+r_{21} \end{aligned} \tag{3.60}$$

Subtracting the second equation of (3.60) from the fourth, we obtain

$$4 = 2r_1$$

or

$$r_1 = 2 \tag{3.61}$$

By substitution of (3.61) into the first equation of (3.60), we obtain

$$0 = 2+r_{22}$$

or

$$r_{22} = -2 \tag{3.62}$$

Next, substitution of (3.61) and (3.62) into the third equation of (3.60) yields

$$-2 = 2-2A-2$$

or

$$A = 1 \tag{3.63}$$

Finally by substitution of (3.61), (3.62), and (3.63) into the fourth equation of (3.60), we obtain

$$4 = 1 + 2 + r_{21}$$

or

$$r_{21} = 1 \tag{3.64}$$

Substitution of these values into (3.57) yields

$$F_7(s) = \frac{-2s+4}{(s^2+1)(s-1)^2} = \frac{2s+1}{(s^2+1)} + \frac{1}{(s-1)^2} - \frac{2}{(s-1)} \tag{3.65}$$

Example 3.8

Use partial fraction expansion to simplify $F_8(s)$ of (3.66) below, and find the time domain function $f_8(t)$ corresponding to $F_8(s)$.

$$F_8(s) = \frac{s+3}{s^3+5s^2+12s+8} \tag{3.66}$$

Solution:

This is the same transform as in Example 3.3, Page 3–6, where we found that the denominator $D(s)$ can be expressed in factored form of a linear term and a quadratic. Thus, we write $F_8(s)$ as

$$F_8(s) = \frac{s+3}{(s+1)(s^2+4s+8)} \tag{3.67}$$

and using the method of clearing the fractions, we express (3.67) as

$$F_8(s) = \frac{s+3}{(s+1)(s^2+4s+8)} = \frac{r_1}{s+1} + \frac{r_2 s + r_3}{s^2+4s+8} \tag{3.68}$$

As in Example 3.3,

$$r_1 = \left.\frac{s+3}{s^2+4s+8}\right|_{s=-1} = \frac{2}{5} \tag{3.69}$$

Next, to compute r_2 and r_3, we follow the procedure of this section and we obtain

$$(s+3) = r_1(s^2+4s+8) + (r_2 s + r_3)(s+1) \tag{3.70}$$

Since r_1 is already known, we only need two equations in r_2 and r_3. Equating the coefficient of s^2 on the left side, which is zero, with the coefficients of s^2 on the right side of (3.70), we obtain

$$0 = r_1 + r_2 \tag{3.71}$$

and since $r_1 = 2/5$, it follows that $r_2 = -2/5$.

To obtain the third residue r_3, we equate the constant terms of (3.70). Then, $3 = 8r_1 + r_3$ or $3 = 8 \times 2/5 + r_3$, or $r_3 = -1/5$. Then, by substitution into (3.68), we obtain

$$F_8(s) = \frac{2/5}{(s+1)} - \frac{1}{5} \cdot \frac{(2s+1)}{(s^2+4s+8)} \tag{3.72}$$

as before.

The remaining steps are the same as in Example 3.3, and thus $f_8(t)$ is the same as $f_3(t)$, that is,

$$f_8(t) = f_3(t) = \left(\frac{2}{5}e^{-t} - \frac{2}{5}e^{-2t}\cos 2t + \frac{3}{10}e^{-2t}\sin 2t\right)u_0(t)$$

3.5 Summary

- The Inverse Laplace Transform Integral defined as

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds$$

  is difficult to evaluate because it requires contour integration using complex variables theory.

- For most engineering problems we can refer to Tables of Properties, and Common Laplace transform pairs to lookup the Inverse Laplace transform.

- The partial fraction expansion method offers a convenient means of expressing Laplace transforms in a recognizable form from which we can obtain the equivalent time–domain functions.

- If the highest power m of the numerator $N(s)$ is less than the highest power n of the denominator $D(s)$, i.e., $m < n$, $F(s)$ is said to be expressed as a proper rational function. If $m \ge n$, $F(s)$ is an improper rational function.

- The Laplace transform $F(s)$ must be expressed as a proper rational function before applying the partial fraction expansion. If $F(s)$ is an improper rational function, that is, if $m \ge n$, we must first divide the numerator $N(s)$ by the denominator $D(s)$ to obtain an expression of the form

$$F(s) = k_0 + k_1 s + k_2 s^2 + \ldots + k_{m-n}s^{m-n} + \frac{N(s)}{D(s)}$$

- In a proper rational function, the roots of numerator $N(s)$ are called the zeros of $F(s)$ and the roots of the denominator $D(s)$ are called the poles of $F(s)$.

- The partial fraction expansion method can be applied whether the poles of $F(s)$ are distinct, complex conjugates, repeated, or a combination of these.

- When $F(s)$ is expressed as

$$F(s) = \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)}$$

  $r_1, r_2, r_3, \ldots, r_n$ are called the residues and $p_1, p_2, p_3, \ldots, p_n$ are the poles of $F(s)$.

- The residues and poles of a rational function of polynomials can be found easily using the MATLAB **residue(a,b)** function. The direct term is always empty (has no value) whenever $F(s)$ is a proper rational function.

- We can use the MATLAB **factor(s)** symbolic function to convert the denominator polynomial form of $F_2(s)$ into a factored form.

- We can use the MATLAB **collect(s)** and **expand(s)** symbolic functions to convert the denominator factored form of $F_2(s)$ into a polynomial form.

- In this chapter we introduced the new transform pair

$$s \Leftrightarrow \delta'(t)$$

and in general,

$$\frac{d^n}{dt^n}\delta(t) \Leftrightarrow s^n$$

- The method of clearing the fractions is an alternate method of partial fraction expansion.

3.6 Exercises

1. Find the Inverse Laplace transform of the following:

a. $\dfrac{4}{s+3}$ b. $\dfrac{4}{(s+3)^2}$ c. $\dfrac{4}{(s+3)^4}$ d. $\dfrac{3s+4}{(s+3)^5}$ e. $\dfrac{s^2+6s+3}{(s+3)^5}$

2. Find the Inverse Laplace transform of the following:

a. $\dfrac{3s+4}{s^2+4s+85}$ b. $\dfrac{4s+5}{s^2+5s+18.5}$ c. $\dfrac{s^2+3s+2}{s^3+5s^2+10.5s+9}$

d. $\dfrac{s^2-16}{s^3+8s^2+24s+32}$ e. $\dfrac{s+1}{s^3+6s^2+11s+6}$

3. Find the Inverse Laplace transform of the following:

a. $\dfrac{3s+2}{s^2+25}$ b. $\dfrac{5s^2+3}{(s^2+4)^2}$ Hint: $\left\{\begin{array}{l} \dfrac{1}{2\alpha}(\sin\alpha t+\alpha t\cos\alpha t) \Leftrightarrow \dfrac{s^2}{(s^2+\alpha^2)^2} \\ \dfrac{1}{2\alpha^3}(\sin\alpha t-\alpha t\cos\alpha t) \Leftrightarrow \dfrac{1}{(s^2+\alpha^2)^2} \end{array}\right\}$

c. $\dfrac{2s+3}{s^2+4.25s+1}$ d. $\dfrac{s^3+8s^2+24s+32}{s^2+6s+8}$ e. $e^{-2s}\dfrac{3}{(2s+3)^3}$

4. Use the Initial Value Theorem to find $f(0)$ given that the Laplace transform of $f(t)$ is

$$\frac{2s+3}{s^2+4.25s+1}$$

Compare your answer with that of Exercise 3(c).

5. It is known that the Laplace transform $F(s)$ has two distinct poles, one at $s = 0$, the other at $s = -1$. It also has a single zero at $s = 1$, and we know that $\lim_{t \to \infty} f(t) = 10$. Find $F(s)$ and $f(t)$.

Chapter 4

Circuit Analysis with Laplace Transforms

This chapter presents applications of the Laplace transform. Several examples are presented to illustrate how the Laplace transformation is applied to circuit analysis. Complex impedance, complex admittance, and transfer functions are also defined.

4.1 Circuit Transformation from Time to Complex Frequency

In this section we will show the voltage–current relationships for the three elementary circuit networks, i.e., resistive, inductive, and capacitive in the time and complex frequency domains. They are shown in Subsections 4.1.1 through 4.1.3 below.

4.1.1 Resistive Network Transformation

The time and complex frequency domains for purely resistive networks are shown in Figure 4.1.

Time Domain:

$$v_R(t) = Ri_R(t) \qquad i_R(t) = \frac{v_R(t)}{R}$$

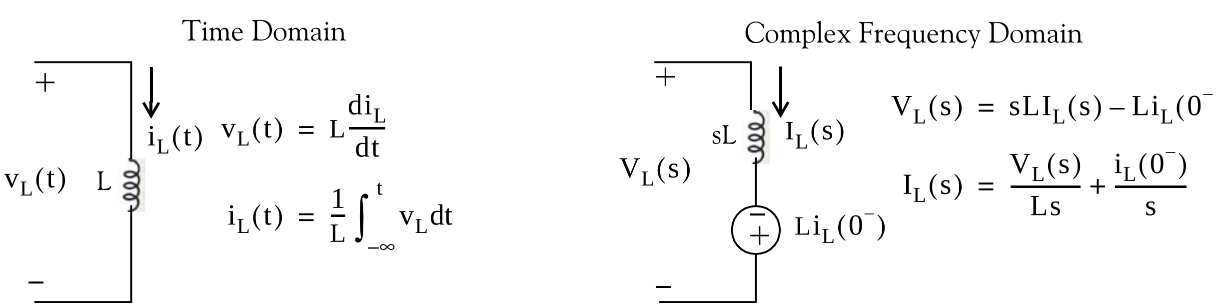
Complex Frequency Domain:

$$V_R(s) = RI_R(s) \qquad I_R(s) = \frac{V_R(s)}{R}$$

Figure 4.1. Resistive network in time domain and complex frequency domain

4.1.2 Inductive Network Transformation

The time and complex frequency domains for purely inductive networks are shown in Figure 4.2.



Time Domain:

$$v_L(t) = L\frac{di_L}{dt} \qquad i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L dt$$

Complex Frequency Domain:

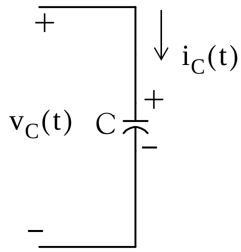
$$V_L(s) = sLI_L(s) - Li_L(0^-) \qquad I_L(s) = \frac{V_L(s)}{Ls} + \frac{i_L(0^-)}{s}$$

Figure 4.2. Inductive network in time domain and complex frequency domain

4.1.3 Capacitive Network Transformation

The time and complex frequency domains for purely capacitive networks are shown in Figure 4.3.

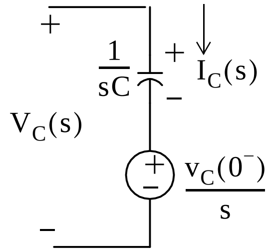
Time Domain



$$i_C(t) = C\frac{dv_C}{dt}$$

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C dt$$

Complex Frequency Domain



$$I_C(s) = sCV_C(s) - Cv_C(0^-)$$

$$V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^-)}{s}$$

Figure 4.3. Capacitive circuit in time domain and complex frequency domain

Note:
In the complex frequency domain, the terms sL and $1/sC$ are referred to as *complex inductive impedance*, and *complex capacitive impedance* respectively. Likewise, the terms and sC and $1/sL$ are called *complex capacitive admittance* and *complex inductive admittance* respectively.

Example 4.1

Use the Laplace transform method and apply Kirchoff's Current Law (KCL) to find the voltage $v_C(t)$ across the capacitor for the circuit of Figure 4.4, given that $v_C(0^-) = 6\ V$.

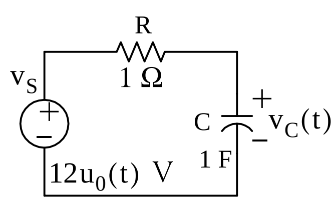


Figure 4.4. Circuit for Example 4.1

Solution:

We apply KCL at node A as shown in Figure 4.5.

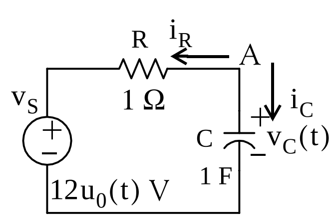


Figure 4.5. Application of KCL for the circuit of Example 4.1

Then,

$$i_R + i_C = 0$$

or

$$\frac{v_C(t) - 12u_0(t)}{1} + 1 \cdot \frac{dv_C}{dt} = 0$$

$$\frac{dv_C}{dt} + v_C(t) = 12u_0(t) \tag{4.1}$$

The Laplace transform of (4.1) is

$$sV_C(s) - v_C(0^-) + V_C(s) = \frac{12}{s}$$

$$(s+1)V_C(s) = \frac{12}{s} + 6$$

$$V_C(s) = \frac{6s+12}{s(s+1)}$$

By partial fraction expansion,

$$V_C(s) = \frac{6s+12}{s(s+1)} = \frac{r_1}{s} + \frac{r_2}{(s+1)}$$

$$r_1 = \left.\frac{6s+12}{(s+1)}\right|_{s=0} = 12$$

$$r_2 = \left.\frac{6s+12}{s}\right|_{s=-1} = -6$$

Therefore,

$$V_C(s) = \frac{12}{s} - \frac{6}{s+1} \Leftrightarrow 12 - 6e^{-t} = (12 - 6e^{-t})u_0(t) = v_C(t)$$

Example 4.2

Use the Laplace transform method and apply Kirchoff's Voltage Law (KVL) to find the voltage $v_C(t)$ across the capacitor for the circuit of Figure 4.6, given that $v_C(0^-) = 6\ V$.

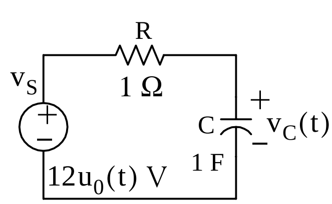


Figure 4.6. Circuit for Example 4.2

Solution:

This is the same circuit as in Example 4.1. We apply KVL for the loop shown in Figure 4.7.

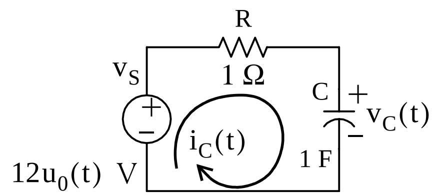


Figure 4.7. Application of KVL for the circuit of Example 4.2

$$Ri_C(t)+\frac{1}{C}\int_{-\infty}^{t} i_C(t)dt = 12u_0(t)$$

and with $R = 1$ and $C = 1$, we obtain

$$i_C(t)+\int_{-\infty}^{t} i_C(t)dt = 12u_0(t) \tag{4.2}$$

Next, taking the Laplace transform of both sides of (4.2), we obtain

$$I_C(s)+\frac{I_C(s)}{s}+\frac{v_C(0^-)}{s} = \frac{12}{s}$$

$$\left(1+\frac{1}{s}\right)I_C(s) = \frac{12}{s}-\frac{6}{s} = \frac{6}{s}$$

$$\left(\frac{s+1}{s}\right)I_C(s) = \frac{6}{s}$$

or

$$I_C(s) = \frac{6}{s+1} \Leftrightarrow i_C(t) = 6e^{-t}u_0(t)$$

Check: From Example 4.1,

$$v_C(t) = (12-6e^{-t})u_0(t)$$

Then,

$$i_C(t) = C\frac{dv_C}{dt} = \frac{dv_C}{dt} = \frac{d}{dt}(12-6e^{-t})u_0(t) = 6e^{-t}u_0(t)+6\delta(t) \tag{4.3}$$

The presence of the delta function in (4.3) is a result of the unit step that is applied at $t = 0$.

Example 4.3

In the circuit of Figure 4.8, switch S_1 closes at $t = 0$, while at the same time, switch S_2 opens. Use the Laplace transform method to find $v_{out}(t)$ for $t>0$.

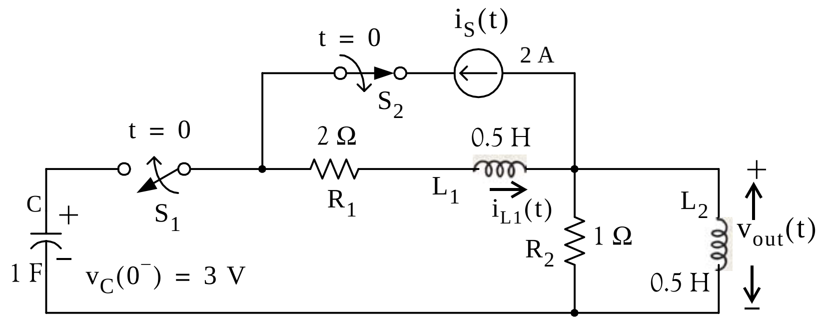


Figure 4.8. Circuit for Example 4.3

Solution:

Since the circuit contains a capacitor and an inductor, we must consider two initial conditions One is given as $v_C(0^-) = 3\ \mathrm{V}$. The other initial condition is obtained by observing that there is an initial current of $2\ \mathrm{A}$ in inductor L_1; this is provided by the $2\ \mathrm{A}$ current source just before switch S_2 opens. Therefore, our second initial condition is $i_{L1}(0^-) = 2\ \mathrm{A}$.

For $t > 0$, we transform the circuit of Figure 4.8 into its *s–domain** equivalent shown in Figure 4.9.

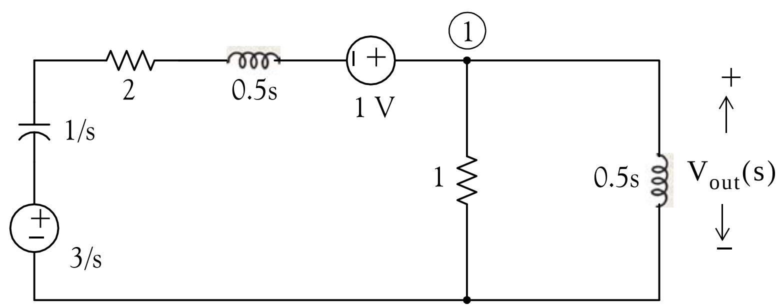


Figure 4.9. Transformed circuit of Example 4.3

In Figure 4.9 the current in inductor L_1 has been replaced by a voltage source of $1\ \mathrm{V}$. This is found from the relation

$$L_1 i_{L1}(0^-) = \frac{1}{2} \times 2 = 1\ \mathrm{V} \tag{4.4}$$

The polarity of this voltage source is as shown in Figure 4.9 so that it is consistent with the direction of the current $i_{L1}(t)$ in the circuit of Figure 4.8 just before switch S_2 opens. The initial capacitor voltage is replaced by a voltage source equal to $3/s$.

* *Henceforth, for convenience, we will refer the time domain as t−domain and the complex frequency domain as s−domain.*

Applying KCL at node ① we obtain

$$\frac{V_{out}(s) - 1 - 3/s}{1/s + 2 + s/2} + \frac{V_{out}(s)}{1} + \frac{V_{out}(s)}{s/2} = 0 \tag{4.5}$$

and after simplification

$$V_{out}(s) = \frac{2s(s+3)}{s^3 + 8s^2 + 10s + 4} \tag{4.6}$$

We will use MATLAB to factor the denominator $D(s)$ of (4.6) into a linear and a quadratic factor.

```
p=[1 8 10 4]; r=roots(p)                              % Find the roots of D(s)

r =
   -6.5708
   -0.7146 + 0.3132i
   -0.7146 - 0.3132i

y=expand((s + 0.7146 – 0.3132j)*(s + 0.7146 + 0.3132j))   % Find quadratic form

y =
s^2+3573/2500*s+3043737/5000000

3573/2500                                             % Simplify coefficient of s

ans =
    1.4292

3043737/5000000                                       % Simplify constant term

ans =
    0.6087
```

Therefore,

$$V_{out}(s) = \frac{2s(s+3)}{s^3 + 8s^2 + 10s + 4} = \frac{2s(s+3)}{(s+6.57)(s^2 + 1.43s + 0.61)} \tag{4.7}$$

Next, we perform partial fraction expansion.

$$V_{out}(s) = \frac{2s(s+3)}{(s+6.57)(s^2 + 1.43s + 0.61)} = \frac{r_1}{s+6.57} + \frac{r_2 s + r_3}{s^2 + 1.43s + 0.61} \tag{4.8}$$

$$r_1 = \left.\frac{2s(s+3)}{s^2 + 1.43s + 0.61}\right|_{s=-6.57} = 1.36 \tag{4.9}$$

The residues r_2 and r_3 are found from the equality

$$2s(s+3) = r_1(s^2+1.43s+0.61)+(r_2 s+r_3)(s+6.57) \tag{4.10}$$

Equating constant terms of (4.10), we obtain

$$0 = 0.61r_1 + 6.57r_3$$

and by substitution of the known value of r_1 from (4.9), we obtain

$$r_3 = -0.12$$

Similarly, equating coefficients of s^2, we obtain

$$2 = r_1 + r_2$$

and using the known value of r_1, we obtain

$$r_2 = 0.64 \tag{4.11}$$

By substitution into (4.8),

$$V_{out}(s) = \frac{1.36}{s+6.57}+\frac{0.64s-0.12}{s^2+1.43s+0.61} = \frac{1.36}{s+6.57}+\frac{0.64s+0.46-0.58}{s^2+1.43s+0.51+0.1}\ *$$

or

$$\begin{aligned} V_{out}(s) &= \frac{1.36}{s+6.57}+(0.64)\frac{s+0.715-0.91}{(s+0.715)^2+(0.316)^2} \\ &= \frac{1.36}{s+6.57}+\frac{0.64(s+0.715)}{(s+0.715)^2+(0.316)^2}-\frac{0.58}{(s+0.715)^2+(0.316)^2} \\ &= \frac{1.36}{s+6.57}+\frac{0.64(s+0.715)}{(s+0.715)^2+(0.316)^2}-\frac{1.84\times 0.316}{(s+0.715)^2+(0.316)^2} \end{aligned} \tag{4.12}$$

Taking the Inverse Laplace of (4.12), we obtain

$$v_{out}(t) = (1.36e^{-6.57t}+0.64e^{-0.715t}\cos 0.316t - 1.84e^{-0.715t}\sin 0.316t)u_0(t) \tag{4.13}$$

From (4.13), we observe that as $t \to \infty$, $v_{out}(t) \to 0$. This is to be expected because $v_{out}(t)$ is the voltage across the inductor as we can see from the circuit of Figure 4.9. The MATLAB script below will plot the relation (4.13) above.

* *We perform these steps to express the term* $\frac{0.64s-0.12}{s^2+1.43s+0.61}$ *in a form that resembles the transform pairs* $e^{-at}\cos\omega t u_0(t) \Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2}$ *and* $e^{-at}\sin\omega t u_0(t) \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$. *The remaining steps are carried out in (4.12).*

```
t=0:0.01:10;...
Vout=1.36.*exp(–6.57.*t)+0.64.*exp(–0.715.*t).*cos(0.316.*t)–1.84.*exp(–0.715.*t).*sin(0.316.*t);...
plot(t,Vout); grid
```

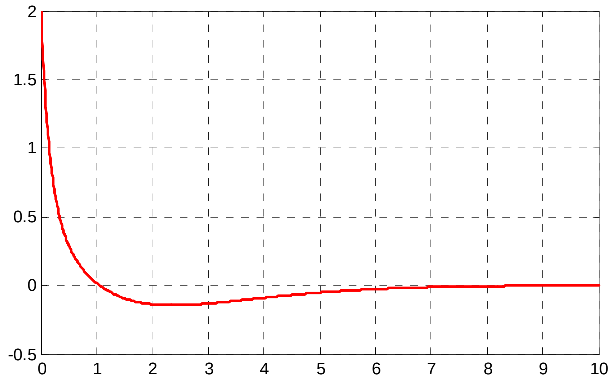


Figure 4.10. Plot of $v_{out}(t)$ for the circuit of Example 4.3

4.2 Complex Impedance Z(s)

Consider the $s-domain$ RLC series circuit of Figure 4.11, where the initial conditions are assumed to be zero.

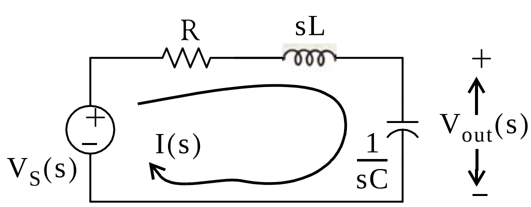


Figure 4.11. Series RLC circuit in s–domain

For this circuit, the sum $R + sL + \frac{1}{sC}$ represents the total opposition to current flow. Then,

$$I(s) = \frac{V_S(s)}{R + sL + 1/sC} \tag{4.14}$$

and defining the ratio $V_s(s)/I(s)$ as $Z(s)$, we obtain

$$\boxed{Z(s) \equiv \frac{V_S(s)}{I(s)} = R + sL + \frac{1}{sC}} \tag{4.15}$$

and thus, the s – domain current $I(s)$ can be found from the relation (4.16) below.

$$\boxed{I(s) = \frac{V_S(s)}{Z(s)}} \tag{4.16}$$

where

$$\boxed{Z(s) = R + sL + \frac{1}{sC}} \tag{4.17}$$

We recall that $s = \sigma + j\omega$. Therefore, $Z(s)$ is a complex quantity, and it is referred to as the *complex input impedance* of an s – domain RLC series circuit. In other words, $Z(s)$ is the ratio of the voltage excitation $V_s(s)$ to the current response $I(s)$ under *zero state* (zero initial conditions).

Example 4.4

For the network of Figure 4.12, all values are in Ω (ohms). Find $Z(s)$ using:

a. nodal analysis

b. successive combinations of series and parallel impedances

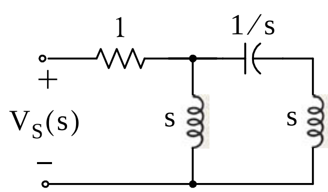


Figure 4.12. Circuit for Example 4.4

Solution:

a.

We will first find $I(s)$, and we will compute $Z(s)$ using (4.15). We assign the voltage $V_A(s)$ at node A as shown in Figure 4.13.

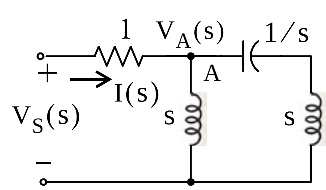


Figure 4.13. Network for finding $I(s)$ in Example 4.4

By nodal analysis,

$$\frac{V_A(s)-V_S(s)}{1}+\frac{V_A(s)}{s}+\frac{V_A(s)}{s+1/s} = 0$$

$$\left(1+\frac{1}{s}+\frac{1}{s+1/s}\right)V_A(s) = V_S(s)$$

$$V_A(s) = \frac{s^3+1}{s^3+2s^2+s+1}\cdot V_S(s)$$

The current $I(s)$ is now found as

$$I(s) = \frac{V_S(s)-V_A(s)}{1} = \left(1-\frac{s^3+1}{s^3+2s^2+s+1}\right)V_S(s) = \frac{2s^2+1}{s^3+2s^2+s+1}\cdot V_S(s)$$

and thus,

$$Z(s) = \frac{V_S(s)}{I(s)} = \frac{s^3+2s^2+s+1}{2s^2+1} \tag{4.18}$$

b.

The impedance $Z(s)$ can also be found by successive combinations of series and parallel impedances, as it is done with series and parallel resistances. For convenience, we denote the network devices as Z_1, Z_2, Z_3 and Z_4 shown in Figure 4.14.

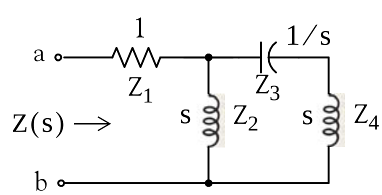


Figure 4.14. Computation of the impedance of Example 4.4 by series – parallel combinations

To find the equivalent impedance $Z(s)$, looking to the right of terminals a and b, we start on the right side of the network and we proceed to the left combining impedances as we would combine resistances where the symbol $\|$ denotes parallel combination. Then,

$$Z(s) = [(Z_3+Z_4)\parallel Z_2]+Z_1$$

$$Z(s) = \frac{s(s+1/s)}{s+s+1/s}+1 = \frac{s^2+1}{(2s^2+1)/s}+1 = \frac{s^3+s}{2s^2+1}+1 = \frac{s^3+2s^2+s+1}{2s^2+1} \tag{4.19}$$

We observe that (4.19) is the same as (4.18).

4.3 Complex Admittance Y(s)

Consider the $s-\text{domain}$ GLC parallel circuit of Figure 4.15 where the initial conditions are zero.

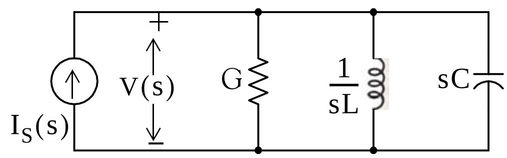


Figure 4.15. Parallel GLC circuit in s–domain

For the circuit of Figure 4.15,

$$GV(s)+\frac{1}{sL}V(s)+sCV(s) = I(s)$$

$$\left(G+\frac{1}{sL}+sC\right)(V(s)) = I(s)$$

Defining the ratio $I_S(s)/V(s)$ as $Y(s)$, we obtain

$$Y(s)\equiv\frac{I(s)}{V(s)} = G+\frac{1}{sL}+sC = \frac{1}{Z(s)} \tag{4.20}$$

and thus the $s-\text{domain}$ voltage $V(s)$ can be found from

$$\boxed{V(s) = \frac{I_S(s)}{Y(s)}} \tag{4.21}$$

where

$$\boxed{Y(s) = G+\frac{1}{sL}+sC} \tag{4.22}$$

We recall that $s = \sigma+j\omega$. Therefore, $Y(s)$ is a complex quantity, and it is referred to as the *complex input admittance* of an $s-\text{domain}$ GLC parallel circuit. In other words, $Y(s)$ is the ratio of the current excitation $I_S(s)$ to the voltage response $V(s)$ under *zero state* (zero initial conditions).

Example 4.5

Compute $Z(s)$ and $Y(s)$ for the circuit of Figure 4.16. All values are in Ω (ohms). Verify your answers with MATLAB.

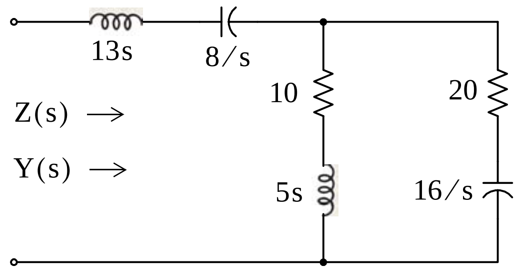


Figure 4.16. Circuit for Example 4.5

Solution:

It is convenient to represent the given circuit as shown in Figure 4.17.

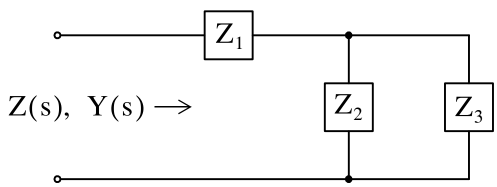


Figure 4.17. Simplified circuit for Example 4.5

where

$$Z_1 = 13s + \frac{8}{s} = \frac{13s^2 + 8}{s}$$

$$Z_2 = 10 + 5s$$

$$Z_3 = 20 + \frac{16}{s} = \frac{4(5s+4)}{s}$$

Then,

$$Z(s) = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{13s^2+8}{s} + \frac{(10+5s)\left(\frac{4(5s+4)}{s}\right)}{10+5s+\frac{4(5s+4)}{s}} = \frac{13s^2+8}{s} + \frac{(10+5s)\left(\frac{4(5s+4)}{s}\right)}{\frac{5s^2+10s+4(5s+4)}{s}}$$

$$= \frac{13s^2+8}{s} + \frac{20(5s^2+14s+8)}{5s^2+30s+16} = \frac{65s^4+490s^3+528s^2+400s+128}{s(5s^2+30s+16)}$$

Check with MATLAB:

```
syms s;                % Define symbolic variable s
z1 = 13*s + 8/s; z2 = 5*s + 10; z3 = 20 + 16/s; z = z1 + z2 * z3 / (z2+z3)

z =
13*s+8/s+(5*s+10)*(20+16/s)/(5*s+30+16/s)
```

z10 = simplify(z)

```
z10 =
(65*s^4+490*s^3+528*s^2+400*s+128)/s/(5*s^2+30*s+16)
```

pretty(z10)

```
          4         3         2
    65 s  + 490 s  + 528 s  + 400 s + 128
    -------------------------------------
                      2
             s (5 s  + 30 s + 16)
```

The complex input admittance $Y(s)$ is found by taking the reciprocal of $Z(s)$, that is,

$$Y(s) = \frac{1}{Z(s)} = \frac{s(5s^2+30s+16)}{65s^4+490s^3+528s^2+400s+128} \tag{4.23}$$

4.4 Transfer Functions

In an $s-\text{domain}$ circuit, the ratio of the output voltage $V_{out}(s)$ to the input voltage $V_{in}(s)$ *under zero state conditions*, is of great interest* in network analysis. This ratio is referred to as the *voltage transfer function* and it is denoted as $G_v(s)$, that is,

$$G_v(s) \equiv \frac{V_{out}(s)}{V_{in}(s)} \tag{4.24}$$

Similarly, the ratio of the output current $I_{out}(s)$ to the input current $I_{in}(s)$ *under zero state conditions*, is called the *current transfer function* denoted as $G_i(s)$, that is,

$$G_i(s) \equiv \frac{I_{out}(s)}{I_{in}(s)} \tag{4.25}$$

The current transfer function of (4.25) is rarely used; therefore, from now on, the transfer function will have the meaning of the voltage transfer function, i.e.,

* *To appreciate the usefulness of the transfer function, let us express relation (4.24) as* $V_{out}(s) = G_v(s) \cdot V_{in}(s)$. *This relation indicates that if we know the transfer function of a network, we can compute its output by multiplication of the transfer function by its input. We should also remember that the transfer function concept exists only in the complex frequency domain. In the time domain this concept is known as the **impulse response**, and it is discussed in Chapter 6 of this text.*

$$\boxed{G(s) \equiv \frac{V_{out}(s)}{V_{in}(s)}} \qquad (4.26)$$

Example 4.6

Derive an expression for the transfer function $G(s)$ for the circuit of Figure 4.18, where R_g represents the internal resistance of the applied (source) voltage V_S, and R_L represents the resistance of the load that consists of R_L, L, and C.

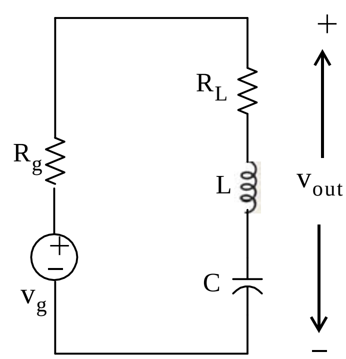


Figure 4.18. Circuit for Example 4.6

Solution:

No initial conditions are given, and even if they were, we would disregard them since the transfer function was defined as the ratio of the output voltage $V_{out}(s)$ to the input voltage $V_{in}(s)$ under zero initial conditions. The $s-\text{domain}$ circuit is shown in Figure 4.19.

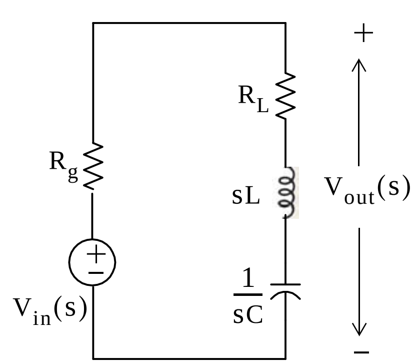


Figure 4.19. The s−domain circuit for Example 4.6

The transfer function $G(s)$ is readily found by application of the voltage division expression of the $s-\text{domain}$ circuit of Figure 4.19. Thus,

$$V_{out}(s) = \frac{R_L + sL + 1/sC}{R_g + R_L + sL + 1/sC} V_{in}(s)$$

Therefore,

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R_L + Ls + 1/sC}{R_g + R_L + Ls + 1/sC} \tag{4.27}$$

Example 4.7

Compute the transfer function $G(s)$ for the circuit of Figure 4.20 in terms of the circuit constants R_1, R_2, R_3, C_1, and C_2 Then, replace the complex variable s with $j\omega$, and the circuit constants with their numerical values and plot the magnitude $|G(s)| = V_{out}(s)/V_{in}(s)$ versus radian frequency ω.

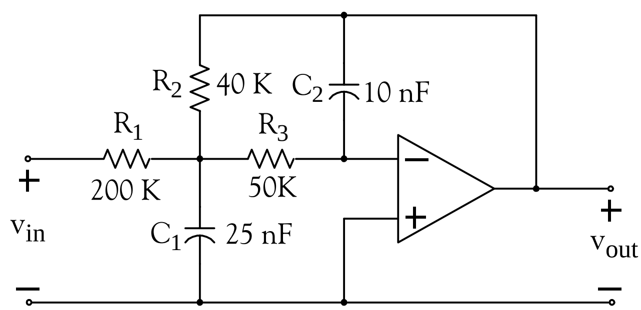


Figure 4.20. Circuit for Example 4.7

Solution:

The complex frequency domain equivalent circuit is shown in Figure 4.21.

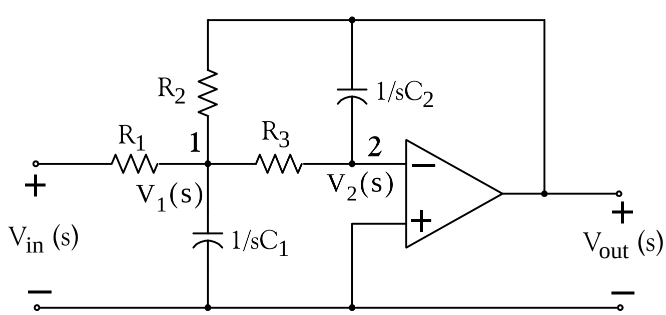


Figure 4.21. The s−domain circuit for Example 4.7

Next, we write nodal equations at nodes **1** and **2**. At node **1**,

$$\frac{V_1(s) - V_{in}(s)}{R_1} + \frac{V_1}{1/sC_1} + \frac{V_1(s) - V_{out}(s)}{R_2} + \frac{V_1(s) - V_2(s)}{R_3} = 0 \tag{4.28}$$

At node **2**,

$$\frac{V_2(s) - V_1(s)}{R_3} = \frac{V_{out}(s)}{1/sC_2} \tag{4.29}$$

Since $V_2(s) = 0$ (virtual ground), we express (4.29) as

$$V_1(s) = (-sR_3C_2)V_{out}(s) \tag{4.30}$$

and by substitution of (4.30) into (4.28), rearranging, and collecting like terms, we obtain:

$$\left[\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + sC_1\right)(-sR_3C_2) - \frac{1}{R_2}\right]V_{out}(s) = \frac{1}{R_1}V_{in}(s)$$

or

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{-1}{R_1\,[(\ 1/R_1 + 1/R_2 + 1/R_3 + sC_1\ )(sR_3C_2) + 1/R_2]} \tag{4.31}$$

To simplify the denominator of (4.31), we use the MATLAB script below with the given values of the resistors and the capacitors.

```
syms s;                    % Define symbolic variable s
R1=2*10^5; R2=4*10^4; R3=5*10^4; C1=25*10^(-9); C2=10*10^(-9);...
DEN=R1*((1/R1+1/R2+1/R3+s*C1)*(s*R3*C2)+1/R2); simplify(DEN)
```

```
ans =
1/200*s+188894659314785825/75557863725914323419136*s^2+5
```

```
188894659314785825/75557863725914323419136     % Simplify coefficient of s^2
```

```
ans =
  2.5000e-006
```

```
1/200                                          % Simplify coefficient of s^2
```

```
ans =
    0.0050
```

Therefore,

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{-1}{2.5 \times 10^{-6}s^2 + 5 \times 10^{-3}s + 5}$$

By substitution of s with $j\omega$ we obtain

$$G(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{-1}{2.5 \times 10^{-6}\omega^2 - j5 \times 10^{-3}\omega + 5} \tag{4.32}$$

We use MATLAB to plot the magnitude of (4.32) on a semilog scale with the following script:

```
w=1:10:10000; Gs=-1./(2.5.*10.^(-6).*w.^2-5.*j.*10.^(-3).*w+5);...
semilogx(w,abs(Gs)); xlabel('Radian Frequency w'); ylabel('|Vout/Vin|');...
title('Magnitude Vout/Vin vs. Radian Frequency'); grid
```

The plot is shown in Figure 4.22. We observe that the given op amp circuit is a second order low-pass filter whose cutoff frequency (-3 dB) occurs at about $700\ r/s$.

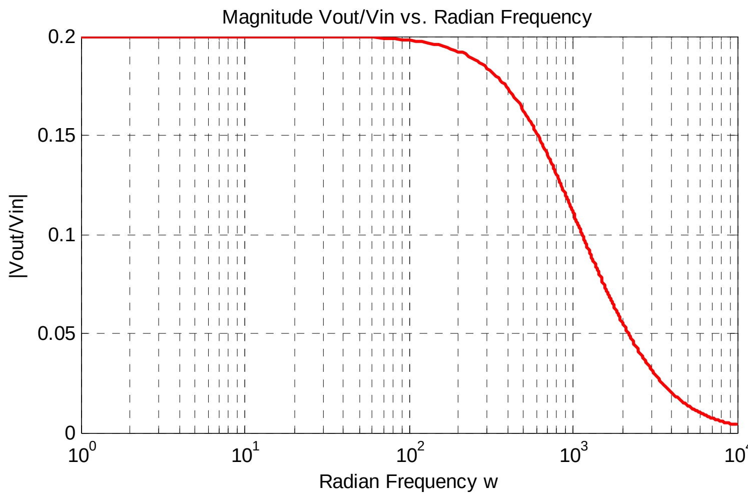
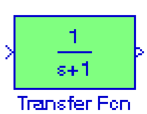


Figure 4.22. $|G(j\omega)|$ versus ω for the circuit of Example 4.7

4.5 Using the Simulink Transfer Fcn Block



The Simulink **Transfer Fcn** block implements a transfer function where the input $V_{IN}(s)$ and the output $V_{OUT}(s)$ can be expressed in transfer function form as

$$G(s) = \frac{V_{OUT}(s)}{V_{IN}(s)} \tag{4.33}$$

Example 4.8

Let us reconsider the active low-pass filter op amp circuit of Figure 4.21, Page 4-15 where we found that the transfer function is

$$G(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{-1}{R_1[(1/R_1+1/R_2+1/R_3+sC_1)(sR_3C_2)+1/R_2]} \tag{4.34}$$

and for simplicity, let $R_1 = R_2 = R_3 = 1\ \Omega$, and $C_1 = C_2 = 1\ F$. By substitution into (4.34) we obtain

$$G(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{-1}{s^2+3s+1} \tag{4.35}$$

Next, we let the input be the unit step function $u_0(t)$, and as we know from Chapter 2, $u_0(t) \Leftrightarrow 1/s$. Therefore,

$$V_{out}(s) = G(s)\cdot V_{in}(s) = \frac{1}{s}\cdot\frac{-1}{s^2+3s+1} = \frac{-1}{s^3+3s^2+s} \tag{4.36}$$

To find $v_{out}(t)$, we perform partial fraction expansion, and for convenience, we use the MATLAB **residue** function as follows:

```
num=–1; den=[1 3 1 0];[r p k]=residue(num,den)
```

```
r =
   -0.1708
    1.1708
   -1.0000
p =
   -2.6180
   -0.3820
         0
k =
     []
```

Therefore,

$$\left(\frac{1}{s}\cdot\frac{-1}{s^2+3s+1}=-\frac{1}{s}+\frac{1.171}{s+0.382}-\frac{0.171}{s+2.618}\right)\Leftrightarrow -1+1.171e^{-0.382t}-0.171e^{-2.618t} = v_{out}(t) \tag{4.37}$$

The plot for $v_{out}(t)$ is obtained with the following MATLAB script, and it is shown in Figure 4.23.

```
t=0:0.01:10; ft=–1+1.171.*exp(–0.382.*t)–0.171.*exp(–2.618.*t); plot(t,ft); grid
```

The same plot can be obtained using the Simulink model of Figure 4.24, where in the Function Block Parameters dialog box for the Transfer Fcn block we enter -1 for the numerator, and $[1\quad 3\quad 1]$ for the denominator. After the simulation command is executed, the Scope block displays the waveform of Figure 4.25.

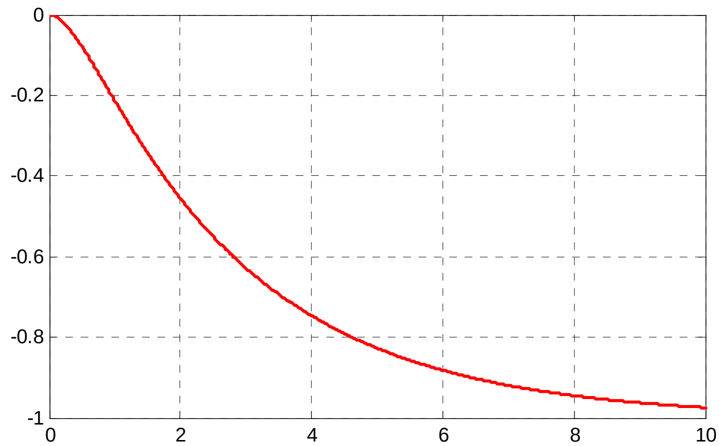


Figure 4.23. Plot of $v_{out}(t)$ for Example 4.8.

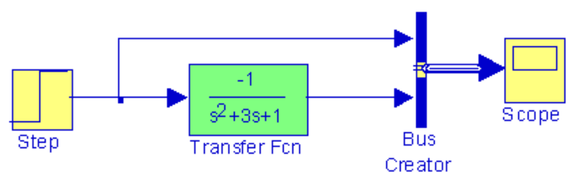


Figure 4.24. Simulink model for Example 4.8

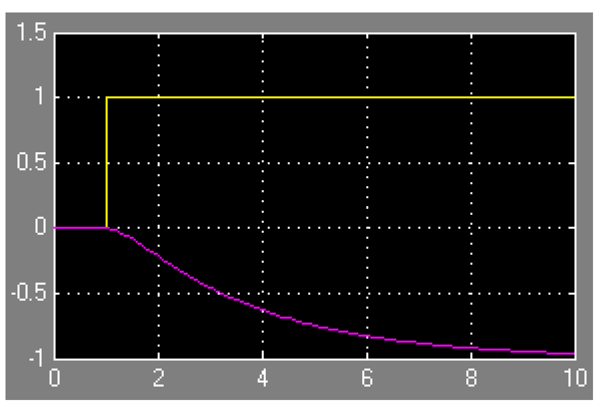


Figure 4.25. Waveform for the Simulink model of Figure 4.24

4.6 Summary

- The Laplace transformation provides a convenient method of analyzing electric circuits since integrodifferential equations in the $t-\text{domain}$ are transformed to algebraic equations in the $s-\text{domain}$.

- In the $s-\text{domain}$ the terms sL and $1/sC$ are called complex inductive impedance, and complex capacitive impedance respectively. Likewise, the terms and sC and $1/sL$ are called complex capacitive admittance and complex inductive admittance respectively.

- The expression

$$Z(s) = R + sL + \frac{1}{sC}$$

is a complex quantity, and it is referred to as the complex input impedance of an $s-\text{domain}$ RLC series circuit.

- In the $s-\text{domain}$ the current $I(s)$ can be found from

$$I(s) = \frac{V_S(s)}{Z(s)}$$

- The expression

$$Y(s) = G + \frac{1}{sL} + sC$$

is a complex quantity, and it is referred to as the complex input admittance of an $s-\text{domain}$ GLC parallel circuit.

- In the $s-\text{domain}$ the voltage $V(s)$ can be found from

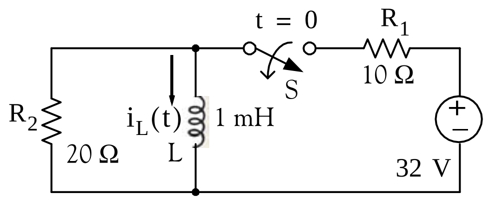
$$V(s) = \frac{I_S(s)}{Y(s)}$$

- In an $s-\text{domain}$ circuit, the ratio of the output voltage $V_{out}(s)$ to the input voltage $V_{in}(s)$ under zero state conditions is referred to as the voltage transfer function and it is denoted as $G(s)$, that is,

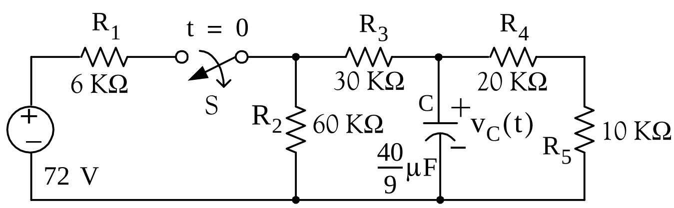
$$G(s) \equiv \frac{V_{out}(s)}{V_{in}(s)}$$

4.7 Exercises

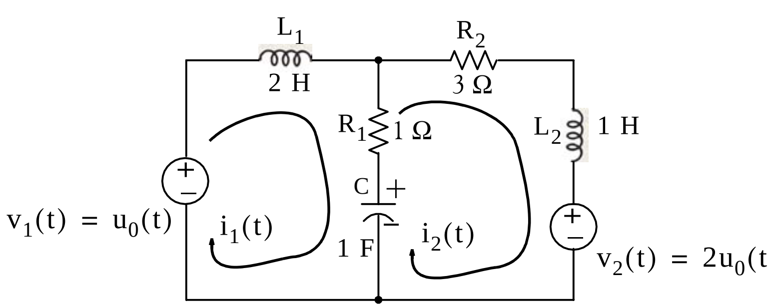
1. In the circuit below, switch S has been closed for a long time, and opens at $t = 0$. Use the Laplace transform method to compute $i_L(t)$ for $t > 0$.



2. In the circuit below, switch S has been closed for a long time, and opens at $t = 0$. Use the Laplace transform method to compute $v_c(t)$ for $t > 0$.



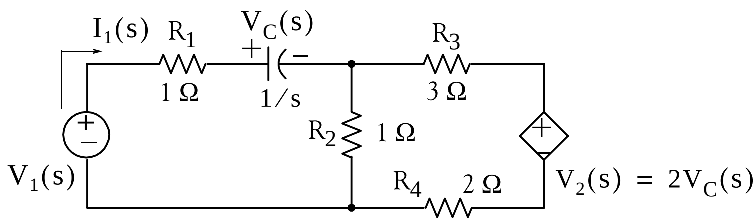
3. Use mesh analysis and the Laplace transform method, to compute $i_1(t)$ and $i_2(t)$ for the circuit below, given that $i_L(0^-) = 0$ and $v_C(0^-) = 0$.



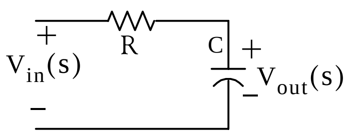
4. For the $s-\text{domain}$ circuit below,

   a. compute the admittance $Y(s) = I_1(s)/V_1(s)$

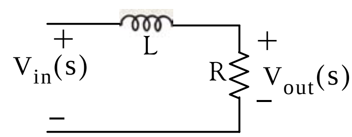
   b. compute the $t-\text{domain}$ value of $i_1(t)$ when $v_1(t) = u_0(t)$, and all initial conditions are zero.



5. Derive the transfer functions for the networks (a) and (b) below.

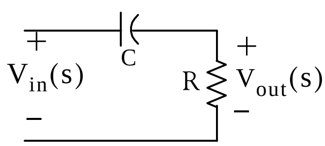


(a)

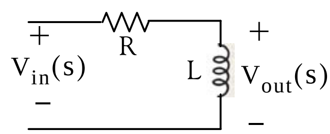


(b)

6. Derive the transfer functions for the networks (a) and (b) below.

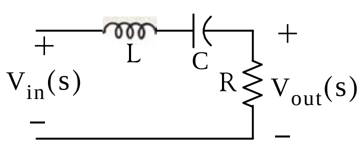


(a)

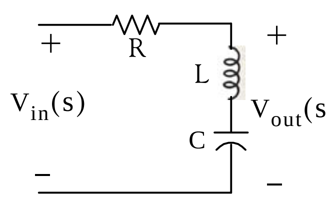


(b)

7. Derive the transfer functions for the networks (a) and (b) below.

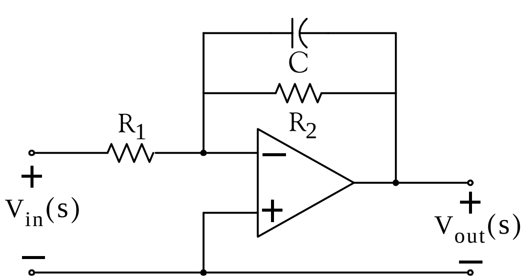


(a)

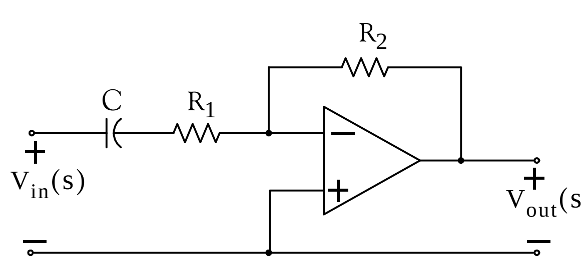


(b)

8. Derive the transfer function for the networks (a) and (b) below.

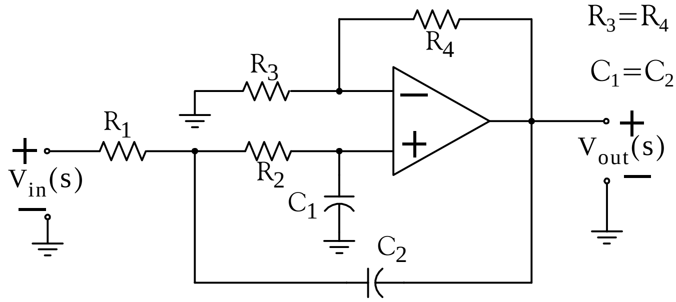


(a)



(b)

9. Derive the transfer function for the network below. Using MATLAB, plot $|G(s)|$ versus frequency in Hertz, on a semilog scale.



$R_1 = 11.3\ k\Omega$

$R_2 = 22.6\ k\Omega$

$R_3 = R_4 = 68.1\ k\Omega$

$C_1 = C_2 = 0.01\ \mu F$

Chapter 5

State Variables and State Equations

This chapter is an introduction to state variables and state equations as they apply in circuit analysis. The state transition matrix is defined, and the state–space to transfer function equivalence is presented. Several examples are presented to illustrate their application.

5.1 Expressing Differential Equations in State Equation Form

As we know, when we apply Kirchoff's Current Law (KCL) or Kirchoff's Voltage Law (KVL) in networks that contain energy–storing devices, we obtain integro–differential equations. Also, when a network contains just one such device (capacitor or inductor), it is said to be a *first–order circuit*. If it contains two such devices, it is said to be *second–order circuit,* and so on. Thus, a first order linear, time–invariant circuit can be described by a differential equation of the form

$$a_1 \frac{dy}{dt} + a_0\, y(t) = x(t) \tag{5.1}$$

A second order circuit can be described by a second–order differential equation of the same form as (5.1) where the highest order is a second derivative.

An *nth–order* differential equation can be resolved to n first–order simultaneous differential equations with a set of auxiliary variables called *state variables*. The resulting first–order differential equations are called *state–space equations,* or simply *state equations*. These equations can be obtained either from the nth–order differential equation, or directly from the network, provided that the state variables are chosen appropriately. The state variable method offers the advantage that it can also be used with non–linear and time–varying devices. However, our discussion will be limited to linear, time–invariant circuits.

State equations can also be solved with numerical methods such as Taylor series and Runge–Kutta methods, but these will not be discussed in this text*. The state variable method is best illustrated with several examples presented in this chapter.

Example 5.1

A series RLC circuit with excitation

$$v_S(t) = e^{j\omega t} \tag{5.2}$$

* *These are discussed in "Numerical Analysis Using MATLAB and Excel", Third Edition, ISBN 978-1-934404-03-4.*

is described by the integro–differential equation

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i dt = e^{j\omega t} \tag{5.3}$$

Differentiating both sides and dividing by L we obtain

$$\frac{d^2t}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}j\omega e^{j\omega t} \tag{5.4}$$

or

$$\frac{d^2t}{dt^2} = -\frac{R}{L}\frac{di}{dt} - \frac{1}{LC}i + \frac{1}{L}j\omega e^{j\omega t} \tag{5.5}$$

Next, we define two state variables x_1 and x_2 such that

$$x_1 = i \tag{5.6}$$

and

$$x_2 = \frac{di}{dt} = \frac{dx_1}{dt} = \dot{x}_1 \tag{5.7}$$

Then,

$$\dot{x}_2 = d^2i/dt^2 \tag{5.8}$$

where $\dot{x}_k$ denotes the derivative of the state variable x_k. From (5.5) through (5.8), we obtain the state equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{R}{L}x_2 - \frac{1}{LC}x_1 + \frac{1}{L}j\omega e^{j\omega t} \end{aligned} \tag{5.9}$$

It is convenient and customary to express the state equations in matrix* form. Thus, we write the state equations of (5.9) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} j\omega e^{j\omega t} \end{bmatrix} u \tag{5.10}$$

We usually write (5.10) in a compact form as

$$\dot{x} = Ax + bu \tag{5.11}$$

where

* *For a review of matrix theory, please refer to Appendix D.*

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \frac{1}{L} j\omega e^{j\omega t} \end{bmatrix}, \text{ and } u = \text{any input} \tag{5.12}$$

The output $y(t)$ is expressed by the state equation

$$y = Cx + du \tag{5.13}$$

where C is another matrix, and d is a column vector.

In general, the state representation of a network can be described by the pair of the of the state–space equations

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{du} \end{aligned}} \tag{5.14}$$

The state space equations of (5.14) can be realized with the block diagram of Figure 5.1.

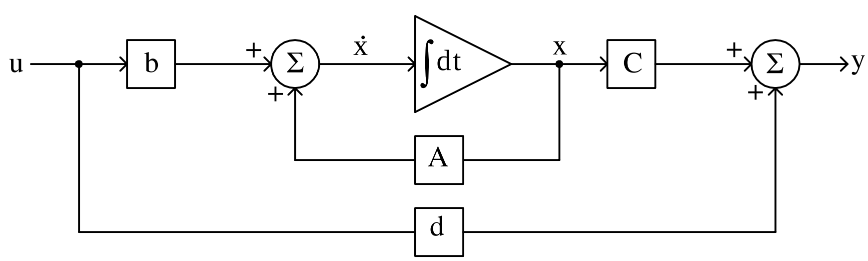


Figure 5.1. Block diagram for the realization of the state equations of (5.14)

We will learn how to solve the matrix equations of (5.14) in the subsequent sections.

Example 5.2

A fourth–order network is described by the differential equation

$$\frac{d^4y}{dt^4} + a_3\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0\, y(t) = u(t) \tag{5.15}$$

where $y(t)$ is the output representing the voltage or current of the network, and $u(t)$ is any input. Express (5.15) as a set of state equations.

Solution:

The differential equation of (5.15) is of fourth–order; therefore, we must define four state variables which will be used with the resulting four first–order state equations.

We denote the state variables as x_1, x_2, x_3, and x_4, and we relate them to the terms of the given differential equation as

$$x_1 = y(t) \qquad x_2 = \frac{dy}{dt} \qquad x_3 = \frac{d^2y}{dt^2} \qquad x_4 = \frac{d^3y}{dt^3} \tag{5.16}$$

We observe that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \frac{d^4y}{dt^4} = \dot{x}_4 &= -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + u(t) \end{aligned} \tag{5.17}$$

and in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{5.18}$$

In compact form, (5.18) is written as

$$\dot{x} = Ax + bu \tag{5.19}$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and } u = u(t)$$

We can also obtain the state equations directly from given circuits. We choose the state variables to represent inductor currents and capacitor voltages. In other words, we assign state variables to energy storing devices. The examples below illustrate the procedure.

Example 5.3

Write state equation(s) for the circuit of Figure 5.2, given that $v_C(0^-) = 0$, and $u_0(t)$ is the unit step function.

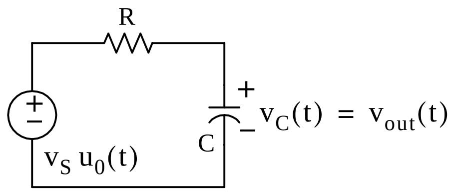


Figure 5.2. Circuit for Example 5.3

Solution:

This circuit contains only one energy–storing device, the capacitor. Therefore, we need only one state variable. We choose the state variable to denote the voltage across the capacitor as shown in Figure 5.3. The output is defined as the voltage across the capacitor.

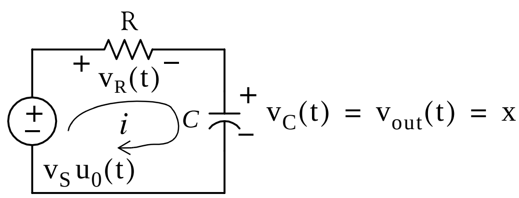


Figure 5.3. Circuit for Example 5.3 with state variable x assigned to it

For this circuit,

$$i_R = i = i_C = C\frac{dv_C}{dt} = C\dot{x}$$

and

$$v_R(t) = Ri = RC\dot{x}$$

By KVL,

$$v_R(t) + v_C(t) = v_S u_0(t)$$

or

$$RC\dot{x} + x = v_S u_0(t)$$

Therefore, the state equations are

$$\begin{aligned} \dot{x} &= -\frac{1}{RC}x + v_S u_0(t) \\ y &= x \end{aligned} \tag{5.20}$$

Example 5.4

Write state equation(s) for the circuit of Figure 5.4 assuming $i_L(0^-) = 0$, and the output y is defined as $y = i(t)$.

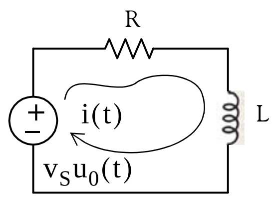


Figure 5.4. Circuit for Example 5.4

Solution:

This circuit contains only one energy–storing device, the inductor; therefore, we need only one state variable. We choose the state variable to denote the current through the inductor as shown in Figure 5.5.

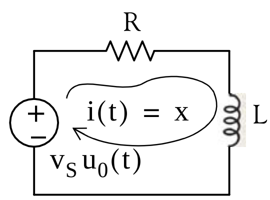


Figure 5.5. Circuit for Example 5.4 with assigned state variable x

By KVL,

$$v_R + v_L = v_S u_0(t)$$

or

$$Ri + L\frac{di}{dt} = v_S u_0(t)$$

or

$$Rx + L\dot{x} = v_S u_0(t)$$

Therefore, the state equations are

$$\begin{aligned} \dot{x} &= -\frac{R}{L}x + \frac{1}{L}v_S u_0(t) \\ y &= x \end{aligned} \tag{5.21}$$

5.2 Solution of Single State Equations

If a circuit contains only one energy–storing device, the state equations are written as

$$\begin{aligned} \dot{x} &= \alpha x + \beta u \\ y &= k_1 x + k_2 u \end{aligned} \tag{5.22}$$

where α, β, k_1, and k_2 are scalar constants, and the initial condition, if non–zero, is denoted as

$$x_0 = x(t_0) \tag{5.23}$$

We will now prove that the solution of the first state equation in (5.22) is

$$x(t) = e^{\alpha(t-t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau \tag{5.24}$$

Proof:

First, we must show that (5.24) satisfies the initial condition of (5.23). This is done by substitution of $t = t_0$ in (5.24). Then,

$$x(t_0) = e^{\alpha(t_0-t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t_0} e^{-\alpha\tau}\beta u(\tau)d\tau \tag{5.25}$$

The first term in the right side of (5.25) reduces to x_0 since

$$e^{\alpha(t_0-t_0)}x_0 = e^0 x_0 = x_0 \tag{5.26}$$

The second term of (5.25) is zero since the upper and lower limits of integration are the same. Therefore, (5.25) reduces to $x(t_0) = x_0$ and thus the initial condition is satisfied.

Next, we must prove that (5.24) satisfies also the first equation in (5.22). To prove this, we differentiate (5.24) with respect to t and we obtain

$$\dot{x}(t) = \frac{d}{dt}(e^{\alpha(t-t_0)}x_0) + \frac{d}{dt}\left\{e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau\right\}$$

or

$$\dot{x}(t) = \alpha e^{\alpha(t-t_0)}x_0 + \alpha e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau + e^{\alpha t}[e^{-\alpha\tau}\beta u(\tau)]\Big|_{\tau=t}$$

$$= \alpha\left[e^{\alpha(t-t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau\right] + e^{\alpha t}e^{-\alpha t}\beta u(t)$$

or

$$\dot{x}(t) = \alpha\left[e^{\alpha(t-t_0)}x_0 + \int_{t_0}^{t} e^{\alpha(t-\tau)}\beta u(\tau)d\tau\right] + \beta u(t) \tag{5.27}$$

We observe that the bracketed terms of (5.27) are the same as the right side of the assumed solution of (5.24). Therefore,

$$\dot{x} = \alpha x + \beta u$$

and this is the same as the first equation of (5.22).

In summary, if α and β are scalar constants, the solution of

$$\dot{x} = \alpha x + \beta u \tag{5.28}$$

with initial condition

$$x_0 = x(t_0) \tag{5.29}$$

is obtained from the relation

$$x(t) = e^{\alpha(t-t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau \tag{5.30}$$

Example 5.5

Use (5.28) through (5.30) to find the capacitor voltage $v_C(t)$ of the circuit of Figure 5.6 for $t>0$, given that the initial condition is $v_C(0^-) = 1\ V$

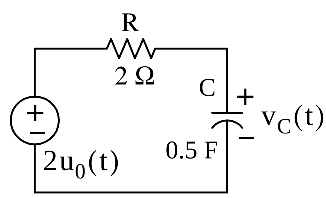


Figure 5.6. Circuit for Example 5.5

Solution:

From (5.20) of Example 5.3, Page 5–5,

$$\dot{x} = -\frac{1}{RC}x + v_S u_0(t)$$

and by comparison with (5.28),

$$\alpha = -\frac{1}{RC} = \frac{-1}{2\times 0.5} = -1$$

and

$$\beta = 2$$

Then, from (5.30),

$$x(t) = e^{\alpha(t-t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau = e^{-1(t-0)}1 + e^{-t}\int_0^t e^{\tau}2u(\tau)d\tau$$

$$= e^{-t} + 2e^{-t}\int_0^t e^{\tau}d\tau = e^{-t} + 2e^{-t}[e^{\tau}]\Big|_0^t = e^{-t} + 2e^{-t}(e^t - 1)$$

or

$$v_C(t) = x(t) = (2 - e^{-t})u_0(t) \tag{5.31}$$

Assuming that the output y is the capacitor voltage, the output state equation is

$$y(t) = x(t) = (2 - e^{-t})u_0(t) \tag{5.32}$$

5.3 The State Transition Matrix

In Section 5.1, relation (5.14), we defined the state equations pair

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx + du \end{aligned} \tag{5.33}$$

where for two or more simultaneous differential equations, A and C are 2×2 or higher order matrices, and b and d are column vectors with two or more rows. In this section we will introduce the *state transition matrix* e^{At}, and we will prove that the solution of the matrix differential equation

$$\dot{x} = Ax + bu \tag{5.34}$$

with initial conditions

$$x(t_0) = x_0 \tag{5.35}$$

is obtained from the relation

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau \tag{5.36}$$

Proof:

Let A be any $n \times n$ matrix whose elements are constants. Then, another $n \times n$ matrix denoted as $\varphi(t)$, is said to be the state transition matrix of (5.34), if it is related to the matrix A as the matrix power series

$$\boxed{\varphi(t) \equiv e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots + \frac{1}{n!}A^nt^n} \tag{5.37}$$

where I is the $n \times n$ identity matrix.

From (5.37), we find that

$$\varphi(0) = e^{A0} = I + A0 + \ldots = I \tag{5.38}$$

Differentiation of (5.37) with respect to t yields

$$\varphi'(t) = \frac{d}{dt}e^{At} = 0 + A \cdot 1 + A^2t + \ldots = A + A^2t + \ldots \tag{5.39}$$

and by comparison with (5.37) we obtain

$$\frac{d}{dt}e^{At} = Ae^{At} \tag{5.40}$$

To prove that (5.36) is the solution of (5.34), we must prove that it satisfies both the initial condition and the matrix differential equation. The initial condition is satisfied from the relation

$$x(t_0) = e^{A(t_0 - t_0)}x_0 + e^{At_0}\int_{t_0}^{t_0} e^{-A\tau}bu(\tau)d\tau = e^{A0}x_0 + 0 = Ix_0 = x_0 \tag{5.41}$$

where we have used (5.38) for the initial condition. The integral is zero since the upper and lower limits of integration are the same.

To prove that (5.34) is also satisfied, we differentiate the assumed solution

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau$$

with respect to t and we use (5.40), that is,

$$\frac{d}{dt}e^{At} = Ae^{At}$$

Then,

$$\dot{x}(t) = Ae^{A(t-t_0)}x_0 + Ae^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau + e^{At}e^{-At}bu(t)$$

or

$$\dot{x}(t) = A\left[e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau\right] + e^{At}e^{-At}bu(t) \tag{5.42}$$

We recognize the bracketed terms in (5.42) as $x(t)$, and the last term as $bu(t)$. Thus, the expression (5.42) reduces to

$$\dot{x}(t) = Ax + bu$$

In summary, if A is an $n \times n$ matrix whose elements are constants, $n \ge 2$, and b is a column vector with n elements, the solution of

$$\dot{x}(t) = Ax + bu \tag{5.43}$$

with initial condition

$$x_0 = x(t_0) \tag{5.44}$$

is

$$\boxed{x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau} \tag{5.45}$$

Therefore, the solution of second or higher order circuits using the state variable method, entails the computation of the state transition matrix e^{At}, and integration of (5.45).

5.4 Computation of the State Transition Matrix e^{At}

Let A be an $n \times n$ matrix, and I be the $n \times n$ identity matrix. By definition, the *eigenvalues* λ_i, $i = 1, 2, \ldots, n$ of A are the roots of the nth order polynomial

$$\boxed{\det[A - \lambda I] = 0} \tag{5.46}$$

We recall that expansion of a determinant produces a polynomial. The roots of the polynomial of (5.46) can be real (unequal or equal), or complex numbers.

Evaluation of the state transition matrix e^{At} is based on the *Cayley–Hamilton theorem*. This theorem states that a matrix can be expressed as an $(n-1)$th degree polynomial in terms of the matrix A as

$$\boxed{e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1}} \tag{5.47}$$

where the coefficients a_i are functions of the eigenvalues λ.

We accept (5.47) without proving it. The proof can be found in Linear Algebra and Matrix Theory textbooks.

Since the coefficients a_i are functions of the eigenvalues λ, we must consider the two cases discussed in Subsections 5.4.1 and 5.4.2 below.

5.4.1 Distinct Eigenvalues (Real of Complex)

If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \ldots \neq \lambda_n$, that is, if all eigenvalues of a given matrix A are distinct, the coefficients a_i are found from the simultaneous solution of the following system of equations:

$$\boxed{\begin{aligned} a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\ a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \ldots + a_{n-1}\lambda_2^{n-1} &= e^{\lambda_2 t} \\ &\ldots \\ a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t} \end{aligned}} \tag{5.48}$$

Example 5.6

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

Solution:

We must first find the eigenvalues λ of the given matrix A. These are found from the expansion of

$$\det[A - \lambda I] = 0$$

For this example,

$$\det[A - \lambda I] = \det\left\{\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \det\begin{bmatrix} -2-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix} = 0$$

$$= (-2-\lambda)(-1-\lambda) = 0$$

or

$$(\lambda + 1)(\lambda + 2) = 0$$

Therefore,

$$\lambda_1 = -1 \text{ and } \lambda_2 = -2 \tag{5.49}$$

Next, we must find the coefficients a_i of (5.47). Since A is a 2×2 matrix, we only need to consider the first two terms of that relation, that is,

$$e^{At} = a_0 I + a_1 A \tag{5.50}$$

The coefficients a_0 and a_1 are found from (5.48). For this example,

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$

$$a_0 + a_1\lambda_2 = e^{\lambda_2 t}$$

or

$$\begin{aligned} a_0 + a_1(-1) &= e^{-t} \\ a_0 + a_1(-2) &= e^{-2t} \end{aligned} \tag{5.51}$$

Simultaneous solution of (5.51) yields

$$\begin{aligned} a_0 &= 2e^{-t} - e^{-2t} \\ a_1 &= e^{-t} - e^{-2t} \end{aligned} \tag{5.52}$$

and by substitution into (5.50),

$$e^{At} = (2e^{-t} - e^{-2t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t})\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

or

$$e^{At} = \begin{bmatrix} e^{-2t} & e^{-t} - e^{-2t} \\ 0 & e^{-t} \end{bmatrix} \tag{5.53}$$

In summary, we compute the state transition matrix e^{At} for a given matrix A using the following procedure:

1. We find the eigenvalues λ from $\det[A - \lambda I] = 0$. We can write $[A - \lambda I]$ at once by subtracting λ from each of the main diagonal elements of A. If the dimension of A is a 2×2 matrix, it will yield two eigenvalues; if it is a 3×3 matrix, it will yield three eigenvalues, and so on. If the eigenvalues are distinct, we perform steps 2 through 4; otherwise we refer to Subsection 5.4.2 below.

2. If the dimension of A is a 2×2 matrix, we use only the first *2* terms of the right side of the state transition matrix

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1} \tag{5.54}$$

   If A matrix is a 3×3 matrix, we use the first 3 terms of (5.54), and so on.

3. We obtain the a_i coefficients from

$$\begin{aligned} a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\ a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \ldots + a_{n-1}\lambda_2^{n-1} &= e^{\lambda_2 t} \\ &\ldots \\ a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t} \end{aligned}$$

   We use as many equations as the number of the eigenvalues, and we solve for the coefficients a_i.

4. We substitute the a_i coefficients into the state transition matrix of (5.54), and we simplify.

Example 5.7

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix} \tag{5.55}$$

Solution:

1. We first compute the eigenvalues from $\det[A - \lambda I] = 0$. We obtain $[A - \lambda I]$ at once, by subtracting λ from each of the main diagonal elements of A. Then,

$$\det[A - \lambda I] = \det\begin{bmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{bmatrix} = 0 \tag{5.56}$$

and expansion of this determinant yields the polynomial

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \tag{5.57}$$

We will use MATLAB **roots(p)** function to obtain the roots of (5.57).

```
p=[1 -6 11 -6]; r=roots(p); fprintf(' \n'); fprintf('lambda1 = %5.2f \t', r(1));...
fprintf('lambda2 = %5.2f \t', r(2)); fprintf('lambda3 = %5.2f', r(3))
```

```
lambda1 = 3.00    lambda2 = 2.00    lambda3 = 1.00
```

and thus the eigenvalues are

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3 \tag{5.58}$$

2. Since A is a 3×3 matrix, we use the first 3 terms of (5.54), that is,

$$e^{At} = a_0 I + a_1 A + a_2 A^2 \tag{5.59}$$

3. We obtain the coefficients a_0, a_1, and a_2 from

$$\begin{aligned} a_0 + a_1\lambda_1 + a_2\lambda_1^2 &= e^{\lambda_1 t} \\ a_0 + a_1\lambda_2 + a_2\lambda_2^2 &= e^{\lambda_2 t} \\ a_0 + a_1\lambda_3 + a_2\lambda_3^2 &= e^{\lambda_3 t} \end{aligned}$$

or

$$\begin{aligned} a_0 + a_1 + a_2 &= e^{t} \\ a_0 + 2a_1 + 4a_2 &= e^{2t} \\ a_0 + 3a_1 + 9a_2 &= e^{3t} \end{aligned} \tag{5.60}$$

We will use the following MATLAB script for the solution of (5.60).

```
B=sym('[1  1  1; 1  2  4; 1  3  9]'); b=sym('[exp(t); exp(2*t); exp(3*t)]'); a=B\b; fprintf(' \n');...
disp('a0 = '); disp(a(1)); disp('a1 = '); disp(a(2)); disp('a2 = '); disp(a(3))
```

```
a0 =
3*exp(t)-3*exp(2*t)+exp(3*t)
a1 =
-5/2*exp(t)+4*exp(2*t)-3/2*exp(3*t)
a2 =
1/2*exp(t)-exp(2*t)+1/2*exp(3*t)
```

Thus,

$$\begin{aligned} a_0 &= 3e^t - 3e^{2t} + e^{3t} \\ a_1 &= -\frac{5}{2}e^t + 4e^{2t} - \frac{3}{2}e^{3t} \\ a_2 &= \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t} \end{aligned} \tag{5.61}$$

4. We also use MATLAB to perform the substitution into the state transition matrix, and to perform the matrix multiplications. The script is shown below.

```
syms t; a0 = 3*exp(t)+exp(3*t)–3*exp(2*t); a1 = –5/2*exp(t)–3/2*exp(3*t)+4*exp(2*t);...
a2 = 1/2*exp(t)+1/2*exp(3*t)–exp(2*t);...
A = [5  7  –5;  0  4  –1;  2  8  -3]; eAt=a0*eye(3)+a1*A+a2*A^2
```

```
eAt =
[-2*exp(t)+2*exp(2*t)+exp(3*t),  -6*exp(t)+5*exp(2*t)+exp(3*t),
4*exp(t)-3*exp(2*t)-exp(3*t)]
[-exp(t)+2*exp(2*t)-exp(3*t),  -3*exp(t)+5*exp(2*t)-exp(3*t),
2*exp(t)-3*exp(2*t)+exp(3*t)]
[-3*exp(t)+4*exp(2*t)-exp(3*t), -9*exp(t)+10*exp(2*t)-exp(3*t),
6*exp(t)-6*exp(2*t)+exp(3*t)]
```

Thus,

$$e^{At} = \begin{bmatrix} -2e^t + 2e^{2t} + e^{3t} & -6e^t + 5e^{2t} + e^{3t} & 4e^t - 3e^{2t} - e^{3t} \\ -e^t + 2e^{2t} - e^{3t} & -3e^t + 5e^{2t} - e^{3t} & 2e^t - 3e^{2t} + e^{3t} \\ -3e^t + 4e^{2t} - e^{3t} & -9e^t + 10e^{2t} - e^{3t} & 6e^t - 6e^{2t} + e^{3t} \end{bmatrix}$$

5.4.2 Multiple (Repeated) Eigenvalues

In this case, we will assume that the polynomial of

$$\det[A - \lambda I] = 0 \tag{5.62}$$

has n roots, and m of these roots are equal. In other words, the roots are

$$\lambda_1 = \lambda_2 = \lambda_3 \ldots = \lambda_m, \quad \lambda_{m+1}, \quad \lambda_n \tag{5.63}$$

The coefficients a_i of the state transition matrix

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1} \tag{5.64}$$

are found from the simultaneous solution of the system of equations of (5.65) below.

$$\boxed{\begin{aligned}
a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\
\frac{d}{d\lambda_1}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d}{d\lambda_1} e^{\lambda_1 t} \\
\frac{d^2}{d\lambda_1^2}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^2}{d\lambda_1^2} e^{\lambda_1 t} \\
&\ldots \\
\frac{d^{m-1}}{d\lambda_1^{m-1}}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^{m-1}}{d\lambda_1^{m-1}} e^{\lambda_1 t} \\
a_0 + a_1\lambda_{m+1} + a_2\lambda_{m+1}^2 + \ldots + a_{n-1}\lambda_{m+1}^{n-1} &= e^{\lambda_{m+1} t} \\
&\ldots \\
a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t}
\end{aligned}} \tag{5.65}$$

Example 5.8

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

Solution:

1. We first find the eigenvalues λ of the matrix A and these are found from the polynomial of $\det[A - \lambda I] = 0$. For this example,

$$\det[A - \lambda I] = \det\begin{bmatrix} -1-\lambda & 0 \\ 2 & -1-\lambda \end{bmatrix} = 0 \qquad (-1-\lambda)(-1-\lambda) = 0 \qquad (\lambda + 1)^2 = 0$$

and thus,

$$\lambda_1 = \lambda_2 = -1$$

2. Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix, that is,

$$e^{At} = a_0 I + a_1 A \tag{5.66}$$

3. We find a_0 and a_1 from (5.65). For this example,

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$
$$\frac{d}{d\lambda_1}(a_0 + a_1\lambda_1) = \frac{d}{d\lambda_1}e^{\lambda_1 t}$$

or

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$
$$a_1 = te^{\lambda_1 t}$$

and by substitution with $\lambda_1 = \lambda_2 = -1$, we obtain

$$a_0 - a_1 = e^{-t}$$
$$a_1 = te^{-t}$$

Simultaneous solution of the last two equations yields

$$\begin{aligned} a_0 &= e^{-t} + te^{-t} \\ a_1 &= te^{-t} \end{aligned} \tag{5.67}$$

4. By substitution of (5.67) into (5.66), we obtain

$$e^{At} = (e^{-t} + te^{-t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-t}\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

or

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 2te^{-t} & e^{-t} \end{bmatrix} \tag{5.68}$$

We can use the MATLAB **eig(x)** function to find the eigenvalues of an $n \times n$ matrix. To find out how it is used, we invoke the **help eig** command.

We will first use MATLAB to verify the values of the eigenvalues found in Examples 5.6 through 5.8, and we will briefly discuss eigenvectors in the next section.

Example 5.6:

```
A= [−2 1; 0 −1]; lambda=eig(A)
```

```
lambda =
     -2
     -1
```

Example 5.7:

```
B = [5 7 −5; 0 4 −1; 2 8 −3]; lambda=eig(B)
```

```
lambda =
    1.0000
    3.0000
    2.0000
```

Example 5.8:

```
C = [−1 0; 2 −1]; lambda=eig(C)
```

```
lambda =
     -1
     -1
```

5.5 Eigenvectors

Consider the relation

$$AX = \lambda X \tag{5.69}$$

where A is an $n \times n$ matrix, X is a column vector, and λ is a scalar number. We can express this relation in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \tag{5.70}$$

We write (5.70) as

$$(A-\lambda I)X = 0 \tag{5.71}$$

Then, (5.71) can be written as

$$\begin{bmatrix} (a_{11}-\lambda)x_1 & a_{12}x_2 & \dots & a_{1n}x_n \\ a_{21}x_1 & (a_{22}-\lambda)x_2 & \dots & a_{2n}x_n \\ \dots & \dots & \dots & \dots \\ a_{n1}x_1 & a_{n2}x_2 & \dots & (a_{nn}-\lambda)x_n \end{bmatrix} = 0 \tag{5.72}$$

The equations of (5.72) will have non–trivial solutions if and only if its determinant is zero*, that is, if

$$\det\begin{bmatrix} (a_{11}-\lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & (a_{nn}-\lambda) \end{bmatrix} = 0 \tag{5.73}$$

Expansion of the determinant of (5.73) results in a polynomial equation of degree n in λ, and it is called the *characteristic equation*.

We can express (5.73) in a compact form as

$$\det(A-\lambda I) = 0 \tag{5.74}$$

As we know, the roots λ of the characteristic equation are the eigenvalues of the matrix A, and corresponding to each eigenvalue λ, there is a non-trivial solution of the column vector X, i.e., $X \neq 0$. This vector X is called *eigenvector*. Obviously, there is a different eigenvector for each eigenvalue. Eigenvectors are generally expressed as *unit eigenvectors*, that is, they are normalized to unit length. This is done by dividing each component of the eigenvector by the square root of the sum of the squares of their components, so that the sum of the squares of their components is equal to unity.

In many engineering applications the unit eigenvectors are chosen such that $X \cdot X^T = I$ where X^T is the transpose of the eigenvector X, and I is the identity matrix.

Two vectors X and Y are said to be *orthogonal* if their inner (dot) product is zero. A set of eigenvectors constitutes an *orthonormal basis* if the set is normalized (expressed as unit eigenvectors) and these vector are mutually orthogonal. An orthonormal basis can be formed with the *Gram-Schmidt Orthogonalization Procedure*; it is beyond the scope of this chapter to discuss this procedure, and therefore it will not be discussed in this text. It can be found in Linear Algebra and Matrix Theory textbooks.

* *This is because we want the vector X in (5.71) to be a non-zero vector and the product* $(A-\lambda I)X$ *to be zero.*

The example below illustrates the relationships between a matrix A, its eigenvalues, and eigenvectors.

Example 5.9

Given the matrix

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$$

a. Find the eigenvalues of A

b. Find eigenvectors corresponding to each eigenvalue of A

c. Form a set of unit eigenvectors using the eigenvectors of part (b).

Solution:

a. This is the same matrix as in Example 5.7, relation (5.55), Page 5−14, where we found the eigenvalues to be

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3$$

b. We start with

$$AX = \lambda X$$

and we let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then,

$$\begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{5.75}$$

or

$$\begin{bmatrix} 5x_1 & 7x_2 & -5x_3 \\ 0 & 4x_2 & -x_3 \\ 2x_1 & 8x_2 & -3x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix} \tag{5.76}$$

Equating corresponding rows and rearranging, we obtain

$$\begin{bmatrix} (5-\lambda)x_1 & 7x_2 & -5x_3 \\ 0 & (4-\lambda)x_2 & -x_3 \\ 2x_1 & 8x_2 & -(3-\lambda)x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (5.77)$$

For $\lambda = 1$, (5.77) reduces to

$$\begin{aligned} 4x_1 + 7x_2 - 5x_3 &= 0 \\ 3x_2 - x_3 &= 0 \\ 2x_1 + 8x_2 - 4x_3 &= 0 \end{aligned} \qquad (5.78)$$

By Crame's rule, or MATLAB, we obtain the indeterminate values

$$x_1 = 0/0 \qquad x_2 = 0/0 \qquad x_3 = 0/0 \qquad (5.79)$$

Since the unknowns x_1, x_2, and x_3 are scalars, we can assume that one of these, say x_2, is known, and solve x_1 and x_3 in terms of x_2. Then, we obtain $x_1 = 2x_2$, and $x_3 = 3x_2$. Therefore, an eigenvector for $\lambda = 1$ is

$$X_{\lambda=1} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 3x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \qquad (5.80)$$

since any eigenvector is a scalar multiple of the last vector in (5.80).

Similarly, for $\lambda = 2$, we obtain $x_1 = x_2$, and $x_3 = 2x_2$. Then, an eigenvector for $\lambda = 2$ is

$$X_{\lambda=2} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 2x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad (5.81)$$

Finally, for $\lambda = 3$, we obtain $x_1 = -x_2$, and $x_3 = x_2$. Then, an eigenvector for $\lambda = 3$ is

$$X_{\lambda=3} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad (5.82)$$

c. We find the unit eigenvectors by dividing the components of each vector by the square root of the sum of the squares of the components. These are:

$$\sqrt{2^2+1^2+3^2} = \sqrt{14}$$

$$\sqrt{1^2+1^2+2^2} = \sqrt{6}$$

$$\sqrt{(-1)^2+1^2+1^2} = \sqrt{3}$$

The unit eigenvectors are

$$\text{Unit } X_{\lambda=1} = \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix} \qquad \text{Unit } X_{\lambda=2} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \qquad \text{Unit } X_{\lambda=3} = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \tag{5.83}$$

We observe that for the first unit eigenvector the sum of the squares is unity, that is,

$$\left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2 = \frac{4}{14} + \frac{1}{14} + \frac{9}{14} = 1 \tag{5.84}$$

and the same is true for the other two unit eigenvectors in (5.83).

5.6 Circuit Analysis with State Variables

In this section we will present two examples to illustrate how the state variable method is used in circuit analysis.

Example 5.10

For the circuit of Figure 5.7, the initial conditions are $i_L(0^-) = 0$, and $v_C(0^-) = 0.5\ V$. Use the state variable method to compute $i_L(t)$ and $v_C(t)$.

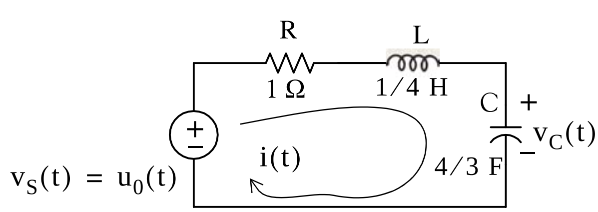


Figure 5.7. Circuit for Example 5.10

Solution:

For this example,

$$i = i_L$$

and

$$Ri_L + L\frac{di_L}{dt} + v_C = u_0(t)$$

Substitution of given values and rearranging, yields

$$\frac{1}{4}\frac{di_L}{dt} = (-1)i_L - v_C + 1$$

or

$$\frac{di_L}{dt} = -4i_L - 4v_C + 4 \tag{5.85}$$

Next, we define the state variables $x_1 = i_L$ and $x_2 = v_C$. Then,

$$\dot{x}_1 = \frac{di_L}{dt} \tag{5.86}$$

and

$$\dot{x}_2 = \frac{dv_C}{dt}$$

Also,

$$i_L = C\frac{dv_C}{dt}$$

and thus,

$$x_1 = i_L = C\frac{dv_C}{dt} = C\dot{x}_2 = \frac{4}{3}\dot{x}_2$$

or

$$\dot{x}_2 = \frac{3}{4}x_1 \tag{5.87}$$

Therefore, from (5.85), (5.86), and (5.87), we obtain the state equations

$$\dot{x}_1 = -4x_1 - 4x_2 + 4$$

$$\dot{x}_2 = \frac{3}{4}x_1$$

and in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}u_0(t) \tag{5.88}$$

We will compute the solution of (5.88) using

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau \tag{5.89}$$

where

$$A = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix} \quad x_0 = \begin{bmatrix} i_L(0) \\ v_C(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \tag{5.90}$$

First, we compute the state transition matrix e^{At}. We find the eigenvalues from

$$\det[A - \lambda I] = 0$$

Then,

$$\det[A - \lambda I] = \det\begin{bmatrix} -4-\lambda & -4 \\ 3/4 & -\lambda \end{bmatrix} = 0 \qquad (-\lambda)(-4-\lambda)+3 = 0 \qquad \lambda^2 + 4\lambda + 3 = 0$$

Therefore,

$$\lambda_1 = -1 \text{ and } \lambda_2 = -3$$

The next step is to find the coefficients a_i. Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix, that is,

$$e^{At} = a_0 I + a_1 A \tag{5.91}$$

The constants a_0 and a_1 are found from

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$
$$a_0 + a_1\lambda_2 = e^{\lambda_2 t}$$

and with $\lambda_1 = -1$ and $\lambda_2 = -3$, we obtain

$$\begin{aligned} a_0 - a_1 &= e^{-t} \\ a_0 - 3a_1 &= e^{-3t} \end{aligned} \tag{5.92}$$

Simultaneous solution of (5.92) yields

$$\begin{aligned} a_0 &= 1.5e^{-t} - 0.5e^{-3t} \\ a_1 &= 0.5e^{-t} - 0.5e^{-3t} \end{aligned} \tag{5.93}$$

We now substitute these values into (5.91), and we obtain

$$e^{At} = (1.5e^{-t} - 0.5e^{-3t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (0.5e^{-t} - 0.5e^{-2t})\begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & 0 \\ 0 & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix} + \begin{bmatrix} -2e^{-t} + 2e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 0 \end{bmatrix}$$

or

$$e^{At} = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix}$$

The initial conditions vector is the second vector in (5.90); then, the first term of (5.89) becomes

$$e^{At}x_0 = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix}\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

or

$$e^{At}x_0 = \begin{bmatrix} -e^{-t} + e^{-3t} \\ 0.75e^{-t} - 0.25e^{-3t} \end{bmatrix} \qquad (5.94)$$

We also need to evaluate the integral on the right side of (5.89). From (5.90)

$$b = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}4$$

and denoting this integral as Int, we obtain

$$Int = \int_{t_0}^{t}\begin{bmatrix} -0.5e^{-(t-\tau)} + 1.5e^{-3(t-\tau)} & -2e^{-(t-\tau)} + 2e^{-3(t-\tau)} \\ \frac{3}{8}e^{-(t-\tau)} - \frac{3}{8}e^{-3(t-\tau)} & 1.5e^{-(t-\tau)} - 0.5e^{-3(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}4d\tau$$

or

$$Int = \int_{t_0}^{t}\begin{bmatrix} -0.5e^{-(t-\tau)} + 1.5e^{-3(t-\tau)} \\ \frac{3}{8}e^{-(t-\tau)} - \frac{3}{8}e^{-3(t-\tau)} \end{bmatrix}4d\tau \qquad (5.95)$$

The integration in (5.95) is with respect to τ; then, integrating the column vector under the integral, we obtain

$$\mathrm{Int} = 4\begin{bmatrix} -0.5e^{-(t-\tau)} + 0.5e^{-3(t-\tau)} \\ 0.375e^{-(t-\tau)} - 0.125e^{-3(t-\tau)} \end{bmatrix}\Bigg|_{\tau=0}^{t}$$

or

$$\mathrm{Int} = 4\begin{bmatrix} -0.5+0.5 \\ 0.375-0.125 \end{bmatrix} - 4\begin{bmatrix} -0.5e^{-t} + 0.5e^{-3t} \\ 0.375e^{-t} - 0.125e^{-3t} \end{bmatrix} = 4\begin{bmatrix} 0.5e^{-t} - 0.5e^{-3t} \\ 0.25 - 0.375e^{-t} + 0.125e^{-3t} \end{bmatrix}$$

By substitution of these values, the solution of

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau$$

is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -e^{-t} + e^{-3t} \\ 0.75e^{-t} - 0.25e^{-3t} \end{bmatrix} + 4\begin{bmatrix} 0.5e^{-t} - 0.5e^{-3t} \\ 0.25 - 0.375e^{-t} + 0.125e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-3t} \\ 1 - 0.75e^{-t} + 0.25e^{-3t} \end{bmatrix}$$

Then,

$$x_1 = i_L = e^{-t} - e^{-3t} \tag{5.96}$$

and

$$x_2 = v_C = 1 - 0.75e^{-t} + 0.25e^{-3t} \tag{5.97}$$

Other variables of the circuit can now be computed from (5.96) and (5.97). For example, the voltage across the inductor is

$$v_L = L\frac{di_L}{dt} = \frac{1}{4}\frac{d}{dt}(e^{-t} - e^{-3t}) = -\frac{1}{4}e^{-t} + \frac{3}{4}e^{-3t}$$

We use the MATLAB script below to plot the relation of (5.97).

```
t=0:0.01:10; x2=1–0.75.*exp(–t)+0.25.*exp(–3.*t);...
plot(t,x2); grid
```

The plot is shown in Figure 5.8.

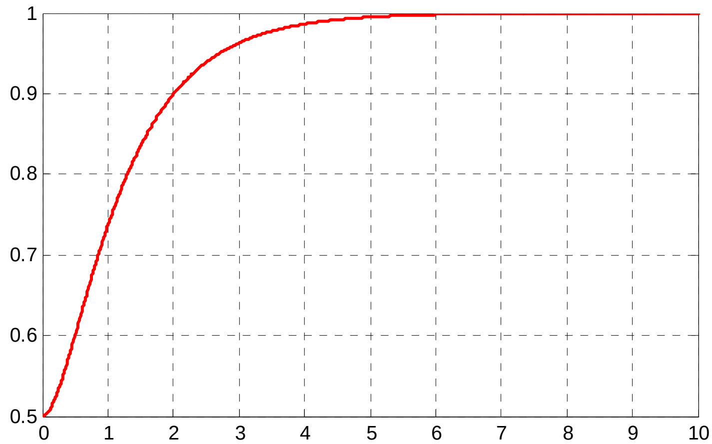


Figure 5.8. Plot for relation (5.97)

We can obtain the plot of Figure 5.8 with the Simulink State–Space block with the unit step function as the input using the **Step** block, and the capacitor voltage as the output displayed on the **Scope** block as shown in the model of Figure 5.9 where for the **State–Space** block Function Block Parameters dialog box we have entered:

A: [−4 −4; 3/4 0]
B: [4 0]’
C: [0 1]
D: [0]
Initial conditions: [0 1/2]

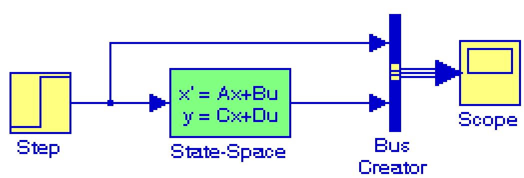


Figure 5.9. Simulink model for Example 5.10

The waveform for the capacitor voltage for the simulation time interval $0 \le t \le 10$ seconds is shown in Figure 5.10 where we observe that the initial condition $v_C(0^-) = 0.5\ V$ is also displayed.

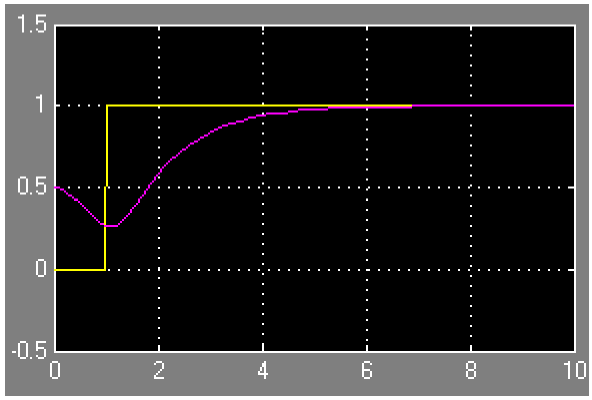


Figure 5.10. Input and output waveforms for the model of Figure 5.9

Example 5.11

A network is described by the state equation

$$\dot{x} = Ax + bu \tag{5.98}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad u = \delta(t) \tag{5.99}$$

Compute the state vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

We compute the eigenvalues from

$$\det[A - \lambda I] = 0$$

For this example,

$$\det[A - \lambda I] = \det\begin{bmatrix} 1-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix} = 0 \qquad (1-\lambda)(-1-\lambda) = 0$$

Then,

$$\lambda_1 = 1 \text{ and } \lambda_2 = -1$$

Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix to find the coefficients a_i, that is,

$$e^{At} = a_0 I + a_1 A \tag{5.100}$$

The constants a_0 and a_1 are found from

$$\begin{aligned} a_0 + a_1\lambda_1 &= e^{\lambda_1 t} \\ a_0 + a_1\lambda_2 &= e^{\lambda_2 t} \end{aligned} \tag{5.101}$$

and with $\lambda_1 = 1$ and $\lambda_2 = -1$, we obtain

$$\begin{aligned} a_0 + a_1 &= e^{t} \\ a_0 - a_1 &= e^{-t} \end{aligned} \tag{5.102}$$

and simultaneous solution of (5.102) yields

$$\begin{aligned} a_0 &= \frac{e^t + e^{-t}}{2} = \cosh t \\ a_1 &= \frac{e^t - e^{-t}}{2} = \sinh t \end{aligned}$$

By substitution of these values into (5.100), we obtain

$$e^{At} = \cosh t\,I + \sinh t\,A = \cosh t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sinh t\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \cosh t + \sinh t & 0 \\ \sinh t & \cosh t - \sinh t \end{bmatrix} \tag{5.103}$$

The values of the vector **x** are found from

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau = e^{At}x_0 + e^{At}\int_{0}^{t} e^{-A\tau}b\delta(\tau)d\tau \tag{5.104}$$

Using the sifting property of the delta function we find that (5.104) reduces to

$$x(t) = e^{At}x_0 + e^{At}b = e^{At}(x_0 + b) = e^{At}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} = e^{At}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cosh t + \sinh t & 0 \\ \sinh t & \cosh t - \sinh t \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \cosh t - \sinh t \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \tag{5.105}$$

5.7 Relationship between State Equations and Laplace Transform

In this section, we will show that the state transition matrix can be computed from the Inverse Laplace transform. We will also show that the transfer function can be found from the coefficient matrices of the state equations.

Consider the state equation

$$\dot{x} = Ax + bu \tag{5.106}$$

Taking the Laplace of both sides of (5.106), we obtain

$$sX(s) - x(0) = AX(s) + bU(s)$$

or

$$(sI - A)X(s) = x(0) + bU(s) \tag{5.107}$$

Multiplying both sides of (5.107) by $(sI - A)^{-1}$, we obtain

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}bU(s) \tag{5.108}$$

Comparing (5.108) with

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}bu(\tau)d\tau \tag{5.109}$$

we observe that the right side of (5.108) is the Laplace transform of (5.109). Therefore, we can compute the state transition matrix e^{At} from the Inverse Laplace of $(sI - A)^{-1}$, that is, we can use the relation

$$\boxed{e^{At} = \mathscr{L}^{-1}\{(sI - A)^{-1}\}} \tag{5.110}$$

Next, we consider the output state equation

$$y = Cx + du \tag{5.111}$$

Taking the Laplace of both sides of (5.111), we obtain

$$Y(s) = CX(s) + dU(s) \tag{5.112}$$

and using (5.108), we obtain

$$Y(s) = C(sI - A)^{-1}x(0) + [C(sI - A)^{-1}b + d]U(s) \tag{5.113}$$

If the initial condition $x(0) = 0$, (5.113) reduces to

$$Y(s) = [C(sI - A)^{-1}b + d]U(s) \tag{5.114}$$

In (5.114), $U(s)$ is the Laplace transform of the input $u(t)$; then, division of both sides by $U(s)$ yields the transfer function

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}b + d} \tag{5.115}$$

Example 5.12

In the circuit of Figure 5.11, all initial conditions are zero. Compute the state transition matrix e^{At} using the Inverse Laplace transform method.

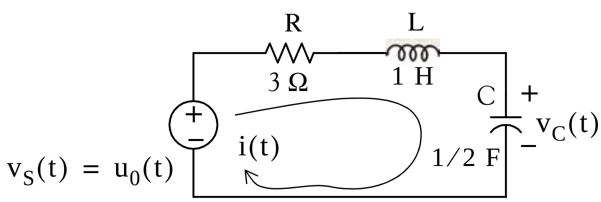


Figure 5.11. Circuit for Example 5.12

Solution:

For this circuit,

$$i = i_L$$

and

$$Ri_L + L\frac{di_L}{dt} + v_C = u_0(t)$$

Substitution of given values and rearranging, yields

$$\frac{di_L}{dt} = -3i_L - v_C + 1 \tag{5.116}$$

Now, we define the state variables

$$x_1 = i_L$$

and

$$x_2 = v_C$$

Then,

$$\dot{x}_1 = \frac{di_L}{dt} = -3i_L - v_C + 1 \tag{5.117}$$

and

$$\dot{x}_2 = \frac{dv_C}{dt}$$

Also,

$$i_L = C\frac{dv_C}{dt} = 0.5\frac{dv_C}{dt} \tag{5.118}$$

and thus,

$$x_1 = i_L = 0.5\frac{dv_C}{dt} = 0.5\dot{x}_2$$

or

$$\dot{x}_2 = 2x_1 \tag{5.119}$$

Therefore, from (5.117) and (5.119) we obtain the state equations

$$\begin{aligned} \dot{x}_1 &= -3x_1 - x_2 + 1 \\ \dot{x}_2 &= 2x_1 \end{aligned} \tag{5.120}$$

and in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 \tag{5.121}$$

By inspection,

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \tag{5.122}$$

Now, we will find the state transition matrix from

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} \tag{5.123}$$

where

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

Then,

$$(sI - A)^{-1} = \frac{adj(sI - A)}{det(sI - A)} = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} = \begin{bmatrix} \dfrac{s}{(s+1)(s+2)} & \dfrac{-1}{(s+1)(s+2)} \\ \dfrac{2}{(s+1)(s+2)} & \dfrac{s+3}{(s+1)(s+2)} \end{bmatrix}$$

We find the Inverse Laplace of each term by partial fraction expansion. Thus,

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} -e^{-t} + 2e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

Now, we can find the state variables representing the inductor current and the capacitor voltage from

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}bu(\tau)d\tau$$

using the procedure of Example 5.11.

MATLAB provides two very useful functions to convert state–space (state equations), to transfer function (s–domain), and vice versa. The function **ss2tf** (state–space to transfer function) converts the state space equations

$$\begin{aligned} \dot{x} &= Ax + Bu^{*} \\ y &= Cx + Du \end{aligned} \qquad (5.124)$$

to the rational transfer function form

$$G(s) = \frac{N(s)}{D(s)} \qquad (5.125)$$

This is used with the statement **[num,den]=ss2tf(A,B,C,D,iu)** where **A, B, C, D** are the matrices of (5.124) and **iu** is 1 if there is only one input. The MATLAB **help** command provides the following information:

help ss2tf

```
 SS2TF  State-space to transfer function conversion.
    [NUM,DEN] = SS2TF(A,B,C,D,iu) calculates the
     transfer function:
                NUM(s)          -1
        G(s) = -------- = C(sI-A) B + D
                DEN(s)
    of the system:
        x = Ax + Bu
        y = Cx + Du
from the iu'th input. Vector DEN contains the coefficients of   the
denominator in descending powers of s. The numerator coefficients are
returned in matrix NUM with as many rows as there     are outputs y.

    See also TF2SS
```

The other function, **tf2ss**, converts the transfer function of (5.125) to the state–space equations of (5.124). It is used with the statement **[A,B,C,D]=tf2ss(num,den)** where **A, B, C**, and **D** are the matrices of (5.124), and num, den are $N(s)$ and $D(s)$ of (5.125) respectively. The MATLAB **help** command provides the following information:

* *We have used capital letters for vectors b and c to be consistent with MATLAB's designations.*

help tf2ss

```
 TF2SS  Transfer function to state-space conversion.
    [A,B,C,D] = TF2SS(NUM,DEN) calculates the state-space
    representation:

        x = Ax + Bu
        y = Cx + Du
    of the system:
                 NUM(s)
        G(s) = --------
                 DEN(s)
from a single input. Vector DEN must contain the coefficients of the
denominator in descending powers of s. Matrix NUM must contain the
numerator coefficients with as many rows as there are outputs y. The
A,B,C,D matrices are returned in controller canonical form. This calcu-
lation also works for discrete systems. To avoid confusion when using
this function with discrete systems, always use a numerator polynomial
that has been padded with zeros to make it the same length as the denom-
inator. See the User's guide for more details.

    See also SS2TF.
```

Example 5.13

For the circuit of Figure 5.12, all initial conditions are zero.

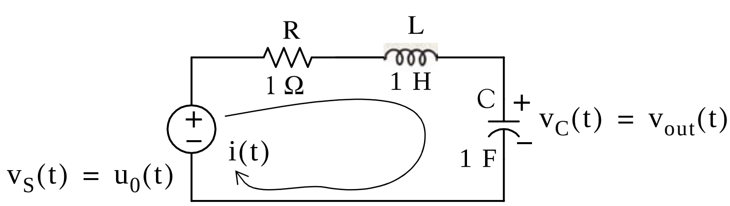


Figure 5.12. Circuit for Example 5.13

a. Derive the state equations and express them in matrix form as

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

b. Derive the transfer function

$$G(s) = \frac{N(s)}{D(s)}$$

c. Verify your answers with MATLAB.

Solution:

a. The differential equation describing the circuit is

$$Ri + L\frac{di}{dt} + v_C = u_0(t)$$

and with the given values,

$$i + \frac{di}{dt} + v_C = u_0(t)$$

or

$$\frac{di}{dt} = -i - v_C + u_0(t)$$

We let

$$x_1 = i_L = i$$

and

$$x_2 = v_C = v_{out}$$

Then,

$$\dot{x}_1 = \frac{di}{dt}$$

and

$$\dot{x}_2 = \frac{dv_c}{dt} = x_1$$

Thus, the state equations are

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 + u_0(t)\\ \dot{x}_2 &= x_1\\ y &= x_2\end{aligned}$$

and in matrix form,

$$\begin{aligned}\dot{x} = Ax + Bu \leftrightarrow \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u_0(t)\\ y = Cx + Du \leftrightarrow y &= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}u_0(t)\end{aligned} \tag{5.126}$$

b. The $s - \mathrm{domain}$ circuit is shown in Figure 5.13 below.

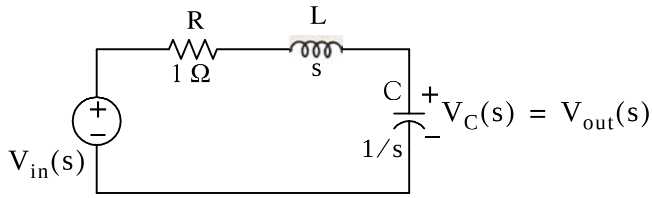


Figure 5.13. Transformed circuit for Example 5.13

By the voltage division expression,

$$V_{out}(s) = \frac{1/s}{1 + s + 1/s} V_{in}(s)$$

or

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{s^2 + s + 1}$$

Therefore,

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{s^2 + s + 1} \tag{5.127}$$

c.

```
A = [−1  −1; 1  0]; B = [1  0]';  C = [0  1]; D = [0];      % The matrices of (5.126)
[num, den] = ss2tf(A, B, C, D, 1)                           % Verify coefficients of G(s) in (5.127)
```

```
num =
     0      0      1
den =
   1.0000    1.0000    1.0000
```

```
num = [0  0  1]; den = [1  1  1];                  % The coefficients of G(s) in (5.127)
[A  B  C  D] = tf2ss(num, den)                     % Verify the matrices of (5.126)
```

```
A =
   -1    -1
    1     0
B =
    1
    0
C =
    0     1
D =
    0
```

The equivalence between the state–space equations of (5.126) and the transfer function of (5.127) is also evident from the Simulink models shown in Figure 5.14 where for the **State–Space** block Function Block Parameters dialog box we have entered:

A: [−1 −1; 3/4 0]
B: [1 0]’
C: [0 1]
D: [0]
Initial conditions: [0 0]

For the **Transfer Fcn** block Function Block Parameters dialog box we have entered:

Numerator coefficient: [1]
Denominator coefficient: [1 1 1]

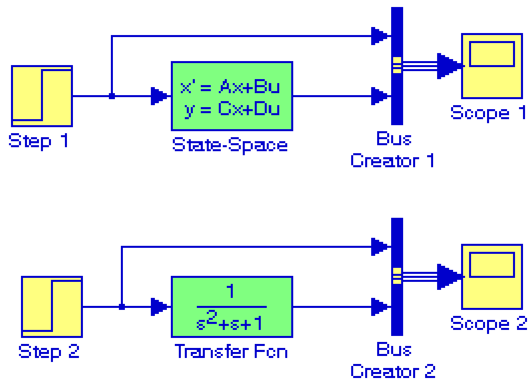


Figure 5.14. Models to show the equivalence between relations (5.126) and (5.127)

After the simulation command is executed, both Scope 1 and Scope 2 blocks display the input and output waveforms shown in Figure 5.15.

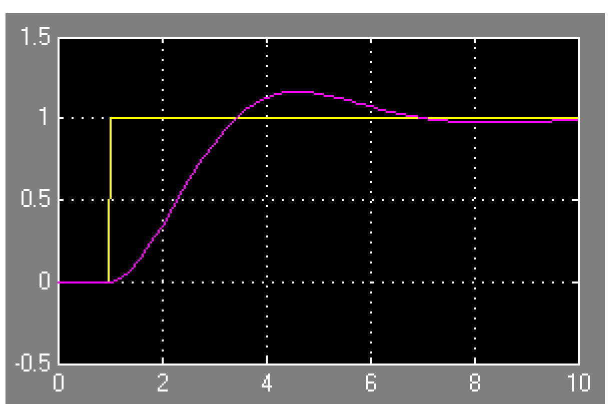


Figure 5.15. Waveforms displayed by Scope 1 and Scope 2 blocks for the models in Figure 5.14

5.8 Summary

- An nth–order differential equation can be resolved to n first–order simultaneous differential equations with a set of auxiliary variables called state variables. The resulting first–order differential equations are called state–space equations, or simply state equations.
- The state–space equations can be obtained either from the nth–order differential equation, or directly from the network, provided that the state variables are chosen appropriately.
- When we obtain the state equations directly from given circuits, we choose the state variables to represent inductor currents and capacitor voltages.
- The state variable method offers the advantage that it can also be used with non–linear and time–varying devices.
- If a circuit contains only one energy–storing device, the state equations are written as

$$\dot{x} = \alpha x + \beta u$$
$$y = k_1 x + k_2 u$$

  where α, β, k_1, and k_2 are scalar constants, and the initial condition, if non–zero, is denoted as

$$x_0 = x(t_0)$$

- If α and β are scalar constants, the solution of $\dot{x} = \alpha x + \beta u$ with initial condition $x_0 = x(t_0)$ is obtained from the relation

$$x(t) = e^{\alpha(t - t_0)} x_0 + e^{\alpha t} \int_{t_0}^{t} e^{-\alpha\tau} \beta u(\tau) d\tau$$

- The solution of the state equations pair

$$\dot{x} = Ax + bu$$
$$y = Cx + du$$

  where A and C are 2×2 or higher order matrices, and b and d are column vectors with two or more rows, entails the computation of the state transition matrix e^{At}, and integration of

$$x(t) = e^{A(t - t_0)} x_0 + e^{At} \int_{t_0}^{t} e^{-A\tau} bu(\tau) d\tau$$

- The eigenvalues λ_i, where $i = 1, 2, \ldots, n$, of an $n \times n$ matrix A are the roots of the nth order polynomial

$$\det[A - \lambda I] = 0$$

  where I is the $n \times n$ identity matrix.

- The Cayley–Hamilton theorem states that a matrix can be expressed as an $(n-1)$th degree polynomial in terms of the matrix A as

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1}$$

where the coefficients a_i are functions of the eigenvalues λ.

- If all eigenvalues of a given matrix A are distinct, that is, if

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \ldots \neq \lambda_n$$

the coefficients a_i are found from the simultaneous solution of the system of equations

$$\begin{aligned}
a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\
a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \ldots + a_{n-1}\lambda_2^{n-1} &= e^{\lambda_2 t} \\
&\ldots \\
a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t}
\end{aligned}$$

- If some or all eigenvalues of matrix A are repeated, that is, if

$$\lambda_1 = \lambda_2 = \lambda_3 \ldots = \lambda_m,\ \lambda_{m+1},\ \lambda_n$$

the coefficients a_i of the state transition matrix are found from the simultaneous solution of the system of equations

$$\begin{aligned}
a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\
\frac{d}{d\lambda_1}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d}{d\lambda_1}e^{\lambda_1 t} \\
\frac{d^2}{d\lambda_1^2}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^2}{d\lambda_1^2}e^{\lambda_1 t} \\
&\ldots \\
\frac{d^{m-1}}{d\lambda_1^{m-1}}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^{m-1}}{d\lambda_1^{m-1}}e^{\lambda_1 t} \\
a_0 + a_1\lambda_{m+1} + a_2\lambda_{m+1}^2 + \ldots + a_{n-1}\lambda_{m+1}^{n-1} &= e^{\lambda_{m+1} t} \\
&\ldots \\
a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t}
\end{aligned}$$

- We can use the MATLAB **eig(x)** function to find the eigenvalues of an $n \times n$ matrix.
- A column vector X that satisfies the relation

$$AX = \lambda X$$

where A is an $n \times n$ matrix and λ is a scalar number, is called an eigenvector.

- There is a different eigenvector for each eigenvalue.
- Eigenvectors are generally expressed as unit eigenvectors, that is, they are normalized to unit length. This is done by dividing each component of the eigenvector by the square root of the sum of the squares of their components, so that the sum of the squares of their components is equal to unity.
- Two vectors X and Y are said to be orthogonal if their inner (dot) product is zero.
- A set of eigenvectors constitutes an orthonormal basis if the set is normalized (expressed as unit eigenvectors) and these vector are mutually orthogonal.
- The state transition matrix can be computed from the Inverse Laplace transform using the relation

$$e^{At} = \mathscr{L}^{-1}\{(sI - A)^{-1}\}$$

- If $U(s)$ is the Laplace transform of the input $u(t)$ and $Y(s)$ is the Laplace transform of the output $y(t)$, the transfer function can be computed using the relation

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}b + d$$

- MATLAB provides two very useful functions to convert state–space (state equations), to transfer function (s-domain), and vice versa. The function **ss2tf** (state–space to transfer function) converts the state space equations to the transfer function equivalent, and the function **tf2ss**, converts the transfer function to state–space equations.

5.9 Exercises

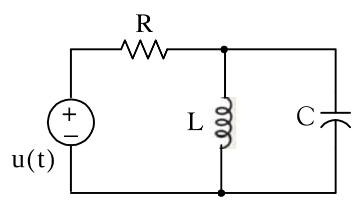
1. Express the integrodifferential equation below as a matrix of state equations where $k_1, k_2,$ and k_3 are constants.

$$\frac{dv^2}{dt^2} + k_3\frac{dv}{dt} + k_2 v + k_1\int_0^t v dt = \sin 3t + \cos 3t$$

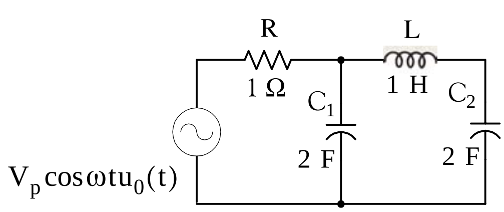
2. Express the matrix of the state equations below as a single differential equation, and let $x(y) = y(t)$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

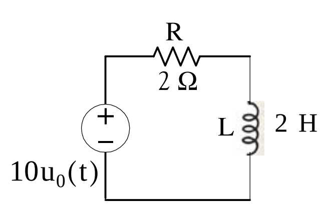
3. For the circuit below, all initial conditions are zero, and $u(t)$ is any input. Write state equations in matrix form.



4. In the circuit below, all initial conditions are zero. Write state equations in matrix form.



5. In the below, $i_L(0^-) = 2$ A. Use the state variable method to find $i_L(t)$ for $t > 0$.



6. Compute the eigenvalues of the matrices A, B, and C below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} a & 0 \\ -a & b \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Hint: One of the eigenvalues of matrix C is -1.

7. Compute e^{At} given that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Observe that this is the same matrix as C of Exercise 6.

8. Find the solution of the matrix state equation $\dot{x} = Ax + bu$ given that

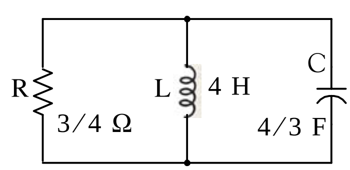
$$A = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad u = \delta(t), \quad t_0 = 0$$

9. In the circuit below, $i_L(0^-) = 0$, and $v_C(0^-) = 1\ V$.

a. Write state equations in matrix form.

b. Compute e^{At} using the Inverse Laplace transform method.

c. Find $i_L(t)$ and $v_C(t)$ for $t > 0$.



Chapter 6

The Impulse Response and Convolution

This chapter begins with the definition of the impulse response, that is, the response of a circuit that is subjected to the excitation of the impulse function. Then, it defines convolution and how it is applied to circuit analysis. Evaluation of the convolution integral using graphical methods is also presented and illustrated with several examples.

6.1 The Impulse Response in Time Domain

In this section we will discuss the impulse response of a network, that is, the output (voltage or current) of a network when the input is the delta function. Of course, the output can be any voltage or current that we choose as the output. The computation of the impulse response assumes zero initial conditions.

We learned in the previous chapter that the state equation

$$\dot{x} = Ax + bu \tag{6.1}$$

has the solution

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_0^t e^{-A\tau}bu(\tau)d\tau \tag{6.2}$$

Therefore, with initial condition $x_0 = 0$, and with the input $u(t) = \delta(t)$, the solution of (6.2) reduces to

$$x(t) = e^{At}\int_0^t e^{-A\tau}b\delta(\tau)d\tau \tag{6.3}$$

Using the sifting property of the delta function, i.e.,

$$\int_{-\infty}^{\infty} f(t)\delta(\tau)d\tau = f(0) \tag{6.4}$$

and denoting the impulse response as $h(t)$, we obtain

$$\boxed{h(t) = e^{At}bu_0(t)} \tag{6.5}$$

where the unit step function $u_0(t)$ is included to indicate that this relation holds for $t > 0$.

Example 6.1

Compute the impulse response of the series RC circuit of Figure 6.1 in terms of the constants R and C, where the response is considered to be the voltage across the capacitor, and $v_C(0^-) = 0$. Then, compute the current through the capacitor.

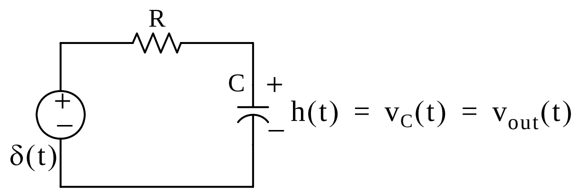


Figure 6.1. Circuit for Example 6.1

Solution:

We assign currents i_C and i_R with the directions shown in Figure 6.2, and we apply KCL.

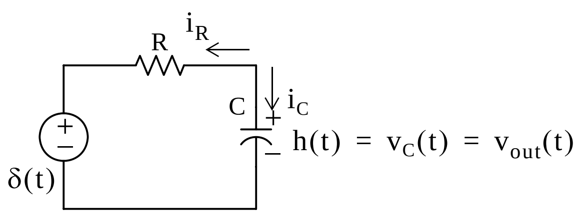


Figure 6.2. Application of KCL for the circuit for Example 6.1

Then,

$$i_R + i_C = 0$$

or

$$C\frac{dv_C}{dt} + \frac{v_C - \delta(t)}{R} = 0 \tag{6.6}$$

We assign the state variable

$$v_C = x$$

Then,

$$\frac{dv_C}{dt} = \dot{x}$$

and (6.6) is written as

$$C\dot{x} + \frac{x}{R} = \frac{\delta(t)}{R}$$

or

$$\dot{x} = -\frac{1}{RC}x + \frac{1}{RC}\delta(t) \tag{6.7}$$

Equation (6.7) has the form

$$\dot{x} = ax + bu$$

and as we found in (6.5),

$$h(t) = e^{At}bu_0(t)$$

For this example,

$$a = -1/RC$$

and

$$b = 1/RC$$

Therefore,

$$h(t) = v_C(t) = e^{-t/RC}\frac{1}{RC}$$

or

$$h(t) = \frac{1}{RC}e^{-t/RC}u_0(t) \tag{6.8}$$

The current i_C can now be computed from

$$i_C = C\frac{dv_C}{dt}$$

Thus,

$$i_C = C\frac{d}{dt}h(t) = C\frac{d}{dt}\left(\frac{1}{RC}e^{-t/RC}u_0(t)\right)$$

$$= -\frac{1}{R^2C}e^{-t/RC} + \frac{1}{R}e^{-t/RC}\delta(t)$$

Using the sampling property of the delta function, we obtain

$$i_C = \frac{1}{R}\delta(t) - \frac{1}{R^2C}e^{-t/RC} \tag{6.9}$$

Example 6.2

For the circuit of Figure 6.3, compute the impulse response $h(t) = v_C(t)$ given that the initial conditions are zero, that is, $i_L(0^-) = 0$, and $v_C(0^-) = 0$.

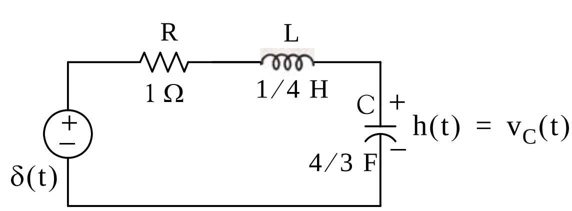


Figure 6.3. Circuit for Example 6.2

Solution:

This is the same circuit as that of Example 5.10, Chapter 5, Page 5−22, where we found that

$$b = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

and

$$e^{At} = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix}$$

The impulse response is obtained from (6.5), Page 6–1, that is,

$$h(t) = x(t) = e^{At}bu_0(t)$$

then,

$$h(t) = x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} u_0(t) = \begin{bmatrix} -2e^{-t} + 6e^{-3t} \\ \frac{3}{2}e^{-t} - \frac{3}{2}e^{-3t} \end{bmatrix} u_0(t) \qquad (6.10)$$

In Example 5.10, Chapter 5, Page 5–22, we defined

$$x_1 = i_L$$

and

$$x_2 = v_C$$

Then,

$$h(t) = x_2 = v_C(t) = 1.5e^{-t} - 1.5e^{-3t}$$

or

$$h(t) = v_C(t) = 1.5(e^{-t} - e^{-3t}) \qquad (6.11)$$

Of course, this answer is not the same as that of Example 5.10, because the inputs and initial conditions were defined differently.

6.2 Even and Odd Functions of Time

A function $f(t)$ is an *even function* of time if the following relation holds.

$$\boxed{f(-t) = f(t)} \qquad (6.12)$$

that is, if in an even function we replace t with $-t$, the function $f(t)$ does not change. Thus, polynomials with even exponents only, and with or without constants, are even functions. For instance, the cosine function is an even function because it can be written as the power series

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots$$

Other examples of even functions are shown in Figure 6.4.

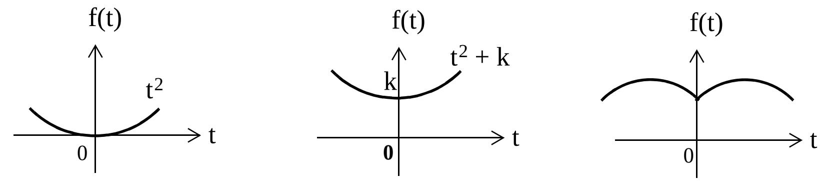


Figure 6.4. Examples of even functions

A function $f(t)$ is an *odd function* of time if the following relation holds.

$$\boxed{-f(-t) = f(t)} \tag{6.13}$$

that is, if in an odd function we replace t with $-t$, we obtain the negative of the function $f(t)$. Thus, polynomials with odd exponents only, and no constants are odd functions. For instance, the sine function is an odd function because it can be written as the power series

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$$

Other examples of odd functions are shown in Figure 6.5.

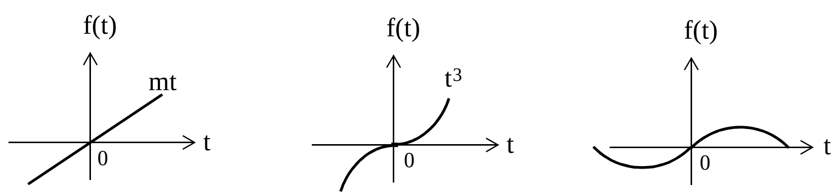


Figure 6.5. Examples of odd functions

We observe that for odd functions, $f(0) = 0$. However, the reverse is not always true; that is, if $f(0) = 0$, we should not conclude that $f(t)$ is an odd function. An example of this is the function $f(t) = t^2$ in Figure 6.4.

The product of *two even* or *two odd* functions is an even function, and the product of an even function times an odd function, is an odd function.

Henceforth, we will denote an even function with the subscript e, and an odd function with the subscript o. Thus, $f_e(t)$ and $f_o(t)$ will be used to represent even and odd functions of time respectively.

For an even function $f_e(t)$,

$$\int_{-T}^{T} f_e(t)dt = 2\int_{0}^{T} f_e(t)dt \tag{6.14}$$

and for an odd function $f_o(t)$,

$$\int_{-T}^{T} f_o(t)dt = 0 \tag{6.15}$$

A function $f(t)$ that is neither even nor odd can be expressed as

$$\boxed{f_e(t) = \frac{1}{2}[f(t) + f(-t)]} \tag{6.16}$$

or as

$$\boxed{f_o(t) = \frac{1}{2}[f(t) - f(-t)]} \tag{6.17}$$

Addition of (6.16) with (6.17) yields

$$\boxed{f(t) = f_e(t) + f_o(t)} \tag{6.18}$$

that is, any function of time can be expressed as the sum of an even and an odd function.

Example 6.3

Determine whether the delta function is an even or an odd function of time.

Solution:

Let $f(t)$ be an arbitrary function of time that is continuous at $t = t_0$. Then, by the sifting property of the delta function

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

and for $t_0 = 0$,

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

Also,

$$\int_{-\infty}^{\infty} f_e(t)\delta(t)dt = f_e(0)$$

and

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0)$$

As stated earlier, an odd function $f_o(t)$ evaluated at $t = 0$ is zero, that is, $f_o(0) = 0$. Therefore, from the last relation above,

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0) = 0 \tag{6.19}$$

and this indicates that the product $f_o(t)\delta(t)$ is an odd function of t. Then, since $f_o(t)$ is odd, it follows that $\delta(t)$ must be an even function of t for (6.19) to hold.

6.3 Convolution

Consider a network whose input is $\delta(t)$, and its output is the impulse response $h(t)$. We can represent the input–output relationship as the block diagram shown below.

$$\xrightarrow{\delta(t)} \boxed{\text{Network}} \xrightarrow{h(t)}$$

In general,

$$\xrightarrow{\delta(t-\tau)} \boxed{\text{Network}} \xrightarrow{h(t-\tau)}$$

Next, we let $u(t)$ be any input whose value at $t = \tau$ is $u(\tau)$. Then,

$$\xrightarrow{u(\tau)\delta(t-\tau)} \boxed{\text{Network}} \xrightarrow{u(\tau)h(t-\tau)}$$

Multiplying both sides by the constant $d\tau$, integrating from $-\infty$ to $+\infty$, and making use of the fact that the delta function is even, i.e., $\delta(t-\tau) = \delta(\tau-t)$, we obtain

$$\left.\begin{matrix}\displaystyle\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau \\ \displaystyle\int_{-\infty}^{\infty} u(\tau)\delta(\tau-t)d\tau\end{matrix}\right\} \longrightarrow \boxed{\text{Network}} \longrightarrow \left\{\begin{matrix}\displaystyle\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \\ \displaystyle\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau\end{matrix}\right.$$

Using the sifting property of the delta function, we find that the second integral on the left side reduces to $u(t)$ and thus

$$u(t) \longrightarrow \boxed{\text{Network}} \longrightarrow \left\{\begin{matrix}\displaystyle\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \\ \displaystyle\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau\end{matrix}\right.$$

The integral

$$\boxed{\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau} \quad \text{or} \quad \boxed{\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau} \tag{6.20}$$

is known as the *convolution integral*; it states that if we know the impulse response of a network, we can compute the response to any input $u(t)$ using either of the integrals of (6.20).

The convolution integral is usually represented as $u(t)*h(t)$ or $h(t)*u(t)$, where the asterisk (*) denotes convolution.

In Section 6.1, we found that the impulse response for a single input is $h(t) = e^{At}b$. Therefore, if we know $h(t)$, we can use the convolution integral to compute the response $y(t)$ of any input $u(t)$ using the relation

$$y(t) = \int_{-\infty}^{\infty} e^{A(t-\tau)}bu(\tau)d\tau = e^{At}\int_{-\infty}^{\infty} e^{-A\tau}bu(\tau)d\tau \tag{6.21}$$

6.4 Graphical Evaluation of the Convolution Integral

The convolution integral is more conveniently evaluated by the graphical evaluation. The procedure is best illustrated with the following examples.

Example 6.4

The signals $h(t)$ and $u(t)$ are as shown in Figure 6.6. Compute $h(t)*u(t)$ using the graphical evaluation.

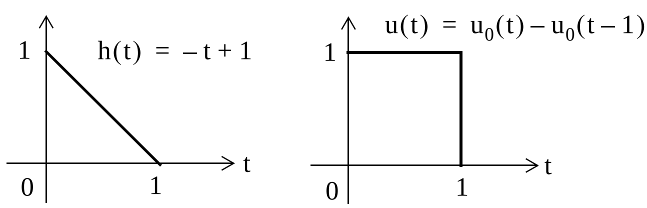


Figure 6.6. Signals for Example 6.4

Solution:

The convolution integral states that

$$h(t)*u(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau \tag{6.22}$$

where τ is a dummy variable, that is, $u(\tau)$ and $h(\tau)$, are considered to be the same as $u(t)$ and $h(t)$. We form $u(t-\tau)$ by first constructing the image of $u(\tau)$; this is shown as $u(-\tau)$ in Figure 6.7.

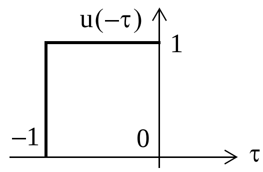


Figure 6.7. Construction of $u(-\tau)$ for Example 6.4

Next, we form $u(t-\tau)$ by shifting $u(-\tau)$ to the right by some value t as shown in Figure 6.8.

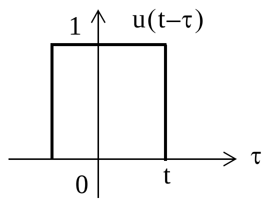


Figure 6.8. Formation of $u(t-\tau)$ for Example 6.4

Now, evaluation of the convolution integral

$$h(t)*u(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

entails multiplication of $u(t-\tau)$ by $h(\tau)$ for each value of t, and computation of the area from $-\infty$ to $+\infty$. Figure 6.9 shows the product $u(t-\tau)h(\tau)$ as point A moves to the right.

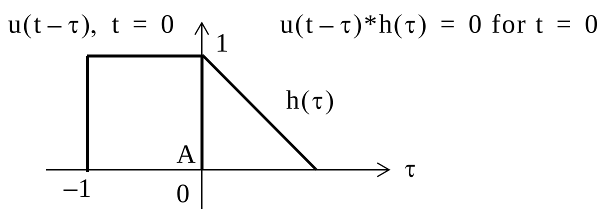


Figure 6.9. Formation of the product $u(t-\tau)*h(\tau)$ for Example 6.4

We observe that $u(t-\tau)|_{t=0} = u(-\tau)$. Shifting $u(t-\tau)$ to the right so that $t>0$, we obtain the sketch of Figure 6.10 where the integral of the product is denoted by the shaded area, and it increases as point A moves further to the right.

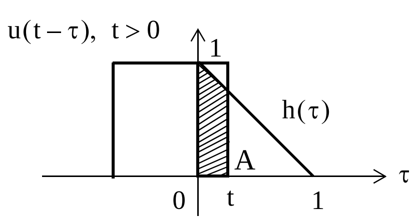


Figure 6.10. Shift of $u(t-\tau)$ for Example 6.4

The maximum area is obtained when point A reaches $t = 1$ as shown in Figure 6.11.

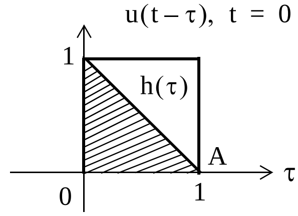


Figure 6.11. Signals for Example 6.4 when $t = 1$

Using the convolution integral, we find that the area as a function of time t is

$$\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau = \int_0^t u(t-\tau)h(\tau)d\tau = \int_0^t (1)(-\tau+1)d\tau = \tau - \frac{\tau^2}{2}\Bigg|_0^t = t - \frac{t^2}{2} \tag{6.23}$$

Figure 6.12 shows how $u(\tau)*h(\tau)$ increases during the interval $0 < t < 1$. This is not an exponential increase; it is the function $t - t^2/2$ in (6.23), and each point on the curve of Figure 6.12 represents the area under the convolution integral.

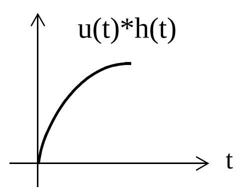


Figure 6.12. Curve for the convolution of $u(\tau)*h(\tau)$ for $0 < t < 1$ in Example 6.4

Evaluating (6.23) at $t = 1$, we obtain

$$t - \frac{t^2}{2}\Bigg|_{t=1} = \frac{1}{2} \tag{6.24}$$

The plot for the interval $0 \le t \le 1$ is shown in Figure 6.13.

As we continue shifting $u(t-\tau)$ to the right, the area starts decreasing, and it becomes zero at $t = 2$, as shown in Figure 6.14.

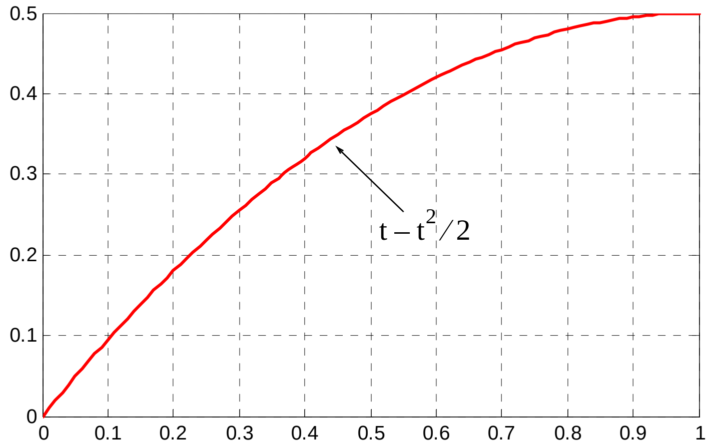


Figure 6.13. Convolution of $u(\tau)*h(\tau)$ at $t = 1$ for Example 6.4

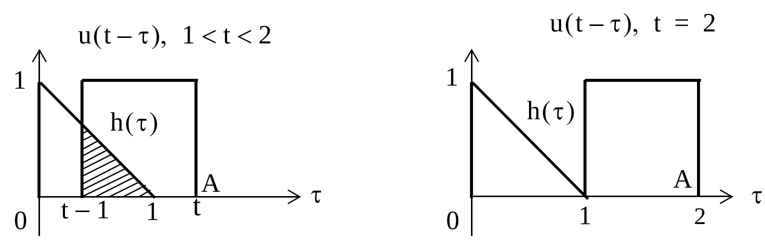


Figure 6.14. Convolution for interval $1 < t < 2$ of Example 6.4

Using the convolution integral, we find that the area for the interval $1 < t < 2$ is

$$\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau = \int_{t-1}^{1} u(t-\tau)h(\tau)d\tau = \int_{t-1}^{1} (1)(-\tau+1)d\tau = \tau - \left.\frac{\tau^2}{2}\right|_{t-1}^{1}$$
$$= 1 - \frac{1}{2} - (t-1) + \frac{t^2 - 2t + 1}{2} = \frac{t^2}{2} - 2t + 2 \qquad (6.25)$$

Thus, for $1 < t < 2$, the area decreases in accordance with $t^2/2 - 2t + 2$.

Evaluating (6.25) at $t = 2$, we find that $u(\tau)*h(\tau) = 0$. For $t > 2$, the product $u(t-\tau)h(\tau)$ is zero since there is no overlap between these two signals. The convolution of these signals for $0 \le t \le 2$, is shown in Figure 6.15.

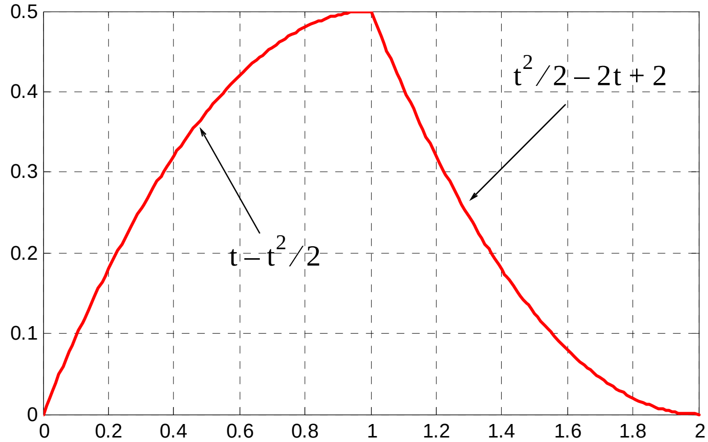


Figure 6.15. Convolution for $0 \le \tau \le 2$ of the signals of Example 6.4

The plot of Figure 6.15 was obtained with the MATLAB script below.

```
t1=0:0.01:1; x=t1–t1.^2./2; axis([0 1 0 0.5]);...
t2=1:0.01:2; y=t2.^2./2–2.*t2+2; axis([1 2 0 0.5]); plot(t1,x,t2,y); grid
```

Example 6.5

The signals $h(t)$ and $u(t)$ are as shown in Figure 6.16. Compute $h(t)*u(t)$ using the graphical evaluation method.

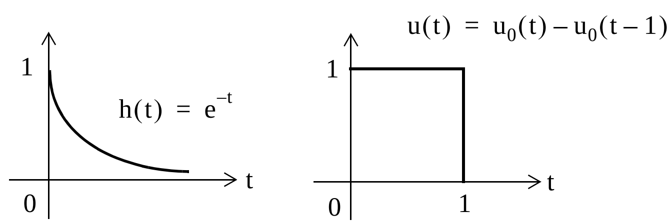


Figure 6.16. Signals for Example 6.5

Solution:

Following the same procedure as in the previous example, we form $u(t-\tau)$ by first constructing the image of $u(\tau)$. This is shown as $u(-\tau)$ in Figure 6.17.

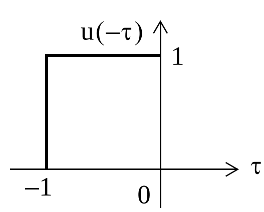


Figure 6.17. Construction of $u(-\tau)$ for Example 6.5

Next, we form $u(t-\tau)$ by shifting $u(-\tau)$ to the right by some value t as shown in Figure 6.18.

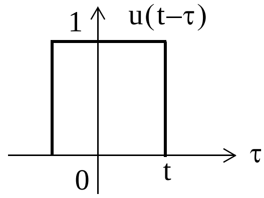


Figure 6.18. Formation of $u(t-\tau)$ for Example 6.5

As in the previous example, evaluation of the convolution integral

$$h(t)*u(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

entails multiplication of $u(t-\tau)$ by $h(\tau)$ for each value ot t, and computation of the area from $-\infty$ to $+\infty$. Figure 6.19 shows the product $u(t-\tau)h(\tau)$ as point A moves to the right.

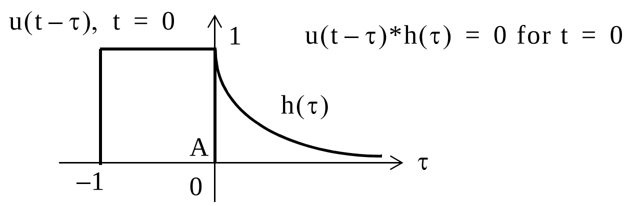


Figure 6.19. Formation of the product $u(t-\tau)*h(\tau)$ for Example 6.5

We observe that $u(t-\tau)|_{t=0} = u(-\tau)$. Shifting $u(t-\tau)$ to the right so that $t > 0$, we obtain the sketch of Figure 6.20 where the integral of the product is denoted by the shaded area, and it increases as point A moves further to the right.

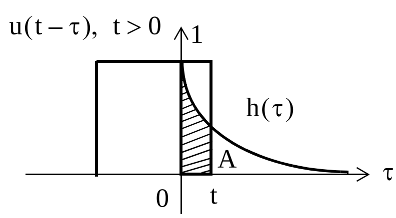


Figure 6.20. Shift of $u(t-\tau)$ for Example 6.5

The maximum area is obtained when point A reaches $t = 1$ as shown in Figure 6.21.

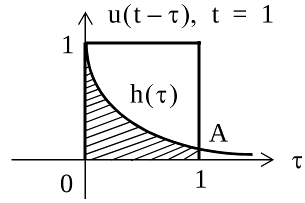


Figure 6.21. Convolution of $u(\tau)*h(\tau)$ at $t = 1$ for Example 6.5

Its value for $0 < t < 1$ is

$$\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau = \int_0^t u(t-\tau)h(\tau)d\tau = \int_0^t (1)(e^{-\tau})d\tau = -e^{-\tau}\Big|_0^t = e^{-\tau}\Big|_t^0 = 1-e^{-t} \tag{6.26}$$

Evaluating (6.26) at $t = 1$, we obtain

$$1-e^{-t}\Big|_{t=1} = 1-e^{-1} = 0.632 \tag{6.27}$$

The plot for the interval $0 \le t \le 1$ is shown in Figure 6.22.

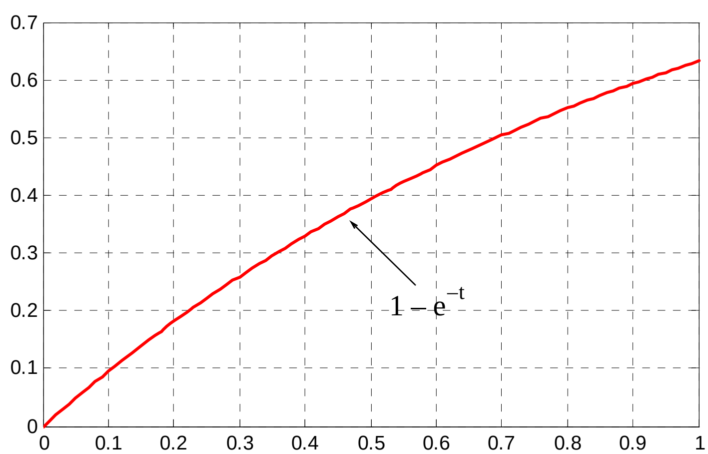


Figure 6.22. Convolution of $u(\tau)*h(\tau)$ for $0 \le t \le 1$ in Example 6.5

As we continue shifting $u(t-\tau)$ to the right, the area starts decreasing. As shown in Figure 6.23, it approaches zero as t becomes large but never reaches the value of zero.

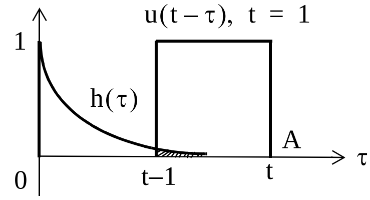


Figure 6.23. Convolution for interval $1 < t < 2$ of Example 6.5

Therefore, for the time interval $t > 1$, we have

$$\int_{t-1}^{t} u(t-\tau)h(\tau)d\tau = \int_{t-1}^{t}(1)(e^{-\tau})d\tau = -e^{-\tau}\Big|_{t-1}^{t} = e^{-\tau}\Big|_{t}^{t-1} = e^{-(t-1)} - e^{-t} = e^{-t}(e-1) \tag{6.28}$$
$$= 1.732e^{-t}$$

Evaluating (6.28) at $t = 2$, we find that $u(\tau)*h(\tau) = 0.233$.

For $t > 2$, the product $u(t-\tau)h(\tau)$ approaches zero as $t \to \infty$. The convolution of these signals for $0 \le t \le 2$, is shown in Figure 6.24.

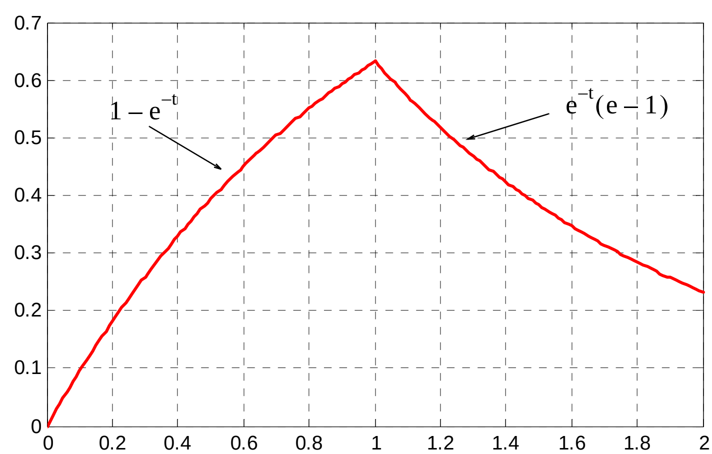


Figure 6.24. Convolution for $0 \le t \le 2$ of the signals of Example 6.5

The plot of Figure 6.24 was obtained with the MATLAB script below.

```
t1=0:0.01:1; x=1–exp(–t1); axis([0 1 0 0.8]);...
t2=1:0.01:2; y=1.718.*exp(–t2); axis([1 2 0 0.8]); plot(t1,x,t2,y); grid
```

Example 6.6

Perform the convolution $v_1(t)*v_2(t)$ where $v_1(t)$ and $v_2(t)$ are as shown in Figure 6.25.

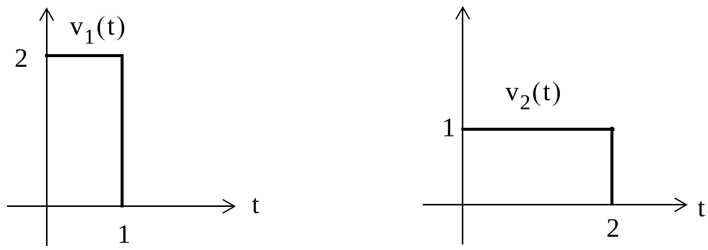


Figure 6.25. Signals for Example 6.6

Solution:

We will use the convolution integral

$$v_1(t)*v_2(t) = \int_{-\infty}^{\infty} v_1(\tau)v_2(t-\tau)d\tau \tag{6.29}$$

The computation steps are as in the two previous examples, and are evident from the sketches of Figures 6.26 through 6.29.

Figure 6.26 shows the formation of $v_2(-\tau)$.

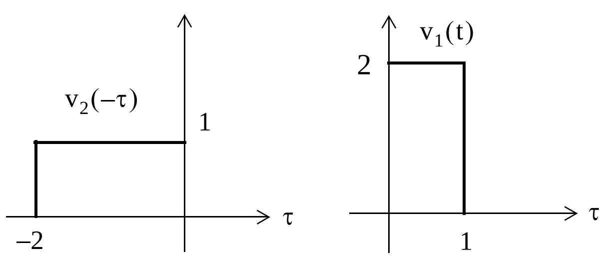


Figure 6.26. Formation of $v_2(-\tau)$ for Example 6.6

Figure 6.27 shows the formation of $v_2(t-\tau)$ and convolution with $v_1(t)$ for $0 < t < 1$.

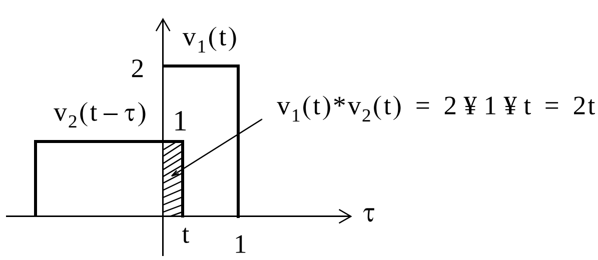


Figure 6.27. Formation of $v_2(t-\tau)$ and convolution with $v_1(t)$

For $0 < t < 1$,

$$v_1(t)*v_2(t) = \int_0^t (1)(2)d\tau = 2\tau\big|_0^t = 2t \tag{6.30}$$

Figure 6.28 shows the convolution of $v_2(t-\tau)$ with $v_1(t)$ for $1 < t < 2$.

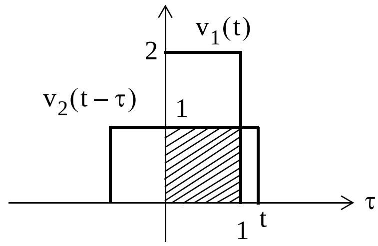


Figure 6.28. Convolution of $v_2(t-\tau)$ with $v_1(t)$ for $1 < t < 2$

For $1 < t < 2$,

$$v_1(t)*v_2(t) = \int_0^1 (1)(2)d\tau = 2\tau\Big|_0^1 = 2 \tag{6.31}$$

Figure 6.29 shows the convolution of $v_2(t-\tau)$ with $v_1(t)$ for $2 < t < 3$.

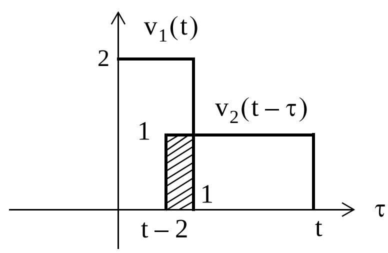


Figure 6.29. Convolution of $v_2(t-\tau)$ with $v_1(t)$ for $2 < t < 3$

For $2 < t < 3$

$$v_1(t)*v_2(t) = \int_{t-2}^1 (1)(2)d\tau = 2\tau\Big|_{t-2}^1 = -2t+6 \tag{6.32}$$

From (6.30), (6.31), and (6.32), we obtain the waveform of Figure 6.30 that represents the convolution of the signals $v_1(t)$ and $v_2(t-\tau)$.

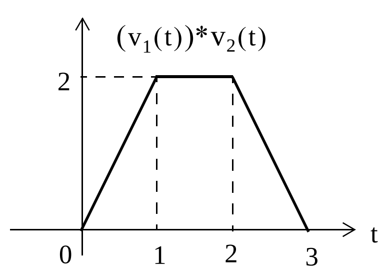


Figure 6.30. Convolution of $v_1(t)$ with $v_2(t)$ for $0 < t < 3$

In summary, the procedure for the graphical evaluation of the convolution integral, is as follows:

1. We substitute $u(t)$ and $h(t)$ with $u(\tau)$ and $h(\tau)$ respectively.
2. We fold (form the mirror image of) $u(\tau)$ or $h(\tau)$ about the vertical axis to obtain $u(-\tau)$ or $h(-\tau)$.
3. We slide $u(-\tau)$ or $h(-\tau)$ to the right a distance *t* to obtain $u(t-\tau)$ or $h(t-\tau)$.
4. We multiply the two functions to obtain the product $u(t-\tau)\,h(\tau)$, or $u(\tau)\,h(t-\tau)$.
5. We integrate this product by varying *t* from $-\infty$ to $+\infty$.

6.5 Circuit Analysis with the Convolution Integral

We can use the convolution integral in circuit analysis as illustrated by the following example.

Example 6.7

For the circuit of Figure 6.31, use the convolution integral to find the capacitor voltage when the input is the unit step function $u_0(t)$, and $v_C(0^-) = 0$.

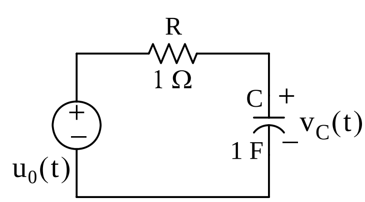


Figure 6.31. Circuit for Example 6.7

Solution:

Before we apply the convolution integral, we must know the impulse response $h(t)$ of this circuit. The circuit of Figure 6.31 was analyzed in Example 6.1, Page 6–2, where we found that

$$h(t) = \frac{1}{RC}e^{-t/RC}u_0(t) \tag{6.33}$$

With the given values, (6.33) reduces to

$$h(t) = e^{-t}u_0(t) \tag{6.34}$$

Next, we use the graphical evaluation of the convolution integral as shown in Figures 6.32 through 6.34.

The formation of $u_0(-\tau)$ is shown in Figure 6.32.

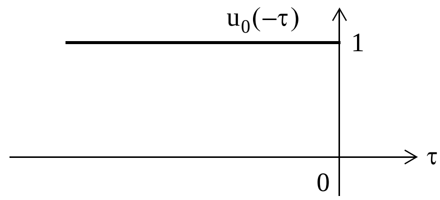


Figure 6.32. Formation of $u_0(-\tau)$ for Example 6.7

Figure 6.33 shows the formation of $u_0(t-\tau)$.

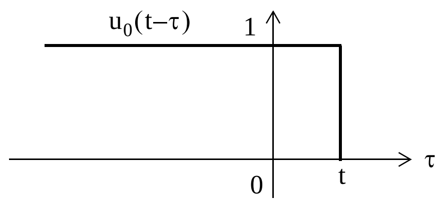


Figure 6.33. Formation of $u_0(t-\tau)$ for Example 6.7

Figure 6.34 shows the convolution $(u_0(t))*h(t)$.

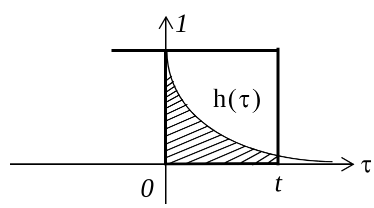


Figure 6.34. Convolution of $u_0(t)*h(t)$ for Example 6.7

Therefore, for the interval $0 < t < \infty$, we obtain

$$u_0(t)*h(t) = \int_{-\infty}^{\infty} u_0(t-\tau)h(\tau)d\tau = \int_0^t (1)e^{-t}d\tau = -e^{-t}\Big|_0^t = e^{-t}\Big|_t^0 = (1-e^{-t})u_0(t) \qquad (6.35)$$

and the convolution $u_0(t)*h(t)$ is shown in Figure 6.35.

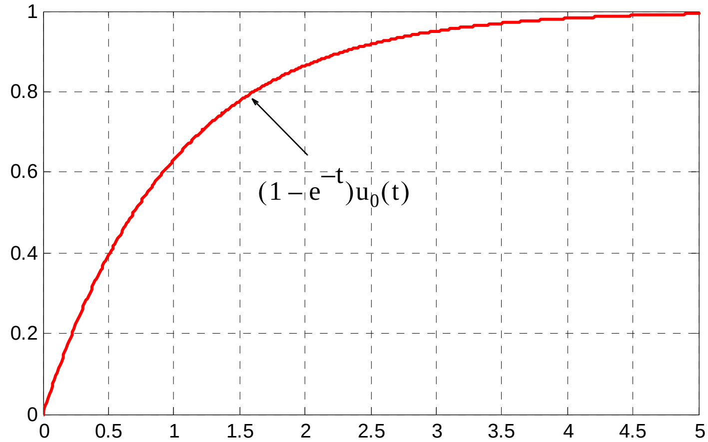


Figure 6.35. Convolution of $u_0(t)*h(t)$ for Example 6.7

6.6 Summary

- The impulse response is the output (voltage or current) of a network when the input is the delta function.
- The determination of the impulse response assumes zero initial conditions.
- A function $f(t)$ is an even function of time if the following relation holds.

$$f(-t) = f(t)$$

- A function $f(t)$ is an odd function of time if the following relation holds.

$$-f(-t) = f(t)$$

- The product of two even or two odd functions is an even function, and the product of an even function times an odd function, is an odd function.
- A function $f(t)$ that is neither even nor odd, can be expressed as

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$$

or as

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$$

where $f_e(t)$ denotes an even function and $f_o(t)$ denotes an odd function.

- Any function of time can be expressed as the sum of an even and an odd function, that is,

$$f(t) = f_e(t) + f_o(t)$$

- The delta function is an even function of time.
- The integral

$$\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

or

$$\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

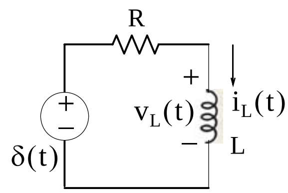
is known as the convolution integral.

- If we know the impulse response of a network, we can compute the response to any input $u(t)$ with the use of the convolution integral.

- The convolution integral is usually denoted as u(t)*h(t) or h(t)*u(t), where the asterisk (*) denotes convolution.
- The convolution integral is more conveniently evaluated by the graphical evaluation method.

6.7 Exercises

1. Compute the impulse response $h(t) = i_L(t)$ in terms of R and L for the circuit below. Then, compute the voltage $v_L(t)$ across the inductor.



2. Repeat Example 6.4, Page 6–8, by forming $h(t-\tau)$ instead of $u(t-\tau)$, that is, use the convolution integral

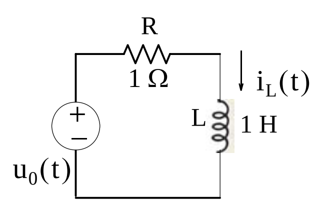
$$\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

3. Repeat Example 6.5, Page 6–12, by forming $h(t-\tau)$ instead of $u(t-\tau)$.

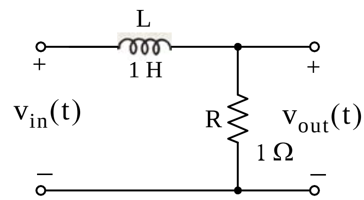
4. Compute $v_1(t)*v_2(t)$ given that

$$v_1(t) = \begin{cases} 4t & t \ge 0 \\ 0 & t < 0 \end{cases}$$

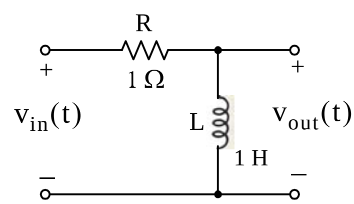
5. For the series RL circuit shown below, the response is the current $i_L(t)$. Use the convolution integral to find the response when the input is the unit step $u_0(t)$.



6. Compute $v_{out}(t)$ for the network shown below using the convolution integral, given that $v_{in}(t) = u_0(t) - u_0(t-1)$.



7. Compute $v_{out}(t)$ for the network shown below given that $v_{in}(t) = u_0(t) - u_0(t-1)$. Using MATLAB, plot $v_{out}(t)$ for the time interval $0 < t < 5$.



Hint: Use the result of Exercise 6.