Differential Drive Mobile Robot along X axis

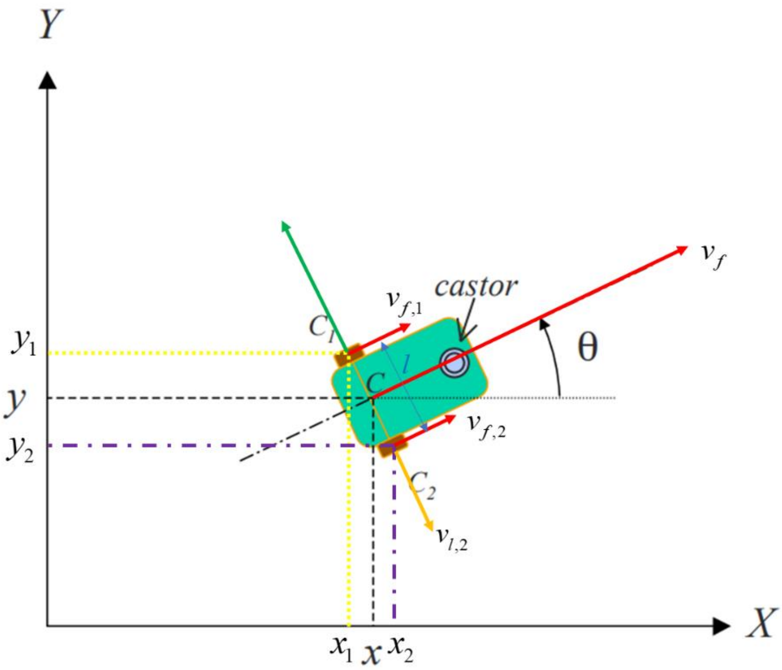


Fig. 1. Differential Driven Mobile Robot

Since conventional wheels are used, there are constraints as follows:

Pure Rolling Constraint:

$$v_{f,i} = r\dot{\phi} = \dot{x}\cos\theta + \dot{y}\sin\theta$$

No Lateral Slip Constraint:

$$v_{l,i} = 0 = \dot{x}\sin\theta - \dot{y}\cos\theta$$

Consider the coordinates of the center C of the axle, (x, y), are half way between C_1 and C_2 as shown in Fig.

$x = \dfrac{x_1 + x_2}{2}$ and $y = \dfrac{y_1 + y_2}{2}$.

Therefore, the velocity of the point C is

$$v_C = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dfrac{\dot{x}_1 + \dot{x}_2}{2} \\ \dfrac{\dot{y}_1 + \dot{y}_2}{2} \end{bmatrix}$$

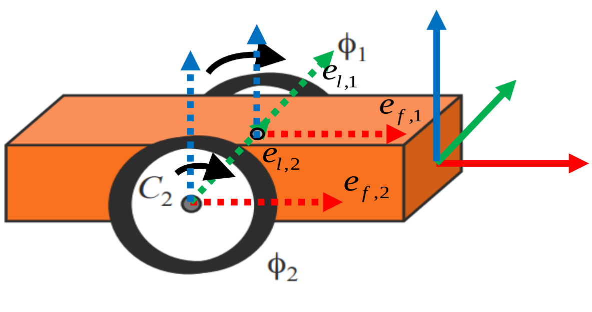


Fig. 2. Frame Assignments

Since $v_C = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$, $e_f = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $e_l = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$,

The linear velocity of the robot is

$$v_f = e_f^{\,T} v_C = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$= \dot{x}\cos\theta + \dot{y}\sin\theta$$

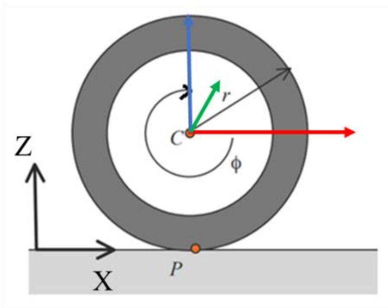
$$= \left(\frac{\dot{x}_1 + \dot{x}_2}{2} \right)\cos\theta + \left(\frac{\dot{y}_1 + \dot{y}_2}{2} \right)\sin\theta$$

$$= \frac{\dot{x}_1\cos\theta + \dot{y}_1\sin\theta}{2} + \frac{\dot{x}_2\cos\theta + \dot{y}_2\sin\theta}{2}$$

$$= \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2}$$

$$v_f = v_x = \frac{r\omega_1}{2} + \frac{r\omega_2}{2}$$

Consider the rolling wheel with a point of contact P with the ground,



(x, y, z) >>> (i, j, k)

Fig. 3. Wheel Frame

Since it is rolling, the velocity of the point P is zero ($v_P = 0$).

The velocity of the point C of the wheel can be obtained from the vector equation:

$$v_C = v_p + \omega \times \overrightarrow{PC} = 0 + \dot{\phi}\,\hat{j} \times r\hat{k} = r\dot{\phi}\hat{i}$$

Therefore, $v_{C_1} = v_p + \omega \times \overrightarrow{PC} = 0 + \dot{\phi}_1\hat{j} \times r\hat{k} = r\dot{\phi}_1\hat{i}$

$$v_{C_2} = v_p + \omega \times \overrightarrow{PC} = 0 + \dot{\phi}_2\hat{j} \times r\hat{k} = r\dot{\phi}_2\hat{i}$$

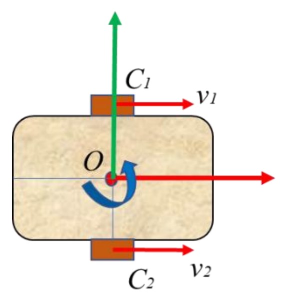


Fig. 4. Frames of the wheels of the differential driven mobile robot

Now if we consider the two points C_1 and C_2 which are rigidly attached to the axle and the mobile robot, the velocities of these two points are related by the equation:

$$v_{C_2} = v_{C_1} + \dot{\theta}\hat{k} \times \overrightarrow{C_1C_2} = v_{C_1} + \dot{\theta}\hat{k} \times \left(-l\right)\hat{j} = v_{C_1} + l\dot{\theta}\hat{i}$$

Substituting $v_{C_1} = r\dot{\phi}_1\hat{i}$ and $v_{C_2} = r\dot{\phi}_2\hat{i}$,

$$r\dot{\phi}_2\hat{i} = r\dot{\phi}_1\hat{i} + l\dot{\theta}\hat{i}$$

$$\dot{\theta} = \frac{r\dot{\phi}_2 - r\dot{\phi}_1}{l} = \frac{r\omega_2 - r\omega_1}{l}$$

Alternatively, we can find the orientation of the robot due to the lateral movement of the mobile robot along y-axis as follows:

In the robot frame, directions of v_1 and v_2 are same as forward direction of the robot, so $v = r\omega$.

since there is no active velocity in the center O ($v_O = 0$) ,

$$v_1 = v_O + \omega \times \overrightarrow{OC_1} = 0 + \dot{\theta}_1\hat{k} \times \overrightarrow{OC_1}\hat{j} = 0 + \dot{\theta}_1\hat{k} \times (\frac{l}{2})\hat{j} = -\frac{l}{2}\dot{\theta}_1\hat{i} = r\omega_1$$

$$\dot{\theta}_1 = -\frac{2r\omega_1}{l}$$

$$v_2 = v_O + \omega \times \overrightarrow{OC_2} = 0 + \dot{\theta}_2\hat{k} \times \overrightarrow{OC_2}\hat{j} = 0 + \dot{\theta}_2\hat{k} \times (-\frac{l}{2})\hat{j} = \frac{l}{2}\dot{\theta}_2\hat{i} = r\omega_2$$

$$\dot{\theta}_2 = \frac{2r\omega_2}{l}$$

$$\dot{\theta} = \frac{\dot{\theta}_1 + \dot{\theta}_2}{2} = \frac{r\omega_2}{l} - \frac{r\omega_1}{l} = \frac{r\omega_2 - r\omega_1}{l}$$

Finally, the equation of the forward kinematics in local coordinate system is

$$v_x = \frac{r\omega_1}{2} + \frac{r\omega_2}{2}$$

$$v_y = 0$$

$$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{r\omega_1}{2} + \frac{r\omega_2}{2} \\ 0 \end{bmatrix}$$

$$\dot{\theta} = \frac{r\omega_2 - r\omega_1}{l}$$

In matrix form,

$$\begin{bmatrix} v \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \\ -\frac{r}{l} & \frac{r}{l} \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

Omitting the zero for better computation,

$$\begin{bmatrix} v \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ -\frac{r}{l} & \frac{r}{l} \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ -\frac{r}{l} & \frac{r}{l} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

Jacobian of forward kinematics is

$$J = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ -\frac{r}{l} & \frac{r}{l} \end{bmatrix}$$

Inverse Jacobian is the inverse of the Jacobian of forward kinematics.

$$J^{-1} = \begin{bmatrix} \frac{1}{r} & -\frac{l}{2r} \\ \frac{1}{r} & \frac{l}{2r} \end{bmatrix}$$

Finally, the inverse kinematic equation is

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & -\frac{l}{2r} \\ \frac{1}{r} & \frac{l}{2r} \end{bmatrix} \begin{bmatrix} v \\ \dot{\theta} \end{bmatrix}$$

Converting the linear velocity of the mobile robot from the local coordinate to the global coordinate,

$$v_G = Qv = Q\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \frac{r\omega_1}{2} + \frac{r\omega_2}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{r\cos\psi}{2} & \frac{r\cos\psi}{2} \\ \frac{r\sin\psi}{2} & \frac{r\sin\psi}{2} \end{bmatrix}$$

Therefore, in global coordinate system, forward Jacobian is

$$J = \begin{bmatrix} \frac{r\cos\psi}{2} & \frac{r\cos\psi}{2} \\ \frac{r\sin\psi}{2} & \frac{r\sin\psi}{2} \\ -\frac{r}{l} & \frac{r}{l} \end{bmatrix}$$

Checking Kinematics

Assume $r = 1$ and $l = 1$. Then, the forward Jacobian is

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{bmatrix},$$

and the inverse Jacobian is

$$J^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$$

If $\omega_1 = 1$ and $\omega_2 = -1$,

$$\begin{bmatrix} v \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

If $v = 0$ and $\dot{\theta} = \omega = 2$,

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$