The Textbook

MATHEMATICS

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Preface

It becomes quite apparant - after years of studying, browsing various forums, and talking to peers - that the cost of academic books and lecture notes (etc.) is not only too high, but on the rise. In most countries the education is already so expensive, that buying a book just becomes another stress factor as students have to balance a job (often more than one), studying, and are already in a lot of debt.

I am so lucky that i was born in Denmark. University is free, and though the books aren't free, we are literally getting payed to study, so buying books isn't a problem.

As i see students - particularly the knowledge they acquire - as one of the most valuable resources in society, i thought that it was perhaps the stupidest idea in the world to start their working life off with crippling debt.

It is true, academic books are often really expensive, and part of that reason is, that professors - in some countries - are paid by the page count. This results in very long, and quite often, very bad books. Often better alternatives are out there, but since its the professors that gets to choose the corriculum, guess whose book is being used.

So the problem we have to solve, is making an academic book, that is cheap (preferably free), well written, has enough to theory to use as the text for the course, and perhaps most importantly, is well known as an alternative to the expensive text written by the professor.

Perhaps only a few, on even no professors at all will choose this textbook as the textbook for their course, but the students should know that this is an alternative which is free, very good, and which covers the same things as the textbook that the professor wrote.

This seems like a daunting task, perhaps even impossible, and that is why its not done. But hopefully in time, when more people are working on it, it will become a much better textbook.

Part I

Measure Theory

Chapter 1

Counting

1.1 Cardinality

Before we dive into measure theory, we will need a short introduction to the concept of counting and cardinality.

In measure theory we are interested in how big sets are, and the most elementary way we can assign a size to a set, is to just count how many elements are in the set. Of course, some set are infinitely big, but as it turns out, there are different kinds of infinities when it comes to sets.

Cardinality is a measure of "how many elements are in the set", so for a set $A = \{1, 2, 3\}$ we see that it has 3 elements, and say that its cardinality is 3, and write $|A| = 3$.

So for a general finite set $A = \{x_1, x_2, \ldots, x_n\}$ we see that $|A| = n$. The concept however is a little bit different for infinite sets, and things start to get a little bit weird.

The natural numbers are infinite, but we will think of them as being countable, this is because, if we were to start counting (i.e. 1, 2, 3, etc.) we would eventually reach every element of the natural numbers (we will not see finite time as a hindrance). So we say that $\mathbb{N}$ is countably infinite, and write $|\mathbb{N}| = \aleph_0$ (aleph zero).

For finite sets, it is easy to compare two sets, by just counting if they have the same number of elements. This procedure is just another way of seeing that there exists a bijection between the two sets. For example if we have two sets $A = \{x_1, \ldots, x_n\}$, $B = \{y_1, \ldots, y_m\}$ we can make the map $f : A \to B$ given by

$$f(x_i) = y_i$$

thus if $n = m$, this defines a bijection, and we will think of the two sets A and B as equally sized. If however $n < m$ this map is only an injection, and in fact no bijection exists, and when this happens, we say that $|A| < |B|$. Along the same line, if $n > m$, the map is only a surjection, and in fact no bijection exists, and when this happens, we say that $|A| > |B|$.

So to compare the size of two sets, we will just see if there is a bijection between the two sets, in which case they are "of equal size", or, if no such bijection exists, we will see if we can find an injection or surjection.

Immediately we see that for any finite set A, we have $|A| < |\mathbb{N}|$, and thus A is called a (countable) finite set. Now let us see where things get a litte weird.

Let $2\mathbb{N} := \{2n | n \in \mathbb{N}\}$, then we can make the map $f : \mathbb{N} \to 2\mathbb{N}$ given by

$$f(n) = 2n.$$

This is a bijection, and hence we see that $|\mathbb{N}| = |2\mathbb{N}|$, i.e. there are "as many" natural numbers as there are even numbers. This feels quite unintuitive, hence the use of the word "cardinality" rather than "size". Using a similar argument, we see that the natural numbers and the *un-even* numbers have the same cardinality.

Potentially we would also have to prove that the uneven and the even numbers have the same cardinality, but it is a well known result that if A, B, C are three sets, and $f : A \to B$ and $g : B \to C$ are two bijections (respectively injections or surjections) then the composition $g \circ f$ is again a bijection (resp. injection or surjection). I.e. the relation of being the same cardinality is transitive.

Let us look at some more examples.

Theorem 1.1.1. *The natural numbers and the integers have the same cardinality, i.e.* $|\mathbb{N}| = |\mathbb{Z}|$.

Proof. We want to show that there is a bijection between the two sets. Our approach will be to make a map that does the following:

$$\begin{aligned} 1 &\to 0 \\ 2 &\to 1 \\ 3 &\to -1 \\ 4 &\to 2 \\ 5 &\to -2 \\ &\vdots \end{aligned}$$

We can define this map $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} 2n, & \text{if } n \text{ is even} \\ -\left(\frac{n-1}{2}\right), & \text{if } n \text{ is odd} \end{cases}$$

Indeed this is a bijection: For every $n \in \mathbb{N}$, $f(n)$ defines a whole number. Thus f is well-defined. Furthermore if $n, m \in \mathbb{N}$ but $n \neq m$ then if n and m have different parity (one is even, the other is uneven) then one of $f(n), f(m)$ will be positive, while the other will be negative. If they have the same parity, then they must be separated by at least two, and thus $\left\lfloor \frac{n}{2} \right\rfloor \neq \left\lfloor \frac{m}{2} \right\rfloor$. Thus f is an injection.

To show that f is a surjection, we see that it has an inverse, $f^{-1} : \mathbb{Z} \to \mathbb{N}$ given by

$$f^{-1}(z) = \begin{cases} 2z, & \text{if } z \text{ is positive} \\ -2z+1, & \text{if } z \text{ is negative or } 0 \end{cases}$$

It is quite easy to verify that this is an inverse. Thus for every $z \in \mathbb{Z}$ there exists an $n \in \mathbb{N}$ such that $f(n) = z$, namely $n = f^{-1}(z)$. Thus f is a surjection, and since it is both a surjection and an injection, it is a bijection. Thus $|\mathbb{N}| = |\mathbb{Z}|$. ■

Theorem 1.1.2. $|\mathbb{N}| = |\mathbb{Q}|$.

Proof. Let us take a look at the positive rational numbers, and align them in the following fashion:

$$\begin{array}{cccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

■

Chapter 2

Real Measures

2.1 Introduction

Much of mathematics is based around taking an everyday notion, such as length, and then making it mathematically vigorous. Thus, mathematicians have taken the notion of length, defined what a length is (specifically what a length *function* should be) and then studied it extensively. This has led to the study of what is called a metric, and all of the broad theory that follows.

So let us now take another notion, or rather a family of related notions, namely: "size", "mass", "area", "volume" and perhaps even more. We are interested in boiling these notions down to their essence, and then beating it with a hammer, and see what tricks it can do. We are, to be precise, interested in defining two things. First we are interested in defining what things (here sets) can be measured; and second we are interested in defining a function which, when given such a set that can be measured, assigns to it a number which we will think of as its size (or in the language, its measure).

This theory, which we will develop over the course of this section, is the theory of measures and σ-algebras (the so-called measurable sets).

For further reading, consult (**schilling2017measures**).

2.2 σ-algebras

Our first task is to define which sets that can be measured. In every day life, there are perhaps things that are to small, or have a much too irregular shape to be measured, and it is these problems we will try to avoid.

It is a difficult task to say exactly which sets can be measured, but we can make a number of observations about the behaviour of sets that can be measured. First of all, we can measure "nothing", i.e. if there is nothing there to measure, we will think of it as having 0 measure (or size). Furthermore if we have a plank of wood, we would like to be able to measure its whole length, i.e. we would like our whole object of interest to be measurable.

Moreover, since we can measure the whole length of the plank, then if we can measure some part of the plank, then we can intuitively measure the remaining part of the plank, since we could just subtract the length of the plank with the length of the part we had measured.

Lastly if we can measure a lot of small non-intersecting parts of the plank of wood, we know that to measure all these small parts together, we can just add all the lengths of the small parts.

It is these three observations that will be the basics of what we will call a σ-algebra.

Definition 2.2.1. *Let X be an arbitrary set. A σ-algebra, $\mathcal{A}$, on the set X, is a family of subsets of X satisfying*

$$X \in \mathcal{A}, \tag{Σ_1}$$

$$A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}, \tag{Σ_2}$$

$$A_1, A_2, \dots \in \mathcal{A} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}. \tag{Σ_3}$$

We will call a set $A \subseteq X$ ($\mathcal{A}$-)measurable if $A \in \mathcal{A}$, and the tuple $(X, \mathcal{A})$ will be called a measurable space.

Proposition 2.2.2. *Here are some introductory results about σ-algebras and their proofs.*

1. $\emptyset \in \mathcal{A}$.
2. $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.
3. $A_1, A_2, \dots \in \mathcal{A} \Rightarrow \bigcap_{i \in \mathbb{N}} A_i \in \mathcal{A}$.

Proof.

1. Since $X \in \mathcal{A}$ by (Σ_1) and $\emptyset = X^c \in \mathcal{A}$ by (Σ_2).
2. Set $A_1 = A, A_2 = B, A_3 = \emptyset, A_4 = \emptyset, \dots$, then $A \cup B = \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$ by (Σ_3).
3. If $A_i \in \mathcal{A}$ then $A_i^c \in \mathcal{A}$ by (Σ_2), so $\bigcup_{i \in \mathbb{N}} A_i^c \in \mathcal{A}$ by (Σ_3) and again by (Σ_2) $\bigcap_{i \in \mathbb{N}} A_i = \left(\bigcup_{i \in \mathbb{N}} A_i^c\right)^c \in \mathcal{A}$.

$\blacksquare$

Example 2.2.3. *Here are some examples of σ-algebras.*

1. *The power-set $\mathcal{P}(X)$ (the set of all subsets of X) is a σ-algebra on X called the maximal σ-algebra on X.*
2. *The set $\{\emptyset, X\}$ is a σ-algebra called the minimal σ-algebra on X.*
3. *For a subset $A \subseteq X$ the family $\{\emptyset, A, A^c, X\}$ is a σ-algebra on X.*

4. $\{\emptyset, A, X\}$ *is not a* σ*-algebra unless* $A = \emptyset$ *or* $A = X$.

5. $\mathcal{A} := \{A \subseteq X : |A| \leq |\mathbb{N}| \text{ or } |A^c| \leq |\mathbb{N}|\}$ *is a* σ*-algebra. Indeed:*

(Σ_1) : $\emptyset = X^c$ *is obviously countable.*

(Σ_2) : *If* $A \in \mathcal{A}$ *one of* A *or* A^c *is by definition countable, hence* $A^c \in \mathcal{A}$.

(Σ_3) : *If* $A_1, A_2, \dots \in \mathcal{A}$ *then one of two cases can occur:*

- *All of the* A_i *are countable. Then* $\bigcup_{i\in\mathbb{N}} A_i$ *is a countable union of countable sets, which by*
- *At least one of the* A_{i_0} *is uncountable. Then* $A_{i_0}^c$ *is countable, hence*

$$\left(\bigcup_{i\in\mathbb{N}} A_i\right)^c = \bigcap_{i\in\mathbb{N}} A_i^c \subseteq A_{i_0}^c.$$

Hence $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{A}$ *since its inverse is countable.*

6. *Let* $(X, \mathcal{A})$ *be a measurable space. For* $E \subseteq X$, *the family*

$$\mathcal{A}_E := \{E \cap B : B \in \mathcal{A}\}$$

is a σ*-algebra called the trace* σ*-algebra of* E *in* $\mathcal{A}$. *This produces the measurable space* $(E, \mathcal{A}_E)$. *Indeed:*

(Σ_1) : *Since* $X \in \mathcal{A}$, *we have* $E = E \cap X \in \mathcal{A}_E$.

(Σ_2) : *Let* $A \in \mathcal{A}_E$, *then by definition of the trace* σ*-algebra, there is an element* $B \in \mathcal{A}$ *such that* $B \cap E = A$. *By* (Σ_2) $B^c \in \mathcal{A}$ *as well, so*

$$\begin{aligned} E \setminus A &= E \setminus (B \cap E) \\ &= E \setminus B \\ &= E \cap B^c \in \mathcal{A}_E \end{aligned}$$

as wanted.

(Σ_3) : *If* $A_1, A_2, \dots \in \mathcal{A}_E$, *then by definition, there exist* $B_1, B_2, \dots \in \mathcal{A}$ *such that* $A_i = E \cap B_i$. *Then*

$$\bigcup_{i\in\mathbb{N}} A_i = \bigcup_{i\in\mathbb{N}} (E \cap B_i) = E \cap \left(\bigcup_{i\in\mathbb{N}} B_i\right) \in \mathcal{A}$$

7. *Let* $(Y, \mathcal{A}_Y)$ *be a measurable space, and let* $f : X \to Y$ *be a mapping. Then*

$$\sigma(f) := \{f^{-1}(A) : A \in \mathcal{A}_Y\}$$

is a σ*-algebra on* X *called the* σ*-algebra generated by* f. *Show this.*

Theorem 2.2.4. *Let $\{\mathcal{A}_1\}_{i\in I}$ be arbitrarily many σ-algebras on X. Then the intersection*

$$\mathcal{A} := \bigcap_{i\in I} \mathcal{A}_i$$

is again a σ-algebra on X.

Proof. (Σ_1): Since $X \in \mathcal{A}_i$ for all $i \in I$, $X \in \mathcal{A}$.

(Σ_2): If $A \in \mathcal{A}$, then $A \in \mathcal{A}_i$ for all $i \in I$, hence $A^c \in \mathcal{A}_i$ for all $i \in I$, and thus $A \in \mathcal{A}$.

(Σ_3): If $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{A}$ then $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{A}_i$ for all $i \in I$, hence $\bigcup_{\mathbb{N}} A_i \in \mathcal{A}_i$ for all $i \in I$, and thus $\bigcup_{\mathbb{N}} A_i \in \mathcal{A}$. ■

An arbitrary family of subsets, $\mathcal{E}$, of X will in general not be a σ-algebra, but we are interested in finding the smallest such σ-algebra that contains it.

Proposition 2.2.5. *For every family $\mathcal{G} \subseteq \mathcal{P}(X)$ there exists a smallest (or: minimal, corsest) σ-algebra containing $\mathcal{G}$, namely the σ-algebra generated by $\mathcal{G}$ denoted by $\sigma(\mathcal{G})$, where $\mathcal{G}$ is called the generator. Furthermore,*

$$\sigma(\mathcal{G}) = \bigcap_{\substack{\mathcal{C}\ \sigma-alg.\\ \mathcal{C}\supseteq\mathcal{G}}} \mathcal{C}.$$

Proof. By **??**

$$\mathcal{A} := \bigcap_{\substack{\mathcal{C}\ \sigma-\text{alg.}\\ \mathcal{C}\supseteq\mathcal{G}}} \mathcal{C}.$$

defines a σ-algebra.

Since $\mathcal{G} \subseteq \mathcal{P}(X)$ and $\mathcal{P}(X)$ is a σ-algebra, the intersection is non-empty. Thus the definition of $\mathcal{A}$ is well-defined, and defines a σ-algebra containing $\mathcal{G}$. If $\mathcal{A}'$ is another σ-algebra containing $\mathcal{G}$, then $\mathcal{A}'$ is in the above intersection and thus $\mathcal{A} \subseteq \mathcal{A}'$. Thus $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{G}$, i.e. in notation $\sigma(\mathcal{G}) = \mathcal{A}$. ■

Remark 2.2.6. *The following are easily proved*

1. *If $\mathcal{G}$ is itself a σ-algebra, then $\mathcal{G} = \sigma(\mathcal{G})$.*
2. *If $A \subseteq X$, we have $\sigma(A) := \sigma(\{A\}) = \{\emptyset, A, A^c, X\}$.*
3. *If $\mathcal{F} \subseteq \mathcal{G} \subseteq A$, then $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{A}) = \mathcal{A}$.*

Recall the definition of an open set $A \subseteq \mathbb{R}$:

$$A \subseteq \mathbb{R} \text{ is open} \Leftrightarrow \forall x \in A, \exists r > 0 : B(x,r) \subseteq A$$

We denote the collection of all open sets of $\mathbb{R}$ by $\mathcal{O}$. This collection is called a topology. A topology is a collection, τ, of subsets of a set X such that

$$\begin{aligned} \emptyset, X &\in \tau \\ A, B \in \tau &\Rightarrow A \cap B \in \tau \\ \{U_i\}_{i \in I} \subseteq \tau &\Rightarrow \bigcup_{i \in I} U_i \in \tau \end{aligned}$$

where I is a, possibly uncountable, index set. The pair (X, τ) is called a topological space. The sets $A \in \tau$ are called open sets. A topology is therefore a generalization of the notion of open sets on a space X.

On $\mathbb{R}$ we have a standard σ-algebra, called the Borel σ-algebra, and denoted $\mathcal{B}(\mathbb{R}) = \mathcal{B}$, which is the σ-algebra generated by the open sets in $\mathbb{R}$, i.e. $\mathcal{B}(\mathbb{R}) := \sigma(\mathcal{O})$. Equivalently, for any topological space (X, τ) the Borel σ-algebra on that set $\mathcal{B}(X)$ is given by $\mathcal{B}(X) := \sigma(\tau)$. We will mostly be working with $\mathcal{B}(\mathbb{R})$, indeed, from now on, if no σ-algebra as given on $\mathbb{R}$ we will by default assume the Borel σ-algebra.

Let us look at some introductory results about $\mathcal{B}(\mathbb{R})$: