SoftMech training day
Introduction to Finite Element Method

SoftMech MP, University of Glasgow

Steven.Roper@glasgow.ac.uk

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Finite Element Method

Principle of virtual work

The equilibrium of a body in some state of deformation is expressed as

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathsf{f} = 0$$

where $\boldsymbol{\sigma}$ is the *Cauchy stress* and f is the body force (per unit mass). The stress $\boldsymbol{\sigma}$ depends on the state of deformation.

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where $\boldsymbol{\sigma}$ is the *Cauchy stress* and f is the body force (per unit mass). The stress $\boldsymbol{\sigma}$ depends on the state of deformation. Let $\delta \mathrm{u}$ be a 'virtual displacement' and define

$$W = \int_B \delta \mathrm{u} \cdot (\operatorname{div}\boldsymbol{\sigma} + \rho \mathrm{f})\ dV.$$

Integrating by parts we have

$$W = \int_{\partial B} \delta \mathrm{u} \cdot (\boldsymbol{\sigma} \mathrm{n})\ dS + \int_B \delta \mathrm{u} \cdot \rho \mathrm{f} - \operatorname{grad} \delta \mathrm{u} \cdot \boldsymbol{\sigma}\, dV$$

Finite elements

Principle of virtual work

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Let ∂B_x be the part of the boundary on which the deformation is specified. We will only consider virtual displacements that have $\delta \mathrm{u} = 0$ on ∂B_x (call these admissible virtual displacements).

Finite elements

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Let ∂B_x be the part of the boundary on which the deformation is specified. We will only consider virtual displacements that have $\delta \mathrm{u} = 0$ on ∂B_x (call these admissible virtual displacements). On the other part of the boundary, ∂B_t, we have specified tractions so $\boldsymbol{\sigma} \mathrm{n} = \mathrm{t}$, then

$$W = \int_{\partial B_t} \delta \mathrm{u} \cdot \mathrm{t} \, dS + \int_B \delta \mathrm{u} \cdot \rho \mathrm{f} - \operatorname{grad} \delta \mathrm{u} \cdot \boldsymbol{\sigma} \, dV$$

the terms represent the virtual work done by the tractions, by the body forces and by the internal forces.

Finite elements

Principle of virtual work

If, for a given $\boldsymbol{\sigma}$, we have that $W = 0$ for every admissible virtual displacement this is equivalent to $\boldsymbol{\sigma}$ satisfying the equilibirum equation is satisfied with traction boundary conditions on ∂B_t.

We can of course use material coordinates to write the principle of virtual work

$$\int_{B_0} (J\boldsymbol{\sigma}) \cdot \mathrm{grad}\, \delta \mathrm{u}\, dV = \int_{B_0} \delta \mathrm{u} \cdot \rho_0 \mathrm{f}\, dV + \int_{\partial B_t} \mathrm{t} \cdot \delta \mathrm{u}\, dS$$

where we have left the surface term (though sometimes this might be best written in material coordinates).

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$$\int_{B_0} \mathrm{S}^T \cdot \mathrm{Grad}\, \delta \mathrm{u}\, dV = \int_{B_0} \delta \mathrm{u} \cdot \rho_0 \mathrm{f}\, dV + \int_{\partial B_t} \mathrm{t} \cdot \delta \mathrm{u}\, dS$$

where S is the nominal stress.

Simple problem

One-dimensional bar

In the theory of linear elasticity in one dimension with one end clamped (at $x = 0$) and a traction t_L applied to the other end at $x = L$, we have

$$W = \int_0^L -\sigma \frac{dv}{dx} + \rho_0 f v \, dx + t_L v \big|_L,$$

where we have used v for the virtual displacements. If $\sigma = E du/dx$ where E is the elastic modulus and u is the displacment then we have exactly the same form as for the Poisson equation.

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Note that we can extend the principle of virtual work to the case in which we have kinetic energy, by use of D'Alembert's principle, then v are interpreted as virtual velocities.

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The next steps are to use this form in a discretisation. As described earlier this consists of the following steps:

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- Selection of a set of basis functions used to approximate u, defined using the mesh. Gives approximation of u in terms of a discrete set of variables (e.g. the values u_i at the nodes of the mesh).

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- Use of $(\star)$, set $W = 0$ for different v, to generate equations for the u_i.

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- Selection of a set of basis functions used to approximate u, defined using the mesh. Gives approximation of u in terms of a discrete set of variables (e.g. the values u_i at the nodes of the mesh). Similarly a set of basis functions to approximate the virtual displacements v.
- Use of $(\star)$, set $W = 0$ for different v, to generate equations for the u_i.
- In general gives nonlinear ODEs.

Simple problem

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- For example, take the 'mesh' as consisting of a single interval $[0, L]$ with $u = u_1 N_1$ and N_1 is the shape function $N_1(x) = x/L$. This approximates u as linear.

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- Note that $u(L) = u_1 N_1(L) = u_1$ so u_1 is the displacement at the node $x = L$. Then $du/dx = u_1/L$.

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- Choose the virtual velocity $v = N_1$.
- Then $W = 0$ in $(\star)$ becomes

$$\left(\frac{\rho_0 L}{3}\right)\ddot{u}_1 + \frac{E}{L}u_1 = t_L + \int_0^L \rho_0 f \frac{x}{L}\, dx$$

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- Choose the virtual velocity $v = N_1$.
- Then $W = 0$ in $(\star)$ becomes

$$M\ddot{u}_1 + Ku_1 = F$$

($M = \rho_0 L/3$ the mass, $K = E/L$ the stiffness).

Simple problem

One-dimensional bar

$$W = \int_0^L -E\frac{du}{dx}\frac{dv}{dx} + \rho_0 fv - \rho_0 \ddot{u}v \, dx + t_L v\big|_L - t_0 v\big|_0. \qquad (\star)$$

- Again take the 'mesh' as consisting of a single interval $[0, L]$ with $u = u_0 N_0 + u_1 N_1$ and $N_0 = (L - x)/L$, $N_1(x) = x/L$. This again approximates u as linear (but now without a clamped boundary at $x = 0$)

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- Choose two different virtual velocities $\tilde{v}_0 = N_0$ and $\tilde{v}_1 = N_1$.

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- Then $W = 0$ in $(\star)$ becomes, using each virtual velocity in turn,

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- Then $W = 0$ in $(\star)$ becomes, using each virtual velocity in turn,

$$\left(\frac{\rho_0 L}{6}\right)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \ddot{u}_0 \\ \ddot{u}_1 \end{pmatrix} + \frac{E}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -t_0 \\ t_L \end{pmatrix} + \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

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mechanical: $\mathrm{M}\ddot{\mathrm{u}} + \mathrm{K}\mathrm{u} = \mathrm{F}$, ($M$ is the mass matrix, K is the stiffness matrix).

Summary

- The principle of virtual work and D'Alembert's principle naturally connect the equation of balance of linear momentum to an integral (weak) form.
- The weak form plus the restriction of the virtual displacements/velocities (test functions) allows us to consider a wider class of solutions than we might be able to consider for the original differential equations.
- Applies to all materials: fluid and solid, hyperelastic or not.