SoftMech training event: Finite element methods

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2021-22

Introduction to SofTMech training

SofTMech training:

LARGE-SCALE EVENTS:

- Introduction to Mathematical modelling (Jan 21)
- Introduction to Bayesian Inference (Jan 21)
- Introduction to Scientific computation (Jan 22)
- Study group (22)

SMALLER-SCALE (informal) EVENTS

- Effective networking (Feb/Mar 22)
- Preparing an effective poster presentation (Apr/May 22)
- Presenting with confidence (Sept/Oct 22)
- Effective figures for scientific papers (Nov/Dec 22)

Welcome

Programme of seven talks:

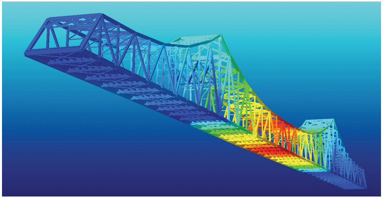
- 9:30-10:00 Theory of the finite element method for PDEs
- 10:00-10:20 Using the FE method for solid mechanics
- 10:20-10:40 FE method using variational principles
- 10:40-11:10 Discretisation of the FE equations
- 11:10-11:30 Break
- 11.30-12:00 Challenges using FE for solid mechanics
- 12:00-12:30 Worked examples in ABAQUS
- 12:30-13:00 Worked examples solving PDEs in FEniCS

Practical session to follow later in the year.

The meeting is being <u>recorded</u> and will be made available.
Slideshows and example codes will all be made available.

Finite element methods

Finite element methods are used widely in engineering applications



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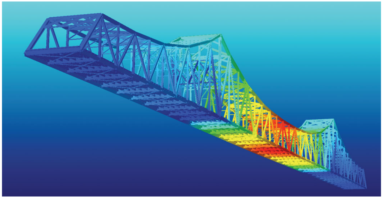
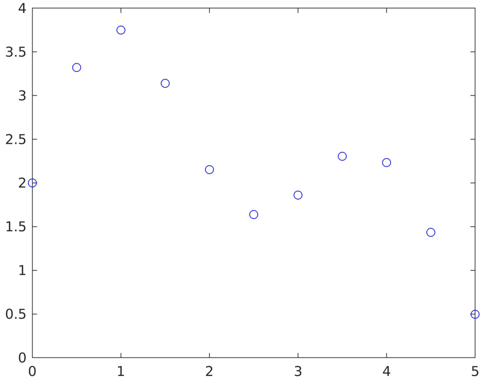


ADVANTAGES:

- Well suited to irregular domains
- Generates sparse matrices: solve large systems efficiently
- Efficient algorithms for adaptively meshing complicated geometries
- Implemented in existing packages (free, commercial)

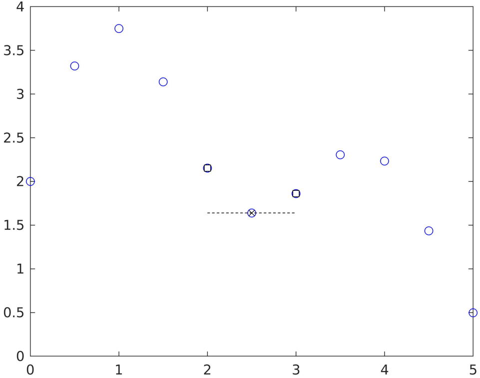
Comparison to other methods

To solve a differential equation on a discretised domain:
Evaluate derivatives of the discretised function at grid points



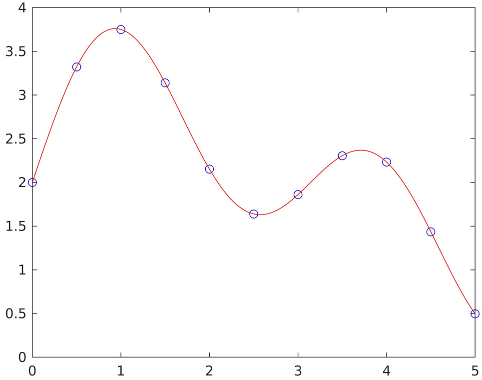
Comparison to other methods

Finite difference methods: estimate derivatives from a local stencil based on Taylor expansions



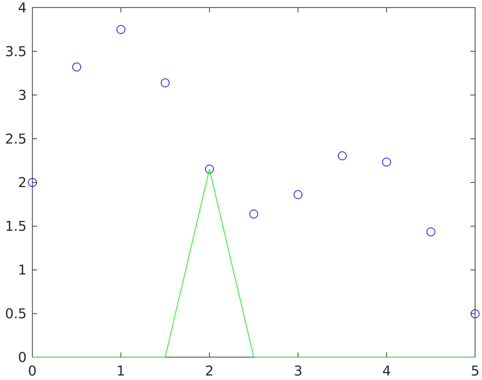
Comparison to other methods

Spectral methods: construct a global interpolant (e.g. series of trigonometric functions) and evaluate derivatives of this interpolant at grid points



Comparison to other methods

Finite element methods: construct a local interpolant at each grid point (e.g. polynomial function) which is only non-zero in mesh intervals (known as elements) touching that grid point



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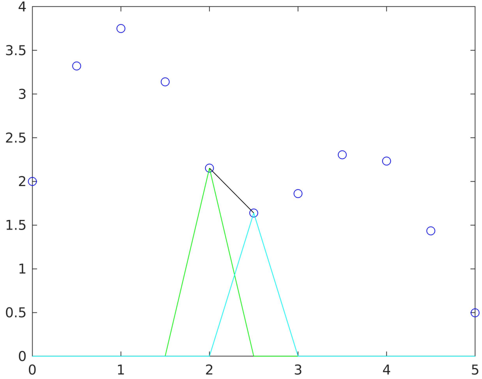
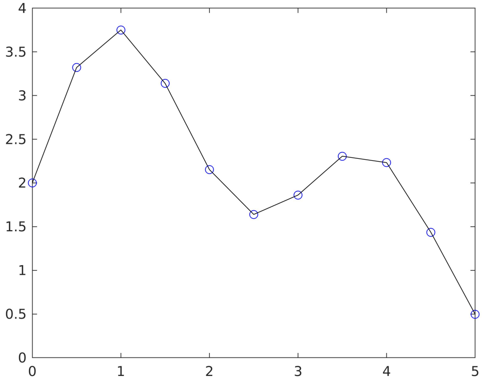


Comparison to other methods

Finite element methods: sum over all local interpolants to form a global interpolant across each element



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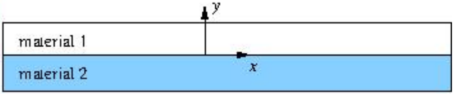
HOWEVER: for non-smooth domains or discontinuous source functions the solution may not be smooth enough (or regular enough) to be regarded as a classical solution

AN EXAMPLE

Consider the heat diffusion problem in two materials which have been fixed together



A related mathematical problem for temperature T:

$$\nabla^2 T = 0, \qquad -1 \le y \le 0, \qquad -L \le x \le L,$$
$$\nabla^2 T = -1, \qquad 0 \le y \le 1, \qquad -L \le x \le L$$

T cannot have a continuous second derivative along the joint $y = 0$ and so there is no classical solution

Poisson equation

FE methods apply to all types of differential equations (elliptic, parabolic, hyperbolic,...)

Poisson equation

Consider the Poisson equation

$$\nabla^2 u = -f \qquad (x \in \Omega)$$

subject to boundary conditions

$$\alpha u + \beta \frac{\partial u}{\partial n} = g \qquad (x \in \partial\Omega)$$

- $\beta = 0$ Dirichlet problem
- $\alpha = 0$ Neumann problem (+ compatability condition)

WEAK FORMULATION

APPROACH: derive a new formulation which is not as restrictive as the CLASSICAL solution

- known as the WEAK solution

Weak formulation

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Consider a set of TEST functions, denoted v

Weak solution

The weak solution to the Poisson equation satisfies

$$\int_{\Omega} (\nabla^2 u + f) v \, \mathrm{d}V = 0$$

subject to the boundary conditions on u.

WEAK FORMULATION

Using the divergence theorem, integration by parts and vector calculus identities, the WEAK formulation

$$-\int_{\Omega} vf \, \mathrm{d}V = \int_{\Omega} v \nabla^2 u \, \mathrm{d}V$$

can be written in the form

$$\int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}V = \int_{\Omega} vf \, \mathrm{d}V + \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, \mathrm{d}s$$

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At boundaries:

- choose u such that Dirichlet conditions are satisfied on Dirichlet portions of the boundary - these functions live in the SOLUTION SPACE
- choose v such that $v = 0$ on Dirichlet portions of the boundary - these functions live in the TEST SPACE

Discretisation into a matrix problem

Identify suitable bases for the solution space and the test space
For a problem with M nodes (degrees of freedom:

- SOLUTION SPACE: basis functions ϕ_j, $(j = 1, .., M)$
- TEST SPACE: basis functions ψ_j, $(j = 1, .., M)$

The finite element problem reduces to identifying the unknown coefficients $a_1, \cdots, a_M$ in the expansion

$$u_h = \sum_{j=1}^{M} a_j \phi_j + \underbrace{\sum_{j=M+1}^{N} a_j \phi_j}_{\text{to satisfy Dirichlet conditions}}$$

GALERKIN Finite Elements - use the same basis functions for the solution and test space $\psi_j = \phi_j$ $(j = 1, \cdots, M)$
Express the problem for $a_1, \cdots, a_M$ as a SPARSE linear system which can be solved efficiently