The Principle of Stationary Potential Energy

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The Mechanics Problem in general

The strong form of the initial boundary-value problem

$$\begin{cases} \mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}} \\ \mathbf{u} = \bar{\mathbf{u}} \qquad\qquad \text{on} \quad \partial B_{\mathbf{u}} \\ \mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n} \qquad\qquad \text{on} \quad \partial B_{\boldsymbol{\sigma}} \\ \mathbf{u}(\mathbf{x}, t)|_{t=0} = \mathbf{u}_0(\mathbf{X}) \\ \dot{\mathbf{u}}(\mathbf{x}, t)|_{t=0} = \dot{\mathbf{u}}_0(\mathbf{X}) \end{cases}$$

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- A *nonlinear* initial boundary value problem for the unknown displacement field $\mathbf{u}$
- $\boldsymbol{\sigma}$, in general, a *nonlinear* function of the displacement field $\mathbf{u}$

The total potential energy of a system

Time-depedent problems in structural mechanics

- System: physical structure, supports all applied loads
- Configuration: set of positions of all particles in a structure
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A conservative system's behaviour depends only on the initial and final configurations, thus it has a potential energy Π that includes

- strain energy of elastic distortion $\rightarrow W_{\text{int}}$
- Potential energy of applied loads $\rightarrow W_{\text{ext}}$,

such that

$$\Pi = W_{\text{int}} + W_{\text{ext}}$$

Principle of Stationary Total Potential Energy

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Extremize the total potential energy $\rightarrow$ Equilibrium (stationary)

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$$\delta\Pi(\mathbf{u}, \delta\mathbf{u}) = \mathfrak{D}_{\delta\mathbf{u}}\Pi(\mathbf{u}) = \frac{d}{d\epsilon}\Pi(\mathbf{u} + \epsilon\delta\mathbf{u})|_{\epsilon=0} = 0,$$

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or

$$\mathfrak{D}_{\delta\mathbf{u}}\Pi(\mathbf{u}) = \lim_{\epsilon\to 0}\frac{\Pi(\mathbf{u} + \epsilon\delta\mathbf{u}) - \Pi(\mathbf{u})}{\epsilon}$$

A linear spring example

Linear Spring with a axial load f and stiffness k, and D is the stretched distance.

$$\Pi = W_{\text{int}} + W_{\text{ext}} = \int_0^D F dx - f\, D = \int_0^D kx dx - f\, D = \frac{1}{2} k\, D^2 - f\, D$$

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Taking the 1^{st} variation and setting to zero

$$\begin{aligned}
\delta\Pi \quad &= \lim_{\epsilon\to 0} \tfrac{[1/2\, k(D+\epsilon\delta D)^2 - f(D+\epsilon\delta D)] - [1/2\, kD^2 - f D]}{\epsilon} \\
&= \lim_{\epsilon\to 0} \left(\tfrac{1}{2}k2D\delta D + \tfrac{1}{2}k\epsilon(\delta D)^2 - f\delta D\right) \\
&= (kD - f)\delta D
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By setting $\delta\Pi = 0$, we have $D = f/k$.

Principle of Stationary Potential Energy for Hyperelastic Material

A formulation based on energy functionals will be very useful, such as development of robust numerical methods that are based on optimization techniques (physics-informed machine learning approaches.)

The total potential energy

$$\Pi = \int_{B_0} \Psi(\mathbf{u}) \mathrm{d}V - \int_{B_0} \mathbf{b}_0 \cdot \mathbf{u} \mathrm{d}V - \int_{\partial B_0} \mathbf{T}_0 \cdot \mathbf{u} \mathrm{d}S \tag{1}$$

where $\Psi(\mathbf{u}) = \Psi(\mathbf{F}(\mathbf{u}))$ is the strain energy function.

The state of equilibrium is when the potential is stationary

This can be achieved by requiring the directional derivative with respect to $\mathbf{u}$ to vanish in all direction $\delta\mathbf{u}$,

$$\delta\Pi(\mathbf{u}, \delta\mathbf{u}) = \mathfrak{D}_{\delta\mathbf{u}}\Pi(\mathbf{u}) = \frac{d}{d\varepsilon}\Pi(\mathbf{u} + \varepsilon\delta\mathbf{u})|_{\varepsilon=0} = 0.$$

In other words, we require the first variation of the total potential energy, denoted $\delta\Pi$, vanishes.

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The second variation of Π

$$\delta^2\Pi(\mathbf{u},\delta\mathbf{u},\Delta\mathbf{u}) = \mathfrak{D}^2_{\delta\mathbf{u},\Delta\mathbf{u}},$$

where $\Delta\mathbf{u}$ is the increment of $\mathbf{u}$. $\mathfrak{D}^2_{\delta\mathbf{u},\Delta\mathbf{u}}$ decides if the solution corresponds to a *maximum*, *minimum* or a *saddle* point.

The first variation of Π

Consider the loads $\mathbf{b}_0$ and $\mathbf{T}_0$ independent on the deformation of body, then

$$
\begin{aligned}
\mathfrak{D}_{\delta\mathbf{u}}\Pi(\mathbf{u}) &= \frac{d}{d\varepsilon}\Pi(\mathbf{u}+\varepsilon\delta\mathbf{u})|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}[\int_{B_0}\Psi\left(\mathbf{F}(\mathbf{u}+\varepsilon\delta\mathbf{u})\right)dV - \int_{B_0}\mathbf{b}_0\cdot(\mathbf{u}+\varepsilon\delta\mathbf{u})dV \\
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Interchanging differentiation and integration and applying the chain rule,

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\mathfrak{D}_{\delta\mathbf{u}}\Psi(\mathbf{u}) = \int_{B_0}\frac{\partial\Psi\left(\mathbf{F}(\mathbf{u})\right)}{\partial\mathbf{F}}:\mathfrak{D}_{\delta\mathbf{u}}\mathbf{F}(\mathbf{u})dV = \int_{B_0}\frac{\partial\Psi\left(\mathbf{F}(\mathbf{u})\right)}{\partial\mathbf{F}}:\frac{d}{d\varepsilon}\mathbf{F}(\mathbf{u}+\varepsilon\delta\mathbf{u})|_{\varepsilon=0}dV
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$$

That is $\mathfrak{D}_{\delta\mathbf{u}}\Psi(\mathbf{u}) = \int_{B_0} \mathbf{P} : \mathrm{Grad}\delta\mathbf{u}\, dV$ with $\mathbf{P}(=\mathbf{S}^T)$ is the first PK stress tensor.

Equivalent to principle of virtual work

Setting $\mathfrak{D}_{\delta\mathbf{u}}\Pi(\mathbf{u}) = 0$, we have

$$\int_{B_0} \mathbf{S}^T : \mathrm{Grad}\delta\mathbf{u}\, dV - \int_{B_0} \mathbf{b}_0 \cdot \delta\mathbf{u}\, dV - \int_{B_{0,\boldsymbol{\sigma}}} \mathbf{T}_0 \cdot \delta\mathbf{u} dS = 0.$$

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The corresponding formula in the spatial description is

$$\int_{B_t} \boldsymbol{\sigma} : \delta\mathbf{e}\, dV - \int_{B_t} \mathbf{b} \cdot \delta\mathbf{u}\, dv - \int_{B_{t,\boldsymbol{\sigma}}} \mathbf{t} \cdot \delta\mathbf{u} ds = 0,$$

where $\delta\mathbf{e}$ is the variation of the Euler-Almansi strain tensor with $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1})$.

Nearly incompressible or incompressible material

Three-field Variational Principles: decomposing Π into volumetric, isochoric and external parts

$$\Pi(\mathbf{u}, p, \tilde{J}) = \int_{B_0} [\Psi_{\text{vol}}(\tilde{J}) + p\left(J(\mathbf{u}) - \tilde{J}\right) + \Psi_{\text{isochoric}}\left(\bar{\mathbf{C}}(\mathbf{u})\right)] dV + W_{\text{ext}},$$

where $\mathbf{u}, p$ are the displacement, pressure fields, $\tilde{J}$ is a third additional kinematic field variable, $\bar{\mathbf{C}} = J^{-2/3}\mathbf{F}^T\mathbf{F}$ with $J = \det \mathbf{F}$.

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The stationary of Π with respect to $(\mathbf{u}, p, \tilde{J})$ requires

$$\begin{cases} \mathfrak{D}_{\delta \mathbf{u}}\Pi(\mathbf{u}, p, \tilde{J}) = 0, \\ \mathfrak{D}_{\delta p}\Pi(\mathbf{u}, p, \tilde{J}) = 0, \\ \mathfrak{D}_{\delta \tilde{J}}\Pi(\mathbf{u}, p, \tilde{J}) = 0, \end{cases} \qquad \forall \delta\mathbf{u}, \quad \delta p, \quad \delta\tilde{J}.$$

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A widely used three-field variational principle is the **Hu-Washizu** Principle $\Pi(\mathbf{u}, \mathbf{F}, \mathbf{P})$.

A taste of FEM in Fenics based on the total potential energy

Twist a cube with displacement boundary

$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu \ln(J) + \frac{\lambda}{2}\ln(J)^2$$

μ and λ are the lame parameters

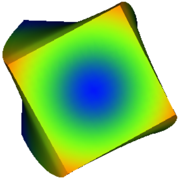
$$\Pi = \int_{\Omega_0} \Psi(\mathbf{u})dV - \int_{\Omega_0} \mathbf{b}_0 \cdot \mathbf{u}dV - \int_{\partial\Omega_0} \mathbf{T}_0 \cdot \mathbf{u}dS$$

$$\mathcal{L}(\mathbf{u};\mathbf{v}) = \mathfrak{D}_{\mathbf{v}}\Pi = \frac{d\Pi(\mathbf{u}+\varepsilon\mathbf{v})}{d\varepsilon}\Big|_{\varepsilon=0} = 0$$

$$\mathcal{J}(\mathbf{u}; d\mathbf{u}, \mathbf{v}) = D_{d\mathbf{u}}\mathcal{L} = \frac{d\mathcal{L}(\mathbf{u}+\varepsilon d\mathbf{u};\mathbf{v})}{d\varepsilon}\Big|_{\varepsilon=0}$$

```
E, nu = 10.0, 0.3
mu, lmbda = Constant(E/(2*(1 + nu))), Constant(E*nu/((1 + nu)*(1 - 2*nu)))

# Stored strain energy density (compressible neo-Hookean model)
psi = (mu/2)*(Ic - 3) - mu*ln(J) + (lmbda/2)*(ln(J))**2

# Total potential energy
Pi = psi*dx - dot(B, u)*dx - dot(T, u)*ds
```

```
# Compute first variation of Pi
F = derivative(Pi, u, v)

# Compute Jacobian of F
J = derivative(F, u, du)

# Solve variational problem
solve(F == 0, u, bcs, J=J,
      form_compiler_parameters=ffc_options)
```



Further readings