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Circumferential Normal Modes in an Empty Elastic Cylinder

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Abstract—The so-called circumferential normal modes propagating in an empty elastic cylinder are considered. A dispersion equation for the wave numbers of these waves, an equation for the critical frequencies, and expressions for the eigenfunctions of such a waveguide are derived. Solutions to these equations are obtained by numerical methods for different values of the parameter *d* representing the relative thickness of the cylinder. An analysis of the solutions is performed, and the main properties of the dispersion curves are described, including those for the low-frequency waves of the new type, which correspond to the branches in the form of open loops. Individual normal modes are identified on the basis of the calculations and subsequent analysis of eigenfunctions. *© 2004 MAIK "Nauka/Interperiodica".*

The waveguide properties of cylindrical elastic bodies are of considerable scientific interest. In the literature, one can find a great number of publications concerned with this subject, including studies of bodies in the form of thick-walled cylindrical shells [1–3]. For the most part, these publications deal with waves in which the elastic fields are periodic functions of the polar angle θ; i.e., in the general case, the solutions are proportional to the factor $\exp(in\theta)$, where n is an integer. Some papers [4–6] consider the asymptotic solutions for waves propagating in the direction of the angular coordinate; in this case, the quantity n is $n = \nu$, where ν plays the role of the angular wave number and, in the general case, is a noninteger. Such waves are taken into account, in particular, in solving the problems of diffraction by cylindrical obstacles [7, 8]. Exact solutions for these waves were obtained in the previous publication [9] devoted to the waveguide properties of a plane ring-shaped plate with flexural waves propagating in it.

The present paper considers the properties of circumferential normal modes propagating in an empty elastic cylinder of infinite length (the wave front is parallel to the z axis).

Let us preset the geometric dimensions of such a cylinder as follows: $r = a$ is the outer boundary, $r = br$ is the inner boundary, and $2L = a - b$ is the thickness of the waveguide.

To solve the problem, we introduce a scalar potential $\varphi(r, \theta)$ and (since the problem is two-dimensional) a single component of the vector potential $\psi_Z(r, \theta) \equiv \psi(r, \theta)$. These functions should satisfy the Helmholtz equations

$$\Delta\varphi(r, \theta) + k_l^2\varphi(r, \theta) = 0, \quad (1)$$

$$\Delta\psi(r, 0) + k_t^2\psi(r, 0) = 0, \quad (2)$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$; k_l is the longitudinal wave number, $k_l^2 = \frac{\rho\omega^2}{\lambda + 2\mu}$; k_t is the shear wave number, $k_t^2 = \frac{\rho\omega^2}{\mu}$; λ and μ are the Lame coefficients; ρ is the density; and ω is the circular frequency.

For boundary conditions, we use the absence of stress on the cylinder surfaces:

$$\sigma_{rr} = 0, \quad \sigma_{r\theta} = 0; \text{ at } r = a, b, \quad (3)$$

where σ_{rr} and $\sigma_{r\theta}$ are the normal and tangential stresses, respectively.

These quantities can be expressed via the potentials φ and ψ (see, e.g., [1]):

$$\sigma_{rr} = -k_l^2\varphi + 2\mu\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right), \quad (4)$$

$$\sigma_{r\theta} = \mu\left(\frac{2}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right). \quad (5)$$

According to the statement of the problem, we seek the solution to the set of equations (1), (2) in the form of elastic waves travelling in the direction of the angle θ:

$$\varphi(r, \theta) = \Phi(r)\exp(i\nu\theta), \quad (6)$$

$$\psi(r, \theta) = \Psi(r)\exp(i\nu\theta). \quad (7)$$

Substituting Eqs. (6) and (7) into Eqs. (1) and (2), we obtain a set of equations for determining the amplitudes of normal modes:

$$\Delta\Phi + \left(k_l^2 - \frac{\nu^2}{r^2}\right)\Phi = 0, \quad (8)$$

$$\Delta\Psi + \left(k_t^2 - \frac{\nu^2}{r^2}\right)\Psi = 0. \quad (9)$$

Equations (4) and (5), with allowance for Eqs. (6) and (7), can be represented as

$$\frac{\sigma_{rr}}{2\mu} = \frac{d^2\Phi}{dr^2} - \frac{\lambda}{2\mu}k_l^2\Phi + i\nu\frac{d}{dr}\left(\frac{\Psi}{r}\right), \quad (10)$$

$$\frac{\sigma_{r\theta}}{2\mu} = -\frac{d^2\Psi}{dr^2} - \frac{1}{2}k_t^2\Psi + i\nu\frac{d}{dr}\left(\frac{\Phi}{r}\right). \quad (11)$$

The solutions to the set of equations (8), (9) have the form

$$\Phi_\nu(r) = AJ_\nu(k_l r) + BY_\nu(k_l r), \quad (12)$$

$$\Psi_\nu(r) = CJ_\nu(k_l r) + DY_\nu(k_l r), \quad (13)$$

where J_ν and Y_ν are the Bessel and Neumann functions, respectively, and A, B, C, and D are arbitrary constants. Let us introduce the following dimensionless parameters: $x = k_t r$, $y = k_l r$, and $\alpha = \frac{k_l}{k_t} = \frac{y}{x}$, where the latter can be expressed via the Poisson ratio σ: $\sigma = \sqrt{\frac{1-2\sigma}{2(1-\sigma)}}$. Note that, in the following treatment, we use the value $\sigma = 0.25$, which yields $\alpha \approx 0.58$. With allowance for this expression and after some cumbersome transformations, Eqs. (10) and (11) take the form

$$\frac{\sigma_{rr}}{2\mu k_t^2} = L_1(x)\Phi_\nu(\alpha x) + T_1(x)\Psi_\nu(x), \quad (14)$$

$$\frac{\sigma_{r\theta}}{2\mu k_t^2} = L_2(x)\Phi_\nu(\alpha x) + T_2(x)\Psi_\nu(x), \quad (15)$$

where the operators L and T are determined by the formulas

$$L_1(x) = -\frac{\alpha}{x}\frac{d}{dx} + \left[\frac{\nu(\alpha - 1 + \nu)}{x^2} - \frac{1}{2}\right];$$

$$T_1(x) = \frac{i\nu}{x}\left(\frac{d}{dx} - \frac{1}{x}\right);$$

$$L_2(x) = \frac{i\nu}{x}\left[\alpha\frac{d}{dx} - \frac{(\alpha-1)\nu + 1}{x}\right];$$

$$T_2(x) = \frac{1}{x}\frac{d}{dx} - \left(\frac{\nu^2}{x^2} - \frac{1}{2}\right).$$

It should be noted that, at $\alpha = 1$ (which is a physically unrealizable case), we have $L_1 = -T_2$ and $L_2 = T_1$.

Substituting expressions (12) and (13) into Eqs. (14) and (15), we finally obtain

$$\frac{\sigma_{rr}}{2\mu k_t^2} = AL_1(x)J_\nu(\alpha x) + BL_1(x)Y_\nu(\alpha x) + CT_1(x)J_\nu(x) + DT_1(x)Y_\nu(x), \quad (16)$$

$$\frac{\sigma_{r\theta}}{2\mu k_t^2} = AL_2(x)J_\nu(\alpha x) + BL_2(x)Y_\nu(\alpha x) + CT_2(x)J_\nu(x) + DT_2(x)Y_\nu(x). \quad (17)$$

Introducing the notation $x_a = k_t a$ and $x_b = k_t b$ and applying boundary conditions (3), we obtain a homogeneous set of algebraic equations for determining the quantities A, B, C, and D:

$$AL_1(x_a)J_\nu(\alpha x_a) + BL_1(x_a)Y_\nu(\alpha x_a) + CT_1(x_a)J_\nu(x_a) + DT_1(x_a)Y_\nu(x_a) = 0, \quad (18)$$

$$AL_2(x_a)J_\nu(\alpha x_a) + BL_2(x_a)Y_\nu(\alpha x_a) + CT_2(x_a)J_\nu(x_a) + DT_2(x_a)Y_\nu(x_a) = 0, \quad (19)$$

$$AL_1(x_b)J_\nu(\alpha x_b) + BL_1(x_b)Y_\nu(\alpha x_b) + CT_1(x_b)J_\nu(x_b) + DT_1(x_b)Y_\nu(x_b) = 0, \quad (20)$$

$$AL_2(x_b)J_\nu(\alpha x_b) + BL_2(x_b)Y_\nu(\alpha x_b) + CT_2(x_b)J_\nu(x_b) + DT_2(x_b)Y_\nu(x_b) = 0. \quad (21)$$

The dispersion equation for the unknown wave number ν is obtained as usual, by equating the determinant of set (18)–(21) to zero:

$$\Delta(\nu) = \begin{vmatrix} L_1(x_a)J_\nu(\alpha x_a) & L_1(x_a)Y_\nu(\alpha x_a) & T_1(x_a)J_\nu(x_a) & T_1(x_a)Y_\nu(x_a) \\ L_2(x_a)J_\nu(\alpha x_a) & L_2(x_a)Y_\nu(\alpha x_a) & T_2(x_a)J_\nu(x_a) & T_2(x_a)Y_\nu(x_a) \\ L_1(x_b)J_\nu(\alpha x_b) & L_1(x_b)Y_\nu(\alpha x_b) & T_1(x_b)J_\nu(x_b) & T_1(x_b)Y_\nu(x_b) \\ L_2(x_b)J_\nu(\alpha x_b) & L_2(x_b)Y_\nu(\alpha x_b) & T_2(x_b)J_\nu(x_b) & T_2(x_b)Y_\nu(x_b) \end{vmatrix} = 0. \quad (22)$$

The equation $\Delta(\nu) = 0$ has many roots $\nu = \nu_n(x_a, x_b)$, where the subscript n determines different branches of the solution to the characteristic equation. For the purpose of their identification and also for testing the validity of boundary conditions (18)–(21), we assume that the roots $\nu = \nu_n(x_a, x_b)$ are calculated and determine the eigenfunctions (modes of vibration) of the waveguide under study. We use the first three equations of set (18)–(21) (the fourth equation is their linear combination). Setting $A = 1$ (without loss of generality), we obtain a set of equations for determining the remaining coefficients, which depend on both ν_n and n:

$$\begin{aligned} &BL_1(x_a)Y_\nu(\alpha x_a) + CT_1(x_a)J_\nu(x_a) \\ &+ DT_1(x_a)Y_\nu(x_a) = -L_1(x_a)J_\nu(\alpha x_a), \end{aligned} \tag{23}$$

$$\begin{aligned} &BL_2(x_a)Y_\nu(\alpha x_a) + CT_2(x_a)J_\nu(x_a) \\ &+ DT_2(x_a)Y_\nu(x_a) = -L_2(x_a)J_\nu(\alpha x_a), \end{aligned} \tag{24}$$

$$\begin{aligned} &BL_1(x_b)Y_\nu(\alpha x_b) + CT_1(x_b)J_\nu(x_b) \\ &+ DT_1(x_b)Y_\nu(x_b) = -L_1(x_b)J_\nu(\alpha x_b). \end{aligned} \tag{25}$$

The solution to set (23)–(25) can be represented in the form

$$B_n = \frac{\Delta_B(\nu_n)}{\Delta(\nu_n)}, \quad C_n = \frac{\Delta_C(\nu_n)}{\Delta(\nu_n)}, \quad D_n = \frac{\Delta_D(\nu_n)}{\Delta(\nu_n)}, \tag{26}$$

where

$$\Delta(\nu_n) = -\begin{vmatrix} L_1(x_a, \nu_n)Y_\nu(\alpha x_a) & T_1(x_a, \nu_n)J_\nu(x_a) & T_1(x_a, \nu_n)Y_\nu(x_a) \\ L_2(x_a, \nu_n)Y_\nu(\alpha x_a) & T_2(x_a, \nu_n)J_\nu(x_a) & T_2(x_a, \nu_n)Y_\nu(x_a) \\ L_1(x_b, \nu_n)Y_\nu(\alpha x_b) & T_1(x_b, \nu_n)J_\nu(x_b) & T_1(x_b, \nu_n)Y_\nu(x_b) \end{vmatrix},$$

$$\Delta_B(\nu_n) = -\begin{vmatrix} L_1(x_a, \nu_n)J_\nu(\alpha x_a) & T_1(x_a, \nu_n)J_\nu(x_a) & T_1(x_a, \nu_n)Y_\nu(x_a) \\ L_2(x_a, \nu_n)J_\nu(\alpha x_a) & T_2(x_a, \nu_n)J_\nu(x_a) & T_2(x_a, \nu_n)Y_\nu(x_a) \\ L_1(x_b, \nu_n)J_\nu(\alpha x_b) & T_1(x_b, \nu_n)J_\nu(x_b) & T_1(x_b, \nu_n)Y_\nu(x_b) \end{vmatrix},$$

$$\Delta_C(\nu_n) = -\begin{vmatrix} L_1(x_a, \nu_n)Y_\nu(\alpha x_a) & L_1(x_a, \nu_n)J_\nu(\alpha x_a) & T_1(x_a, \nu_n)Y_\nu(x_a) \\ L_2(x_a, \nu_n)Y_\nu(\alpha x_a) & L_2(x_a, \nu_n)J_\nu(\alpha x_a) & T_2(x_a, \nu_n)Y_\nu(x_a) \\ L_1(x_b, \nu_n)Y_\nu(\alpha x_b) & L_1(x_b, \nu_n)J_\nu(\alpha x_b) & T_1(x_b, \nu_n)Y_\nu(x_b) \end{vmatrix},$$

$$\Delta_D(\nu_n) = -\begin{vmatrix} L_1(x_a, \nu_n)Y_\nu(\alpha x_a) & T_1(x_a, \nu_n)J_\nu(x_a) & L_1(x_a, \nu_n)J_\nu(\alpha x_a) \\ L_2(x_a, \nu_n)Y_\nu(\alpha x_a) & T_2(x_a, \nu_n)J_\nu(x_a) & L_2(x_a, \nu_n)J_\nu(\alpha x_a) \\ L_1(x_b, \nu_n)Y_\nu(\alpha x_b) & T_1(x_b, \nu_n)J_\nu(x_b) & L_1(x_b, \nu_n)J_\nu(\alpha x_b) \end{vmatrix}.$$

With the coefficients obtained above, the eigenfunctions of the waveguide for stresses can be represented as

$$\begin{aligned} \frac{\sigma_{rr}}{2\mu k_t^2} &= L_1(x)J_{\nu_n}(\alpha x) + B_nL_1(x)Y_{\nu_n}(\alpha x) \\ &+ C_nT_1(x)J_{\nu_n}(x) + D_nT_1(x)Y_{\nu_n}(x), \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{\sigma_{r\theta}}{2\mu k_t^2} &= L_2(x)J_{\nu_n}(\alpha x) + B_nL_2(x)Y_{\nu_n}(\alpha x) \\ &+ C_nT_2(x)J_{\nu_n}(x) + D_nT_2(x)Y_{\nu_n}(x). \end{aligned} \tag{28}$$

The eigenfunctions for the radial u_r and angular u_θ displacements can be determined via the scalar and vector potentials:

$$u_r = \frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} - \frac{\partial\psi}{\partial r}. \tag{29}$$

Introducing the operators $R_1 = \frac{d}{dx}$ and $R_2(\nu) = \frac{i\nu}{x}$, we represent displacements (29) corresponding to the root ν_n of characteristic equation (22) in the form

$$\frac{u_r}{k_t} = R_1\Phi_{\nu_n}(\alpha x) + R_2(\nu_n)\Psi_{\nu_n}(x), \tag{30}$$

$$\frac{u_\theta}{k_t} = R_2(\nu_n)\Phi_{\nu_n}(\alpha x) - R_1\Psi_{\nu_n}x, \tag{31}$$

where

$$\Phi_{\nu_n}(\alpha x) = J_{\nu_n}(\alpha x) + B_nY_{\nu_n}(\alpha x),$$

$$\Psi_{\nu_n}(x) = C_nJ_{\nu_n}(x) + D_nY_{\nu_n}(x).$$

Before discussing the results of the calculations, it is necessary to make some comments concerning their representation. Above, it was noted that the desired

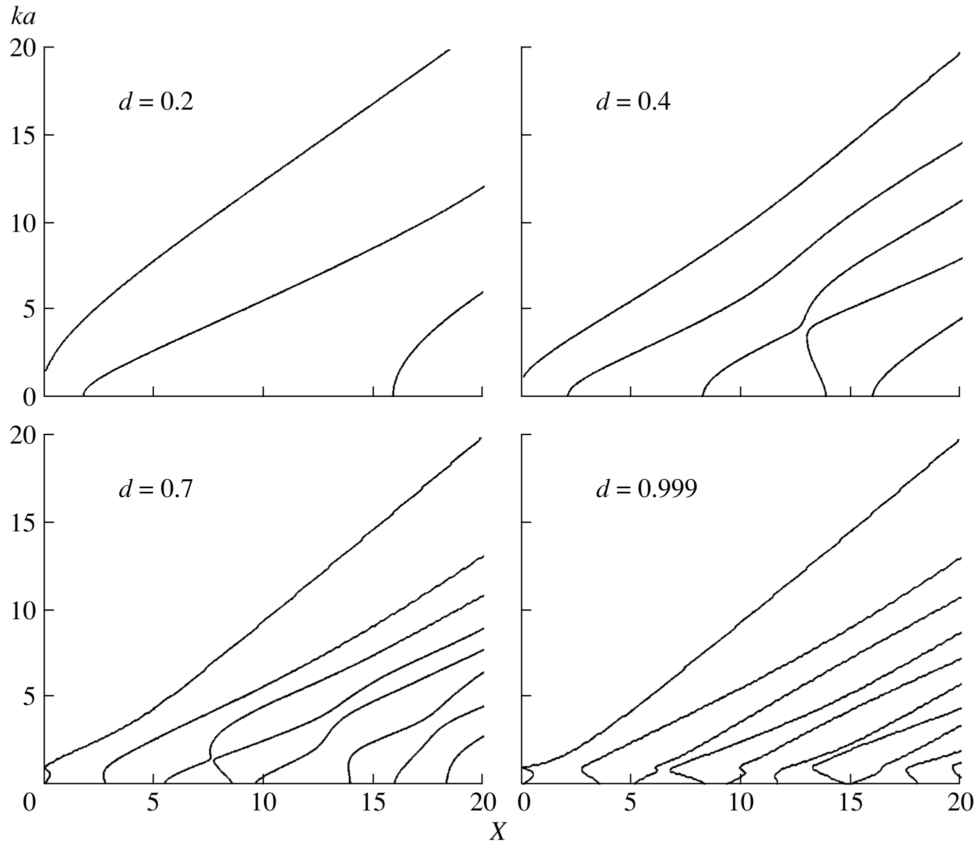

Fig. 1. Dispersion curves for the circumferential normal modes in an empty elastic cylinder; $X = k_t a$.

quantity ν plays the role of the wave number involved in the expression for the phase of the normal mode: $\varphi = \nu\theta$. As in the previous paper [9], this expression can be transformed as $\varphi = \frac{\nu}{r} r\theta = k_r S$, where $k_r = \frac{\nu}{r}$ is the "linear" wave number and $S = r\theta$ is the arc length traveled by the wave. Choosing this form of the result representation, we set $r = a$ for the sake of definiteness; i.e., we consider the wave number corresponding to the outer boundary of the empty cylinder. Then, in the equations given above, the unknown quantity ν can be replaced by $\nu = ka$ (the subscript of k is omitted). The arguments involved in the equations given above can be represented in the form $x_a = k_t a$ and $x_b = k_t(1 - d)$, where $d = \frac{L}{R_0}$ is the dimensionless thickness of the cylinder. The velocity of wave propagation c also depends on the radius. Choosing the quantity $X = k_t a$ as the main argument, we represent the desired dimensionless velocity in the form $C = \frac{c}{c_t} = \frac{k_t a}{\nu} = \frac{X}{ka}$, where c_t is the velocity of shear waves in the medium.

To solve Eq. (22) for the unknown quantity $\nu = ka$ as a function of the dimensionless frequency $X = k_t a$ for different values of the dimensionless thickness of the cylinder d, a special computer program was used.

Figure 1 shows the results of calculations for different values of d. The plot is typical of all values of the parameter $0 < d < 1$. One can see three groups of dispersion curves: (a) single branches ($n = 0$) that begin at zero frequency and have the value $\nu = 1$ at this point; (b) single branches ($n = 0'$) in the form of open loops lying in the immediate vicinity of the origin of coordinates (the latter branches are shown in more detail in Fig. 2 for some of the values of the parameter d: branches corresponding to the waves with $n = 0'$ lie in the region $ka \leq 1$, and waves with $n = 0$ are partially represented in the region $ka \geq 1$; similar branches were also obtained for normal modes in [9]); and (c) an infinite number of branches corresponding to the higher-order normal modes, as in any waveguide (the number of these waves increases with increasing d).

Figure 3 shows the plots for the dimensionless phase velocity $C = \frac{c}{c_t}$ of the wave with $n = 0$ for certain values of d. One can see that, near the origin of coordinates, this velocity is proportional to the frequency. When d is large, the velocity as a function of frequency exhibits maxima and, then, at $X \longrightarrow \infty$, tends to a constant value

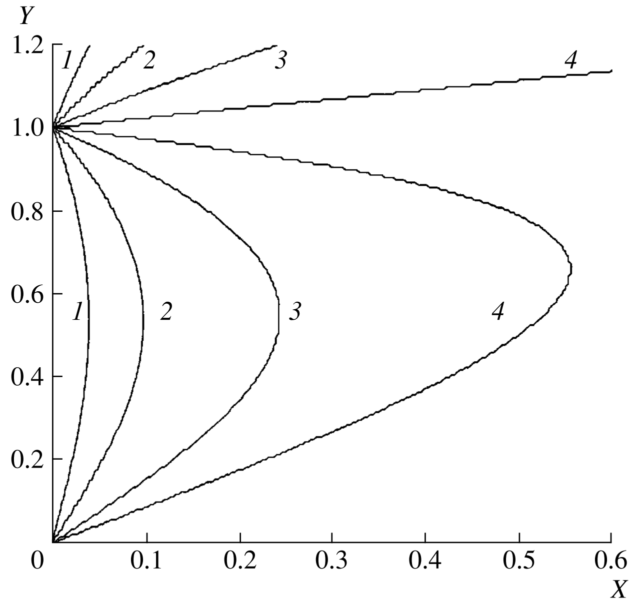

Fig. 2. Low-frequency half-closed loops; $Y = ka$ and d = (*1*) 0.2, (*2*) 0.4, (*3*) 0.7, and (*4*) 0.9.

of $C_\infty \simeq 0.933$, which does not depend on the parameter d. When d is small, such maxima are absent and the velocity tends to the same constant C_∞ but at much higher frequencies. In its physical meaning, the velocity $C = C_\infty$ is close to the Rayleigh wave velocity.

To identify different branches of the dispersion curves, eigenfunctions were calculated for the displacements u_r and u_θ and for the stresses σ_{rr} and $\sigma_{r\theta}$ by using formulas (28)–(31) with different values of the parameter d at certain frequencies. The aforementioned quantities were normalized to their maximal values, and the dimensionless coordinate x was given in the form $x = \frac{r}{a} - \frac{a+b}{2a}$, so that the origin of coordinates corresponded to the median surface of the cylinder and the quantity x varied within $-\frac{d}{2} \le x \le \frac{d}{2}$. In all subsequent plots the factor i is omitted in the quantities u_θ and $\sigma_{r\theta}$.

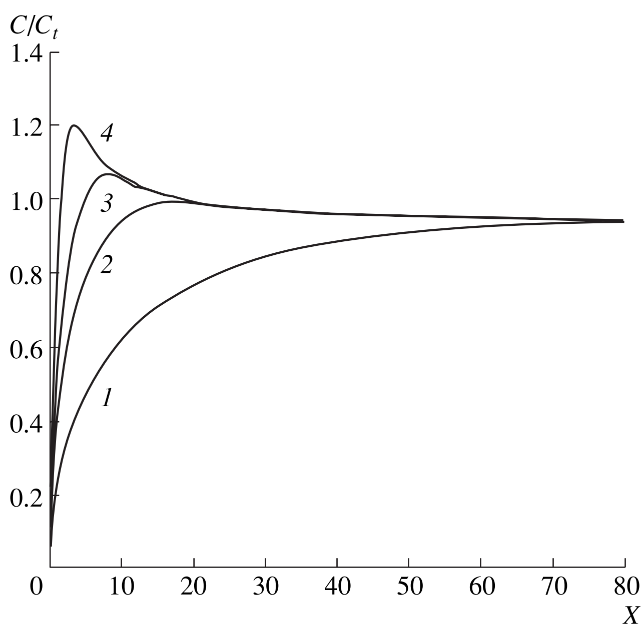

Fig. 3. Phase velocities of normal modes with $n = 0$; d = (*1*) 0.1, (*2*) 0.3, (*3*) 0.5, and (*4*) 0.9.

Figure 4 shows the plots for $d = 0.9$ (other parameters are indicated in the figure caption). One can see that, at low frequencies (Fig. 3a), the radial displacement u_r is almost independent of the radius; i.e., a quasi-flexural wave propagates in the body. In this case, the angular displacement u_θ and the stresses σ_{rr} and σ_{r2} are maximal near the inner surface of the cylinder and decrease toward the outer surface. Other cases (Figs. 3b–3d) correspond to the transformation of the eigenmodes with increasing frequency, which consists of the sequential displacement of all quantities toward the outer surface and their concentration near it.

Figure 5 shows similar data for $d = 0.1$. The behavior of the quantities under consideration is approximately the same as for $d = 0.9$, except that the maximal stress values initially occur at the median surface of the cylinder ($x = 0$), and that the concentration of displacements and stresses near the outer surface occurs at a much higher frequency.

Figure 6 represents (for $d = 0.9$) the eigenfunctions for the normal modes with $n = 0'$, which are described by open loops (Fig. 2). One can see that, independently

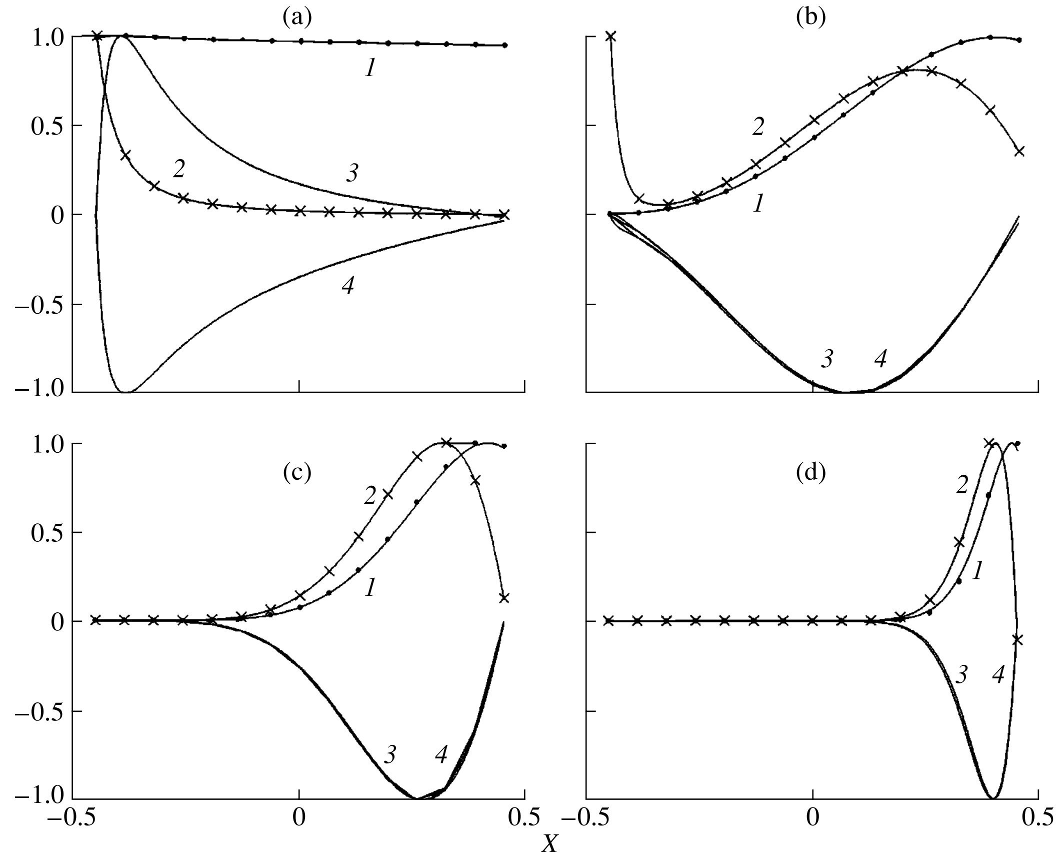

Fig. 4. Eigenfunctions for a normal mode with $n = 0$ at $d = 0.9$: (*1*) displacement u_r, (*2*) displacement u_θ, (*3*) stress σ_{rr}, and (*4*) stress $\sigma_{r\theta}$; X = (a) 0.5, (b) 5, (c) 10, and (d) 32.

of frequency, all the functions differ little from each other and from the corresponding functions for the waves with $n = 0$ (Fig. 1a), except that the displacement u_r changes its phase for the opposite one. Cases (a) and (c) correspond to the lower half of the loop, and cases (b) and (d), to the upper half.

The data shown in Figs. 2 and 6 suggest the following basic qualitative conclusions about the properties of low-frequency waves with $n = 0'$:

(i) these waves have two different phase velocity values at the same frequency (except for the frequency at which the upper and lower branches merge);

(ii) the group velocity determined by the formula $c_g = \dfrac{d\omega}{dk} = c_t\dfrac{dX}{dY}$ is negative for the upper (descending) branch and positive for the lower (ascending) branch;

(iii) the waves are of quasi-flexural nature, with a concentration of stresses near the inner surface of the cylindrical body.

From Fig. 1, one can see that normal modes of higher orders ($n \geq 1$) originate at $\nu_n = 0$. The quantities describing the corresponding normal mode field do not depend on the polar angle θ. At the instant of origination of each of these waves, the wave front has the form of a circle coaxial with the waveguide boundaries $r = a$ and $r = b$. In this case, the displacements are expressed via cylindrical functions of zero order. In plane waveguides, the process of normal mode origination is analogous to that considered above: the wave front is parallel to the plane waveguide boundary.

As in the plane waveguides, critical frequencies (the frequencies of wave origination) exist in the cylinder under study. The equation for the critical frequencies can be easily obtained from Eq. (22) by setting $\nu = \nu_n = 0$. Then, the operators $L_{1,2}(x)$ and $T_{1,2}(x)$ have the form

$$L_1(x) = -\left(\frac{\alpha}{x}\frac{d}{dx} + \frac{1}{2}\right); \quad L_2(x) = T_1(x) = 0;$$

$$T_2(x) = \frac{1}{x}\frac{d}{dx} + \frac{1}{2},$$

and Eq. (22) takes the form

$$\begin{vmatrix} J(\alpha x_a) & Y(\alpha x_a) \\ J(\alpha x_b) & Y(\alpha x_b) \end{vmatrix} \begin{vmatrix} I(x_a) & K(x_a) \\ I(x_b) & K(x_b) \end{vmatrix} = 0, \qquad (32)$$

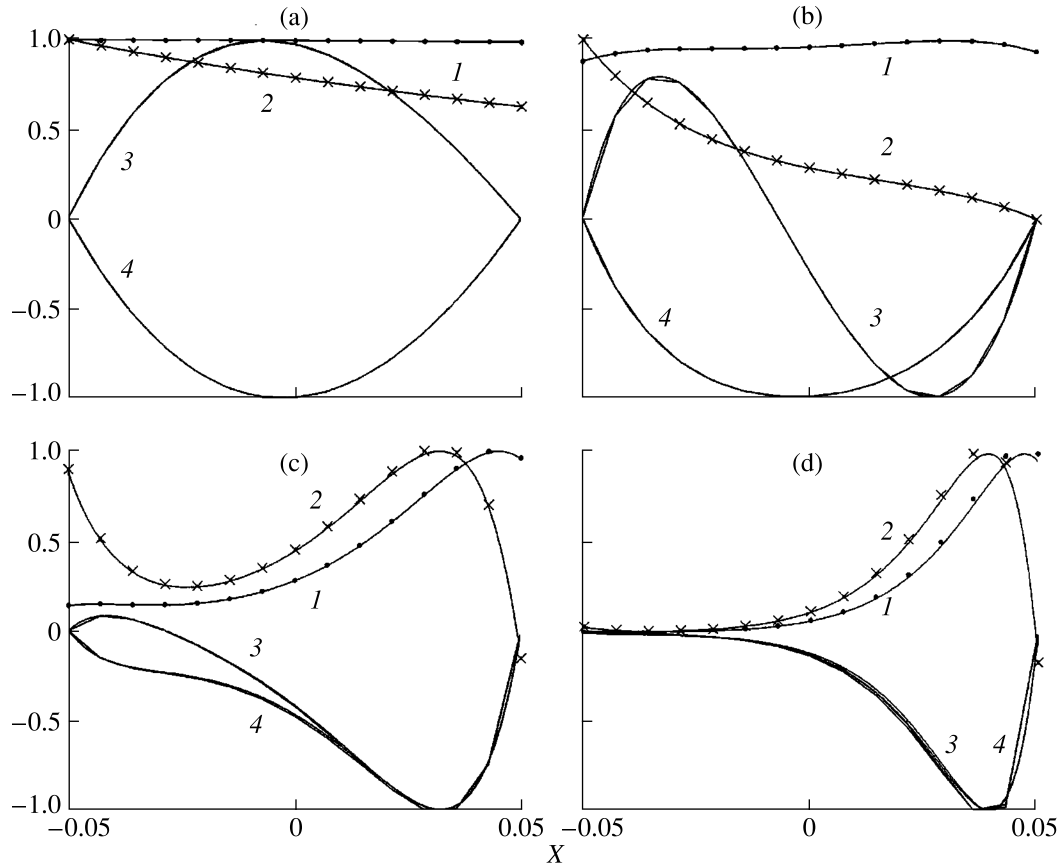

Fig. 5. The same as in Fig. 4 but for $d = 0.1$; X = (a) 0.5, (b) 30, (c) 90, and (d) 148.

where additional notations are introduced:

$$J(x) = \frac{\alpha}{x}J_1(x) - \frac{1}{2}J_0(x);$$

$$Y(x) = \frac{\alpha}{x}Y_1(x) - \frac{1}{2}Y_0(x);$$

$$I(x) = -\frac{1}{x}J_1(x) + \frac{1}{2}J_0(x);$$

$$K(x) = -\frac{1}{x}Y_1(x) + \frac{1}{2}Y_0(x).$$

Thus, as one would expect, the equation for the critical frequencies consists of two independent equations, the first of which refers to the waves originating as longitudinal, and the second to the waves originating as transverse.

Figure 7 shows the dimensionless wave numbers $\frac{k}{k_t}$ for the first four modes (n = 1–4) at $d = 0.9$. According to Eq. (32), $n = 1$ and $n = 4$ correspond to the modes originating as longitudinal, and $n = 2$ and $n = 3$ to the modes originating as transverse. It is essential that, when the wave size of the cylinder tends to infinity, the velocity of all the normal modes tends to that of transverse waves, which testifies that they belong to the quasi-Lamb wave type. The difference between them and the Lamb waves of an elastic layer consists, in particular, in that the mode with $n = 1$ originates as a transverse wave in the case of the Lamb waves and as a longitudinal wave in the case under study.

Figure 8 represents the eigenmodes of the displacements for n = 1–4 at a high dimensionless frequency $X = 80$. Here, only positive values of the x coordinate are used, because, for $x < 0$, both displacements and forces are fairly small at this frequency, and all fields concentrate near the outer surface of the cylinder. From this figure, one can see that the number of a mode (as in plane waveguides) corresponds to the number of intersections of the corresponding curves with the x axis.

The results obtained above can be used for studying the characteristics of circumferential normal modes of a shear-longitudinal type in a ring-shaped thin plate (such a problem for flexural waves was solved in [9]). For this purpose, it is necessary to replace the elastic modulus of a plane wave $\lambda + 2\mu$ by the longitudinal elastic modulus

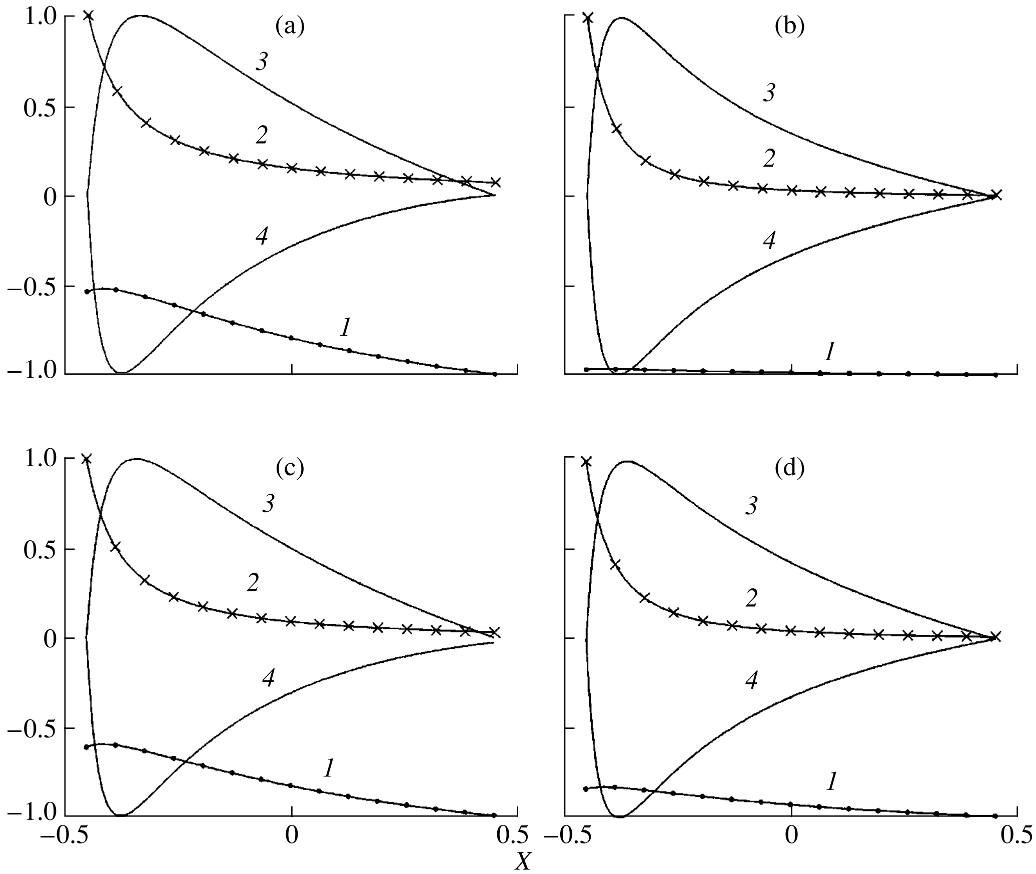

Fig. 6. Eigenfunctions for the normal mode with $n = 0'$ at $d = 0.9$; X = (a) 0.5, (b) 5, (c) 10, and (d) 32.

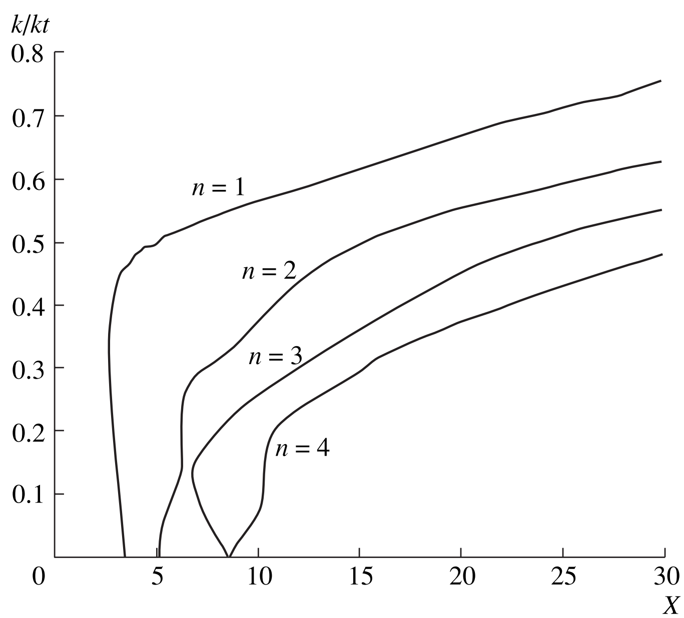

Fig. 7. Frequency characteristics of the dimensionless wave numbers for $n \geq 1$ ($d = 0.9$).

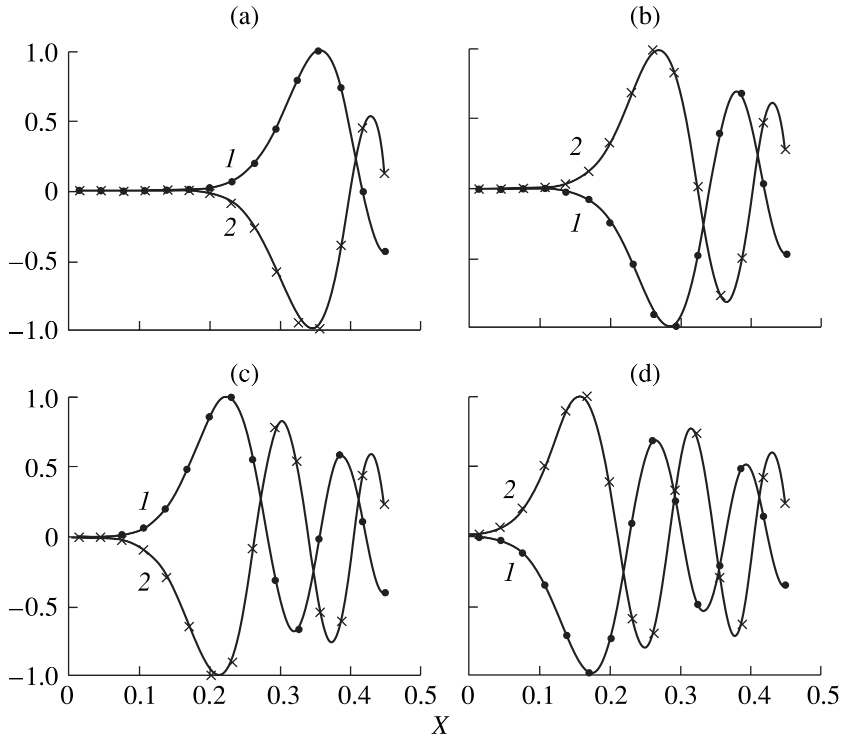

Fig. 8. Displacement eigenfunctions for normal modes with $n \geq 1$; $d = 0.9$, $X = 80$, and n = (a) 1, (b) 2, (c) 3, and (d) 4.

of a plate $E_p = \dfrac{E}{1-\sigma^2}$ and to represent the factor α in the form $\alpha = \dfrac{k_p}{k_t} = \sqrt{\dfrac{1-\sigma}{2}}$. For the value $\sigma = 0.25$ used in the calculations, this factor is $\alpha \cong 0.612$.

In closing, it should be noted that the normal modes studied in this paper can be considered as helical waves propagating in an empty elastic cylinder at an angle $\vartheta = \dfrac{\pi}{2}$ to its axis. For helical waves with other values of this angle, the aforementioned wave numbers of circumferential waves can be determined as "critical wave numbers" (by analogy with critical frequencies).

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