TALLER #3

→ Ejercicios sección 3.14

6. $\{e_i\}$ define un sistema de coordenadas no necesariamente ortogonal, entonces, demuestre que:

a). $$e^i = \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)}$$

$\rightarrow e^i \cdot e_i = 1$

• $e^i = \alpha(e_j \times e_k) \rightarrow e^i$ es ortonormal a e_j y e_k.

$$\rightarrow \alpha = \frac{e^i}{e_j \times e_k}$$

Partiendo de eso

$e^i \cdot e_i = \alpha(e_j \times e_k) e_i$

$1 = \alpha e_i (e_j \times e_k)$ → Reemplazando α

$1 = \frac{e^i}{e_j \times e_k} e_i(e_j \times e_k)$ → Despejando e^i

$$\boxed{\frac{(e_j \times e_k)}{e_i(e_j \times e_k)} = e^i}$$

b). Si los volúmenes $V = e_1 \cdot (e_2 \times e_3)$, $\tilde{V} = e^1 \cdot (e^2 \times e^3) \Rightarrow V\tilde{V} = 1$

$\tilde{V} = e^1 \cdot (e^2 \times e^3)$ → teniendo que $e^1 = e_2 \times e_3$

$\tilde{V} = \underbrace{(e_2 \times e_3)(e^2 \times e^3)}_{1}$

$V = e_1(e_2 \times e_3)$

$V = (e^2 \times e^3)(e_2 \times e_3)$

$V = 1$

$VV = 1$

c). ¿Qué vector satisface $a \cdot e^i = 1$?

$a \cdot e^i = 1$

$$a \cdot \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)} = 1$$

$a \cdot e_j \times e_k = e_i \cdot (e_j \times e_k)$ → $a \cdot e_j \times e_k - e_i(e_j \times e_k) = 0$

$e_j \times e_k\, (a - e_i) = 0$

$\neq 0$

$\boxed{a = e_i}$

Primavera

d. Encuentre el producto vectorial de 2 vectores a y b que están representados. Dada la base, $w_1 = 4\hat{i} + 2\hat{j} + \hat{k}$. $w_2 = 3\hat{i} + 3\hat{j}$, $w_3 = 2\hat{k}$. Encuentre

→ Bases recíprocas.

$$w_1 = 4i + 2j + k$$
$$w_2 = 3\hat{i} + 3\hat{j}$$
$$w_3 = 2\hat{k}$$

$$w^1 = \frac{(w_2 \times w_3)}{w_1 \cdot (w_2 \times w_3)}$$

$$w^1 = (w_2 \times w_3) \quad \begin{vmatrix} i & j & k \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6\hat{i} - 6\hat{j}$$

$w_1 (w_2 \times w_3)$

$$(4\hat{i} + 2\hat{j} + k)(6\hat{i} - 6j)$$
$$= 24\hat{i} - 12j = \underline{12}$$

$$w^1 = \frac{(6\hat{i} - 6j)}{12} = \boxed{\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}}$$

$$w^2 = \frac{(w_1 \times w_3)}{w_1 \cdot (w_2 \times w_3)}$$

$w_1 \times w_3$

$$\begin{vmatrix} i & j & k \\ 4 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 4\hat{i} - 8\hat{j}$$

$$w^2 = \frac{4\hat{i} - 8\hat{j}}{12} = \boxed{\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j}}$$

$$w^3 = w_1 \times w_2$$

$$\begin{vmatrix} i & j & k \\ 4 & 2 & 1 \\ 3 & 3 & 0 \end{vmatrix} = -3\hat{i} + 3\hat{j} + 6\hat{k}$$

$$w^3 = \frac{-3\hat{i} + 3j + 6\hat{k}}{12} = \boxed{-\frac{1}{4}\hat{i} + \frac{1}{4}\hat{j} + \frac{1}{2}\hat{k}}$$

$$w^1 = \left(\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}\right), \ w^2 = \left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j}\right), \ w^3 = \left(-1/4\,\hat{i} + 1/4\,\hat{j} + 1/2\,k\right)$$

→ Componentes covariantes y contravariantes del vector $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

componente contravariante

$$\begin{pmatrix} \hat{i} \\ 2\hat{j} \\ 3\hat{k} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4\alpha + 3\beta \\ 2\alpha + 3\beta \\ 1\alpha + 0 + 2\gamma \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 3 & 0 & 1 \\ 2 & 3 & 0 & 2 \\ 1 & 0 & 2 & 3 \end{array}\right] \quad \begin{array}{l} \alpha = -1/2 \\ \beta = 1 \\ \gamma = 5/4 \end{array}$$

$$a = -1/2 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + 5/4 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

componente covariante

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha_2 \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 1/3 \\ -2/3 \\ 0 \end{pmatrix} + \gamma_3 \begin{pmatrix} -1/4 \\ 1/4 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2\alpha & 1/3\beta & -1/4\gamma \\ -1/2\alpha & -2/3\beta & 1/4\gamma \\ 0 & 0 & 1/2\gamma \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1/2 & 1/3 & -1/4 & 1 \\ -1/2 & -2/3 & 1/4 & 2 \\ 0 & 0 & 1/2 & 3 \end{array}\right] \quad \begin{array}{l} \alpha_2 = 11/3 \\ \beta_2 = 2 \\ \gamma = 6 \end{array}$$

$$a = 11/3 \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1/3 \\ -2/3 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

7. $\langle a|b\rangle \Rightarrow Tr(A^{+}B)$. encuentre la base dual. asociada a la base de Pauli

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

→ Si ortonormalizamos las bases de Pauli

$$\|\sigma_0\| = \sqrt{\langle \sigma_0|\sigma_0\rangle} = \sqrt{Tr\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \sqrt{2}$$

$$\|\sigma_1\| = \sqrt{\langle \sigma_1|\sigma_1\rangle} = \sqrt{Tr\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \sqrt{2}$$

$$\|\sigma_2\| = \sqrt{\langle \sigma_2|\sigma_2\rangle} = \sqrt{Tr\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}} = \sqrt{-2} = i\sqrt{2}$$

$$\|\sigma_3\| = \sqrt{\langle \sigma_3|\sigma_3\rangle} = \sqrt{Tr\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \sqrt{2}$$

$$\langle a|b\rangle \to Tr(A^{\dagger}B)$$

$$\langle a^1|\sigma_1\rangle = Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix}^{\dagger} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^3 & a^1 \\ a^4 & a^2 \end{pmatrix} = \frac{1}{\sqrt{2}}(a^3 + a^2) = 1$$

$$\langle a^1,\sigma_2\rangle = Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix}^{\dagger} \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{\sqrt{2}\, i} Tr\begin{pmatrix} ia^3 & -ia^1 \\ ia^4 & -ia^2 \end{pmatrix} = \frac{1}{i\sqrt{2}}(ia^3 - ia^2) = 0$$

$$\langle a^1|\sigma_3\rangle = Tr\begin{pmatrix} a^1 & a^3 \\ a^1 & a^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^1 & -a^3 \\ a^2 & a^4 \end{pmatrix} = \frac{1}{\sqrt{2}}(a^1 + a^2) = 0$$

$$\langle a^1|\sigma_0\rangle = Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix} = \frac{1}{\sqrt{2}}(a^1 + a^4) = 0$$

$$\langle b^2|\sigma_1\rangle = Tr\begin{pmatrix} b^1 & b^3 \\ b^2 & b^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} Tr\begin{pmatrix} b^3 & b^1 \\ b^4 & b^2 \end{pmatrix} = \frac{1}{\sqrt{2}}(b^3 + b^2) = 0$$

$$\langle b^2|\sigma_2\rangle = Tr\begin{pmatrix} b^1 & b^3 \\ b^2 & b^4 \end{pmatrix} \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{\sqrt{2}\, i} Tr\begin{pmatrix} ia^3 & -ia^1 \\ ia^4 & -ia^2 \end{pmatrix} = \frac{1}{i\sqrt{2}}(ia^3 - ia^2) = 1$$

$$\langle b^2|\sigma_3\rangle = Tr\begin{pmatrix} b^1 & b^3 \\ b^2 & b^4 \end{pmatrix} \frac{1}{i\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}\, i} Tr\begin{pmatrix} b^1 & -b^3 \\ b^2 & b^4 \end{pmatrix} = \frac{1}{i\sqrt{2}}(b^1 + b^2) = 0$$

$$\langle b^2|\sigma_0\rangle = Tr\begin{pmatrix} b^1 & b^3 \\ b^2 & b^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix} = \frac{1}{\sqrt{2}}(a^1 + a^4) = 0$$

$$\langle c^3|\sigma_1\rangle = Tr\begin{pmatrix} c^1 & c^3 \\ c^2 & c^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^3 & a^1 \\ a^4 & a^2 \end{pmatrix} = \frac{1}{\sqrt{2}}(a^3 + a^2) = 0$$

$$\langle c^3|\sigma_2\rangle = Tr\begin{pmatrix} c^1 & c^3 \\ c^2 & c^4 \end{pmatrix} \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{i\sqrt{2}} Tr\begin{pmatrix} ia^3 & -ia^1 \\ ia^4 & -ia^2 \end{pmatrix} = \frac{1}{i\sqrt{2}}(ia^3 - ia^2) = 0$$

$$\langle c^3|\sigma_3\rangle = Tr\begin{pmatrix} a^1 & c^3 \\ c^2 & c^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^1 & -a^3 \\ a^2 & a^4 \end{pmatrix} = \frac{1}{\sqrt{2}}(a^1 + a^2) = 1$$

$$\langle c^3|\sigma_0\rangle = Tr\begin{pmatrix} c^1 & c^3 \\ c^2 & c^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix} = \frac{1}{\sqrt{2}}(a^1 + a^4) = 0$$

$$\langle d^0|\sigma_1\rangle = Tr\begin{pmatrix} d^1 & d^3 \\ a^2 & d^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^3 & a^1 \\ a^4 & a^2 \end{pmatrix} = \frac{1}{\sqrt{2}}(a^3 + a^2) = 0$$

$$\langle d^0|\sigma_2\rangle = \frac{1}{i\sqrt{2}}(id^3 - id^2) = 0$$

$$\langle d^0|\sigma_3\rangle = \frac{1}{\sqrt{2}}(a^1 + a^2) = 0$$

$$\langle d^0|\sigma_0\rangle = \frac{1}{\sqrt{2}}(d^1 + d^4) = 1$$

$a^1 = 0 \qquad b^1 = 0 \qquad c^1 = \sqrt{2}/2 \qquad d^1 = \sqrt{2}/2$

$a^2 = \sqrt{2}/2 \qquad b^2 = -\sqrt{2}/2 \qquad c^2 = 0 \qquad d^2 = 0$

$a^3 = \sqrt{2}/2 \qquad b^3 = \sqrt{2}/2 \qquad c^3 = 0 \qquad d^3 = 0$

$a^4 = 0 \qquad b^4 = 0 \qquad c^4 = -\sqrt{2}/2 \qquad d^4 = \sqrt{2}/2$

$$\langle \Theta_0 | = \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 \end{pmatrix}, \quad \langle \Theta_1 | = \begin{pmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix}, \quad \langle \Theta_2 | = \begin{pmatrix} 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix},$$

$$\langle \Theta_3 | = \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 \end{pmatrix}$$

→ Vector genérico del espacio dual

$$a = \alpha \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 \end{pmatrix} + \beta \begin{pmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix} + \gamma \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 \end{pmatrix}$$

$$a = \left(\begin{matrix} \alpha\sqrt{2}/2 + \gamma\sqrt{2}/2 & \beta\sqrt{2}/2 - c\sqrt{2}/2 \\ \beta\sqrt{2}/2 + c\sqrt{2}/2 & \alpha\sqrt{2}/2 - \gamma\sqrt{2}/2 \end{matrix} \middle| \begin{matrix} 1 \\ 1 \end{matrix} \right)$$

$a =$