“Machine Learning and Computational Statistics”

1st Homework

Exercise 1 (optional)**:**

Name a classifier whose associate *f*(.) is of (a) **parametric** and (b) **non parametric** nature.

Exercise 2:

(a) Define the parametric set of the **quadratic** functions $f_{\boldsymbol{\vartheta}}: R \rightarrow R$ and give two instances of it. What is the dimensionality of $\boldsymbol{\theta}$?

(b) Define the parametric set of the **3rd degree polynomials** $f_{\boldsymbol{\vartheta}}: R^2 \rightarrow R$ and give two instances of it. What is the dimensionality of $\boldsymbol{\theta}$?

(c) Define the parametric set of the **3rd degree polynomials** $f_{\boldsymbol{\vartheta}}: R^3 \rightarrow R$ and give two instances of it. What is the dimensionality of $\boldsymbol{\theta}$?

(d) Consider the function $f_{\boldsymbol{\theta}}(\boldsymbol{x}): R^5 \rightarrow R,\ f_{\theta}(\boldsymbol{x}) = \frac{1}{1+\exp(-\boldsymbol{\theta}^T\boldsymbol{x})}$. Define the associated parametric set and give two instances of it. What is the dimensionality of $\boldsymbol{\theta}$?

(e) In which of the above cases $f_{\boldsymbol{\theta}}$ is linear with respect to $\boldsymbol{\theta}$?

Exercise 3:

Verify that for two l-dimensional column vectors $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_l]^T$ and $\boldsymbol{x} = [x_1, x_2, \dots, x_l]^T$ it holds: $(\boldsymbol{\theta}^T\boldsymbol{x})\boldsymbol{x} = (\boldsymbol{x}\,\boldsymbol{x}^T)\boldsymbol{\theta}$.

Exercise 4:

Consider the vectors $\boldsymbol{x}_n = [x_{n1}, x_{n2}, \dots, x_{nl}]^T, n = 1, \dots, N$. Define the Nxl matrix X and N-dimensional column vector $\boldsymbol{y}$ as follows:

$$X = \begin{bmatrix} \boldsymbol{x}_1{}^T \\ \boldsymbol{x}_2{}^T \\ \vdots \\ \boldsymbol{x}_N{}^T \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1l} \\ x_{21} & x_{22} & \cdots & x_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{Nl} \end{bmatrix} \text{ and } \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

(Note that the rows of X are the vectors $\boldsymbol{x}_n, n = 1, \dots, N$).

Verify the following identities:

$X^TX = \sum_{n=1}^{N} \boldsymbol{x}_n\boldsymbol{x}_n{}^T$ and $X^T\boldsymbol{y} = \sum_{n=1}^{N} y_n\boldsymbol{x}_n$.

Exercise 5 (no grades): Write explicitly the derivation of the Least square estimator, following the line of proof given in the slides of the 1st lecture.

Exercise 6: A body moves on a straight line and performs a smoothly accelerating motion. In the following table is given the velocity at certain time instances

t (sec)	1	2	3	4	5
v (m/sec)	5.1	6.8	9.2	10.9	13.1

(a) Estimate the initial velocity and the acceleration of the body, based on the above measurements, utilizing the least squares error criterion.

(b) Estimate the velocity of the body at t=2.3.

Hints:

(i) The velocity v of a body moving on a straight line and performing a smoothly accelerating motion as is

$$v = v_0 + a \cdot t$$

where t is the time, a is the acceleration and v_0 the initial speed.

(ii) The previous table of values is associated with the following data set

$$\begin{aligned}\{(y_i, x_i), i = 1, \dots ,5\} \equiv & \ \{(v_i, t_i), i = 1, \dots ,5\} \\ & = \{(5.1, 1), (6.8, 2), (9.2, 3), (10.9, 4), (13.1, 5)\}\end{aligned}$$

(iii) Define $\theta = [v_0, a]^T$, construct the matrix X and the vector y (slide 49 of the 1st lecture), and utilize the equation that gives the Least squares estimation (slide 50 of the 1st lecture).

(iv) The inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $D = a \cdot d - b \cdot c$