Homework 7

Exercise 1

Since class ω_1 is modeled by the normal distribution $N(0,1)$ we have that:

$$p(x|\omega_1) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

Similarly, since class ω_2 is modeled by the normal distribution $N(0,5)$ we have that:

$$p(x|\omega_2) = \frac{1}{\sqrt{2\pi}\sqrt{5}} \exp\left(\frac{-x^2}{10}\right)$$

We know that if x lies in the decision region R_1, then it holds that:

$$p(x|\omega_1)P(\omega_1) > p(x|\omega_2)P(\omega_2) \quad (1)$$

But $P(\omega_1) = P(\omega_2)$, because classes are equiprobable.

$$(1) \Rightarrow \; p(x|\omega_1) > p(x|\omega_2) \Rightarrow$$

$$\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) > \frac{1}{\sqrt{2\pi}\sqrt{5}} \exp\left(\frac{-x^2}{10}\right) \Rightarrow$$

$$\exp\left(\frac{-x^2}{2}\right) > \frac{1}{\sqrt{5}} \exp\left(\frac{-x^2}{10}\right) \Rightarrow$$

$$ln\left(\exp\left(\frac{-x^2}{2}\right)\right)) > ln\left(\frac{1}{\sqrt{5}} \exp\left(\frac{-x^2}{10}\right)\right) \Rightarrow$$

$$-\frac{x^2}{2} > -\frac{1}{2}\ln(5) - \frac{x^2}{10} \Rightarrow$$

$$x^2 < \ln(5) + \frac{x^2}{5} \Rightarrow$$

$$x^2 < \ln(5) + \frac{x^2}{5} \Rightarrow$$

$$x^2 < \frac{5}{4}\ln(5)$$

$$\Rightarrow -\sqrt{5/4\, ln\,(5)} < x < \sqrt{5/4\, ln\,(5)}$$

Therefore, decision regions are:

R_1: $\{x: -\sqrt{5/4\, ln\,(5)} < x < \sqrt{5/4\, ln\,(5)}\}$ and R_2: $\{x: x < -\sqrt{\frac{5}{4}\ln(5)} \cup x > \sqrt{5/4\, ln\,(5)}\}$

Exercise 2

According to the Bayesian classifier, the border between two classes is given by the equation: $p(x|\omega_1)P(\omega_1) = p(x|\omega_2)P(\omega_2)$. Since the two classes are equiprobable it holds that $P(\omega_1) = P(\omega_2)$. Combining the two equations we get:

$p(x|\omega_1) = p(x|\omega_2)$

$$\Rightarrow \frac{1}{(2\pi)^{\frac{-l}{2}}} exp\left(-\frac{(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)}{2}\right) = \frac{1}{(2\pi)^{\frac{-l}{2}}} exp\left(-\frac{(x-\mu_2)^T\Sigma^{-1}(x-\mu_2)}{2}\right)$$

$$\Rightarrow exp\left(-\frac{(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)}{2}\right) = exp\left(-\frac{(x-\mu_2)^T\Sigma^{-1}(x-\mu_2)}{2}\right)$$

$$\Rightarrow -\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1) = -\frac{1}{2}(x-\mu_2)^T\Sigma^{-1}(x-\mu_2)$$

$$\Rightarrow -\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1) = -\frac{1}{2}(x-\mu_2)^T\Sigma^{-1}(x-\mu_2)$$

$$\Rightarrow -\frac{1}{2}(x^T\Sigma^{-1}x - 2x^T\Sigma^{-1}\mu_1 + {\mu_1}^T\Sigma^{-1}\mu_1) = -\frac{1}{2}(x^T\Sigma^{-1}x - 2x^T\Sigma^{-1}\mu_2 + {\mu_2}^T\Sigma^{-1}\mu_2)$$

$$\Rightarrow -\frac{1}{2}x^T\Sigma^{-1}x + {\mu_1}^T\Sigma^{-1}x - \frac{1}{2}{\mu_1}^T\Sigma^{-1}\mu_1 = -\frac{1}{2}x^T\Sigma^{-1}x + {\mu_2}^T\Sigma^{-1}x - \frac{1}{2}{\mu_2}^T\Sigma^{-1}\mu_2$$

$$\Rightarrow {\mu_1}^T\Sigma^{-1}x - \frac{1}{2}{\mu_1}^T\Sigma^{-1}\mu_1 = {\mu_2}^T\Sigma^{-1}x - \frac{1}{2}{\mu_2}^T\Sigma^{-1}\mu_2$$

$$\Rightarrow {\mu_1}^T\begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix} x - \frac{1}{2}{\mu_1}^T\begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix}\mu_1$$
$$= {\mu_2}^T\begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix} x - \frac{1}{2}{\mu_2}^T\begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix}\mu_2$$

$$\Rightarrow \sigma^2{\mu_1}^T\begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix} x - \sigma^2\frac{1}{2}{\mu_1}^T\begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix}\mu_1$$
$$= \sigma^2{\mu_2}^T\begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix} x - \sigma^2\frac{1}{2}{\mu_2}^T\begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix}\mu_2$$

$$\Rightarrow {\mu_1}^T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x - \frac{1}{2}{\mu_1}^T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mu_1 = {\mu_2}^T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x - \frac{1}{2}{\mu_2}^T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mu_2$$

$$\Rightarrow ({\mu_1}^T - {\mu_2}^T)x - \frac{1}{2}{\mu_1}^T\mu_1 + \frac{1}{2}{\mu_2}^T\mu_2 = 0$$

$\Rightarrow (\mu_1 - \mu_2)^T x - \frac{1}{2}||\mu_1||^2 + \frac{1}{2}||\mu_2||^2 = 0$, which is the perpendicular bisector of the line segment whose endpoints are $\mu_1$and μ_2.

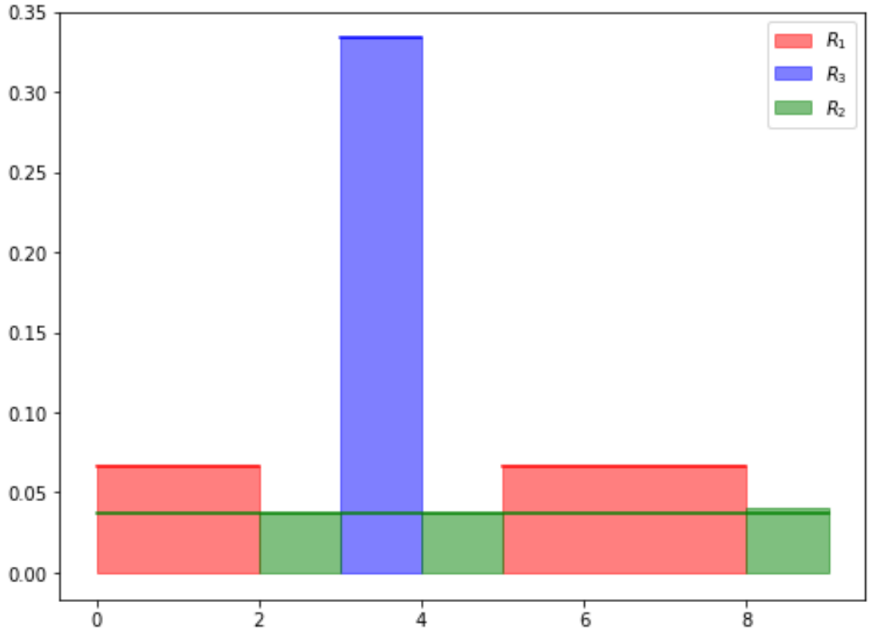
Exercise 3

I. Classes are equiprobable: $P(\omega_1) = P(\omega_2) = P(\omega_3) = \frac{1}{3}$

(i)
$$p(x|\omega_1) = \begin{cases} \frac{1}{5}, & x \in (0,2) \cup (5,8) \\ 0, & otherwise \end{cases}$$
$$p(x|\omega_2) = \begin{cases} \frac{1}{9}, & x \in (0,9) \\ 0, & otherwise \end{cases}$$
$$p(x|\omega_3) = \begin{cases} 1, & x \in (3,4) \\ 0, & otherwise \end{cases}$$

In the diagram bellow, horizontal lines are the probabilities $p(x|\omega_1)P(\omega_1)$(red), $p(x|\omega_2)P(\omega_2)$ (green), $p(x|\omega_3)P(\omega_3)$ (blue).

The blocks under the lines are the decision regions of the classes.



(ii)
$$P_e = \left(\int_{R_1}(p(x|\omega_2) + p(x|\omega_3))dx + \int_{R_2}(p(x|\omega_1) + p(x|\omega_3))dx + \int_{R_3}(p(x|\omega_1) + p(x|\omega_2))dx\right)\frac{1}{3} = \left(\int_0^2 (p(x|\omega_2) + p(x|\omega_3))dx + \int_5^8 (p(x|\omega_2) + p(x|\omega_3))dx + \int_2^3(p(x|\omega_1) + p(x|\omega_3))dx + \int_4^5(p(x|\omega_1) + p(x|\omega_3))dx + \int_8^9(p(x|\omega_1) + p(x|\omega_3))dx + \int_3^4(p(x|\omega_1) + p(x|\omega_2))dx\right)\frac{1}{3} = \left(\int_0^2 (\tfrac{1}{9} + 0)dx + \int_5^8 (\tfrac{1}{9} + 0)dx + \int_2^3(0 + 0)dx + \int_4^5(0 + 0)dx + \int_8^9(0 + 0)dx + \int_3^4(0 + \tfrac{1}{9})dx\right)\frac{1}{3} = \left(\frac{1}{9}(2-0) + \frac{1}{9}(8-5) + 0 + 0 + 0 + \frac{1}{9}(4-3)\right)\frac{1}{3} = \left(\frac{2}{9} + \frac{3}{9} + \frac{1}{9}\right)\frac{1}{3}$$
$$\Rightarrow P_e = 0.2222$$

Error classification probability is $P_e = 0.2222$.

(iii) To classify the point $x' = 3.5$ we will use the Bayes classifier:

- $p(3.5|\omega_1)\frac{1}{3} = 0$
- $p(3.5|\omega_2)\frac{1}{3} = \frac{1}{27}$
- $p(3.5|\omega_3)\frac{1}{3} = \frac{1}{3}$, which is the greatest value

Thus, $x' = 3.5$ is classified to class ω_3.

II. Classes are not equiprobable.

(i)

- Case ω_2 over ω_1

$x' = 3.5$ can never be assigned to class ω_1 since $p(3.5|\omega_1) = 0$. Thus, for every value of $P(\omega_1), P(\omega_2)$ the point will be assigned to ω_2.

- Case ω_2 over ω_3

For $x' = 3.5$ to be assigned to ω_2 over ω_3 it needs to hold that

$$p(3.5|\omega_2)P(\omega_2) > p(3.5|\omega_3)P(\omega_3) \Rightarrow \frac{1}{9}P(\omega_2) > 1P(\omega_3)$$
$$\Rightarrow P(\omega_2) > 9P(\omega_3)$$

Thus, $P(\omega_2)$ needs to be 9 times greater than $P(\omega_3)$.

(ii) Since $p(3.5|\omega_1) = 0$ then $p(\omega_1|3.5) = 0$ for every value of $P(\omega_1), P(\omega_2), P(\omega_3)$. In conclusion, there is no combination of the a priori probabilities that guarantees that it will be assigned to ω_1.