Flow in a cylindrical pipe

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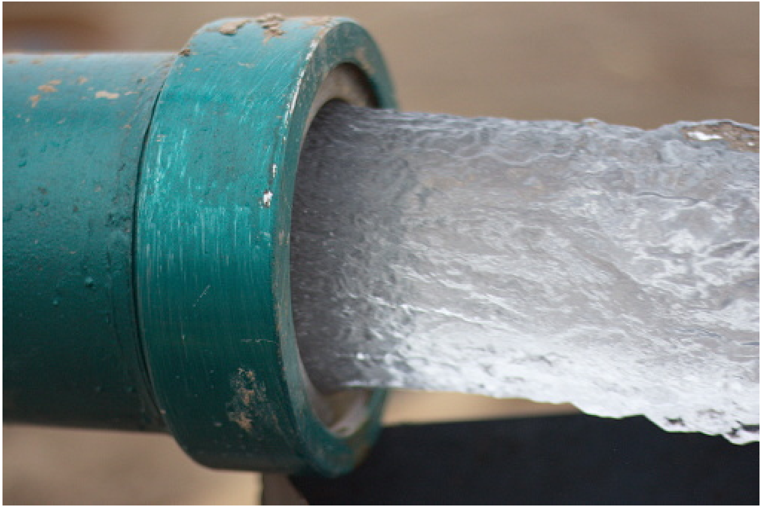
Scientific/engineering background of the specific problem



Presentation of the problem

- Modeling the problem
- Discretization
- Numerical treatment of the model
- Determine how the flow velocity varies in the pipe

Mathematical model

Velocity vector

$$(u, v)^T,$$

where u(r,z) is the velocity in the length direction and v(r,z) in the radial direction.
Navier- Stokes Eq.

$$u\frac{\partial u}{\partial z} = \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - v\frac{\partial u}{\partial r} - \frac{1}{\rho}\frac{dP}{dz} \tag{1}$$

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial (vr)}{\partial r} = 0 \tag{2}$$

Limitations/Assumptions:

- Flow is cylindrical symmetric thus u(r,z) and v(r,z)
- Density of fluid remains constant with $\rho = 1000\ [kg/m^3]$.
- The Reynolds number which is important for fluid problems i.e. $Re = u_0\frac{R}{\mu}$ is bigger than 1. Possible to approximate Navier Stokes Eq. as above

Some other given values

- R = 0.1 m
- viscosity $\mu = 4 \times 10^{-5} \frac{m^2}{s}$
- Boundary conditions

$$\frac{\partial u}{\partial r}(0, z) = 0, \quad v(0, z) = 0$$

and at r = R :

$$u(R, z) = v(R, z) = 0$$

- Initial Condition

$$u(r, 0) = 0.15[m/s], \quad v(r, 0) = 0[m/s]$$

The preassure of the fluid is $P = 10^4$ [Pa] at $z = 0$ where the preassure is computed with

$$\sigma(z) = \frac{1}{\rho}\frac{dP}{dz}$$

along the length of pipe.

Stationary distribution with parabolic shape some meters away

$$u = 2u_0\left(1 - \left(\frac{r}{R}^2\right)\right), v = 0$$

at r $= 0$, specific treatment shown with L'Hôpitals rule:

$$u\frac{\partial u}{\partial z} = 2\mu\frac{\partial^2 u}{\partial r^2} - \frac{1}{\rho}\frac{dP}{dz}, \quad \frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r} = 0$$

- Classification: Our numerical model is a continuous one, a parabolic model, which is solved with finite difference method : Method of lines.
- Special Diffucilities:

Had to take in account that we have nonlinear PDE:s and have to account for it at every length step z. And one of BC was Neumann was implemented by Skewed approximation.

l'Hôpitals Rule at r = 0

$$u\frac{\partial u}{\partial z} = \mu(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}) - v\frac{\partial u}{\partial r} - \frac{1}{\rho}\frac{dP}{dz}$$

$$\frac{\partial u}{\partial r}(0, z) = 0, \quad r = 0$$

$$\Rightarrow \lim_{r \mapsto 0} \frac{\frac{\partial u}{\partial r}(r, z)}{r} = \lim_{r \mapsto 0} \frac{\frac{\partial}{\partial r}(\frac{\partial u}{\partial r})(r, z)}{\frac{\partial}{\partial r}(r)} = \frac{\partial^2 u}{\partial r^2}(0, z)$$

$$\Rightarrow u\frac{\partial u}{\partial z} = 2\mu\frac{\partial^2 u}{\partial r^2} - \frac{1}{\rho}\frac{dP}{dz}, \quad (at \;\; r = 0)$$

l'Hôpitals Rule at r = 0

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial (vr)}{\partial r} = 0 \quad \Rightarrow \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{1}{r}v = 0$$

$$v(0,z) = 0, \quad r = 0$$

$$\Rightarrow \lim_{r \mapsto 0} \frac{v(r,z)}{r} = \lim_{r \mapsto 0} \frac{\frac{\partial}{\partial r}(v)(r,z)}{\frac{\partial}{\partial r}(r)} = \frac{\partial v}{\partial r}(0,z)$$

$$\Rightarrow \frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r} = 0, \quad (at \ \ r = 0)$$

Numerical methods used

- Method of lines
- Euler's implicit method
- Newtons method

Discretization

Central difference:

$$\frac{\partial u_j}{\partial r} = \frac{u_{j+1} - u_{j-1}}{2h_r}, \quad \frac{\partial^2 u_j}{\partial r^2} = \frac{u_{j+1} - 2u_j + u_{j-1}}{h_r^2}$$

Forward difference:

$$\frac{\partial u_j^n}{\partial z} = \frac{u_j^{n+1} - u_j^n}{h_z}$$

Discretization

We use the difference approximations to rewrite equation (1) and (2) to the following:

$$u_j^{n+1}\frac{u_j^{n+1}-u_j^n}{h_z} - \mu\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h_r^2} + \left(v_j^{n+1} - \frac{\mu}{r_j}\right)\left(\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2h_r}\right) + \sigma^{n+1} = 0$$

$$\frac{u_j^{n+1} - u_j^n}{h_z} + \frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2h_r} + \frac{v^{n+1}}{r_j} = 0$$

Discretization

We know by description that by using the mentioned approximations (central difference and forward difference), equation (1) and (2) can be approximated by

$$u_j^{n+1}\frac{u_j^{n+1}-u_j^n}{h_z}-F_j(u_{j-1}^{n+1},u_j^{n+1},u_{j+1}^{n+1},\sigma^{n+1},v_j^{n+1})=0,$$

$$\frac{u_j^{n+1}-u_j^n}{h_z}+G_j(v_{j-1}^{n+1},v_j^{n+1},vn+1_{j+1})=0$$

For $1\leq j\leq N$, we set

$$\hat{F}_j=u_j^{n+1}\frac{u_j^{n+1}-u_j^n}{h_z}-\mu\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h_r^2}+\left(v_j^{n+1}-\frac{\mu}{r_j}\right)\left(\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2h_r}\right)+\sigma^{n+1}=0$$

$$\hat{G}_j=\frac{u_j^{n+1}-u_j^n}{h_z}+\frac{v_{j+1}^{n+1}-v_{j-1}^{n+1}}{2h_r}+\frac{v^{n+1}}{r_j}=0$$

Discretization of boundaries

Here we use skewed-difference

$$\frac{\partial u}{\partial r}(0, z_{n+1}) = \frac{-3u_0^{n+1} + 4u_1^{n+1} - u_2^{n+1}}{2h_r} = 0$$

$$\Rightarrow \hat{F}_0 = -3u_0^{n+1} + 4u_1^{n+1} - u_2^{n+1} = 0$$

$$\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r} = 0, \quad (at\ r = 0)$$

$$\frac{u_0^{n+1} - u_0^n}{h_z} + 2\frac{-3v_0^{n+1} + 4v_1^{n+1} - v_2^{n+1}}{2h_r} = \{v_0^{n+1} = 0\} = \frac{u_0^{n+1} - u_0^n}{h_z} + \frac{4v_1^{n+1} - v_2^{n+1}}{h_r}$$

$$\Rightarrow \hat{G}_0 = \frac{u_0^{n+1} - u_0^n}{h_z} + \frac{4v_1^{n+1} - v_2^{n+1}}{h_r} = 0$$

Newtons method

Goal is to compute

$$\mathbf{x}^{k+1} = \mathbf{x}^k - [\mathbf{J}(\mathbf{x}^k)]^{-1}\mathbf{g}(\mathbf{x}^k)$$

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} \hat{\mathbf{F}} \\ \hat{\mathbf{G}} \end{bmatrix}$$

$$\mathbf{x}(z) = \begin{bmatrix} \mathbf{u}(z) \\ \sigma(z) \\ \mathbf{v}(z) \end{bmatrix}, \quad \mathbf{u}(z) = \begin{bmatrix} u_0(z) \\ u_1(z) \\ \vdots \\ u_N(z) \end{bmatrix}, \quad \mathbf{v}(z) = \begin{bmatrix} v_1(z) \\ v_2(z) \\ \vdots \\ v_N(z) \end{bmatrix}$$

From this calculate Jacobian: $\mathbf{J} = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{\mathbf{F}}}{\partial u} & \begin{bmatrix} \frac{\partial \hat{\mathbf{F}}}{\partial \sigma} & \frac{\partial \hat{\mathbf{F}}}{\partial v} \end{bmatrix} \\ \frac{\partial \hat{\mathbf{G}}}{\partial u} & \begin{bmatrix} \frac{\partial \hat{\mathbf{G}}}{\partial \sigma} & \frac{\partial \hat{\mathbf{G}}}{\partial v} \end{bmatrix} \end{bmatrix}$

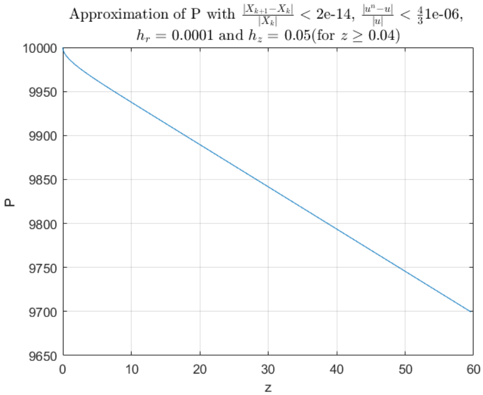
$$
J_1 = \begin{bmatrix}
-3 & 4 & -1 & 0 & \cdots & 0 \\
-a_1 & \frac{2u_1^{n+1} - u_1^n}{h_z} - b_1 & -c_1 & 0 & \cdots & \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & & 0 \\
0 & 0 & 0 & -a_{N-1} & \frac{2u_{N-1}^{n+1} - u_{N-1}^n}{h_z} - b_{N-1} & -c_{N-1} \\
0 & 0 & 0 & 0 & -a_N & \frac{2u_N^{n+1} - u_N^n}{h_z} - b_N
\end{bmatrix}
$$

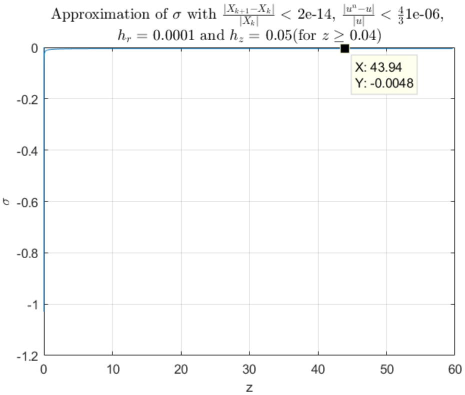
$$
a_j = \frac{\mu}{h_r^2} - \frac{\mu}{2r_j h_r} + v_j^{n+1}\frac{1}{2h_r}, \quad b_j = -\frac{2\mu}{h_r^2}, \quad c_j = \frac{\mu}{h_r^2} - \frac{\mu}{2r_j h_r} - v_j^{n+1}\frac{1}{2h_r}
$$

$$
J_2 = \begin{bmatrix}
0 & \cdots & & & \cdots & 0 \\
1 & \frac{-u_0^{n+1}}{2h_r} + \frac{u_2^{n+1}}{2h_r} & 0 & \cdots & & 0 \\
 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & & \\
 & 0 & 0 & & \frac{-u_{N-1}^{n+1}}{2h_r} + \frac{u_N^{n+1}}{2h_r} & \\
1 & 0 & 0 & 0 & \cdots & \frac{-u_{N-1}^{n+1}}{2h_r}
\end{bmatrix}
$$

$$
J_3 = \begin{bmatrix}
\frac{1}{h_z} & 0 & & & 0 & \\
0 & \ddots & \vdots & 0 & \cdots & 0 \\
\vdots & & & & & \vdots \\
 & & & & & 0 \\
\vdots & & & & \ddots & \vdots \\
0 & & & & & \frac{1}{h_z}
\end{bmatrix}
\quad
J_4 = \begin{bmatrix}
0 & \frac{4}{h_r} & \frac{-1}{h_r} & 0 & \cdots & 0 \\
0 & \frac{1}{h_r} & \frac{1}{2h_r} & 0 & \cdots & 0 \\
\vdots & \frac{-1}{2h_r} & \frac{1}{2h_r} & \frac{1}{2h_r} & 0 & 0 \\
 & & \ddots & \ddots & \ddots & \vdots \\
0 & & & \frac{-1}{2h_r} & \frac{1}{(N-1)h_r} & \frac{1}{2h_r} \\
0 & & & 0 & \frac{-1}{2h_r} & \frac{1}{Nh_r}
\end{bmatrix}
$$

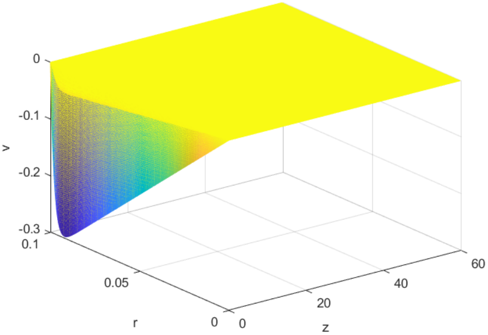
Results


Approximation of P with $\frac{|X_{k+1}-X_k|}{|X_k|} < 2\text{e-}14$, $\frac{|u^n-u|}{|u|} < \frac{4}{3}1\text{e-}06$,
$h_r = 0.0001$ and $h_z = 0.05$(for $z \geq 0.04$)
P
z
10000
9950
9900
9850
9800
9750
9700
9650
0
10
20
30
40
50
60

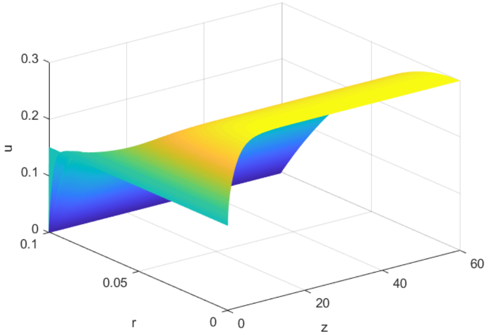

Approximation of σ with $\frac{|X_{k+1}-X_k|}{|X_k|} < 2\text{e-}14$, $\frac{|u^n-u|}{|u|} < \frac{4}{3}1\text{e-}06$,
$h_r = 0.0001$ and $h_z = 0.05$(for $z \geq 0.04$)
σ
z
X: 43.94
Y: -0.0048
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
0
10
20
30
40
50
60

Results

Approximation of v with $\frac{|X_{k+1}-X_k|}{|X_k|} < 2\text{e-}14$, $\frac{|u^n-u|}{|u|} < \frac{4}{3}1\text{e-}06$, $h_r = 0.0001$ and $h_z = 0.05$(for $z \geq 0.04$)

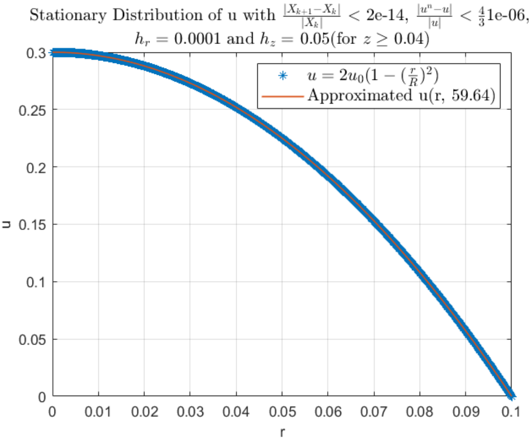


Approximation of u with $\frac{|X_{k+1}-X_k|}{|X_k|} < 2\text{e-}14$, $\frac{|u^n-u|}{|u|} < \frac{4}{3}1\text{e-}06$, $h_r = 0.0001$ and $h_z = 0.05$(for $z \geq 0.04$)



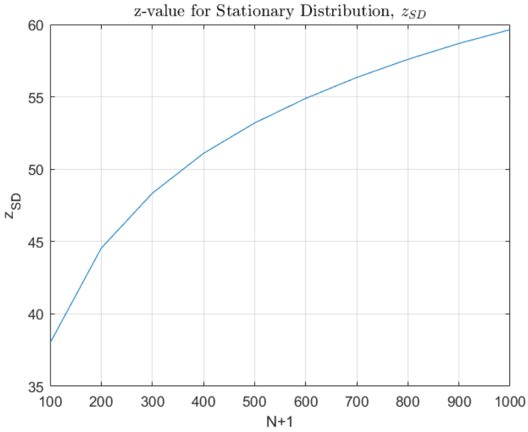
Results

- $\frac{|u^n - u|}{|u|} < \frac{4}{3(N+1)^2}$
- $h_z = 0.05$ and $N + 1 = 1000$
- z-value for Stationary Distribution, $z_{SD} = 59.64$



Results

$N+1$	z_{SD}
100	37.99
200	44.54
300	48.34
400	51.09
500	53.19
600	54.89
700	56.34
800	57.59
900	58.69
1000	59.64



Thanks!