Advanced Numerical Methods
SF2520

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Computer Exercise 2

Elliptic Equations

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Part 1: Finite difference approximation in 1D

(a)

In this case we are using the Skewed approximation to solve the boundary value problem. Discretizing the z interval with the grid points $z_j = jh$ where $j = 0, 1, ..., N$ with N+1 points and the length L of the pipe as $L = Nh$. Rewriting $T_j = T(z_j)$, $Q_j = Q(z_j)$ and defining $z_0 = 0$ and $z_N = L$ where T_j is the temperature for every j and $Q(z)$ is the driving function.

We are given the convection-diffusion equation

$$-\frac{d^2T}{dz^2} + v\frac{dT}{dz} = Q(z). \tag{1}$$

with the boundary conditions $T(0) = T_0$ and at $z = L$

$$-\frac{dT}{dz}(L) = \alpha(v)(T(L) - T_{\text{out}}). \tag{2}$$

where $\alpha(v) = \sqrt{\frac{v^2}{4} + \alpha_0^2} - \frac{v}{2}$

We need to approximate the derivatives with second order differences defining the first and second derivatives as

$$\frac{dT}{dz}(z_j) = \frac{T_{j+1} - T_{j-1}}{2h} + O(h^2) \tag{3}$$

$$\frac{d^2T}{dz^2}(z_j) = \frac{T_{j+1} - 2T_j + T_{j-1}}{2h} + O(h^2). \tag{4}$$

for $j = 1, \ldots, N-1$.

Adding equation (3) and (4) to (1) and sorting the terms by steps of T gives

$$\Big(-\frac{1}{h^2} - \frac{v}{2h}\Big)T_{j+1} + \frac{2}{h^2}T_{j-1} + \Big(-\frac{1}{h^2} + \frac{v}{2h}\Big)T_{j+1} = k_3T_{j+1} + k_2T_j + k_1T_{j-1} = Q_j. \tag{5}$$

where we introduced the constant coefficients $k_1 = -\frac{1}{h^2} - \frac{v}{2h}$, $k_2 = \frac{2}{h^2}$ and $k_3 = -\frac{1}{h^2} + \frac{v}{2h}$. Before writing down the matrix, we want to take care of the boundary condition. Starting with the fact that at $z = 0$ the fluid has a temperature $T_0 = 50$:

$$k_1T_0 + k_2T_1 + k_3T_2 = Q_1 \tag{6}$$

Rewriting the forcing terms to insert it later in the matrix yields

$$k_2T_1 + k_3T_2 = Q_1 - k_1T_0 \tag{7}$$

Now, set up equation (5) near the right boundary at $z = L - h = z_{N-1}$:

$$k_1T_{N-2} + k_2T_{N-1} + k_3T_N = Q_{N-1} \tag{8}$$

Now we use the skewed asymmetric formula to approximate the first derivative which yields

$$-\frac{dT}{dz}(z_N) = \frac{-3T_N + 4T_{N-1} - T_{N-2}}{2h} + O(h^2) = \alpha(v)(T_n - T_{\text{out}}) \tag{9}$$

Neglecting the term $O(h^2)$ and solving for T_N, which is what we are looking fore, alone to the left side we get:

$$T_N = \frac{-2h\alpha}{-3-2h\alpha}T_{\text{out}} + \frac{-4}{-3-2h\alpha}T_{N-1} + \frac{1}{-3-2h\alpha}T_{N-2} \tag{10}$$

$$= d_0 + d_1T_{N-1} + d_2T_{N-2} \tag{11}$$

where $d_0 = \frac{-2h\alpha}{-3-2h\alpha}T_{\text{out}}$, $d_1 = \frac{-4}{-3-2h\alpha}$ and $d_2 = \frac{1}{-3-2h\alpha}$.

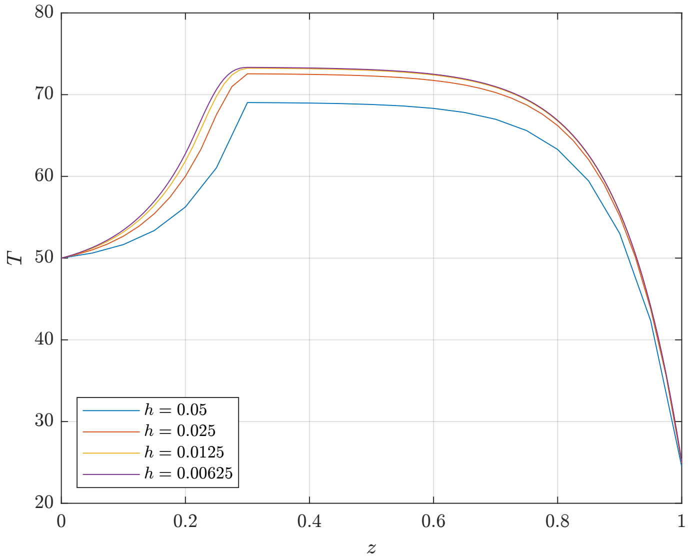


Figure 1: Solutions of the convection-diffusion equation for $T(z)$ for the different steps sizes h. It was $L = 1$, $v = 10$ and $T_0 = 50$.

Plugging this in into equation (8) gives us

$$k_1 T_{N-2} + k_2 T_{N-1} + k_3(d_0 + d_1 T_{N-1} + d_2 T_{N-2}) = Q_{N-1} \tag{12}$$

Finally, we rewrite the equation for the purpose of the matrix which gives us

$$(k_1 + k_3 d_2)T_{N-2} + (k_2 + k_3 d_1)T_{N-1} = Q_{N-1} - k_3 d_0 \tag{13}$$

We have just evaluated the equation for the boundary condition at $z = L$ and $z = 0$ and we are ready do write out our matrix.

$$A = \begin{pmatrix} k_2 & k_3 & 0 & \dots & 0 \\ k_1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & k_2 & k_3 & 0 \\ \vdots & \ddots & k_1 & k_2 & k_3 \\ 0 & \dots & 0 & k_1 + k_3 d_2 & k_2 + k_3 d_1 \end{pmatrix} \tag{14}$$

$$T = \begin{pmatrix} T_1 \\ \vdots \\ T_{N-1} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} Q_1 - k_1 T_0 \\ Q_2 \\ \vdots \\ Q_{N-2} \\ Q_{N-1} - k_3 d_0 \end{pmatrix} \tag{15}$$

The solution is then found by solving the system $AT = Q$ for T and then also adding T_0, which is given, and T_N from equation (11).

For the solutions that are shown in figure 1 the parameters were $L = 1$, $T_0 = 50$, $T_{\text{out}} = 20$, $\alpha_0 = 100$, and for the forcing part $Q_0 = 4000$, $a = 0.2$ $b = 0.3$ and

$$Q(z) = \begin{cases} 0 & 0 \le z < a \\ Q_0 \sin\left(\frac{\pi(z-a)}{b-a}\right) & a \le z \le b \\ 0 & b < z \le L \end{cases} \tag{16}$$

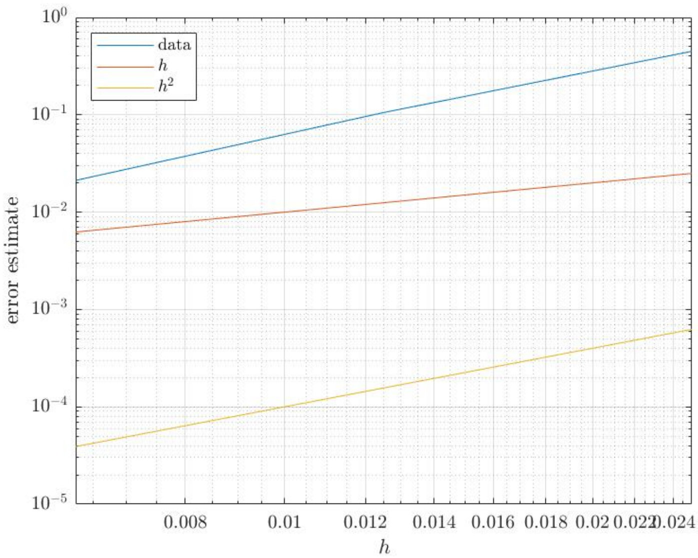


Figure 2: Convergence of our solution which states that it is of second order.

You can see in figure 1 that as h gets smaller the solutions converge.

Actually checking the convergence of second order we plot the convergence for when $z = L$ and comparing with the corresponding h^2. Observing figure 2 we see that the slope of data and h^2 are comparable and comparing both of them with the slope of h, corresponding the first order convergence, confirms the accuracy of our data slope of second order convergence.

(b)

Varying the velocity v of the pipe should change the solution because it is a main part of transferring the heat from the coil by convection. For the 4 different velocities $v = 1, 10, 30, 100$ the solutions are shown in figure 3. The step size was in all cases $h = 0.003125$ with $N = 320$ points. For all solutions the effect of the coil starting at $z = 0.2$ is visible as the temperature rises. For $v = 1$ the temperature has its maximum there as the fluid flows very slowly so it heats up at the coil and then cools down rather quickly.

When the velocity of the fluid increases the temperature reaches a constant value after the coil which makes sense as the heat that is dissipated is replenished by the convectional heat coming from the fluid that just passed the coil. In the end the temperature drops as well (the outside temperature is just 20) but not as low if the velocity is larger.

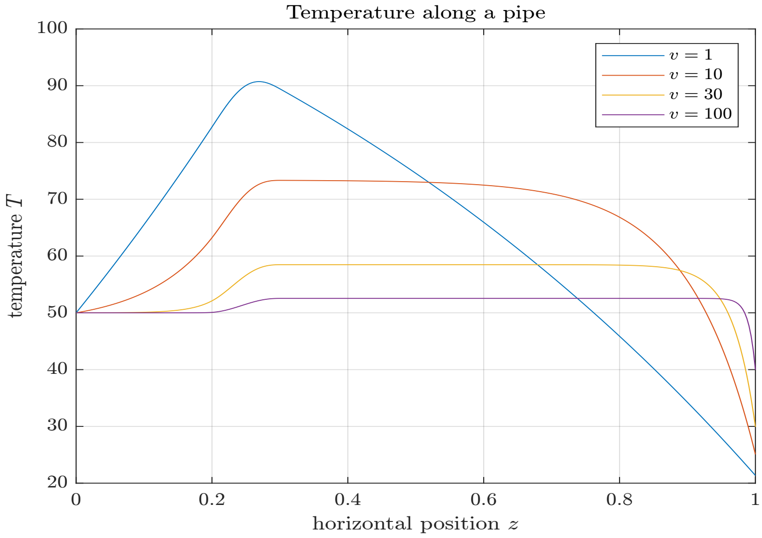


Figure 3: Solutions of the steady state heat equation along a pipe of length $L = 1$. The velocity v of the fluid inside the pipe (flowing in the positive z-direction) was increased from 1 to 100 see the legend.

Part 2: Finite difference approximation in 2D

We are given a rectangle $\Omega = [0, 5] \times [0, 2] \subset \mathbb{R}^2$. The temperature in the rectangle is determined by the Poisson equation

$$-\Delta T(x, y) = f(x, y) \quad \forall (x, y) \in \Omega \tag{17}$$

with boundary conditions

$$T(0, y) = 40 \quad \forall y \in (0, 2) \tag{18}$$

$$T(5, y) = 400 \quad \forall y \in (0, 2) \tag{19}$$

$$\frac{\partial T}{\partial y}(x, 0) = \frac{\partial T}{\partial y}(x, 2) = 0 \quad \forall x \in (0, 5) \tag{20}$$

We discretize the rectangle with a rectangular grid of step size h in x- and y-direction, so $x_i = ih$, $i = 0, \ldots, N$ and $y_j = jh$, $j = 0, \ldots, M$. The Laplacian is then approximated by the standard 5-point stencil of 2nd order accuracy. The approximated PDE is for $Ti, j = T(x_i, y_j)$ and $f_{i,j} = f(x_i, y_j)$:

$$\frac{1}{h^2}(4T_{i,j} - T_{i+1,j} - T_{i-1,j} - T_{i,j+1} - T_{i,j-1}) = f_{i,j} \tag{21}$$

where $i = 2, \ldots, N-2$ and $j = 1, \ldots, M-1$. The Dirichlet boundary conditions affect the points $T_{0,j}$ and $T_{N,j}$. So the difference approximations are modified.

$$\frac{1}{h^2}(4T_{1,j} - T_{1,j} - T_{i,j+1} - T_{i,j-1}) = f_{1,j} + \frac{40}{h^2} \qquad \forall j = 1, \ldots, M-1 \tag{22}$$

$$\frac{1}{h^2}(4T_{N-1,j} - T_{N-2,j} - T_{i,j+1} - T_{i,j-1}) = f_{N-1,j} + \frac{400}{h^2} \qquad \forall j = 1, \ldots, M-1 \tag{23}$$

For the Neumann boundary conditions we use the Ghost point method to replace the values $T_{i,-1}$ and $T_{i,M+1}$ in the stencils. The 2nd order approximation of the 1st derivative is

$$\frac{\partial T}{\partial y}(x, 0) \approx \frac{1}{2h}(T_{i,1} - T_{i,-1}) = 0 \tag{24}$$

$$\frac{\partial T}{\partial y}(x, 2) \approx \frac{1}{2h}(T_{i,M+1} - T_{i,M-1}) = 0 \tag{25}$$

Thus we simply have $T_{i,-1} = T_{i,1}$ and $T_{i,M+1} = T_{i,M-1}$ for $i = 1, \ldots, N-1$. So we can write the difference approximations at the boundaries $y = 0$ and $y = 2$ as:

$$\frac{1}{h^2}(4T_{i,0} - T_{i+1,0} - T_{i-1,0} - 2T_{i,1}) = f_{i,0} \qquad \forall i = 1, \ldots, N-1 \tag{26}$$

$$\frac{1}{h^2}(4T_{i,M} - T_{i+1,M} - T_{i-1,M} - 2T_{i,M-1}) = f_{i,M} \qquad \forall i = 1, \ldots, N-1 \tag{27}$$

In total we now have $(N-1)(M+1)$ equations as we have so many unknowns in the $(N+1)(M+1)$ points grid where $2(M+1)$ points are already given by the Dirichlet boundary conditions.

Ordering the values $T_{i,j}$ in lexicographic order of (i, j) in the vector $\bar{u}$ the total system can be written using the following matrix.

$$A = \begin{pmatrix} T_{N-1} & -2I_{N-1} & & & \\ -I_{N-1} & \ddots & -I_{N-1} & & \\ & \ddots & \ddots & \ddots & \\ & & -I_{N-1} & \ddots & -I_{N-1} \\ & & & -2I_{N-1} & T_{N-1} \end{pmatrix} \tag{28}$$

which has $(M+1) \times (M+1)$ blocks where we the the $(N-1) \times (N-1)$ matrix T_{N-1} is

$$T_{N-1} = \begin{pmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix} \tag{29}$$

and I_{N-1} the $(N-1) \times (N-1)$ identity matrix. And the right hand side is the vector $\bar{f}$.

$$\bar{f} = \begin{pmatrix} f_{1,0} + \frac{40}{h^2} \\ f_{2,0} \\ \vdots \\ f_{N-2,0} \\ f_{N-1,0} + \frac{400}{h^2} \\ f_{1,1} + \frac{40}{h^2} \\ f_{2,1} \\ \vdots \\ f_{N-2,M-1} \\ f_{N-1,M-1} + \frac{400}{h^2} \\ f_{1,M} + \frac{40}{h^2} \\ f_{2,M} \\ \vdots \\ f_{N-2,M} \\ f_{N-1,M} + \frac{400}{h^2} \end{pmatrix} \tag{30}$$

The linear system that we have to then solve is $A\bar{u} = \bar{f}$. To actually do the computation is is helpful to write A as the following.

$$A = I_{M+1} \otimes S_{N-1} + \tilde{S}_{M+1} \otimes I_{N-1} \tag{31}$$

where we use the matrix $S_k \in \mathbb{R}^{k \times k}$ from the 1D finite difference part.

$$S_k = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \tag{32}$$

and $\tilde{S}_{M+1}$ is just a slightly changed version of S_{M+1} to account for the two extra -2 values on the off diagonals.

$$\tilde{S}_{M+1} = \begin{pmatrix} 2 & 2 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix} \tag{33}$$

(a)

Use the scheme above to solve the PDE with the constant inhomogeneity $f(x, y) = 100$, step size $h = 0.1$ and thus $N = 50$ and $M = 20$.

The solution we obtained is shown in figure 4 and shows a parabolic distribution along x and a constant temperature along y. The temperature at the point $(3, 1)$ is 556.000.

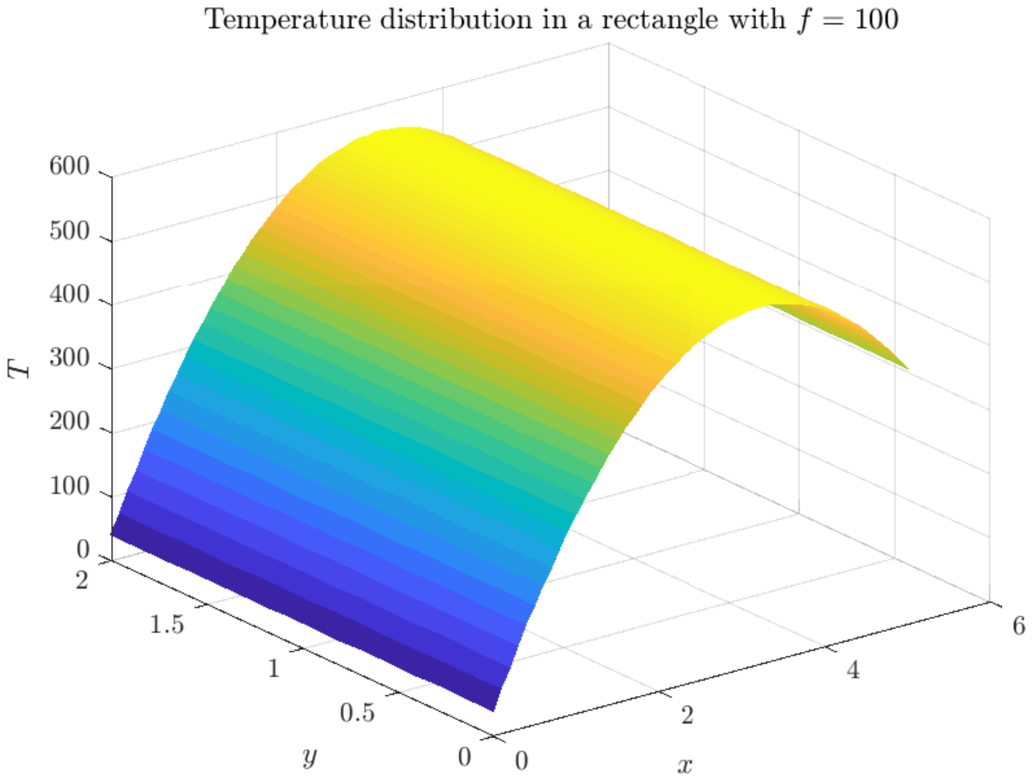


Figure 4: Temperature distribution of a rectangle $[0, 5] \times [0, 2]$ that is heated with constant $f = 100$ and has boundary conditions 40 at $x = 0$ 400 at $x = 5$ and a vanishing derivative $\frac{\partial T}{\partial y}$ at $y = 0$ and $y = 2$. The step size to solve the Poisson equation with finite differences was $h = 0.1$.

(b)

To derive the exact solution split the process in finding a particular solution T_{p} and a linear combination of homogeneous solutions T_{h}. A particular solution to equation (17) is

$$T_{\mathrm{p}}(x, y) = -50x^2 + 322x + 40 \tag{34}$$

which satisfies $-\frac{\partial T_{\mathrm{p}}}{\partial x} = 100$, $T_{\mathrm{p}}(0, y) = 40$ and $T_{\mathrm{p}}(5, y) = 400$. This is the actual solution that was also obtained in part (a) which you can also see by computing $T_{\mathrm{p}}(3, y) = 556$.

The reason is that the homogeneous solutions solutions have to be zero as the boundary conditions require it which we are going to see now. Since we have the non-zero boundary conditions already absorbed, the ones remaining for the homogeneous solution are:

$$T_{\mathrm{h}}(0,y) = T_{\mathrm{h}}(5,y) = 0 \qquad \text{and} \qquad \frac{\partial T_{\mathrm{h}}}{\partial y}(x,0) = \frac{\partial T_{\mathrm{h}}}{\partial y}(x,2) = 0 \tag{35}$$

Use the method of separation of variables, that is make the ansatz $T_{\mathrm{h}}(x,y) = v(x)w(y)$ on the homogeneous PDE:

$$\frac{\partial^2 T_{\mathrm{h}}}{\partial x^2} + \frac{\partial^2 T_{\mathrm{h}}}{\partial y} = 0 \tag{36}$$

Then divide by $v(x)y(y)$ to obtain:

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{y(y)} \tag{37}$$

Since both sides depend on different variables they have to be constant, name this constant λ. Then we need to solve the two ODE with their boundary conditions.

$$v''(x) = \lambda v(x) \qquad v(0) = v(5) = 0 \tag{38}$$

$$w''(y) = -\lambda w(y) \qquad w'(0) = w'(2) = 0 \tag{39}$$

The solutions to the two ODEs are:

$$\lambda > 0: \qquad v(x) = A\mathrm{e}^{\sqrt{\lambda}x} + B\mathrm{e}^{-\sqrt{\lambda}x} \tag{40}$$

$$w(y) = C\cos(\sqrt{\lambda}y) + D\sin(\sqrt{\lambda}y) \tag{41}$$

$$\lambda < 0: \qquad v(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x) \tag{42}$$

$$w(y) = C\mathrm{e}^{\sqrt{-\lambda}y} + D\mathrm{e}^{-\sqrt{-\lambda}y} \tag{43}$$

Impose the boundary conditions in the next step. The case $\lambda > 0$ has the following conditions:

$$A + B = 0 \tag{44}$$

$$\mathrm{e}^{5\sqrt{\lambda}}A + \mathrm{e}^{-5\sqrt{\lambda}}B = 0 \tag{45}$$

$$\cos(0)D = 0 \tag{46}$$

$$-\sin(2\sqrt{\lambda})C + \cos(2\sqrt{\lambda})D = 0 \tag{47}$$

For A and B when only have a non-zero solution if $\mathrm{e}^{-5\sqrt{\lambda}} - \mathrm{e}^{5\sqrt{\lambda}} = -2\sinh(5\sqrt{\lambda}) = 0$ which means $\lambda = 0$. But for D we have a non-zero solution if $\cos(2\sqrt{\lambda}) = 0$ which means $2\sqrt{\lambda} = \pi(\frac{1}{2} + n)$ where $n \in \mathbb{Z}$. Both conditions on λ cannot be fulfilled simultaneously, so the only solution is $v(x) = w(y) = 0$.

In the other case $\lambda < 0$ we have these conditions:

$$A = 0 \tag{48}$$

$$\cos(5\sqrt{-\lambda})A + \sin(5\sqrt{-\lambda})B = 0 \tag{49}$$

$$C - D = 0 \tag{50}$$

$$\mathrm{e}^{2\sqrt{-\lambda}}C - D\mathrm{e}^{-2\sqrt{-\lambda}} = 0 \tag{51}$$

The last two equations only have a non-trivial solution if $\mathrm{e}^{-2\sqrt{-\lambda}} + \mathrm{e}^{2\sqrt{-\lambda}} = 2\cosh(2\sqrt{-\lambda}) = 0$ which is never fulfilled. Thus the only solution is $w(y) = 0$

In the end we have for all values of λ the only solution $T_{\mathrm{h}}(x,y) = 0$. So the particular solution is the only solution which is also why we saw it in the numerical part (a).

(c)

In this part we have a localized heat source:

$$f(x, y) = 6000 \exp(-5(x-1)^2 - 10(y-1.5)^2) \tag{52}$$

We used the same scheme to solve the PDE as before but reduced the step sizes from $h = 0.1$ in factors of 2. Table 1 lists the step sizes and the extracted temperature value at $(3, 1)$. The logarithmic plot showing the 2nd order accuracy is shown in figure 5. The blue line shows the estimated error from the differences of the $T(3, 1)$ values and the orange line is the reference h^2 for the 2nd order.

Table 1: List of step sizes in the 2D finite difference method and the temperature values at the point $x = 3$, $y = 1$.

Step size h	$T(3,1)$
0.1	781.789798
0.05	782.275079
0.025	782.397217
0.012500	782.427801
0.006250	782.435451
0.003125	782.437363
0.001563	782.437841

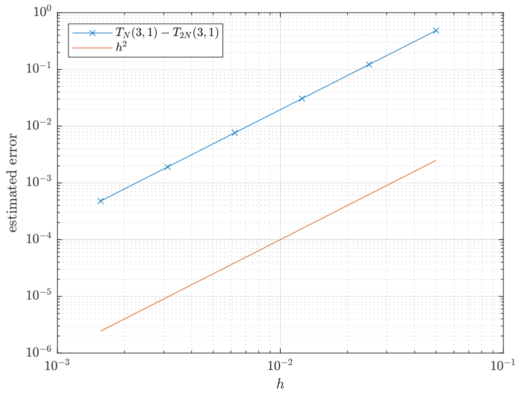
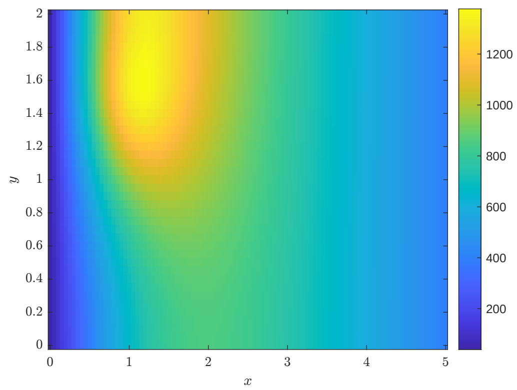


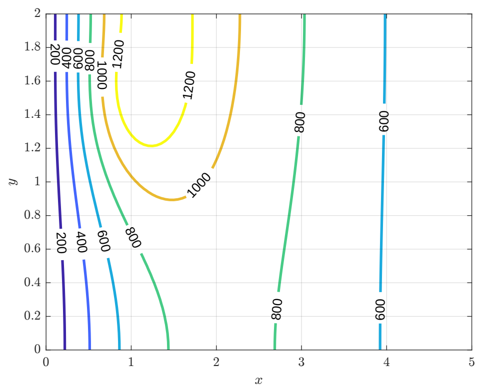
Figure 5: Order of accuracy plot for the 2D finite differences method. Blue is the estimated error computed by taking the difference of the same temperature value when halving the step size h. Orange is the reference line for 2nd order accuracy.

Finally we visualize the solution we computed with the step size $h = 0.05$. In figure 1 we have three different methods of visualization. The command `imagesc` shows a color plot of the temperature T in the rectangle. The contour lines of this plot are visualized with the `contour` command and at last we have again a `mesh` plot which shows T in a 3D plot.

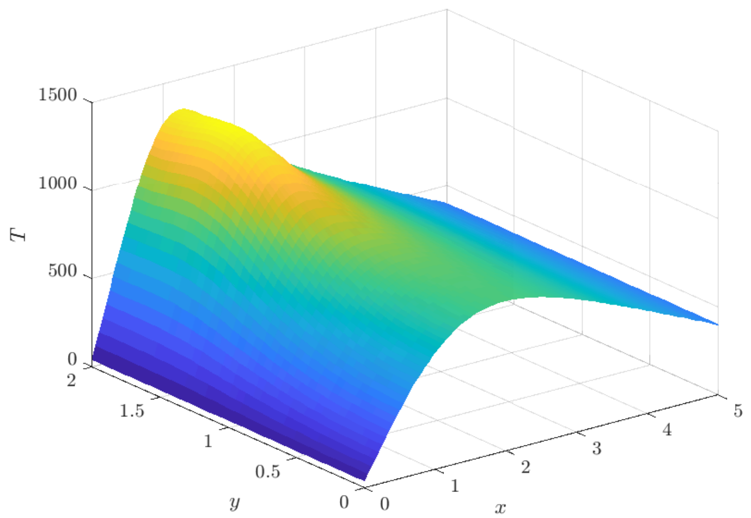
The solution for this localized heat souce clearly shows a maximum at the heat source at $(1, 1.5)$ and then the temperature decreases until it matches the boundary conditions on the left and right side. They are as expected as we assumed the temperature would only be very high near the heat source. The behavior near the top boundary of the heat source is also expected since it is insulated so the heat stays trapped between the boundary and the heat source. On the other side of the heat source ($y < 1.5$) this behavior is not visible because there the heat can more easily spread to the left and right sides.



(a) `imagesc`



(b) `contour`



(c) `mesh`

Figure 6: Different visualizations for the solution of the steady state heat equation with a localized heat source $f(x, y) = 6000 \exp(-5(x-1)^2 - 10(y-1.5)^2)$. The step size of the grid is $h = 0.5$.

Part 3: Comsol Multiphysics

(a)

We set up the rectangle $[0,5]\times[0,2] \subset \mathbb{R}^2$ in Comsol and solved the Poisson equation $-\Delta T = f$ with $f(x,y) = 6000\exp\left(-5(x-1)^2 - 10(y-1.5)^2\right)$ and Dirichlet boundary conditions $T(0,y) = 40$, $T(5,y) = 400$ and Neumann boundary conditions $\frac{\partial T}{\partial y}(x,0) = \frac{\partial T}{\partial y}(x,2) = 0$.

To look at the convergence we extracted the temperature at the point $(3,1)$ and refined the mesh. The results are shown in table 2 which states the setting of the mesh, the number of domain elements, i. e. the triangles, the number of boundary elements and the temperature. The temperature was already correct to 3 decimal points ($T(3,1) = 782.439$) after refining the mesh once. For all later exercises we then kept the mesh at this setting `Fine`.

Table 2: List of the mesh setting in Comsol with the domain and boundary elements and the corresponding temperature values at $(3,1)$.

Mesh (domain elements/ boundary elements)	$T(3,1)$
Normal (240 / 42)	782.4394287501322
Fine (386 / 54)	782.439665452663

The contour plot of the solution with mesh size `Fine` is shown in figure 7. It looks the same as the solution we obtained in part 2 (c) (see figure 6). The temperature values are also quite similar with 782.437 for $h = 0.001563$ of the finite differences and 782.439 for the finite elements solution in Comsol. They are correct within 2 decimal points.

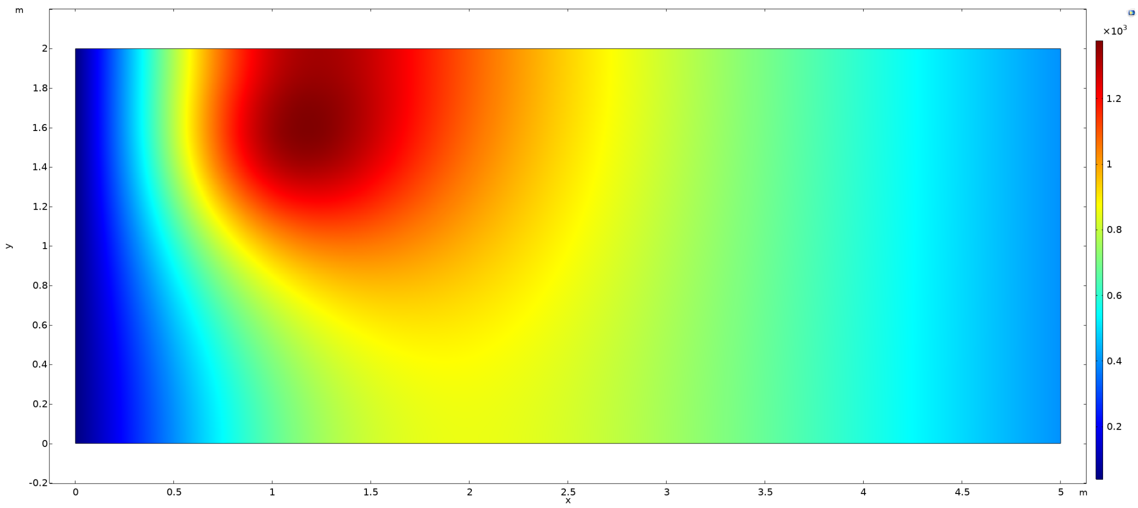


Figure 7: Contour plot of the temperature in the rectangle where the top and bottom part are insulated and the left and right part are set to the temperatures 40 and 400. The heating is a Gaussian heat profile with its center at $(1,1.5)$ and peak temperature 6000.

(b)

Now we insulate the right boundary so it has a Neumann condition $\frac{\partial T}{\partial x}(5, y) = 0$. With the same settings as in part (a) we then computed the solution which you can see in figure 8. The average value of the temperature at the right border was in this case $T = 1356$. We also note that the whole area to the right of the heating source is now much hotter (similar to the temperature at the right boundary) than in part (a).

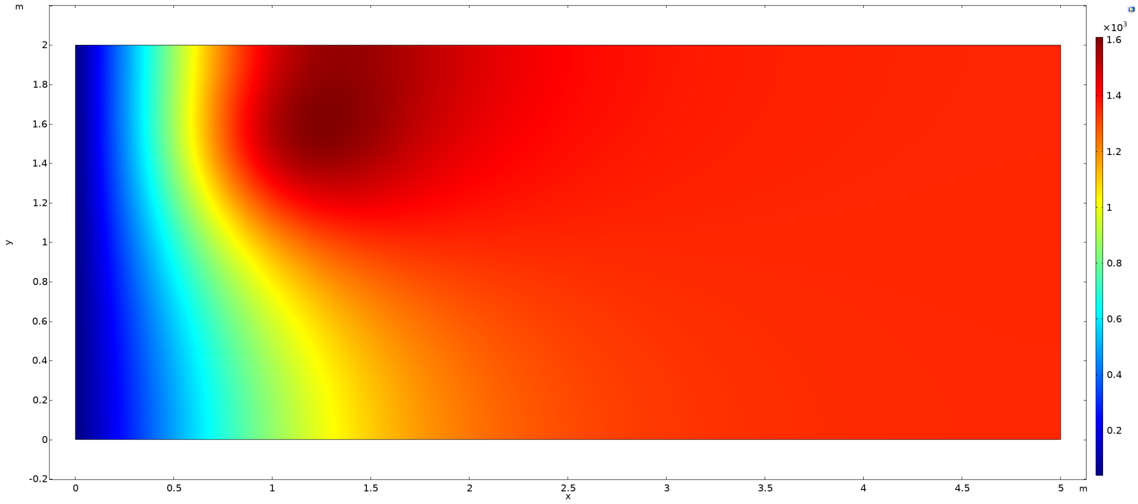


Figure 8: Temperature profile when 3 sides (top, bottom, right) are insulated and the left kept at 40 while we have a Gaussian profile heating at $(1, 1.5)$ with peak temperature 6000.

(c)

In this part we are inserting a hole with a cool liquid flushing through. It is located at $(3, 1)$ with a radius of 0.5 and has the boundary condition:

$$\frac{dT}{dn} = T_0 - T$$

where $T_0 = 20$. The solution can be seen in figure 9 below. The temperature at the right boundary is now reduced to 179.8 with still using the mesh size `Fine`.

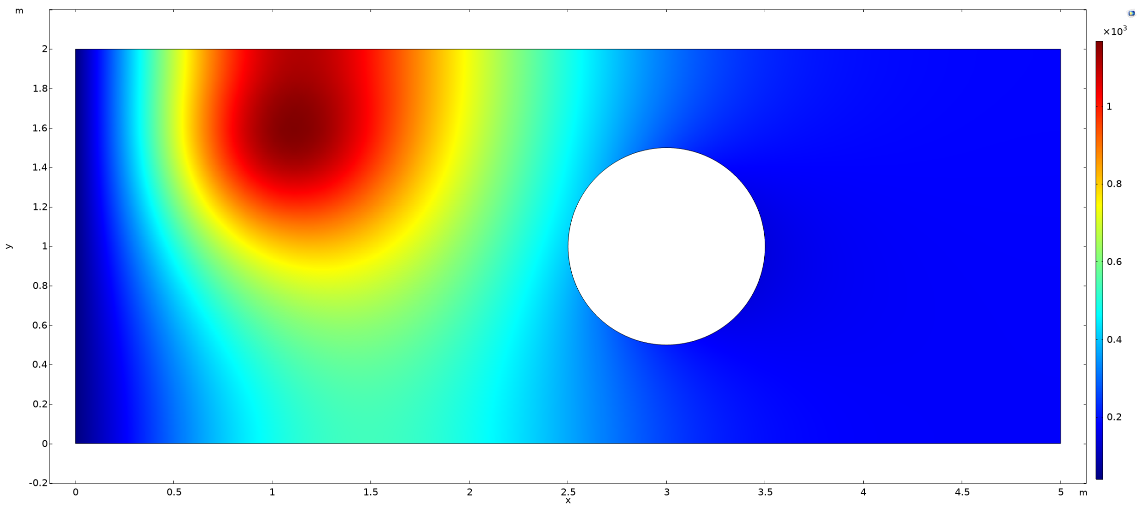


Figure 9: Temperature profile after adding a hole with cooling liquid. The whole has a radius of 0.5 and its center at $(3, 1)$.

(d)

For more efficient cooling, several holes are inserted. In this part we have in total of four smaller holes in the region around $(3, 1)$. They have a radius of 0.2 and their centers at $(3 \pm 0.25, 1 \pm 0.25)$ and the same boundary condition as in part (c). The solution plotted as a contour plot together with the mesh is shown in figure 10. The temperature in this case with 4 holes is again lowered, now 147.9 (mesh setting still at `Fine`).

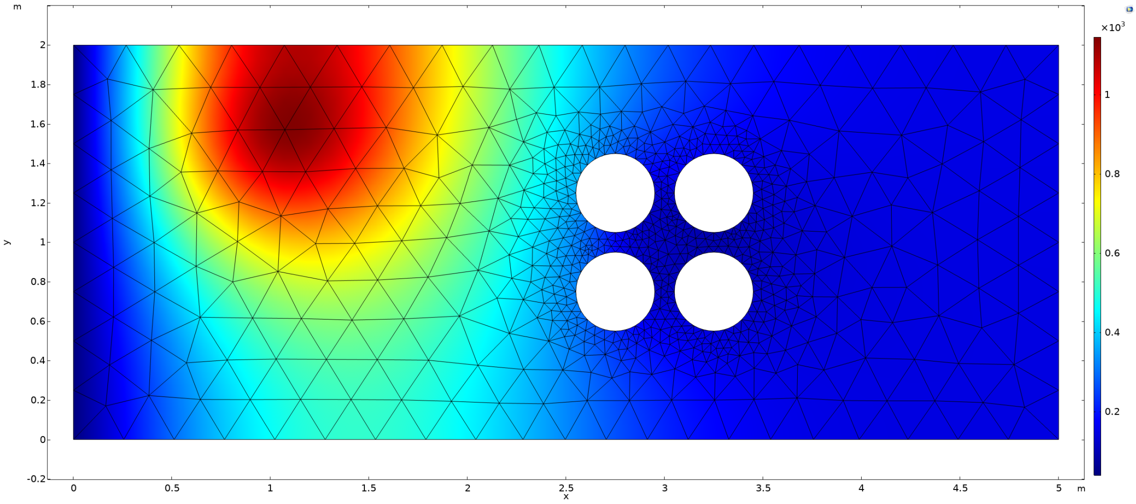


Figure 10: Temperature profile with four smaller holes of radius 0.2 and centers at $(3\pm0.25, 1\pm0.25)$ with cooling liquid passing in the same rate as before. The mesh is also plotted.

We have plotted the mesh in the figure which has smaller triangles in the regions closer to the holes and bigger ones further away. This can be explained by the changing geometry close to the circles as oppose to the other regions. Between the circles the distances are less than in the other parts of the rectangle so you can think of the length scale having to be refined in that area. This is visible by the smaller triangles there. The larger triangles correspond to the original length scale as we had in in part (a) and (b).

(e)

Our own configuration is shown in figure 11 where we used 25 smaller holes of radius 0.09. The bottom left circle is centered at $(2.6, 0.6)$ and the centers of the other circles are offset by 0.2 in x and y. Thus the smallest distance between the circles is 0.02. This configuration results in an average temperature on the right boundary of 78.9. Having many holes increases the surface area which allows a better cooling. The main difficulty is in making the radius of the circles not to small which would be negative on the cooling effect. A further improvement would be to increase the number of holes further while trying to keep the radius of the circles as large as possible. The physical relevance of our results is that it shows that a larger surface area allows for a better cooling.

If the length unit is in centimeters, i. e. the rectangle is of size 5 cm times 2 cm the holes would have a diameter of 1.8 mm and the separation between them is 0.2 mm. This would be difficult to manufacture as it would be likely that during the manufacturing process of cutting the holes in the metal sheet, the 0.2 mm between the holes just breaks away not leaving a space for the heat to travel through.

Limits in the numerical simulation are the conditioning of the finite element problem which scales with the step size but more so is the difference between the relative length scales important which occurs when generating the mesh between the holes. The mesh then need to be refined as the number of holes increases leading to growing number of triangles in order to get a reasonable solution.

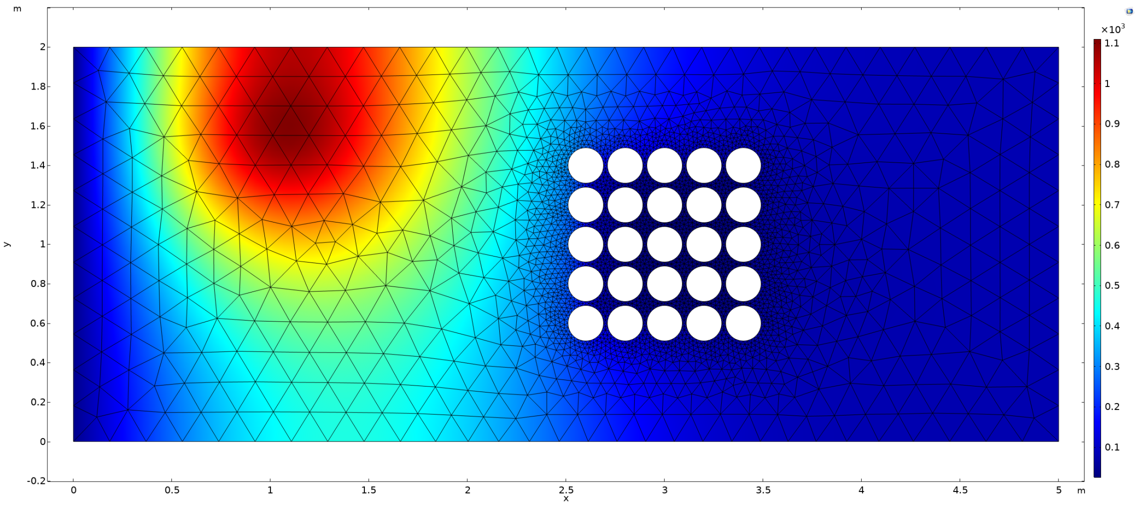


Figure 11: Temperature profile with our own configuration of 25 holes to decrease the average temperature on the right boundary below 100. The holes have a radius of 0.09 and their centers are 0.2 apart with the center circle being centered at $(3, 1)$.