Resolution of HANK models in discrete and continuous time

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November 18, 2025

Introduction

Heterogeneous agents (HA) models introduce household inequalities in DSGE models:

- There is a mass of households.
- Households face heterogeneity in labor endowment.
- To protect themselves against possible future low endowments, households can invest in assets.
- Intertemporal problem: how much to consume and to save today given that today's savings will be tomorrow's return.

Introduction

Advantages:

- More realistic.
- Household inequalities can alter the transmission of macroeconomic mechanisms.
- Macroeconomic policies can affect the distribution of household inequalities.

Literature Review

HA models are complex to compute and time-expensive to solve.

1. Solve the individual consumption and asset policy choices (dynamic, intertemporal problem).
   - Value function iteration (Huggett (1993))
   - Endogenous grid-point method (EGM) (Carroll (2006))
2. Execute this process for a given interest rate r, solve the model, update r and iterate.
3. Repeat for every time period t.

Literature Review

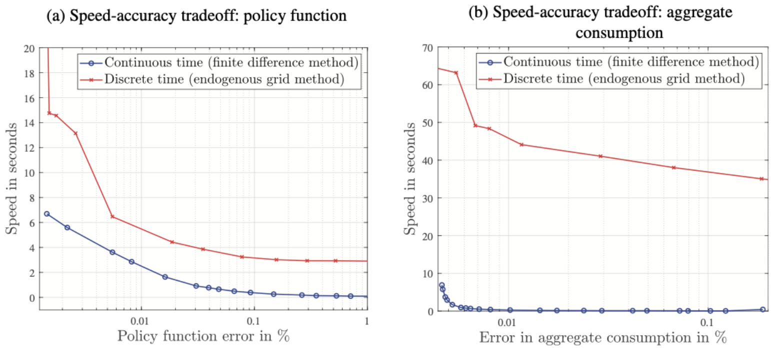


Figure 1: Computational speed and accuracy: continuous versus discrete time (EGM), figure from Achdou et al. (2022).

Literature Review

How to reduce this cost?

- Continuous-time method (Achdou et al. (2022)): **reduce the time needed to solve the household's problem**.
  - Simplify the household's problem by turning the problem "static".
  - Use Mean Field Games to reduce the amount of computation required.

- Discrete-time Sequence-Space Jacobian method (Auclert et al. (2021)): **bypass the household problem in dynamic time**.
  - The SSJ matrix directly knows how aggregates respond to shocks by exploiting the linearization of the household's problem around the steady-state.
  - Less effective for nonlinear shocks and loss of individual-level details during the transition.
  - Python library: SSJ toolkit.

Objective of the master thesis

Replicate the continuous-time Heterogeneous Agent New Keynesian (HANK) model with energy shortages of Pieroni (2023) using the SSJ method of Auclert et al. (2021) with the help of the SSJ toolkit.

Goals:

- Develop a methodological framework to translate continuous-time HA models into discrete-time using the SSJ method.
- Illustrate how to use the SSJ toolkit to implement HANK models with energy shortages.

Replication outline

Presentation and discretization of the model

Implementation of the model using the SSJ toolkit

Adjusting the calibration

Analysis of the results

Presentation of the model

- Households face idiosyncratic risk in labor productivity z_t, and earn labor endowment equal to $w_t n_t z_t$.
- Intertemporal choice between consumption and saving c_t and a_t.
- Intratemporal choice for households between energy consumption and another composite good, c_e and c_g. Households must all consume an incompressible level of energy $\underline{c}$, identical for all.
- Asset supply B is fixed and constant through time.

Presentation of the model

- Intermediate firms produce intermediate inputs using energy $E_{f,t}$ and labor $N:t$. Final firms earn dividends D_t that are distributed to households proportionally to productivity.
- Wage and price rigidities.
- Monetary policy follows a Taylor rule with 1 pillar: price inflation.
- Energy supply E_t is exogenous.
- The asset, labor and energy markets and the resource constraint close the economy.

Shock: 10% negative energy supply shock.

Discretization of the model

Table 1: Discretization principles

	Continuous time	**Discrete time**
Intertemporal variable growth	Differential equation: $\dot{x}_t = \frac{dx_t}{dt}$	Difference equation: $\dot{x}_t \approx x_{t+1} - x_t$
Future discounting	Instantaneous exponential discount rate $\rho \leq 0.1$	Per-period discount factor $\beta \geq 0.95$
Productivity process	Log-normal (continuous) process	Markov chain process (discrete)
Borrowing constraint	Non-binding: $a_t \geq -1$	Binding: $a_t \geq 0$

Implementing the model

About Pieroni (2023)'s continuous-time algorithm:

- The overarching structure of the economy is intuitive and similar to all macroeconomic models.
- Code is difficult to understand:
  - Complex mathematics (Trapezoidal integration).
  - Busy code: too many lines and variables.

The SSJ toolkit: seeks to simplify the task.

- Uses widely known language and libraries (Python, NumPy, SciPy, Numba).
- All computation is automatically handled by the toolkit's functions.
- The code is kept clean and structured.
- Available notebooks to guide the user.

Implementing the model

Using the SSJ toolkit

There are different steps to replicating a HA model in discrete time:

1. Decide how to structure the economy.
    - Divide the economy into blocks (Python functions). Their structure will form a Directed Acyclic Graph (DAG).
        - Notebook's usual template: Firm, households, MP, market clearing.
        - Use `drawdag` function for additional support.
    - Using the DAG, simultaneously choose the model's unknowns, targets, exogenous and endogenous variables.
        - Unknowns: r, w, p_e, Y instead of r, N, E_f for more flexibility.
        - Targets: asset market, NKWPC, energy market, resource constraint.

Implementing the model

Using the SSJ toolkit

2. Understand how to use the toolkit to specify the different kinds of blocks of the economy:
   - Household problem: `Hetblock` function (automatically solve policy choices and distribution, compute the aggregates).
   - Create asset and productivity grids: `markov_rouwenhorst` and `agrid` functions.
   - To compute specific individual-level variables from aggregate variables and vice versa: `Hetinput` and `Hetoutput` functions.
   - To use lagged variables: `Simple blocks`.

3. Define the model by gathering all blocks with `create_model`.

4. Compute the steady state and responses to shocks with functions `solve_steady_state`,`solve_impulse_linear`, `solve_impulse_nonlinear`.

Implementing the model

In summary: an assessment of the SSJ toolkit

Advantages:

- Fast and efficient:
  - On a mac M1 silicone chip laptop, maximum 3 minutes to find the steady-state.
  - 0.1 s to compute the linearized shock.
- Clean and user-friendly structure (compared to continuous-time implementation):
  - Clearly separated blocks in Python function form.
  - Complex computation is run in the background.
  - Proposes useful functions (decorators, making asset and productivity grids and processes, `drawdag`, `create_model`, `solve_impulse_linear`, `solve_impulse_nonlinear`)

Implementing the model

In summary: an assessment of the SSJ toolkit

Disadvantages: Need advanced understanding of Python to understand source of possible problems.

- Naming conventions.
- Convergence errors.

Implementing the model

Adjusting the calibration

We adjust the calibration to fit the discrete-time model.

Table 2: Model parameters in continuous and discrete time

Parameter	Description	Continuous-time value	Discrete-time value
Asset and Productivity grids			
n_a	Number of asset states	40	200
n_z	Number of productivity states	25	7
ρ_z	Mean reversion parameter / Persistence	0.0263	0.9
σ_z	Standard deviation	0.2	0.4
Households and policy			
ρ (β)	Discount rate	0.08	0.995

Implementing the model

Adjusting the calibration

We calibrate the model to obtain a set of targeted stationary statistics, and compare it to the continuous-time statistics:

Table 3: Model parameters in continuous and discrete time

Parameter	**Description**	**Continuous-time value**	**Discrete-time value**
Households and policy			
$\underline{c}$	Minimum energy consumption	0.0015	0.04
B	Net asset supply	5.8	5
E_s	Energy supply	0.067	0.124

Table 4: Target statistics of the steady-state equilibrium

Statistic	**Target**	**Continuous-time**	**Discrete-time**
Average wealth-to-income share	4.2	4.4178	4.9315
Total energy as a share of output	0.04	0.0403	0.0472
Average energy expenditure share of consumption	[0.06:0.12]	0.0892	0.076
Gini coefficient	0.35	0.4812	0.4791
Average marginal propensity to consume	[0.15:0.25]	0.1069	0.1288

Implementing the model

Adjusting the calibration

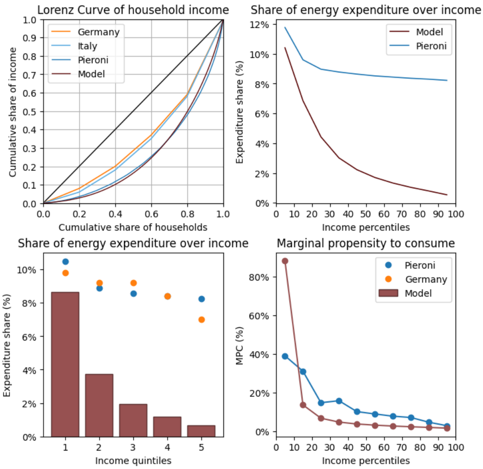
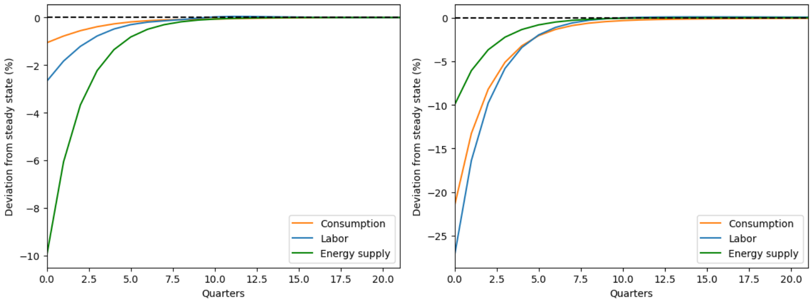


Figure 2: Steady-state income distribution, energy consumption and MPCs across the income distribution.

Results: response to a large energy supply shock



(a) IRF of labor and consumption, continuous time

(b) IRF of labor and consumption, discrete time

Figure 3: IRFs of consumption and labor to a 10% negative energy supply shock

Results: response to a large energy supply shock

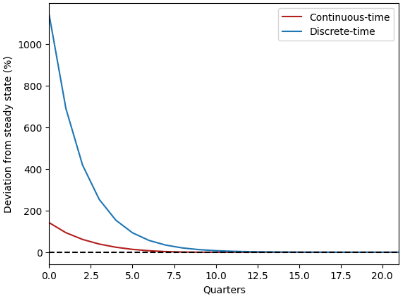


Figure 4: IRF of the price of energy in both continuous and discrete-time models

Conclusion

We were able to build a methodology for replicating a continuous-time model with the SSJ method. The results present responses of correct sign.

However, there are still several challenges:

- Inaccurate stationary distribution of inequalities.
- High magnitude of responses compared to continuous-time model.
- High sensibility of dynamic results to calibration.
- Difficulty to use the function `solve_impulse_nonlinear`.
  - Increasing divergence of the residuals.
  - Unknowns and targets might be too nonlinear and interdependent with each-other.

Conclusion

Possible solutions:

- Declaring different unknowns and targets to reduce non-linearity and interdependency.
- Obtaining a better calibration to faithfully replicate the original results.
- Studying the effect of smaller shocks firsts.

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