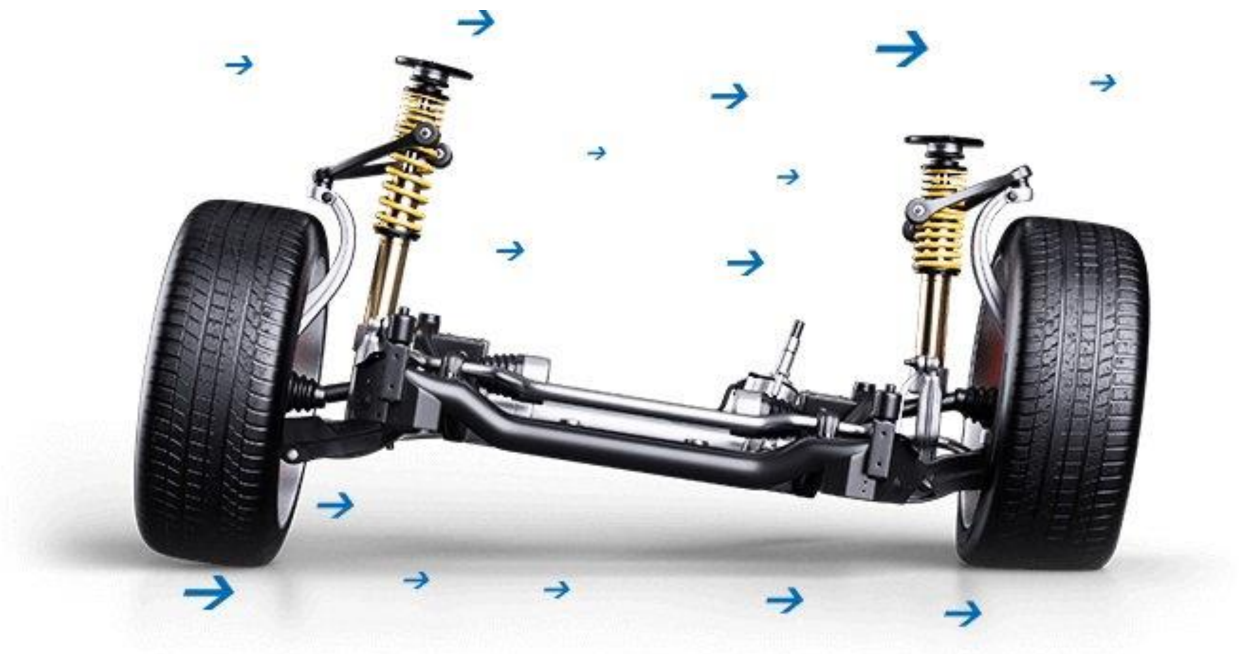
UNIVERSITÀ DELLA CALABRIA

Dipartimento di INFORMATICA , MODELLISTICA , ELETTRONICA e SISTEMISTICA

Master In Robotics and Automation

DAMPING OF VEHICLE ROLL DYNAMICS BY ACTIVE STEERING CONTROL

Final Project of Vehicle Control Mod_ 1



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2. Introduction

Anti-roll control, also known as sway control or stability control, is used in vehicles to improve stability and reduce lateral roll or body lean during driving manoeuvres.

Side sway occurs when a vehicle turns a curve or performs quick evasive manoeuvres. During these actions, centrifugal force exerts an outward force on the vehicle, which can cause the chassis to lean to one side. This can affect the stability of the vehicle and have negative consequences, such as loss of traction on the wheels, loss of control and risk of rollover.

Anti-roll control uses different mechanisms to counteract roll. One of the most common methods is through the use of stabilizer bars or anti-roll bars. These bars are connected to the front and rear axles of the vehicle and are used to link the opposing suspensions. When the vehicle leans to one side, the stabilizer bar transfers part of the suspension load from the outside wheel to the inside wheel, thus reducing roll and maintaining vehicle stability.

In addition to anti-roll bars, many modern vehicles also use electronic stability control systems, such as Electronic Stability Control (ESC) or Dynamic Stability Control (DSC). These systems use sensors to constantly monitor the behaviour of the vehicle and, if an imbalance or loss of traction is detected, they can selectively apply brakes to individual wheels and adjust engine torque to stabilize the vehicle.

In short, anti-roll control is used in vehicles to reduce lateral roll and improve stability during driving manoeuvres, helping to maintain vehicle control and safety.

When resumed, the anti-roll control is used in the vehicles to reduce lateral balance and improve stability during driving manoeuvres, which helps maintain control and safety of the vehicle.

Three main schemes concerned with the possible active intervention into the vehicle dynamics have been proposed: active anti-roll bars, active steering, and active brake.

A stability control system focusing on rollover prevention by active steering is presented. An actuator sets a small auxiliary front wheel steering angle in addition to the steering angle commanded by the driver as it is shown in the figure 1. The aim is to decrease the rollover risk due to the transient roll overshoot of the vehicle when changing lanes or avoiding obstacles. The advantage of the active steering control is that it affects the lateral acceleration directly. However, the active steering control influences, not only the roll dynamics of the vehicle, but also modifies the desired path of the vehicle, so it affects the yaw motion. Then, for improve the control also a gain scheduling method, which considers the change in vehicle velocity.

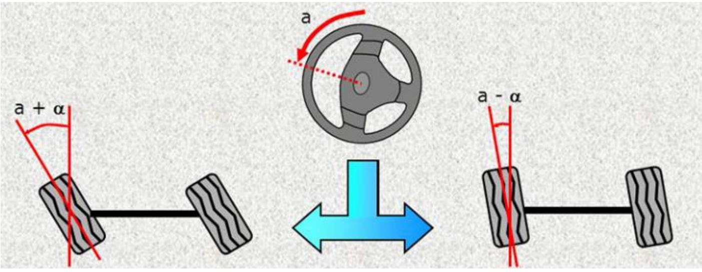


Fig1. Active steering.

The primary purpose of rollover controls is to detect and counteract conditions that may lead to a rollover. They use sensors to constantly monitor vehicle motion, lateral acceleration, lean, and other stability-related parameters. If a risk of rollover is detected, the system activates corrective measures to keep the vehicle in a safe condition.

Active steering control is an important part of these systems. It consists of adjusting the direction of the wheels to counteract the forces that contribute to rollover. When a risk of rollover is detected, the system applies a slight spin to the wheels to change the trajectory of the vehicle and reduce the possibility of rollover. This is accomplished by applying additional torque through electric or hydraulic power steering.

A general overview of how a four-wheel steering system works:

The steering system of a four-wheeled vehicle is made up of various components that work together to allow control and manoeuvrability of the vehicle.

When the driver turns the steering wheel, the movement is transmitted through the steering column and the steering box. The tie rods transmit that movement to the knuckles of the front wheels, which causes the wheels to spin in the desired direction.

In a rear-wheel-drive vehicle, the rear wheels are normally not connected to the steering system and follow the direction imposed by the front wheels due to the suspension geometry. However, on some performance vehicles or vehicles with four-wheel-steer systems, there may be an additional system to control the rear wheels.

There are two main types of rear wheel steering systems:

Mechanical System: In some older or simpler 4WS systems, mechanical linkages are used to connect the front and rear wheels. As the driver turns the steering wheel, the mechanical linkage transfers the motion to the rear wheels, causing them to turn in the desired direction.

Electronic System: Modern 4WS systems often use electronic actuators to control the rear wheel steering. The control unit sends signals to the electronic actuators located at the rear wheels, which adjust the steering angles of the rear wheels based on the calculated values. This can be achieved through electric motors or hydraulic systems. The control unit continuously monitors the vehicle's speed, lateral acceleration, and other parameters to determine the optimal steering angles.

In this project, several controllers for rollover control using active steering were implemented. In the first part, the mathematical model is described. Then, it was developed the feedback state controller using optimal control techniques to compare them. Finally, the centre of gravity heigh of sprung mass parameters and the speed will be changed to observe the robustness of the system.

3. Modelling

2.1. System model

Figure 2 illustrates a simple model of the vehicle which will be used for the considerations in this project. This multibody system consists of two rigid bodies. Body 1 with mass m_1 is composed of the front and rear axles, the four wheels and the frame. Body 2 is the sprung mass m_2. The position of the vehicle's roll axis depends on the suspension kinematics. The model assumes a fixed roll axis parallel to the road plane in the longitudinal direction of the vehicle at a height h_R above the street. Hence de body 2 is linked to body 1 with a one degree of freedom joint. The roll movement of the roll body is damped and sprung by suspensions and stabilizers with an effective roll damping coefficient $d_{\dot{\emptyset}}$ and roll stiffness $c_{\emptyset}$. The CG of body 1 (CG1), is assumed to be in the road plane below CG2, since its contribution to the roll movement is considered negligible. Linear tire characteristics were assumed. All model parameters are compiled in Tab. 1.

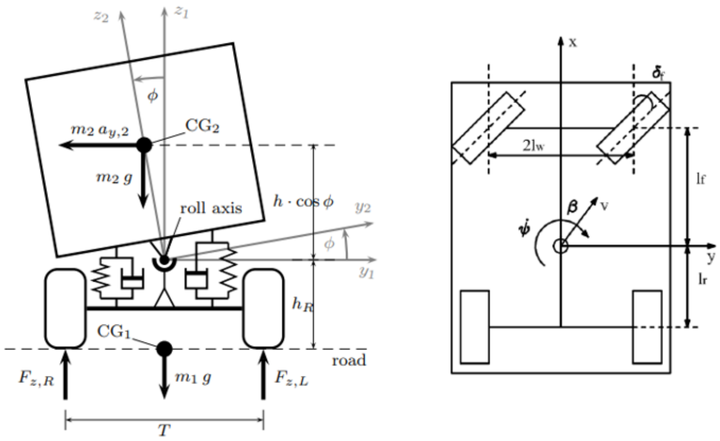


Fig2. Vehicle rollover model.

C_f = 582 [kN/rad]	Front cornering stiffness
C_r = 783 [kN/rad]	Rear cornering stiffness
c_ϕ = 457 [kN m/rad]	Roll stiffness of passive suspension
$d_{\dot\phi}$ = 100 [kN/rad]	Roll damping of passive suspension
g = 9.81 [m/s^2]	Acceleration due to gravity
h_R = 0.68 [m]	Height of roll axis over ground
h = 1.15 [m]	Nominal height of CG2 over roll axis
$J_{2,x,}$ = 24201 [kg m^2]	Roll moment of inertia, sprung mass
J_z = 34917 [kg m^2]	Overall yaw moment of inertia
l_f = 1.95 [m]	Distance front axle to CG1
l_r = 1.54 [m]	Distance rear axle to CG1
m = 14300 [kg]	Overall vehicle mass
m_2 = 12487 [kg]	Sprung mass
μ = 1	Road adhesion coefficient
T = 1.86 [m]	Track width

Table 1. Parameters values

Systems equations describing the nonlinear model are as follows:

$$m\dot\beta - m_2 h\ddot\phi\cos\phi = m_2 h r^2\sin\phi - mvr + m_2 h\dot\phi^2\sin\phi + F_{yf} + F_{yr}$$

$$J_z\dot r = -r\left(m_2 h^2\dot\phi\sin 2\phi + m_2 h\phi\sin\phi\right) + l_f F_{yf} - l_r F_{yr}$$

$$m_2 h\dot\beta\cos\phi + \left(J_{2,x} + m_2 h^2\right)\ddot\phi = -m_2 hv\cos\phi + \frac{m_2 h^2}{2m} r^2\sin 2\phi - d_{\dot\phi}\dot\phi + m_2 g h\sin\phi - c_\phi\phi$$

Where m is the total vehicle mass, h is the nominal height of CG2 from the roll axis, v is the vehicle speed, J_z is the overall yaw moment of inertia, $J_{2,x}$ is the roll moment of inertia of the suspended mass, lf is the distance from the front axle to CG1, lr is the distance from the rear axle to CG1, F_{yf} is the lateral force of the front tires, F_{yr} is the lateral force of the rear tires, β is the lateral velocity, is the roll angle of the suspended mass relative to the unsprung mass, is the roll of the suspended mass relative to the unsprung mass, γ is the yaw.

Under the following assumptions:

- Constant vehicle speed.
- Neglect all second- and higher-order angular terms.
- Neglect all angular terms of travel.
- Apply the approximation of working with small angles:

From the three nonlinear equations of the model, the three corresponding linear equations:

- $m\dot{\beta} - m_2 h\ddot{\phi} = -mvr + F_{yf} + F_{yr}$
- $J_z\dot{r} = l_f F_{yf} - l_r F_{yr}$
- $m_2 h\dot{\beta} + (J_{2,x} + m_2 h^2)\ddot{\phi} = -m_2 hvr - d_{\dot{\phi}}\dot{\phi} + m_2 gh\phi - c_{\phi}\phi$

The lateral force of the front wheels F_{yf} and rear wheels F_{yr} can be expressed as follows:

$$F_{yf} = c_f\mu(\delta - \theta_{vf})$$
$$F_{yr} = c_r\mu(-\theta_{vr})$$

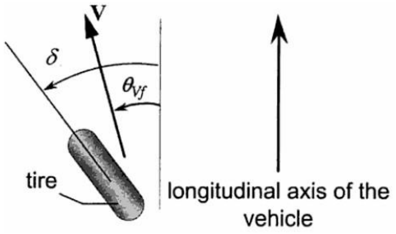


Fig3. Tire slip angle

Where c_f is a constant indicates the front stiffness in cornering, c_r is a constant indicates the rear stiffness in cornering, μ is the coefficient of road grip, θ_{vf} is the speed angle of the front wheels, and $\theta_{vr}$is the speed angle of the rear wheels. From the following relationships:

$$\tan(\theta_{vf}) = \frac{\beta + l_f r}{v}$$
$$\tan(\theta_{vr}) = \frac{\beta - l_r r}{v}$$

It is possible to derive the terms θ_{vf} and θ_{vr} assuming we consider small angles:

$$\theta_{vf} = \frac{\beta + l_f r}{v}$$
$$\theta_{vr} = \frac{\beta - l_r r}{v}$$

Substituting into the expressions of F_{yf} and F_{yr} it is obtained:

$$F_{yf} = c_f\mu\left(\delta - \frac{\beta + l_f r}{v}\right)$$
$$F_{yr} = c_r\mu\left(-\frac{\beta - l_r r}{v}\right)$$

From which the new linear equations of the model are:

- $m\dot{\beta} - m_2 h\ddot{\phi} = -mvr + c_f\mu\left(\delta - \frac{\beta + l_f r}{v}\right) + c_r\mu\left(-\frac{\beta - l_r r}{v}\right)$
- $J_z\dot{r} = l_f c_f\mu\left(\delta - \frac{\beta + l_f r}{v}\right) - l_r c_r\mu\left(-\frac{\beta - l_r r}{v}\right)$
- $m_2 h\dot{\beta} + (J_{2,x} + m_2 h^2)\ddot{\phi} = -m_2 hvr - d_{\dot{\phi}}\dot{\phi} + (m_2 gh - c_{\phi})\phi$

Thus, the three linear differential equations of the model are:

- $m\dot{\beta} - m_2 h\ddot{\phi} = -(c_f + c_r)\frac{\mu}{v}\beta - (c_f l_f - c_r l_r)\frac{\mu}{v}r - mvr + c_f\mu\delta$
- $J_z\dot{r} = -(c_f l_f - c_r l_r)\frac{\mu}{v}\beta - (c_f l_f^2 - c_r l_r^2)\frac{\mu}{v}r + c_f l_f\mu\delta$
- $m_2 h\dot{\beta} + (J_{2,x} + m_2 h^2)\ddot{\phi} = -m_2 hvr - d_{\dot{\phi}}\dot{\phi} + (m_2 gh - c_{\phi})\phi$

From the linear equations of the model, the corresponding representation in the space of states can be derived. The following vector of state and input was chosen for the rollover model:

$$x = \begin{bmatrix} \emptyset \\ \beta \\ \gamma \\ \dot{\emptyset} \end{bmatrix}, u = \delta$$

The equations of the model as a function of state variables will be:

- $\dot{x}_1 = x_4$
- $m\dot{x}_2 - m_2 h\dot{x}_4 = -(c_f + c_r)\frac{\mu}{v}x_2 - (c_f l_f - c_r l_r)\frac{\mu}{v}x_3 - mvx_3 + c_f\mu u$
- $J_z\dot{x}_3 = -(c_f l_f - c_r l_r)\frac{\mu}{v}x_2 - (c_f l_f^2 - c_r l_r^2)\frac{\mu}{v}x_3 + c_f l_f\mu u$
- $m_2 h\dot{x}_2 + (J_{2,x} + m_2 h^2)\dot{x}_4 = (m_2 gh - c_{\phi})x_1 - m_2 hvx_3 - d_{\dot{\phi}}x_4$

To simplify the equations the following substitutions are proposed:

$$Y_\beta = (c_f + c_r)\mu \qquad N_\beta = (c_f l_f - c_r l_r)\mu \qquad N_r = (c_f l_f^2 - c_r l_r^2)\mu$$
$$I_{\dot{\phi}} = (J_{2,x} + m_2 h^2) \qquad \sigma = (m_2 gh - c_{\phi})$$

The state space representation of the system is:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -hm_2 \\ 0 & 0 & J_z & 0 \\ 0 & -m_2 h & 0 & I_{\dot{\emptyset}} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{Y_\beta}{v} & -\left(\frac{N_\beta}{v} + mv\right) & 0 \\ 0 & -\frac{N_\beta}{v} & -\frac{N_r}{v} & 0 \\ \sigma & 0 & -m_2 hv & -d_{\dot{\emptyset}} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -hm_2 \\ 0 & 0 & J_z & 0 \\ 0 & -m_2 h & 0 & I_{\dot{\emptyset}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ c_f\mu \\ 0 \\ c_f l_f\mu \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The input for the system is considering the steering angle. This angle is the sum of the steering angle given by the driver δ_s and the steering angle given by the controller δ_c. As it is shown in the following equation:

$$\delta = \delta_s + \delta_c$$

In our model the steering angle δ_s is considering the disturbance and the matrix that represent this behavior in our model is W but in this case it affects in the same way that the matrix B because the above relation.

Then, the state space representation becomes:

$$\begin{cases} \dot{x} = A \begin{bmatrix} \emptyset \\ \beta \\ \gamma \\ \dot{\emptyset} \end{bmatrix} + B\delta_c + W\delta_s \\ y = C \begin{bmatrix} \emptyset \\ \beta \\ \gamma \\ \dot{\emptyset} \end{bmatrix} \end{cases}$$

Rollover coefficient:

From the equilibrium of vertical forces, i.e. gravitation forces of body 1 and 2, $m_1 g$ and $m_2 g$ respectively, and wheel vertical loads $F_{Z,L}$ and $F_{Z,R}$ (front and rear), and balance of moments w.r.t. CG1 (see Fig. 2) a rollover coefficient is defined as:

$$\begin{aligned} R &= \frac{F_{Z,R} - F_{Z,L}}{F_{Z,R} + F_{Z,L}} \\ &= \frac{2m_2}{mT}\left((h_R + h\,cos\emptyset)\frac{a_{y,2}}{g} + h\,sin\emptyset\right) \end{aligned}$$

with $a_{y,2}$ derived from the dynamical model, i.e.

$$a_{y,2} = \dot{\beta} + v\gamma - h\ddot{\emptyset}$$

When $F_{Z,R} = 0$ ($F_{Z,L}$=0) the right (left) wheels lift off the road and the rollover coefficient takes on the value R = −1 (R = 1). For straight driving on a horizontal road for the tire vertical loads it holds that $F_{Z,R} = F_{Z,L}$ which means that R = 0. Note, that the vehicle model is only valid until one or more wheels lift off the road.

Assuming $m_1 \ll m_2$ and ∅ to be small, the equation results in:

$$R \approx \frac{2(h_R + h)\frac{a_{y,2}}{g}}{Tg} + \frac{2h\emptyset}{T}$$

The objective of the control is try to maintain rollover coefficient value to zero. As it can be seen in the above equation this coefficient depends directly on all the states of our system, consequently all the states must try to become zero.

2.2. Modelling in MATLAB/Simulink environment

To carry out the simulation, a Simulink diagram was designed, which can reproduce the model of the system:

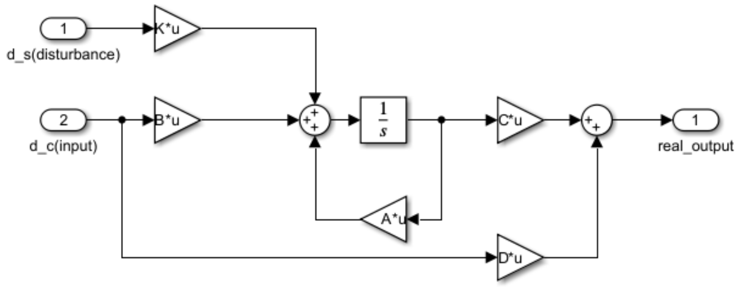


Fig.4 Simulink diagram of the Model Dynamics

Disturbance signals

Front steering angle corresponds to a disturbance signal. The signals considered as system disturbances is shown in the Fig.5 and the movement desired is shown in the Fig.6.

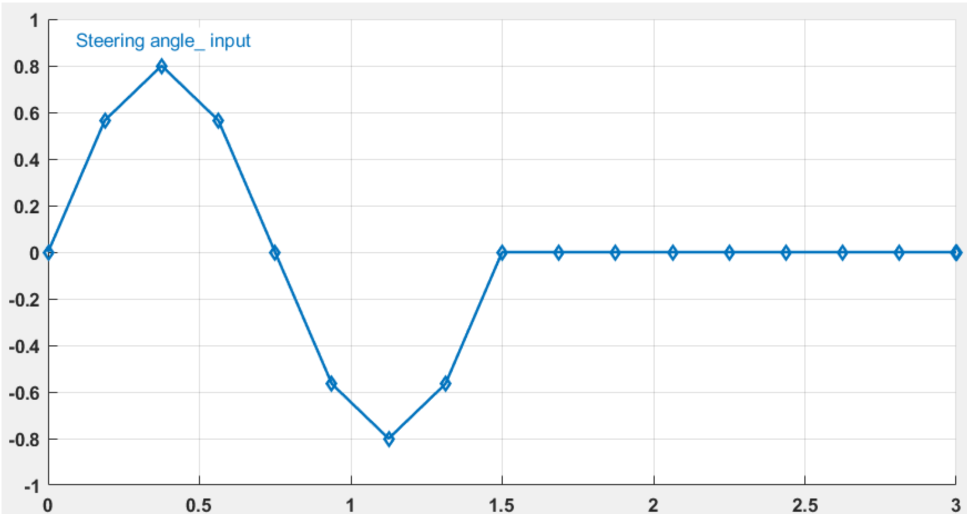


Fig.5 Front steering angle disturbance.

Fig.6 Movement desired.

4. System analysis

System analysis allows us to understand the behavior and characteristics of a system. By studying the system's inputs, outputs, and internal dynamics, we can gain insights into how the system responds to different stimuli and how its components interact with each other. This understanding is crucial for designing effective control strategies and optimizing system performance. Also helps in evaluating the performance of a system. By analyzing its response to inputs, we can assess whether the system meets the desired specifications and performance requirements. This evaluation can involve analyzing stability, transient response, steady-state behavior, robustness, and other performance metrics.

3.1. Model analysis

In this part, it is analyzed the eigenvalues of the dynamic matrix A corresponding to state space representation of the system. To calculate the eigenvalues, the roots of the characteristic polynomial are computed.

$$\det(\lambda I - A) = 0$$

The eigenvalues are:

$$\lambda_1 = -1.5494 + 8.091i$$
$$\lambda_2 = -1.5494 - 8.091i$$
$$\lambda_3 = -0.9162 + 2.1972i$$
$$\lambda_2 = -0.9162 - 2.1972i$$

The eigenvalues are complex conjugates. The real part $\mathbb{R}e(\lambda i)$<0, i=1,2,3,4. Therefore the system is stable

3.2. Free evolution

This evolution corresponds to the behavior of the states and outputs when the input is null, and the system only depends on the initial conditions. The following figures shows the dynamic behavior of the system starting from different initial conditions for roll angle, slip angle, yaw rate, and roll rate are:

$$x_0 = [0.02, 0.05, 0.01, 0.06]$$

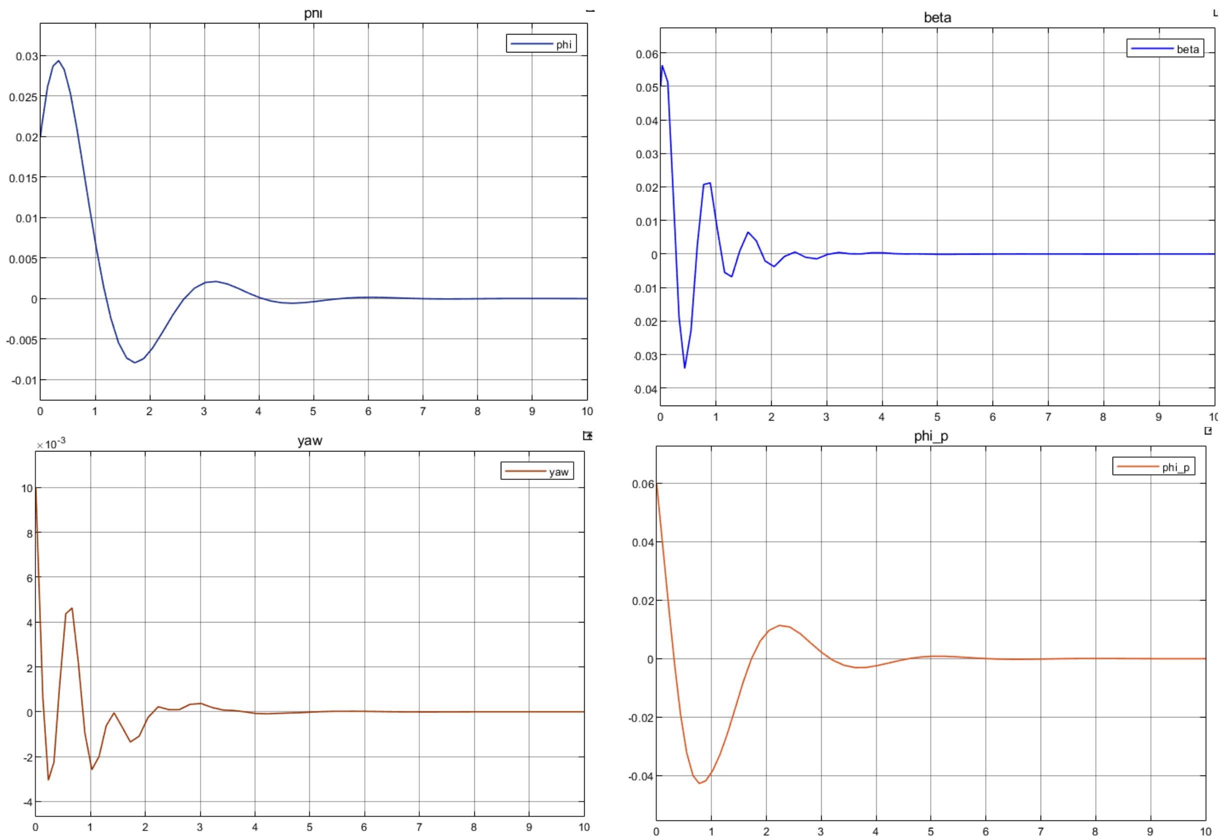


Fig.7. Free response for x_0 as initial condition

As we can see in the picture below the state tends to zero “Equilibrium point” after a few seconds, this is because the system is asymptotically stable.

3.3. Force response

The forced response focuses on how the system responds and adjusts to a specific external force or input and initial conditions equal to 0. It is analyzed the case in which a step of amplitude equal to 0.0524 [rad] is applied to the input of the steer angle. That means an angle of 3 degree.

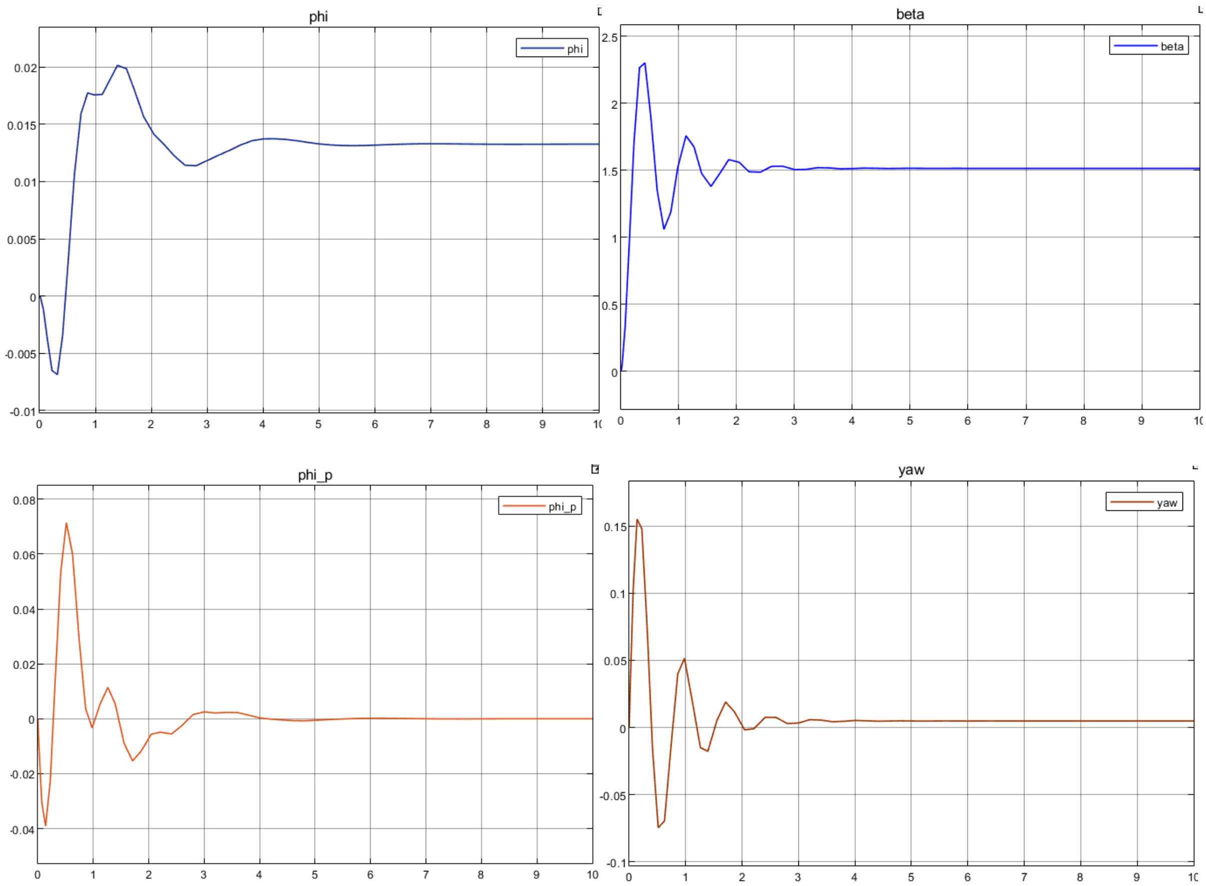


Fig.8. Force response for a step input of 0.0524 [rad] corresponding to δ.

The states react to inputs distinct from zero but tend toward a constant value in steady-state conditions.

3.4. Structural properties

A fundamental step is to analyze the structural properties of the dynamical system. In particular, we will pay attention to the reachability and observability of the system. Reachability is input-state properties and related to the control law and therefore to the position of the actuators within the system to obtain the desired results, from it. While observability is closely related to the sensor side and are properties exploited for the design of state observers, in fact it is input-output property. Structural properties of LTI systems are necessary and sufficient conditions for the synthesis of control laws and observers.

Reachability

The reachability concept refers to the real possibility of being able to modifier the estate of the system acting on its inputs so that it assumes arbitrary conditions.

For this reason, it is studied if u is well located, and it is sufficient to get complete reachability of the system and satisfied any control prescriptions that we want. To know if the system is reachable is necessary to compute the reachability matrix which is related to matrices A and B in the following way:

$$R = [B\ AB\ A^2B \dots A^{n-1}B] \in R^{n\ x\ n.m}$$

The system is completely reachable if and only if the rank of the reachable matrix is full rank, to certificate this, the rank command was used resulting in rank(R)=4. It means that the system is completely reachable.

Reachability concept refers to the real possibility of being able to modifier the estate of the system acting on its inputs so that it assumes arbitrary conditions. To know if the system is reachable in $[0\ ,T]$ is necessary to compute the reachability matrix which is related to matrices A and B in the following way.

$$R = [B\ AB\ A^2B \dots A^{n-1}B] \in R^{n\ x\ n.m}$$

Using *ctrb* command in MATLAB is possible to get the matrix R which is:

$$R = 1 * 10^5 \begin{bmatrix} 0 & -0.0001 & 0.0005 & 0.0067 \\ 0.0003 & 0.0187 & -0.0784 & -1.0249 \\ 0.0003 & -0.0010 & -0.0190 & 0.1245 \\ -0.0001 & 0.0005 & 0.0067 & -0.0612 \end{bmatrix}$$

The system is completely reachable because the reachable matrix is full rank $rank(R) = 4$.

Lastly, Reachability and Controllability in CT-case are the same, complete Reachability implies complete controllability and vice versa.

Observability

Observability is the property under which given a sequence of inputs $u(\mathrm{t})$ and sequence de outputs $y(\mathrm{t})$ it is possible estimate the initial condition $x(t_0)$ of the system.

It is said that a system is complete observable if the knowledge of inputs and outputs sequence allow to compute the initial state without errors. Besides, there are not indistinguishable states in the future.

The observability matrix is linked to A and B matrix in the following way:

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \in R^{p.n\ x\ n}$$

In the case of study, the observability matrix is the following:

$$O = 1 * 10^3 \begin{bmatrix} 0.0010 & 0 & 0 & 0 \\ 0 & 0.0010 & 0 & 0 \\ 0 & 0 & 0.0010 & 0 \\ 0 & 0 & 0 & 0.0010 \\ 0 & 0 & 0 & 0.0010 \\ -0.0058 & -0.0012 & 0.0580 & -0.0018 \\ 0 & -0.0011 & -0.0019 & 0 \\ -0.0057 & 0.0004 & 0.0007 & -0.0018 \\ -0.0057 & 0.0004 & 0.0007 & -0.0018 \\ 0.0172 & -0.0642 & -0.1821 & -0.0003 \\ -0.0057 & 0.0004 & 0.0007 & -0.0018 \\ 0.0064 & 0.0035 & -0.0610 & 0.0020 \\ 0.0080 & -0.0020 & 0.0214 & -0.0032 \\ 0.3712 & 0.2787 & -3.3669 & 0.1346 \\ -0.0317 & 0.0649 & 0.3219 & -0.0036 \\ 0.0300 & -0.0228 & -0.1616 & 0.0175 \end{bmatrix}$$

A linear system is complete reachable if and only if the observability matrix is full rank. Using MATLAB is verified that the $rank(O) = 4$, It means that the system is complete observable.

Constructability is a weaker condition than observability and consist in given a sequence of inputs $u(\mathrm{t})$ and $y(\mathrm{t})$ it is possible estimate $x(t)$ of the system. However, complete observability implies complete constructability. It can be said that in CT-case the constructability and observability are the same concept.

5. Linear Matrix Inequalities

Linear Matrix Inequalities is an inequality involving linear matrices and is represented in matrix form. It is used to set constraints on system matrices and their linear combinations, allowing controllers to be analysed and synthesized to meet certain performance and robustness requirements.

LMIs are used in various areas of control theory, such as: Robust control, stability, analysis of discrete and continuous time systems and others.

LMIs are solved using convex optimization techniques such as semi-definite programming (PSD) or linear programming (PL), ensuring computational efficiency and the ability to find feasible solutions. Are used in control theory to formulate constraints and conditions in the design and analysis of control systems. They are a powerful tool for addressing robust control, stability, and optimization problems in dynamic systems.

Linear Matrix Inequality is defined as an affine combination of matrices constrained to be positive or negative definite, formally.

Given a set of square and symmetric matrices $F_i = F_i^T \in \mathbb{R}^{nxn}, i = 0,1 \dots m$ and a vector $x \in \mathbb{R}^m$, a LMI is defined as:

$$F(x) = F_0 + \sum_{i=1}^{m} (x_i F_i) < 0 \; or > 0$$

4.1. Stability using LMI.

A way to prove stability is using *Lyapunov Algebraic equation*. This theorem state that the system $\dot{x} = Ax$ is asymptotical stable if and only if, for some symmetrical matrix $Q = Q^T > 0$, the ALE admits a unique solution $P = P^T > 0$ such that:

$$A^T P + PA = -Q$$

Using LMI and Lyapunov stability criterion the problem can be stated as following:

Let $V(x) = x^T P x \,, P \in \mathbb{R}^{nxn}, P = P^T > 0$ be a positive define quadratic Lyapunov function. The sufficient condition for asymptotical stability is $\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^n / 0_x$

In LIM term must be verified the conditions:

$$\exists P = P^T > 0 \qquad st \qquad A^T P + PA < 0$$

Using Schur Complement it is translated in the following LMI:

$$\begin{bmatrix} -P & 0 \\ 0 & A^T P + PA \end{bmatrix} < 0$$

Which is a feasibility problem of LMI with P unknown. Taking advantage of MATLAB, it is possible to determine a matrix P that satisfies the Lyapunov condition. The result obtained is:

$$P = 1.0e - 10 \begin{bmatrix} -0.7340 \\ -0.0313 \\ -0.0017 \\ -0.0004 \end{bmatrix}$$

To proof if the matrix obtained fulfil the condition for stability it is replace the matrix P:

$$A^T P + PA < 0$$

The eigenvalues resulting of this matrix are:

$$\text{eigs}(A^T P + PA) = 1.0\text{e} - 10 \begin{bmatrix} -0.7340 \\ -0.0313 \\ -0.0017 \\ -0.0004 \end{bmatrix}$$

As shown, the eigenvalues of the matrix are negative, therefore the matrix is negatively defined. Hence, the system is asymptotically stable as we discovered in the previous analysis with the classical technique.

4.2. R-region

The dynamic properties of LTI systems are determined by the location of the poles of its transfer function in the complex plane. There is a relationship between the pole location and damping ratio, rise time and settling time. Locate the poles of the closed-loop system should achieve a good performance. The constraints can be expressed as a region in the complex plane where the closed-loop poles should be located. A typical pole conical region $S(\alpha, \mathrm{r}, \theta)$ in the complex plane is shown in Fig. 9. The condition that the poles of a system are located within a given region in the complex plane can be formulated as an LMI constrains.

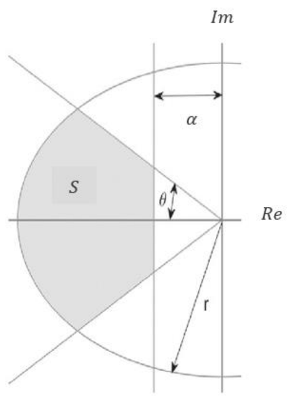


Fig.9 Region $S(\alpha, \mathrm{r}, \theta)$, composed by the intersection of three convex LMI regions.

Where:

α: It is related with the setting time, is the largest of the real parts of the numbers belonging to the cone. In particular, in our case we obtain a $\alpha = 0.9162$.

$$\alpha = -max\big(Re(\lambda)\big);\ \lambda \in\ sp(A)$$

θ: It is related to the overshoot and is the largest of the phases of these numbers. The value obtained in m is $\theta = 1.9659$.

$$\theta = max\big(\angle\,(\lambda)\big);\ \lambda \in\ sp(A)$$

r: It is related to the natural pulsation. Is the largest module. The value obtained is $r = 2.3806$.

$$r = max(||\lambda||);\ \lambda \in sp(A)$$

4.3. R-Stability Lyapunov criteria

Let $S(\alpha, r, \theta)$ be an LMI region, the matrix A is R-Stabile if and only if $\exists\ P = P^T > 0$ such that the following equation is satisfied.

$$\begin{cases} [PA + A^{\mathrm{T}}P + 2\alpha P] < 0 \\ \begin{bmatrix} -rP & PA \\ A^TP & -rP \end{bmatrix} < 0 \\ \begin{bmatrix} \sin(\theta)(PA + A^{\mathrm{T}}P) & \cos(\theta)(PA - A^{\mathrm{T}}P) \\ \cos(\theta)(-PA + A^{\mathrm{T}}P) & \sin(\theta)(PA + A^{\mathrm{T}}P) \end{bmatrix} < 0 \end{cases}$$

Using MATLAB, it is identified the matrix $P = P^T > 0$ such that the above condition is satisfied, then it confirms that the values of ($\boldsymbol{\alpha}$,$\boldsymbol{r}$,$\boldsymbol{\theta}$) are correct.

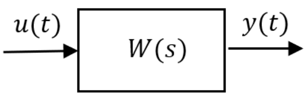
$$P = 1.0\mathrm{e} - 0.3 \begin{bmatrix} 0.1552 \\ 0.0253 \\ 0.0001 \\ 0.0002 \end{bmatrix}$$

4.4. Norms of a System.

System norms are mathematical tools used to quantify and analyze different aspects of systems. They are measurements that provide information about the properties and characteristics of the system under study.

System norms provide quantitative measures that help understand and compare different aspects of systems behavior. They are useful tools in the analysis, design, and optimization of control systems, allowing to evaluate the stability, convergence, sensitivity, performance and other relevant aspects of the systems. In addition, the standards can be used to establish design criteria and compare different solutions in terms of their magnitude or general behavior.

Consider the simple case:



Let $y(t)\epsilon\mathbb{R}^p$ be the system response to a causal input $u(t)\epsilon\mathbb{R}^m$. with the following representation

$$Time - domain:\ y(t) = \int_{-\infty}^{t} W(t-\tau)u(\tau)d\tau\,, \qquad W(t) \in \mathbb{R}^{pxm}$$

$$Frecuency - domain: \quad \hat{y}(s) = W(s)\,\hat{u}(s), \qquad W(s) \in \mathbb{R}(s)^{pxm}$$

Then, the space and their norms are the natural candidates to measure the size of the system. And it would be found an expression for the induces norm W such that:

$$\|y\|_p \le \|W\|_q \|u\|_p,\ \ \forall u \in \mathcal{L}_p$$

$\mathcal{L}_1 - norm$, It is called the *peak-to-peak gain* and relates the max amplitude of the output with the max amplitude of the input.

$$\|w\|_{\mathcal{L}_1} = \max_{1 \le i \le p} \int_0^{\infty} \sum_{j=1}^{m} \left|W_{ij}(t)\right| dt$$

$$where\ w(t) = \left[W_{ij}(t)\right]$$

$H_2 - norm$, It represents the energy of the system signals.

$$\|w\|_{H_2} = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} tr\{W^T(-jw)W(jw)\} dw\right]^{1/2}$$

$$where\ w^*(jw) = w^T(-jw)$$

$H_\infty - norm$, It is called *RMS gain* and relates the energy/power of the output with the energy/output of the input.

$$\|w\|_{H_\infty} = sup_w \bar{\sigma}(W(jw))$$

$$where\ \bar{\sigma}(W(jw) = \sqrt{\bar{\lambda}(w^T(-jw)\left(W(jw)\right)}\ , \bar{\lambda} = \max eigenvalue$$

Induced norms.

The induced norms are used to measure the magnitude of a matrix or a linear operator, considering the relationship between the input and output spaces. Induced norms provide mathematical tools for measuring the magnitude and performance of linear systems and linear operators. They are used in the analysis, design, and optimization of control systems, allowing to evaluate the stability, performance, and robustness of the systems.

Consider a linear system with input u, output y, and transfer function $\widehat{W}$, assumed stable and strictly proper. The tables 2 and 3 tables tell us how much u affects y according to various measures. First, in the case of known inputs and then the case when $u(t)$ is unknown, but it belongs to a specific signal space $\mathcal{L}_p = \{u(t) \in \mathcal{L}_p\ iff\ \|u\|_p < \infty\}$.

Table.2. Induced norms with known inputs.

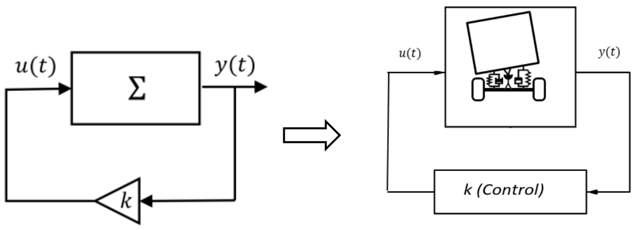
$u(t)$ / $y(t)$	*Dirac impulse* $u(t) = \delta(t)$	*sinusoidal* $u = \sin(wt)$	*White noise* $u(t) = w \sim (0, {\sigma_u}^2)$
$\|y\|_2$	$\|w\|_{H_2}$	∞	∞
$\|y\|_\infty$	$\|w\|_{H_\infty}$	$\|W(jw)\|$	$\le \infty$
$\|y\|_{RMS}$	0	$\frac{1}{\sqrt{2}}\|W(jw)\|$	${\sigma_y}^2 = \|w\|_{H_2}^2 {\sigma_u}^2$

Table.3. induced norms with unknow inputs.

$y(t)$ \ $u(t)$	$\mathcal{L}_2$ $\lVert u \rVert_2$	$\mathcal{L}_\infty$ $\lVert u \rVert_\infty$	$\mathcal{L}_{RMS}$ $\lVert u \rVert_{RMS}$
$\lVert y \rVert_2$	$\lVert w \rVert_{H_\infty}$	∞	∞
$\lVert y \rVert_\infty$	$\lVert w \rVert_{H_2}$	$\lVert w \rVert_{\mathcal{L}_1}$	$\leq \infty$
$\lVert y \rVert_{RMS}$	0	$\leq \lVert w \rVert_{H_\infty}$	$\lVert w \rVert_{H_\infty}$

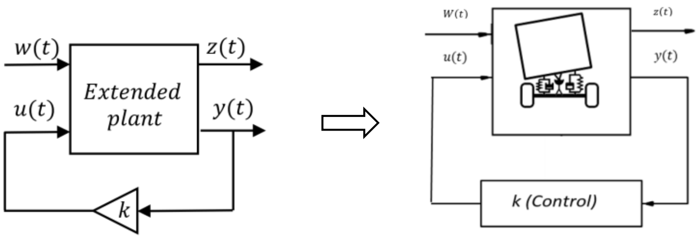
6. Control design

The state static feedback law is of the type:



Basic regulation problem, the given plant $\dot{x}(t) = Ax(t) + Bu(t)$, consist in determining a matrix $k \in \mathbb{R}^{mxn}$ such that, the control law $u(t) = kx(t)$ makes the close loop system $\dot{x}(t) = (A + Bk)x(t)$ asymptotically stable

We want to create a static controller which, in addition to guaranteeing the asymptotic stability of the system, assure the specifications on the induced norms of the system. Therefore, starting from the open loop system described by the following equations:



$$\Sigma ol : \begin{cases} \dot{x}(t) = Ax(t) + B_1u(t) + B_2w(t) \\ z_\infty(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t) \\ z_2(t) = C_2x(t) + D_{22}u(t) \\ z_1(t) = C_3x(t) + D_{31}w(t) + D_{32}u(t) \end{cases}$$

Where z_∞, z_2, z_1 are the control objectives or the performance outputs on which is necessary to find an optimal k so that the feedback system results in the best possible performance.

$$\Sigma cl : \begin{cases} \dot{x}(t) = (A + B_1k)x(t) + B_2w(t) \\ Z_\infty = (C_1 + D_{12}k)x(t) + D_{11}w(t) \\ Z_2 = (C_2 + D_{22}k)x(t) \\ Z_\infty = (C_3 + D_{32}k)x(t) + D_{31}w(t) \end{cases}$$

The control actions have been projected to make a zero regulation of the roll index of the vehicle, but all the states were considered to regulation because them influence on the objective. The states are roll angle, lateral slip angle, yaw rate, and roll rate.

For feedback gain, the control action will be aimed at minimizing the effects of the disturbance in all the states to setting the roll index to zero. For the objective are proposed different types of controls which are described in the following section.

5.1. $\boldsymbol{H_2}$ - Optimal Control

The objective of using optimal H2 control is to minimize the H2 norm of the system's response to disturbances or errors. The H2 norm measures the energy or power of the system's response, taking into account the frequency content of the disturbances. By minimizing the H2 norm, the control design aims to achieve good disturbance rejection and precise tracking performance.

Optimal H2 control utilizes optimization techniques to find the controller that minimizes the H2 norm of the system's response. This involves adjusting the controller parameters to minimize the energy of the error signal between the system's output and a desired reference signal.

By using induced norms, optimal H2 control takes into account the sensitivity of the system's response to different disturbances. This allows designing a controller that is robust against disturbances with different frequency and amplitude characteristics. Additionally, by considering the H2 norm, optimal H2 control focuses on minimizing the total energy of the system's response, which can be beneficial in applications where energy minimization is important.

The objective of this control is to minimize the norm H_2, i.e., the **amplitude of the output** of interest, considering the input signals as **limited in energy**, through the following relationship:

$$\|y\|_\infty \leq \|W\|_{H_2}\|u\|_2$$

The zero regulation of the states is the main objective of the control design, following this idea the controller H_2 allows to have outputs with limited amplitudes. Therefore, the controller that make the closed loop $(A + Bk)$ asymptotically stable and minimizes the $H_2 - norm$ is obtained by stablishing the following LMI condition:

$$\begin{cases} [x^*, y^*] = \min\limits_{X,Q,Y,\gamma} \gamma \\ s.t \\ \begin{bmatrix} AX + B_1Y + (AX + B_1Y)^T & B_2 \\ B_2^T & -I \end{bmatrix} < 0 \\ \begin{bmatrix} Q & (C_2X + D_{22}Y) \\ (C_2X + D_{22}Y)^T & X \end{bmatrix} > 0 \\ tr\{Q\} < \gamma \\ X > 0 \\ Q > 0 \end{cases}$$

If a solution exists, it is unique and the optimal control H_2 is given by.

$$k_2^* = y^* x^{*-1}$$

The results using MATLAB value of optimal control is:

$$k_2^* = [\,12.849608459950865 \quad -7.116713834769314 \quad -3.033030460092959 \quad 4.371201404601038]$$

The related eigenvalues of the close loop plant are:

$$eig\ H_2 = 1.0\mathrm{e}+02 \begin{bmatrix} -3.175521690725549 & + & 0.000000000000000i \\ -0.444155942790632 & + & 0.000000000000000i \\ -0.008855323485581 & + & 0.021854555771508i \\ -0.008855323485581 & - & 0.021854555771508i \end{bmatrix}$$

And the stability of the controller is verified.

The H_2 controller has been designed with the main objective to make zero all the states to reduce de roll index. Therefore, the signals of the states with and without control are shown in the figure 10. The disturbance input signal was explained in the sections before, and it is the front steering angle.

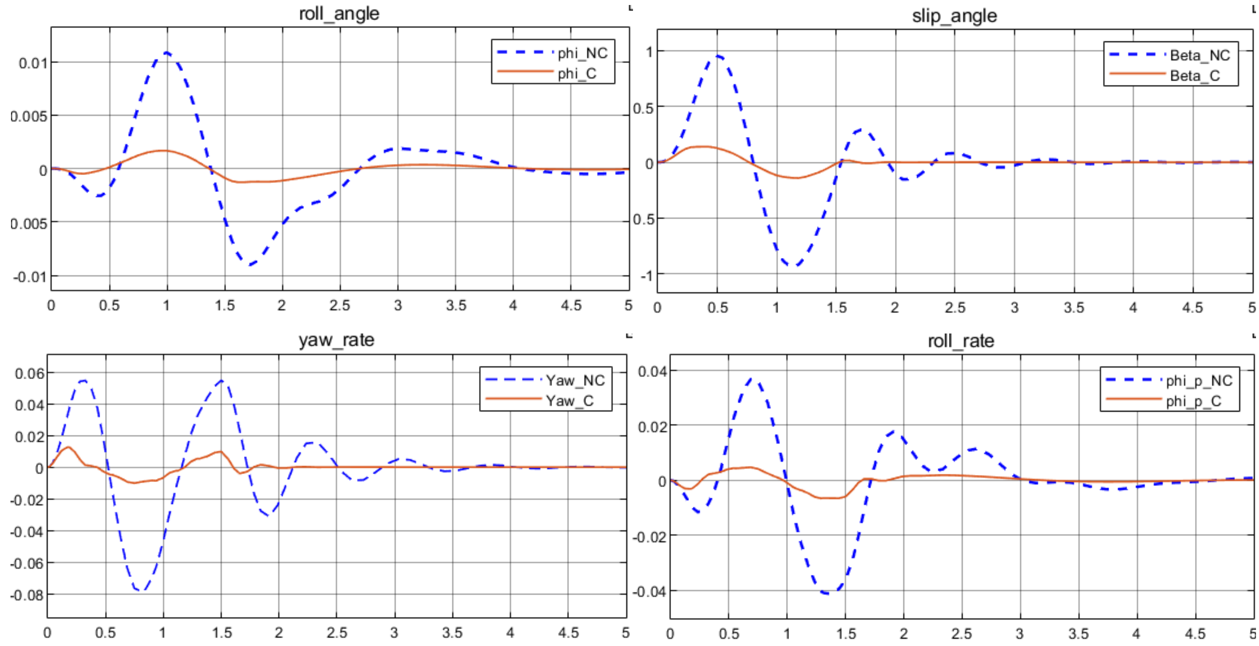


Fig.10 Controlled H_2states $x = [\emptyset \quad \beta \quad \gamma \quad \dot{\emptyset}]$ of the vehicle vs non-controlled.

From Fig.10 it is possible to notice that all the states of the controlled system are more contained with respect to the non-controlled one. The states tend to be regularized but not arise to be zero because the controller try to contain as best as possible the states with this optimal gain.

All these results are obtained when the vehicle is moving at 60 km/h. Looking at the results the response becomes closer to the desired response moreover.

5.2. H_∞ - Optimal Control

The objective of using H∞ control is to achieve maximum attenuation of possible disturbances across all frequencies. H∞ control focuses on minimizing the H∞ norm of the system's response to disturbances, where the H∞ norm represents the maximum amplification gain of a disturbance across all frequencies.

H∞ control aims to design a controller that provides optimal performance and maximum robustness against uncertainties and disturbances in the system. By considering the H∞ norm, the worst-case disturbance scenario is taken into account, and the goal is to minimize its impact on the system's response across all frequencies.

In other words, the objective of H∞ control using induced norms is to ensure that the system is robust and capable of handling disturbances throughout the frequency range, providing optimal attenuation under all conditions.

This approach is particularly useful in systems where significant uncertainties exist in the model parameters or where precise characteristics of the disturbances are unknown. By considering the H∞ norm and optimizing the controller based on this norm, stability and performance of the system in the presence of adverse disturbances are sought to be guaranteed.

The objective of this control is to minimize the norm H_∞ of the transfer function $W(s)$. From the induced norms in the table 3 is possible found the relationship that link the input and output with H_∞ norm

$$\|y\|_2 \leq \|W\|_{H_\infty} \|u\|_2$$

Minimizing the H_∞ norm result in **minimize the energy of the output**. In fact, H_∞ control, is a wide used technique in control system theory to design robust and optimal controllers. Due to, its main objective is to ensure satisfactory performance of the control system even in the presence of disturbances, uncertainties in system parameters, and noise. In other words, the goal is to minimize the influence of the most unfavorable disturbances while ensuring a desired response for the signals of interest.

Given a finite dimensional LTI system, it is necessary to find static feedback of the state such that in closed loop the transfer function *W* has the norm H_∞ smaller than an upper limit fixed a priori.

$$\|W\|_{H_\infty} < \gamma$$

It is wanted to minimize the output of the error with respect to the disturbance according to the above equation. Then, it is possible set the following LMI problem:

$$\begin{cases} [x^*, y^*] = \min\limits_{X,Y,\gamma} \gamma \\ s.t \\ \begin{bmatrix} AX + B_1Y + (AX + B_1Y)^T & B_2 & (C_1X + D_{12}Y)^T \\ B_2^T & -\gamma I & D_{11}^T \\ (C_1X + D_{12}Y) & D_{11} & -\gamma I \end{bmatrix} < 0 \\ \gamma > 0 \\ X = X^T > 0 \end{cases}$$

If a solution exists, it is unique and the optimal control H_∞ is given by.

$$k_\infty^* = y^* x^{*-1}$$

The results using MATLAB are:

$$k_\infty^* = 1.0\mathrm{e}+2\,[0.316650436981548 \;\; -0.068288494413070 \;\; -2.924960793418407 \;\; 0.098111554439284]$$

The related eigenvalues of the close loop plant are:

$$eig\ \ H_\infty = 1.0\mathrm{e}+3 \begin{bmatrix} -9.817773633041284 + 0.000000000000000\mathrm{i} \\ -0.000913065469109 + 0.002195018536241\mathrm{i} \\ -0.000913065469109 - 0.002195018536241i \\ -0.001509139178280 + 0.000000000000000i \end{bmatrix}$$

And the stability of the controller is verified.

The following figures shown the response of the system due to the H_∞-controller and without controller for a disturbance related to the front steer angle.

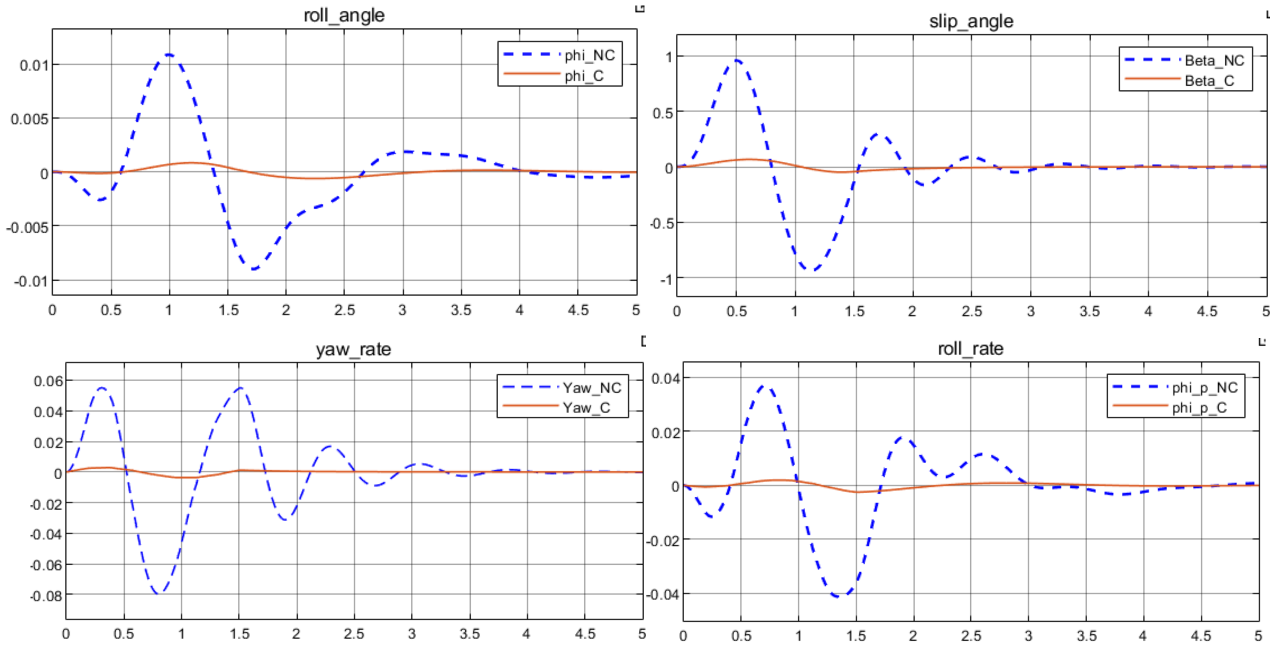


Fig.11 Controlled H_∞ states $x = [\emptyset \quad \beta \quad \gamma \quad \dot{\emptyset}]$ of the vehicle vs non-controlled

In this case it can be highlighted that the states are much more accurate than the before control. This control aims are rejecting the disturbance effect with a gain related to the energy of output with the energy of the input. All the response were obtained when the vehicle is moving at 60 km/h.

5.3. L_1 - Optimal Control

The objective of using L1 control is to minimize the L1 norm of the system's response to disturbances. The L1 norm represents the sum of the absolute values of the system's response in the time domain or frequency domain.

L1 control aims to design a controller that provides a system response as close to zero as possible in terms of the L1 norm. This involves minimizing the total contribution of disturbances to the system's response, without focusing on a specific frequency.

The focus of L1 control is useful when the goal is to minimize the overall magnitude of the system's response to disturbances, regardless of the frequency at which they occur. It is particularly suitable for applications where the aim is to limit the total energy or maximum amplitude of the system's response.

By considering induced norms, L1 control takes into account the contribution of disturbances across all frequencies and aims to minimize their overall impact. This allows designing a controller that is robust against disturbances with different frequency and amplitude characteristics.

The objective of this control is to minimize the norm L_1 of the transfer function $W(s)$. From the induced norms in the table 3 is possible found the relationship that link the input and output with L_1 norm

$$\|y\|_\infty \leq \|W\|_{L_1}\|u\|_\infty$$

This induced norm relates to the amplitude of the output of interest, also considering a small amplitude of the control signal. The zero regulation of the states (errors signals) is the main objective of the control design, following this idea the controller L_1 allows to have outputs with limited amplitudes. Therefore, the controller that make the closed loop $(A + Bk)$ asymptotically stable and minimizes the $L_1 - norm$ is obtained by stablishing the following LMI condition:

$$\begin{cases} [x^*, y^*] = \min\limits_{X,Y,\gamma,\mu} \gamma \\ s.t \\ \begin{bmatrix} \lambda X & 0 & (C_3X + D_{32}Y)^T \\ 0 & (\gamma - \mu)I & D_{31}^T \\ (C_3X + D_{32}Y) & D_{31} & -\gamma I \end{bmatrix} < 0 \\ \begin{bmatrix} AX + B_1Y + (AX + B_1Y)^T + \lambda X & B_2 \\ B_2^T & -\mu I \end{bmatrix} > 0 \\ \gamma > 0 \\ X = X^T > 0 \end{cases}$$

If a solution exists, it is unique and the optimal control H_∞ is given by.

$$k_1^* = y^* x^{*-1}$$

The results using MATLAB are:

$$k_1^* = 1.0\mathrm{e}+02[-1.487086439923369 \quad -0.048146417208758 \quad -0.323029889048422 \quad -0.795391422949707]$$

The related eigenvalues of the close loop plant are:

$$eig\ \ L_1 = 1.0\mathrm{e}+02 \begin{bmatrix} -3.245010939748220 + 0.000000000000000i \\ -0.287139146760042 + 0.000000000000000i \\ -0.016142325507136 + 0.022054996330548i \\ -0.016142325507136 - 0.022054996330548i \end{bmatrix}$$

And the stability of the controller is verified.

The following shown the response of the system due to the L_1-controller and without controller for a disturbance related to moose test in the front steer angle.

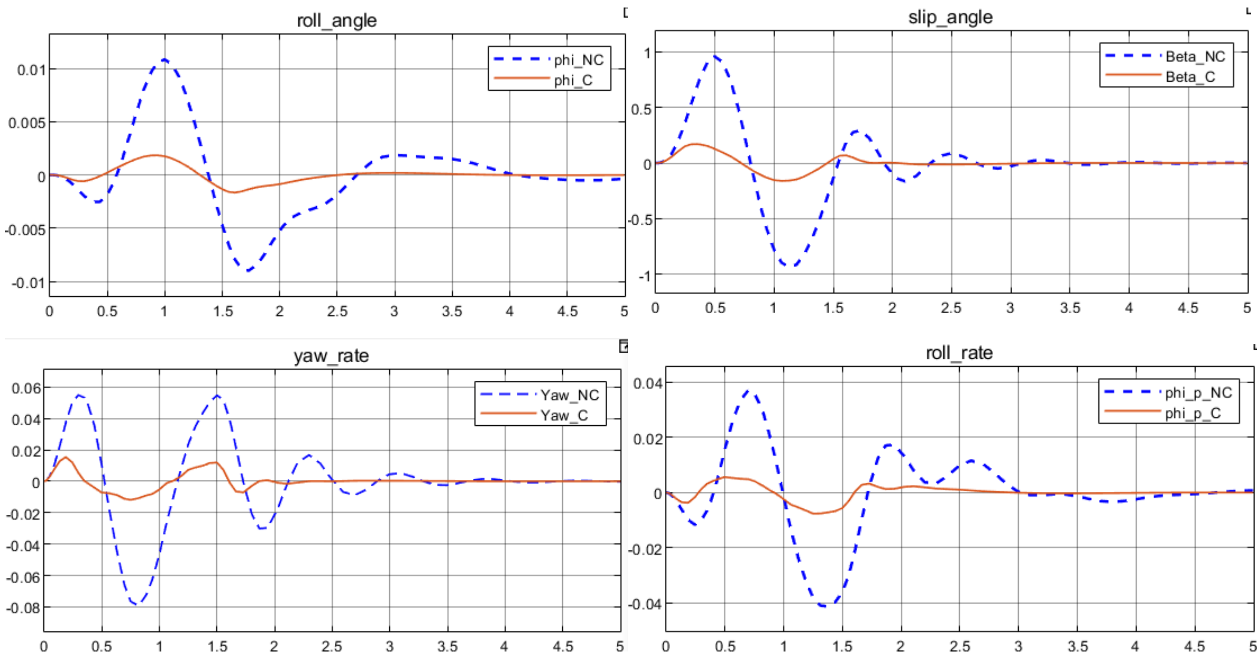


Fig.12 Controlled L_1 states $x = [\emptyset \quad \beta \quad \gamma \quad \dot{\emptyset}]$ of the vehicle vs non-controlled

All the results were obtained with a constant vehicle velocity of 60 km/h. With L_1-controller the results become better but not better than H_∞. The behavior is quite similar to the H_2 control.

5.4. *LQ*-Optimal control

The aim of this control is to find a state feedback control $u(t) = kx(t), k \in \mathbb{R}^{mxn}$ that stabilize the system and minimize the following cost function.

$$J = \int_0^{\infty} x^T(t)Qx(t) + u^T(t)Ru(t)dt$$

With the following assumptions R and Q are symmetric and positive defined matrices the matrices A, B must be stabilizable. The first term of cost J measures the energy of the state (energy of the evolution), while the second term measure the energy of the input. The smaller the sum of these two contributions, the lower the value of the cost J: the objective of the LQ control is precisely minimize both terms. However, the terms imposer opposition to the other, in fact, minimize the energy of the input u leads to a deterioration of the system performance consequently produce an increase in the states energy.

Formulation of the LQ control in terms of LMI:

$$\begin{cases} [x^*, y^*] = \min_{X,Y} \gamma \\ s.t \\ \begin{bmatrix} AX + B_1Y + (AX + B_1Y)^T & X^T & Y^T \\ X & -Q^{-1} & 0 \\ Y & 0 & -R^{-1} \end{bmatrix} < 0 \\ \begin{bmatrix} \gamma & X_0^T \\ X_0 & X \end{bmatrix} > 0 \\ X = X^T > 0 \end{cases}$$

If a solution exists, it is unique and the optimal control LQ is given by.

$$k_{LQ}^* = y^* x^{*-1}$$

The results using MATLAB are:

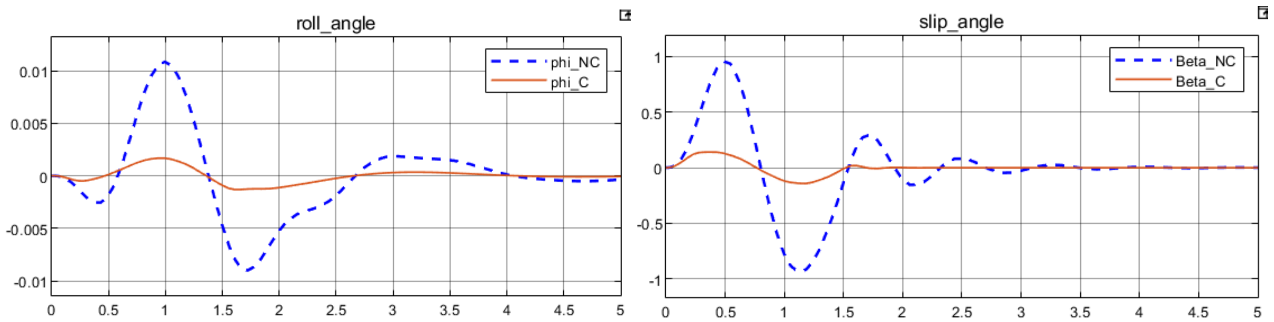
$$k_{LQ}^* = [0.509742072109162 \quad -7.005474619453144 \quad -4.541854759159881 \quad -0.001040223923047]$$

The related eigenvalues of the close loop plant are:

$$eig\ \ LQ = 1.0\mathrm{e}+02 \begin{bmatrix} -3.177114627081616 + 0.000000000000000\mathrm{i} \\ -0.435515186130985 + 0.000000000000000\mathrm{i} \\ -0.009135744111485 + 0.021987737563341i \\ -0.009135744111485 - 0.021987737563341i \end{bmatrix}$$

And the stability of the controller is verified.

The zero regulation of the states is the main objective of the control design. On the following figure it is shown the response of the system due to the LQ-controller.



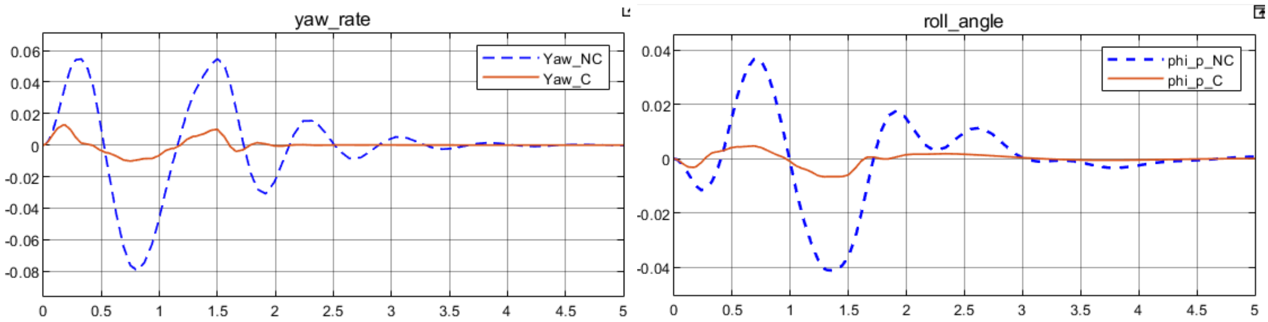


Fig.13 Controlled LQ states $x = [\emptyset \quad \beta \quad \gamma \quad \dot{\emptyset}]$ of the vehicle vs non-controlled

It can be highlighted that the states signal become very close to zero. The LQ control minimizes a quadratic cost function not a norm like the controllers before used in this project.

MULTIOBJECTIVE CONTROL

The previous LMI conditions that describe the optimization problems underlaying the synthesis of the various controllers are all convex in the variables X and Y. Then, they can be combined because the intersection of convex constrains is still a convex constrain.

5.5. H_2/H_∞ Optimal control

The objective of this multi-objective controller is made the closed loop $(A + Bk)$ asymptotically stable and minimize the norm H_2/H_∞. It seeks to find a balance between these two approaches. The goal is to design a controller that achieves good performance in terms of disturbance attenuation.

Given $a, b > 0$ fixed scalars the solution of this control is:

$$k^*_{H_2/H_\infty} = \min_{k \in S} a \, \|W\|_{H_\infty} + \|W\|_{H_2}$$

Formulation of this control in terms of LMI:

$$[x^*, y^*] = \min_{X,Y,Q} a\gamma_\infty + b\gamma_2$$
$$s.t$$
$$\begin{bmatrix} AX + B_1Y + (AX + B_1Y)^T & B_2 & (C_1X + D_{12}Y)^T \\ B_2^T & -\gamma_\infty I & D_{11}^T \\ (C_1X + D_{12}Y) & D_{11} & -\gamma_\infty I \end{bmatrix} < 0$$
$$\begin{bmatrix} AX + B_1Y + (AX + B_1Y)^T & B_2 \\ B_2^T & -I \end{bmatrix} < 0$$
$$\begin{bmatrix} Q & (C_2X + D_{22}Y) \\ (C_2X + D_{22}Y)^T & X \end{bmatrix} > 0$$
$$\gamma_2, \gamma_\infty > 0$$
$$tr\{Q\} < \gamma_2$$
$$X = X^T > 0$$

If a solution exists, it is unique and the optimal control H_2/H_∞ is given by.

$$k^*_{H_2/H_\infty} = y^* x^{*-1}$$

The results using MATLAB are:

$$k^*_{H_\infty/H_2} = [1.241078537219437 \quad -19.985269472971822 \quad -10.818704287812579 \quad -0.079373759479495]$$

The related eigenvalues of the close loop plant are:

$$eig\ H_\infty/H_2 = 1.0\text{e}+02\begin{bmatrix} -9.113873736070204 & + & 0.000000000000000i \\ -0.431633167275672 & + & 0.000000000000000i \\ -0.009135839125317 & + & 0.021987615824777i \\ -0.009135839125317 & - & 0.021987615824777i \end{bmatrix}$$

And the stability of the controller is verified.

Simulation

The Fig.15 shown the response of the system due to the H_2/H_∞-controller and without controller for a disturbance related to moose test in the front steer angle.

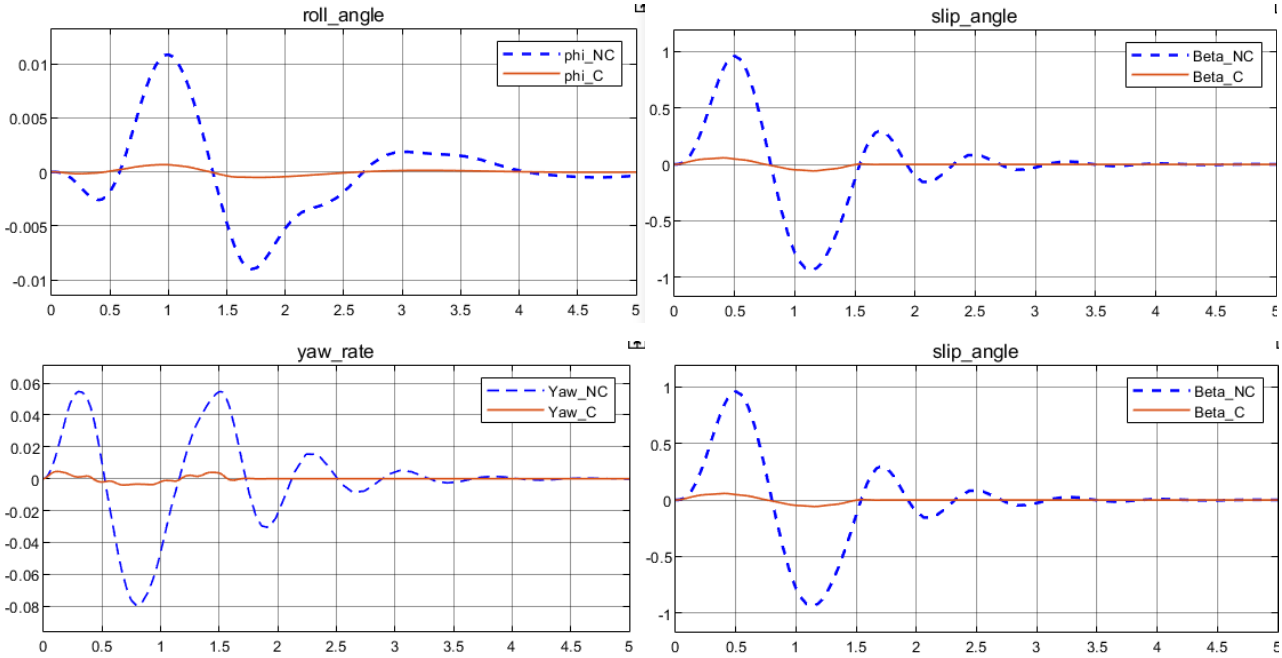


Fig.15 States $x = [\emptyset \quad \beta \quad \gamma \quad \dot{\emptyset}]$ controlled by multi-objective control vs. no controlled states.

It can be highlighted that the states are more contained with respect to the uncontrolled ones due to the H_2/H_∞-controller. This result is obtained when the model has a steering disturbance as it is shown before and a constant velocity of 60 km/h.

5.6. Comparison between the controllers

It is presented a comparation between the different controllers implemented in the previous section, to see which of them offers the best features. The comparation refers to the performance of the model feed with the disturbance presented before and a constant velocity of 60 km/h.

Regulation of roll angle (∅)

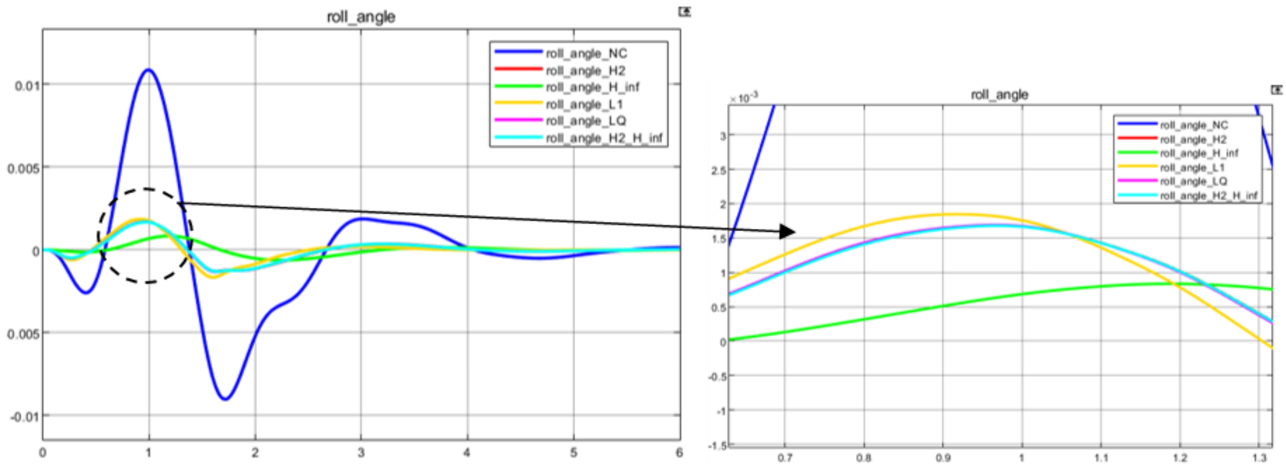


Fig.16 Comparation of the regulated roll angle state

Regulation of slip angle (β)

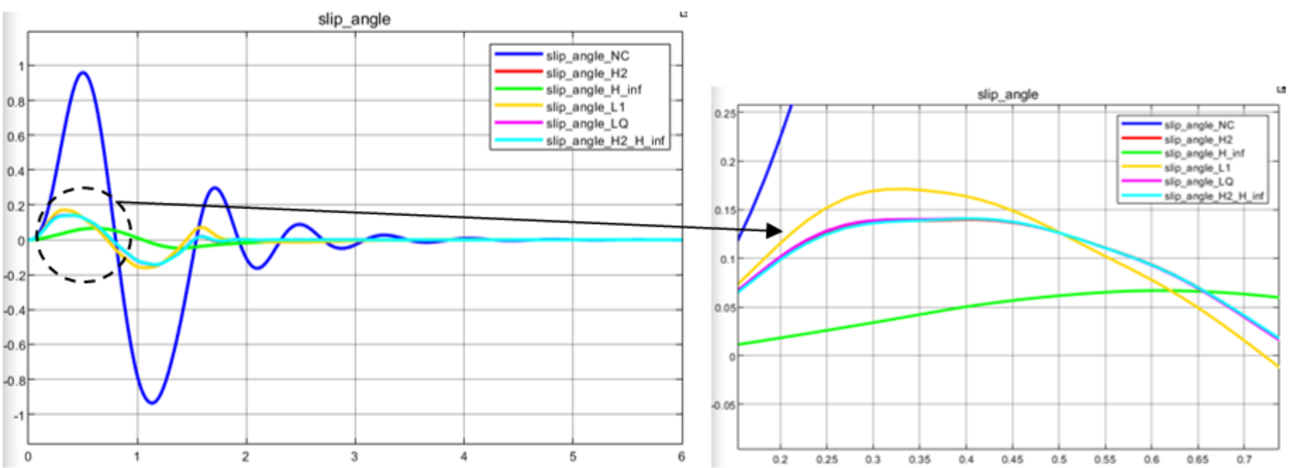


Fig.17 Comparation of the regulated slip angle state

Regulation of yaw (γ)

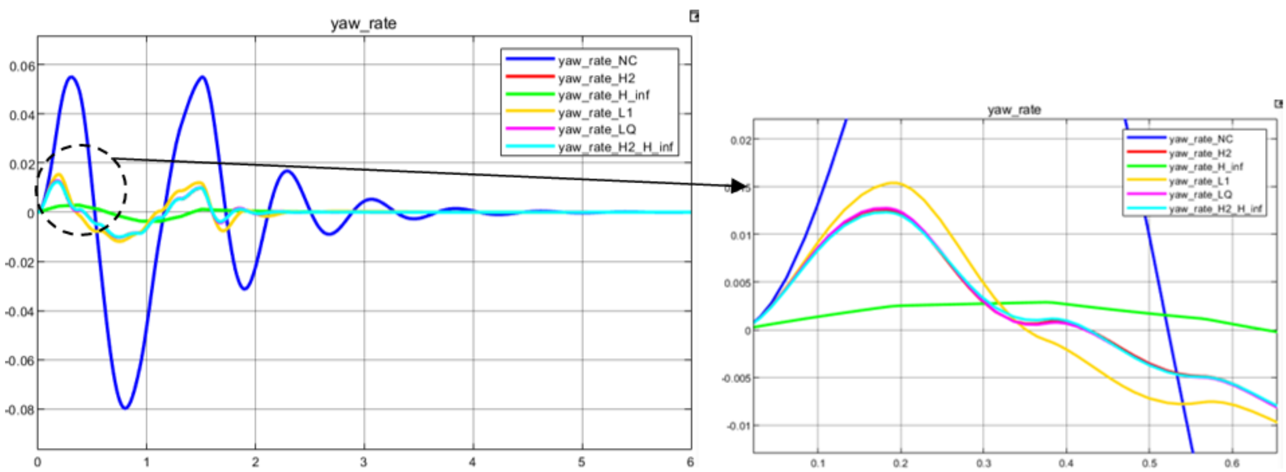


Fig.18 Comparation of the regulated yaw state

Regulation of roll rate ($\dot{\phi}$)

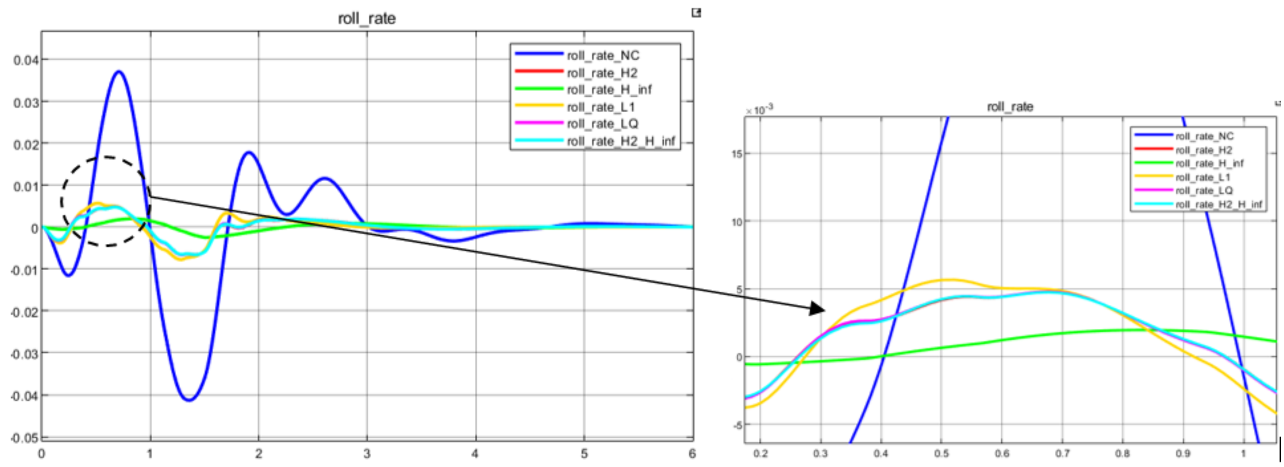


Fig.19 Comparation of the regulated roll rate state

From Fig.16, 17,18, and 19 can be seen that the regulation of all the states of $H_2, H_2 - H_\infty$ and LQ controllers have the same performance but not the best. The control H_∞ has the best performance and make the states achieve a good behavior and reach their desired value. The L_1 control has a behavior bit worse than the first three ones, but the difference is not big. All the controllers can regulate the states in a good way, there is a big difference between the states controlled and no controlled.

7. Robust Control

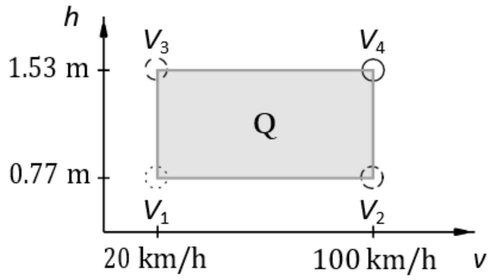


Fig.20 Operating domain

The control design objective is to find a control law which decreases the overshoot of the rollover coefficient during transient maneuvers. The challenge is the robustness of the control law with respect to a wide speed range v ∈ [20 km/h,100 km/h] and an interval of the CG height, i.e., h ∈ [0.77 m,1.53 m]. Fig.20 shows the respective operating domain Q.

The idea is having a model and a mathematical description of the model uncertainty design a control law able to ensure stability and/or performance for all values of allowed uncertainty. However, robust control has less performance with respect to optimal controls for LIT systems.

Until now the control has been designed for a constant speed and CG height of the vehicle. Because the system strongly dependent on these two parameters the robust control has the purpose of controlling the change of these.

The uncertainties are parametric because we know the state space representation of the system, but now the velocity and CG height are uncertain parameters which vary.

Define the state space uncertain representation.

It is used the Polytropic model to represent the uncertain system.

$$\dot{x}(t) = A(p)x(t) + B(p)\delta_f(t) + W(p)\delta_s(t)$$
$$y = Cx$$

Where:

$$p = \begin{bmatrix} p1 \\ \vdots \\ pn \end{bmatrix}, \qquad 0 \le pi \le 1, \quad \sum_{i=1}^{l} pi = 1$$

Because the analyzed problem only has two uncertain parameters the number of vertices is four. Then, the related matrices are obtained by:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -0.77m_2 \\ 0 & 0 & J_z & 0 \\ 0 & -0.77m_2 & 0 & I_{\dot{\phi}} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{Y_\beta}{20} & -\left(\frac{N_\beta}{20} + m20\right) & 0 \\ 0 & -\frac{N_\beta}{20} & -\frac{N_r}{20} & 0 \\ \sigma & 0 & -15.4m_2 & -d_{\dot{\phi}} \end{bmatrix}$$

$$B_1 = W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -0.77m_2 \\ 0 & 0 & J_z & 0 \\ 0 & -0.77m_2 & 0 & I_{\dot{\phi}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ c_f\mu \\ 0 \\ c_f l_f \mu \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -0.77m_2 \\ 0 & 0 & J_z & 0 \\ 0 & -0.77m_2 & 0 & I_{\dot{\phi}} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{Y_\beta}{100} & -\left(\frac{N_\beta}{100} + m100\right) & 0 \\ 0 & -\frac{N_\beta}{100} & -\frac{N_r}{100} & 0 \\ \sigma & 0 & -77m_2 & -d_{\dot{\phi}} \end{bmatrix}$$

$$B_2 = W_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -0.77m_2 \\ 0 & 0 & J_z & 0 \\ 0 & -0.77m_2 & 0 & I_{\dot{\phi}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ c_f\mu \\ 0 \\ c_f l_f \mu \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -1.53m_2 \\ 0 & 0 & J_z & 0 \\ 0 & -1.53m_2 & 0 & I_{\dot{\phi}} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{Y_\beta}{20} & -\left(\frac{N_\beta}{20} + m20\right) & 0 \\ 0 & -\frac{N_\beta}{20} & -\frac{N_r}{20} & 0 \\ \sigma & 0 & -30.6m_2 & -d_{\dot{\phi}} \end{bmatrix}$$

$$B_3 = W_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -1.53m_2 \\ 0 & 0 & J_z & 0 \\ 0 & -1.53m_2 & 0 & I_{\dot{\phi}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ c_f\mu \\ 0 \\ c_f l_f \mu \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -1.53m_2 \\ 0 & 0 & J_z & 0 \\ 0 & -1.53m_2 & 0 & I_{\dot{\emptyset}} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{Y_\beta}{100} & -\left(\frac{N_\beta}{100} + m100\right) & 0 \\ 0 & -\frac{N_\beta}{100} & -\frac{N_r}{100} & 0 \\ \sigma & 0 & -153m_2 & -d_{\dot{\emptyset}} \end{bmatrix}$$

$$B_4 = W_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & -1.53m_2 \\ 0 & 0 & J_z & 0 \\ 0 & -1.53m_2 & 0 & I_{\dot{\emptyset}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ c_f\mu \\ 0 \\ c_f l_f \mu \end{bmatrix}$$

Due to the good performance of H_inf controller, it will be implemented for robust control.

The optimal H_inf robust control is achieved if $\exists\, k \epsilon R^{mxn}$ such that $A(p)x(t) + B(p)u(t) + E(p)\delta_f(t)$ is asymptotically stable and $\|T_\infty\|_{H_\infty}^{wc} < \gamma \; if \; \exists\; x = x^T > 0, \exists\; Y \epsilon R^{mxn}$ and the following optimal problem has solution.

$$\begin{cases} [x^*, y^*] = \min_{X,Y,\gamma} \gamma \\ s.t \\ \begin{bmatrix} A_i X + B_{1,i} Y + (A_i X + B_{1,i} Y)^T & B_{2,i} & (C_{1,i} X + D_{12,i} Y)^T \\ B_{2,i}^T & -\gamma I & D_{11,i}^T \\ (C_{1,i} X + D_{12,i} Y) & D_{11,i} & -\gamma I \end{bmatrix} < 0 \\ i = 1 \ldots l \\ \gamma > 0 \\ X = X^T > 0 \end{cases}$$

If a solution exists, it is unique and the robust optimal control H_∞ is given by.

$$k_\infty^* = y^* x^{*-1}$$

The results using MATLAB are:

$$k_\infty^* = [0.316650436981548 \;\; -0.068288494413070 \;\; -2.924960793418407 \;\; 0.098111554439284]$$

The related eigenvalues of the close loop plant are:

$$eig \;\; H_\infty = \begin{bmatrix} -0.916243567002336 \; + \; 2.197219301606751i \\ -0.916243567002336 \; - \; 2.197219301606751i \\ -1.549377739537914 \; + \; 8.091015270731472i \\ -1.549377739537914 \; - \; 8.091015270731472i \end{bmatrix}$$

And the stability of the controller is verified.

To check the performance of the robust control, the values of heigh takes the values of 0.69 1.45 [m] and the velocity 30 and 90 km/h as it is shown in Fig.21. The proofs of the performance of the control were made on the four possibles combination of these values. The disturbance signal is the same showed in the Fig.5.

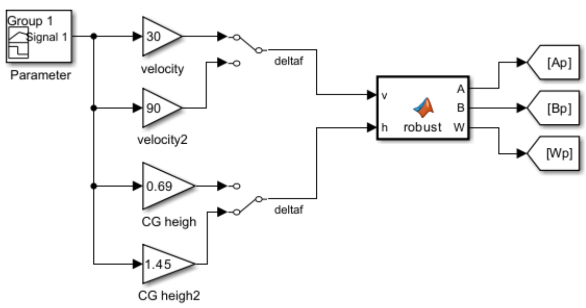


Fig.21 Velocity and heigh signals

Simulations

The Fig.22 shown the response of the system due to the $robust\ H_{\infty}$-controller versus the no-controlled plant for a disturbance with a velocity of 30 km/h and a heigh of 0.69 m.

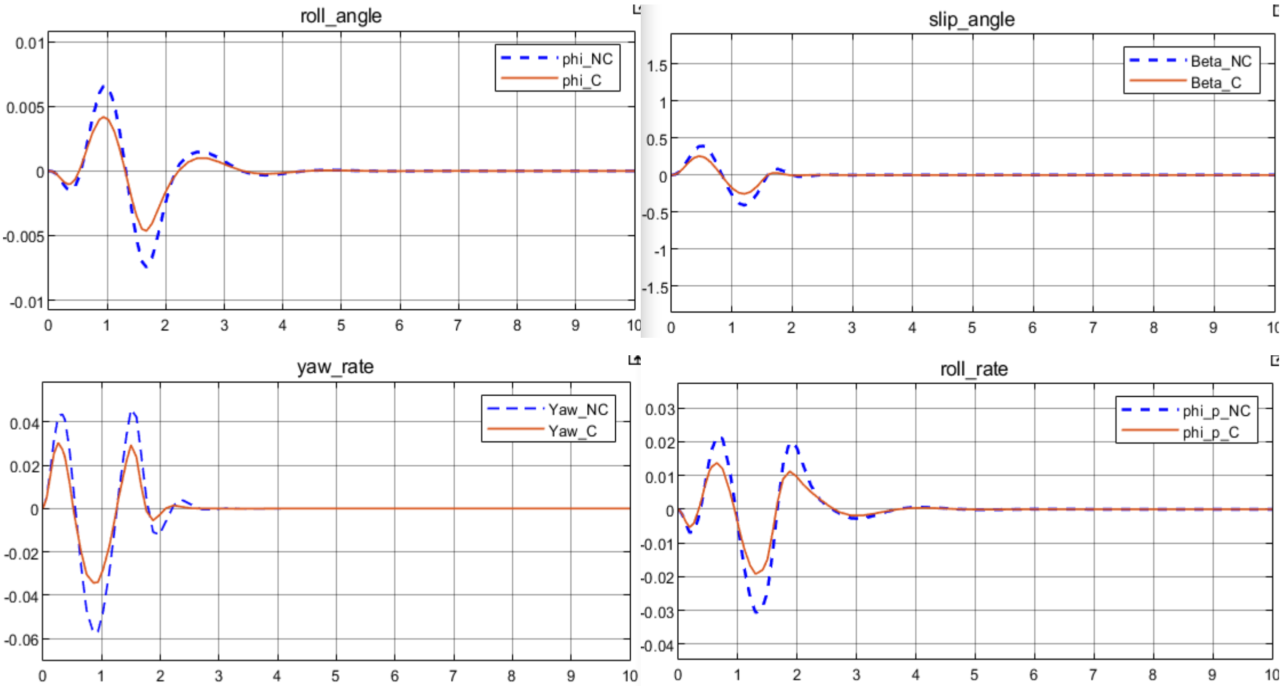


Fig.22 States $x = [\emptyset \quad \beta \quad \gamma \quad \dot{\emptyset}]$ controlled by robust control vs. no controlled states with a velocity of 30 km/h and a heigh of 0.69 m.

The Fig.23 shown the response of the system due to the $robust\ H_{\infty}$-controller versus the no-controlled plant for a disturbance with a velocity of 30 km/h and a heigh of 1.45 m.

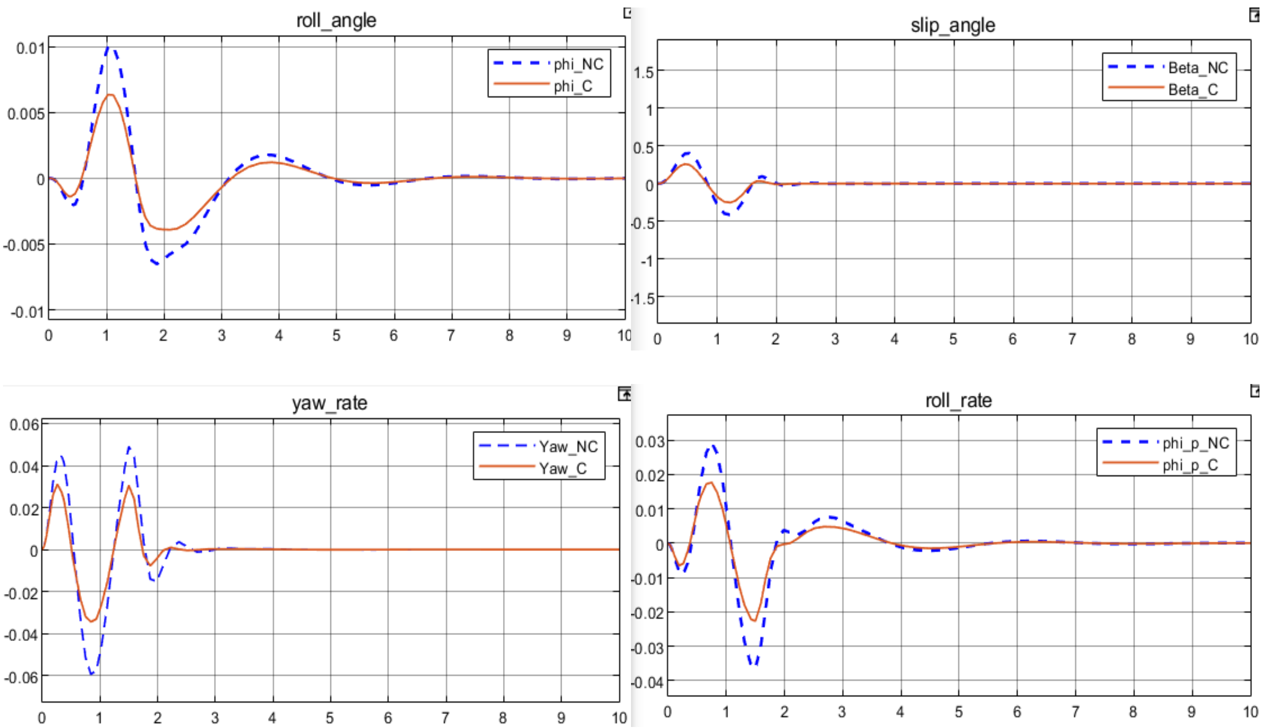


Fig.23 States $x = [\emptyset \quad \beta \quad \gamma \quad \dot{\emptyset}]$ controlled by robust control vs. no controlled states with a velocity of 30 km/h and a heigh of 1.45 m.

The Fig.24 shown the response of the system due to the $robust\ H_{\infty}$-controller versus the no-controlled plant for a disturbance with a velocity of 90 km/h and a heigh of 0.69 m.

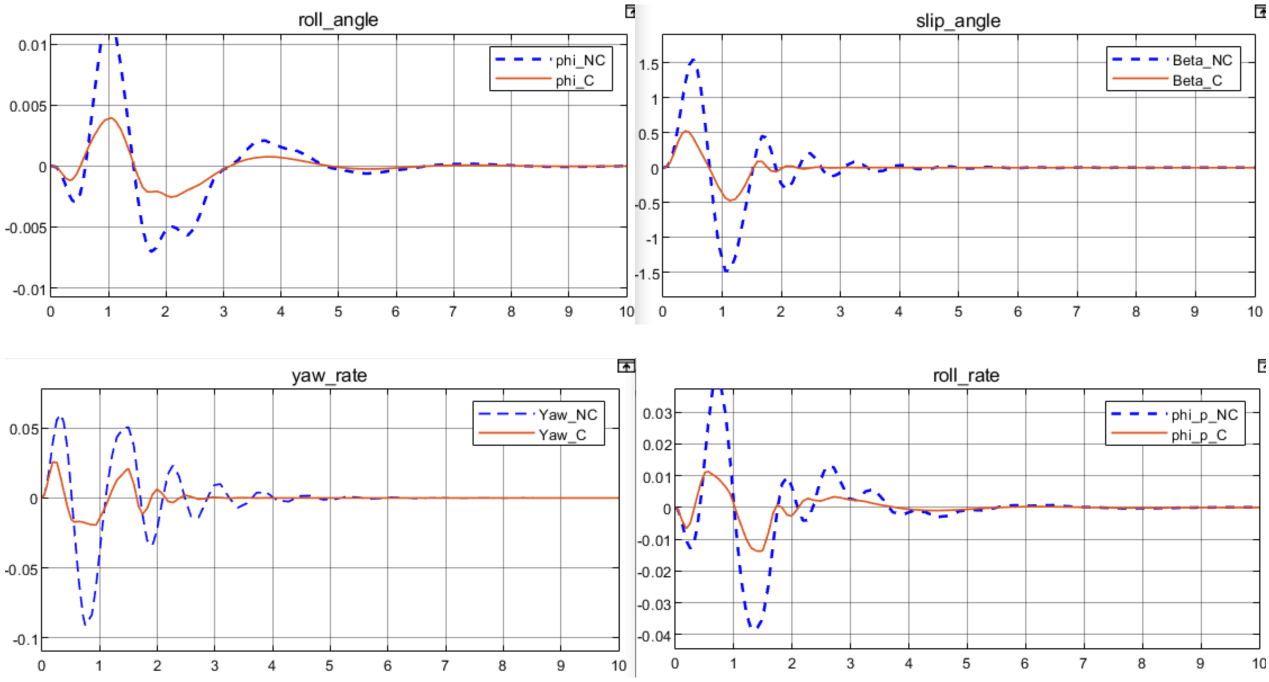


Fig.24 States $x = [\emptyset \quad \beta \quad \gamma \quad \dot{\emptyset}]$ controlled by robust control vs. no controlled states with a velocity of 90 km/h and a heigh of 0.69 m.

The Fig.25 shown the response of the system due to the $robust\ H_{\infty}$-controller versus the no-controlled plant for a disturbance with a velocity of 90 km/h and a heigh of 1.45 m.

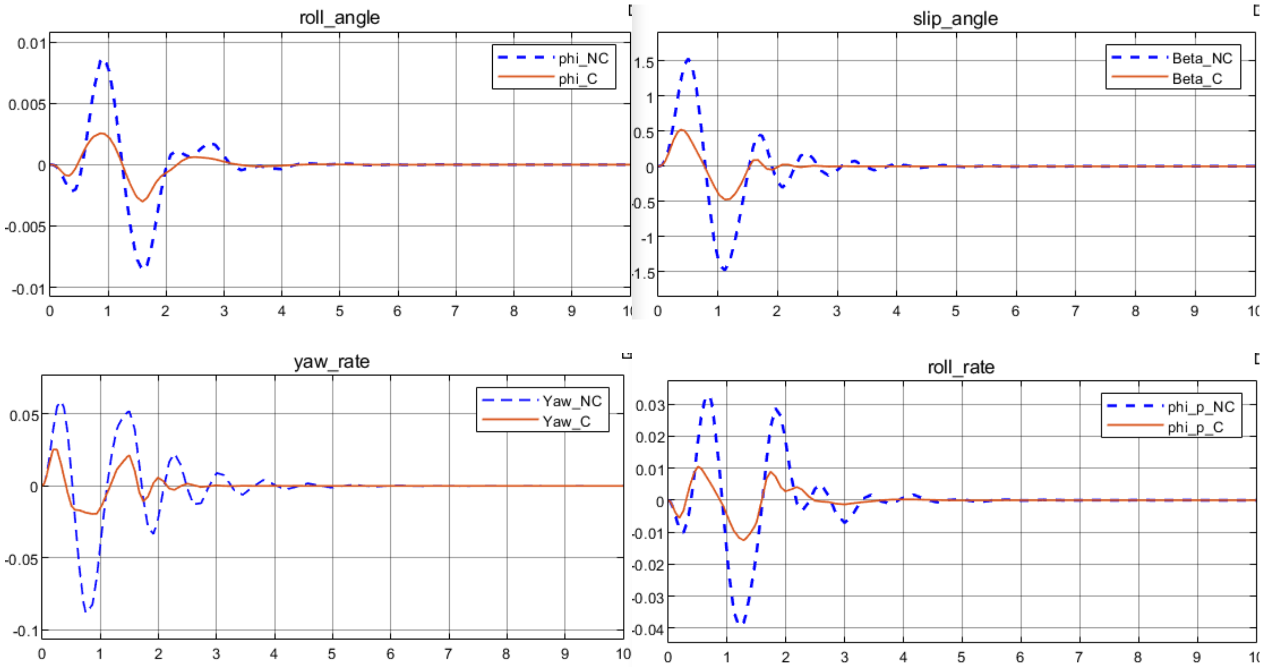


Fig.25 States $x = [\emptyset \quad \beta \quad \gamma \quad \dot{\emptyset}]$ controlled by robust control vs. no controlled states with a velocity of 90 km/h and a heigh of 1.45 m.

It can be highlighted that the controlled states are much more contained with respect to the uncontrolled one. Moreover, it also can be seen that the system behavior is strongly connected with the velocity than the heigh, but both change it. The control action makes better the performance of the system in all the cases, so it was proof its robustness, but its performance is not good enough as the static optimal controllers.

6.1. LPV control

LPV (Linear Parameter-Varying) control is a control methodology that is specifically designed for systems whose dynamics vary with different operating conditions or parameter variations. It is a form of adaptive control that allows the control strategy to adapt to changes in the system's parameters or operating points.

LPV control involves selecting appropriate control laws or controller parameters based on the values of the varying parameters. This is known as parameter scheduling. The control laws can be predefined or determined through adaptive mechanisms that continuously estimate the system parameters and adjust the control accordingly.

The uncertain modelling is equal to the robust case, with the difference of hypotheses that we can measure the uncertain parameter online, which varies over time. Such information will be exploited in the synthesis by building a time-varying gain.

Therefore p = $p(t)$ consequently we will have $A(p(t))$ where:

$$\dot{x}(t) = A(p(t))x(t) + B(p(t))u(t) + E(p(t))\delta_f(t)$$
$$y(t) = C\big(p(t)\big)x(t) + D\big(p(t)\big)u(t) + D_w(p(t))\,\delta_f(t)$$

It must be remembered that the values of the speed and heigh are determined through sensors in real time, which can measure them at any instant of time.

The matrices of the system depend on the ρn parameter through the following relationship.

$$A(\rho n) = (1-\rho n)\underline{A} + \rho n\overline{A} \qquad B(\rho n) = (1-\rho n)\underline{B} + \rho n\overline{B}$$
$$E(\rho n) = (1-\rho n)\underline{E} + \rho n\overline{E} \qquad D(\rho n) = (1-\rho n)\underline{D} + \rho n\overline{D}$$
$$D_w(\rho n) = (1-\rho n)\underline{D_w} + \rho n\overline{D_w} \qquad C(\rho n) = (1-\rho n)\underline{C} + \rho n\overline{C}$$
$$\rho_1 = \frac{1}{2}\frac{v - v_{min}}{v_{max} - v_{min}} \;, \qquad \rho_2 = \frac{1}{2}\frac{h - h_{min}}{h_{max} - h_{min}}$$

Then, the control gain will be obtained with the following equation.

$$k(\rho_1,\rho_2) = (0.5-\rho_1)\underline{k} + \rho_1\overline{k} + (0.5-\rho_2)\underline{k} + \rho_2\overline{k}$$

Where the control gains $\underline{k},\ \overline{k}$ are obtained front the matrices in the vertices $v_{max},\ v_{min}$. The operating domain Q were divided into four regions is it is shown in the figure bellow:

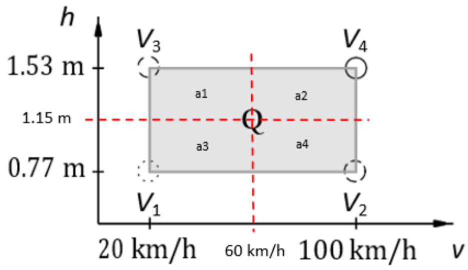


Fig.26 Operating domain for LPV

With MATLAB's StateFlow the finite state machine has been implemented which allows to select the feedback control gain *k* based on the speed and heigh. This gain is computed according to H_∞-controller due to its good performance.

Fig.27 shows the StateFlow machine configuration implemented in the control. The control is the so-called Gain scheduling control.

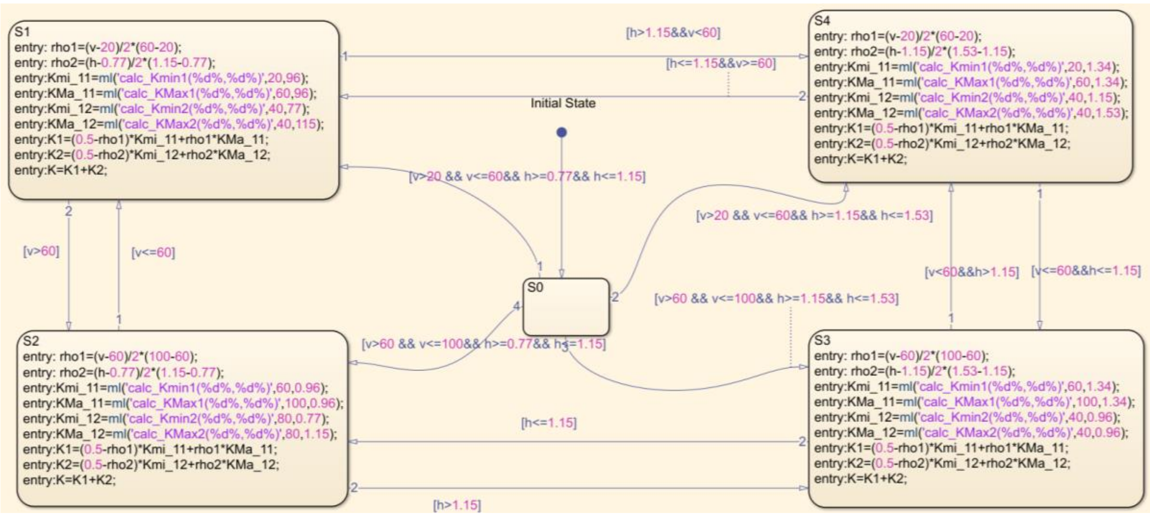


Fig.27 StateFlow machine configuration.

Simulations

To proof the performance of the control for the different four regions the velocity will take the values of 30 and 90 km/h and the heigh will take the values of 0.89 and 1.45 m, each combination of these values belongs to an area of the figure 27. The response of the system due to the $Gain\ Scheduling\ LPV\ H_\infty$-controller and without controller for a disturbance related to the front steer angle for different velocities and heights are shown in the figures bellow.

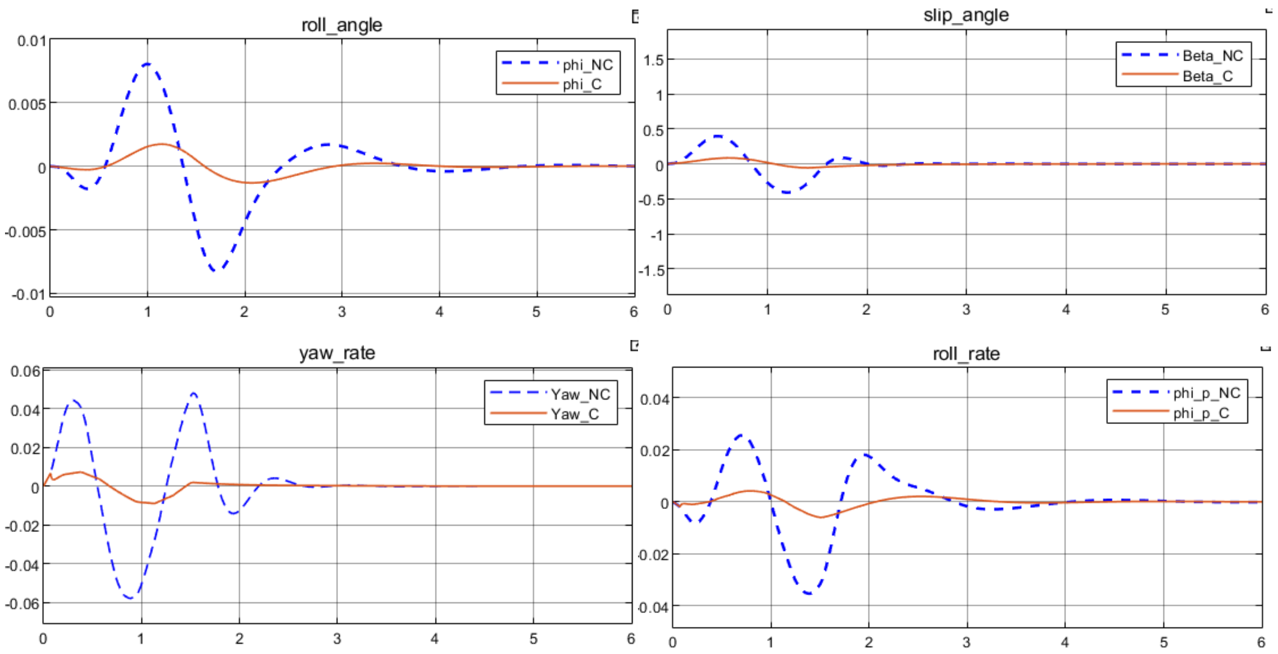


Fig.28 States $x = [\phi \quad \beta \quad \gamma \quad \dot{\phi}]$ controlled by LPV control vs. no controlled states with a velocity of 90 km/h and a heigh of 0.69 m.

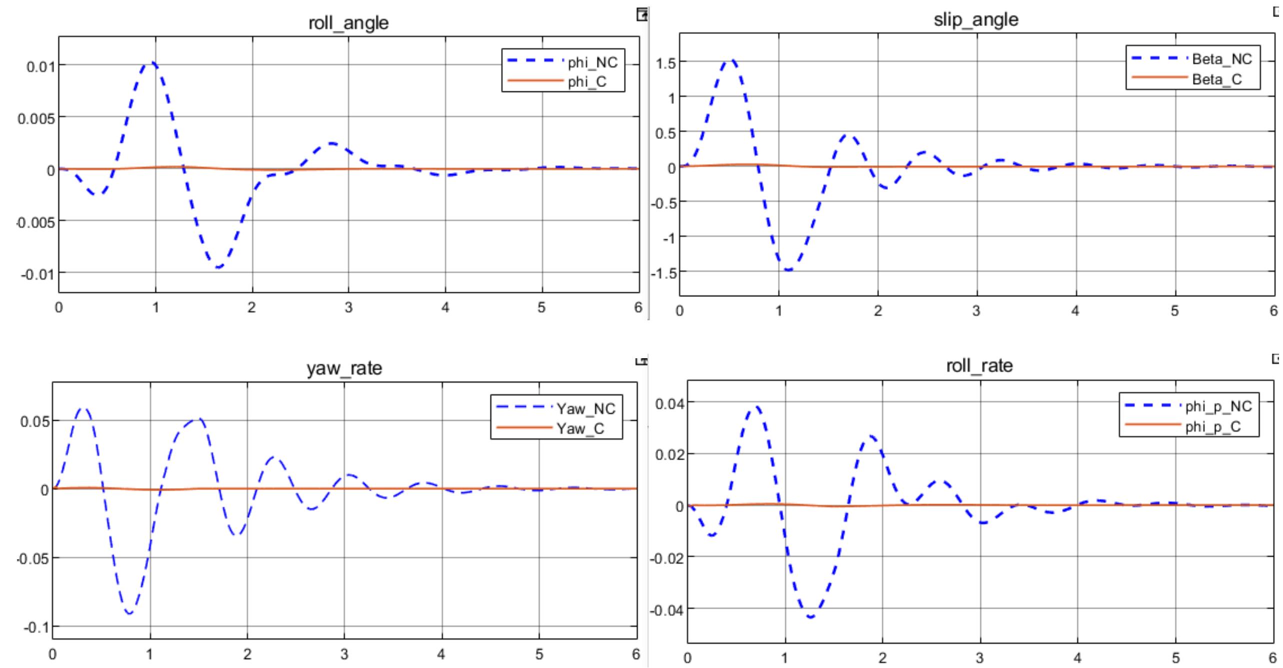
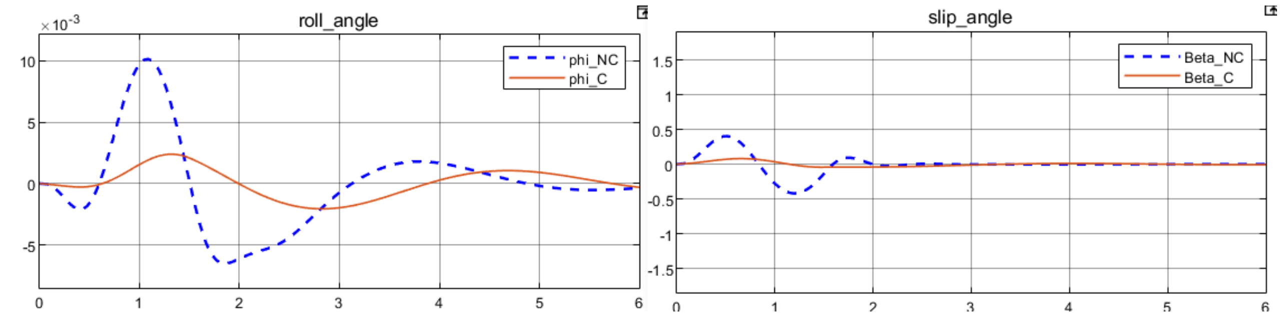


Fig.29 States $x = [\phi \quad \beta \quad \gamma \quad \dot{\phi}]$ controlled by LPV control vs. no controlled states with a velocity of 90 km/h and a heigh of 1.45 m.



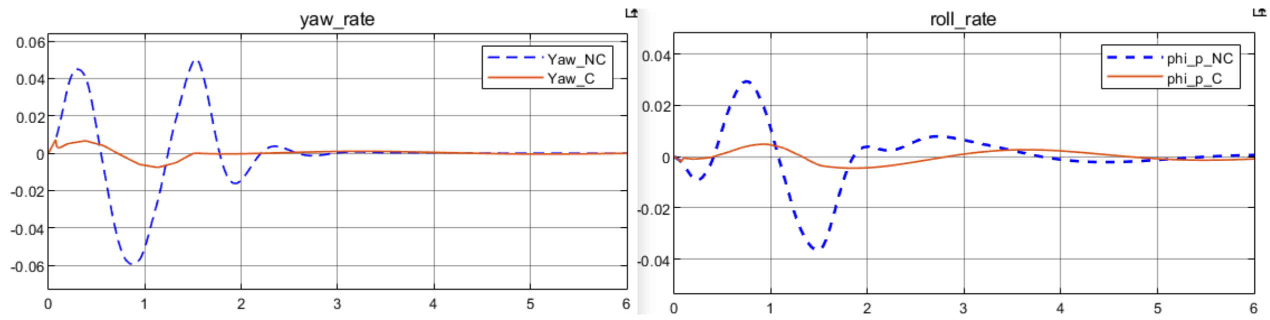


Fig.30 States $x = [\phi \quad \beta \quad \gamma \quad \dot{\phi}]$ controlled by LPV control vs. no controlled states with a velocity of 30 km/h and a heigh of 1.45 m

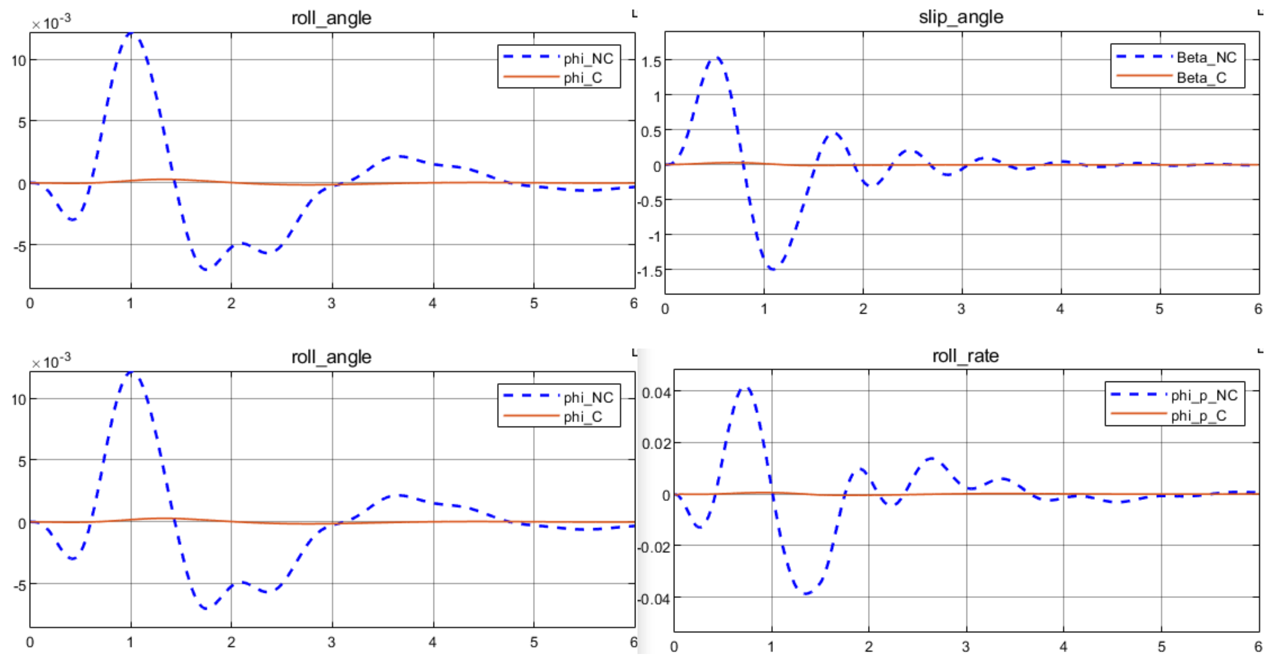


Fig.31 States $x = [\phi \quad \beta \quad \gamma \quad \dot{\phi}]$ controlled by LPV control vs. no controlled states with a velocity of 90 km/h and a heigh of 1.45 m

It can be highlighted that the states are much more contained with respect to the uncontrolled one. As we can see in the previous simulations $Gain\ Scheduling\ LPV\ H_{\infty}$-controller works for all the combination of velocities and heights in a better way than the robust control.

8. Conclusions

In this project, the model of a single unit heavy vehicle including active steering is used. Some control schemes are developed to maximize its roll stability in order to prevent rollover. Simulation results demonstrate that the H∞ active steering control almost completely reduce the roll angle. Even though, also all the other controllers applied reduce the rollover risk significatively. It can be seen that the response of a vehicle with a high center of gravity (CG) in transient steering maneuvers change by the controllers. The roll dynamics is robustly improved despite the height of the CG is uncertain due to varying payloads. For that it was also study a robust controller, in this case H∞, considering this parameter and the velocity. The control could control the plant in the presence of uncertain parameters however, it presents a poor performance. This was greatly improved by using an LPV plant design and considering that velocity is a parameter that we can measure online. This concept must be expanded by an emergency rollover avoidance system based on simultaneous active steering and braking in future works.

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