1. Consider the model,

$$Y_t = \mu + \epsilon_t, \quad \epsilon_t = \sigma_t z_t$$

Where $z_t \sim i.i.d.\ N(0,1)$.

(a) Suppose Y_t stationary and $\sigma_t^2 = w + \alpha\epsilon_{t-1}^2$, i.e. an ARCH(1) model. Show that the variance of Y_t is given by:

$$\sigma_y^2 = Var(Y_t) = \frac{w}{1-\alpha}$$

What are the restrictions for the variance to be well defined?

(b) Define $v_t \equiv \epsilon_t^2 - \sigma_t^2$. Show that v_t is a white noise process.

(c) Now suppose an ARCH(q) process. Show that if $\sum_{i=1}^{q} \alpha_i < 1$ then the process is stationary and with a well-defined second moment. Use v_t defined in b) to link an ARCH process with an AR process.

Consider the following ARCH(2) model for the stock returns r_t:

$$\begin{aligned} r_t &= \mu + \epsilon_t; \\ \epsilon_t &= \sigma_t z_t;\ z_t \sim iidN(0,1), \\ \sigma_t^2 &= \alpha_0 + \alpha_1^2\epsilon_{t-1}^2 + \alpha_2\epsilon_{t-2}^2; \end{aligned} \tag{1}$$

where r_t is the return on a stock. Assume that all necessary restrictions hold on the coefficients so that the process is stationary and σ_t^2 is always non-negative. Answer all the following questions.

1. Derive the conditional mean and variance of r_t, conditional on information up to time t-1.

2. Derive the unconditional mean and unconditional variance of r_t.

3. Show that ϵ_t^2 follows an AR(2) model, while ϵ_t is not autocorrelated. Which stylized fact in the data is consistent with this finding?

3. Assume now the following model:

$$\begin{aligned} Y_t &= \mu + \epsilon_t \quad (1) \\ \epsilon_t &= \sigma_t z_t \quad (2) \end{aligned}$$

$$\sigma_t^2 = w + \alpha_1 \mathbb{I}(\epsilon_{t-1} < 0)\epsilon_{t-1}^2 + \alpha_2 \mathbb{I}(\epsilon_{t-1} \geq 0)\epsilon_{t-1}^2 \quad (3)$$

where z_t is i.i.d. $(0, 1)$, and $\mathbb{I}(.)$ is the indicator function. That is, $\mathbb{I}(Y_{t-1} < 0) = 1$ if $Y_{t-1} < 0$ and zero otherwise.

(a) Show that if $\alpha_1 = \alpha_2$, the model collapses to the usual lineal ARCH. Use this to interpret the coefficients α_1 and α_2.

(b) Show that the volatility process can be express as:

$$\sigma_t^2 = \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \gamma_2 \mathbb{I}(\epsilon_{t-1} \geq 0)\epsilon_{t-1}^2 \quad (4)$$

and get an expression for γ_0, γ_1 and γ_2 as a function of w, α_1 and α_2.

(c) Explain how you would formally test for asymmetric news impact using either model (3) or (4).