02450: Introduction to Machine Learning and Data Mining

Decision trees and linear regression

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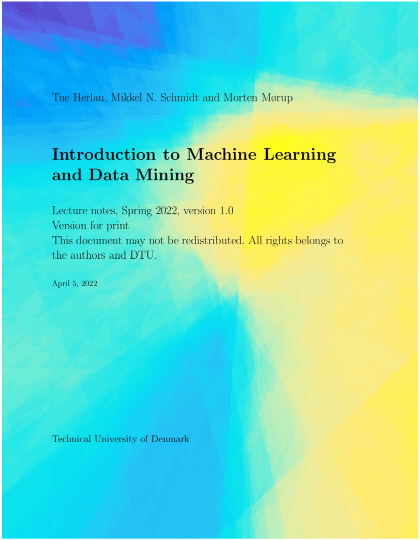
Today

Feedback Groups of the day:

Emma Faval, Adrian Dahlerup Fazlagic, Rasmus Højmark Fischer, Jonathan Bendix Fjendsbo, Yu Fan Fong, Jennifer Fortuny I Zhan, Simon Holmelund Frandsen, Lorenzo Fratini, Christian Frost, Julia Gabriela Makulec, Madiha Nour-El-Houda Bent Abdelkader Gam, Jian Gao, Evangelia Giagka, Eirini Giannakopoulou, Line Glade, Anton Egeskov Grier, Isabelle Marie Grimaldi, Niels Christian Grønlykke, Mads Dalsgaard Grunnan, Simone Øst Grunnell, Toma Guci, Eren Güldal, Liming Guo, Alvaro Gutierrez Leon, Maria Louise Thorup Hagedorn, Marc Sabater Hansen, Viktor Hinding Hansen, Jonas Bolvig Hansen, Monica Diaz Hansen, Christopher Sonne Hansen, Stefan Smedegaard Hansen, Matilde Yde Hansen, Maria Hansen, Mikkel Piester Hansen, Waygal Hashemiar

Reading material:

Chapter 8, Chapter 9



Lecture Schedule

1. Introduction
   31 January: C1

Data: Feature extraction, and visualization

2. Data, feature extraction and PCA
   7 February: C2, C3
3. Measures of similarity, summary statistics and probabilities
   14 February: C4, C5
4. Probability densities and data visualization
   21 February: C6, C7

Supervised learning: Classification and regression

5. **Decision trees and linear regression**
   28 February: C8, C9
6. Overfitting, cross-validation and Nearest Neighbor
   7 March: C10, C12 **(Project 1 due before 13:00)**
7. Performance evaluation, Bayes, and Naive Bayes
   14 March: C11, C13
8. Artificial Neural Networks and Bias/Variance
   21 March: C14, C15
9. AUC and ensemble methods
   28 March: C16, C17

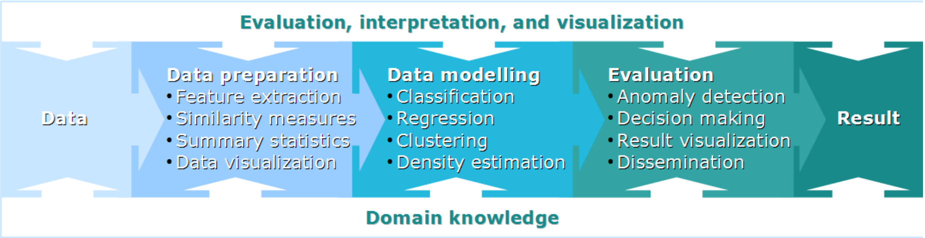
Unsupervised learning: Clustering and density estimation

10. K-means and hierarchical clustering
    11 April: C18
11. Mixture models and density estimation
    18 April: C19, C20 **(Project 2 due before 13:00)**
12. Association mining
    25 April: C21

Recap

13. Recap and discussion of the exam
    2 May: C1-C21

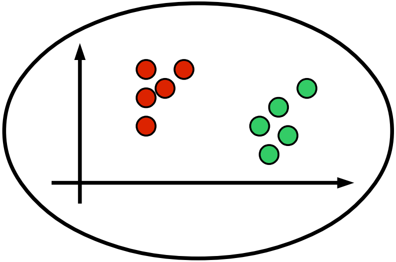
Online 24/7 help: Discussion Forum/Piazza
Streaming & Videos: `https://panopto.dtu.dk/`
Online exercises: MS Teams



Learning Objectives

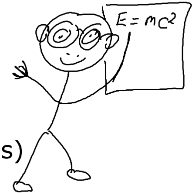
- Explain what supervised learning is
- Explain the difference between classification and regression
- Be able to evaluate classifiers in terms of the confusion matrix, error rate and accuracy
- Understand the principle behind decision trees and Hunt's algorithm
- Apply and interpret decision trees, linear regression and logistic regression

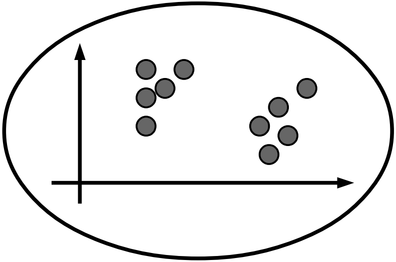
Supervised and Unsupervised learning



Supervised Learning
Input data $\mathbf{x}_n$ and output y_n

(Generalize from known examples)

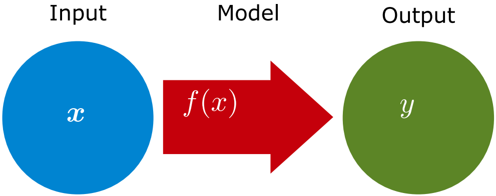




Unsupervised Learning
Input data $\mathbf{x}_n$ alone

(Exploratory analysis)

Supervised learning



- **Data**
  - Inputs and outputs *(this is what we are given)*

$$\{\boldsymbol{x}_n, y_n\}_{n=1}^{N}$$

- **Model**
  - Function that maps inputs to outputs *(what we are trying to determine)*

$$f(\boldsymbol{x})$$

- **Cost function**
  - Dissimilarity measure between observation and prediction *(how we tell if a model is good or bad)*

$$d\big(y, f(\boldsymbol{x})\big)$$

- **Types of supervised learning**
  - Regression: Continuous output **y**
  - Classification: Discrete output **y**

Classification

- **Definition:** Learning a function that maps a data object to a discrete class

- **Why classify?**
  - Descriptive modeling
    - Explain / understand the relation between attributes and class
  - Predictive modeling
    - Predict the class of a new data object

Decision trees

- Remember the game “20 questions to the professor”? (see also www.20q.net)

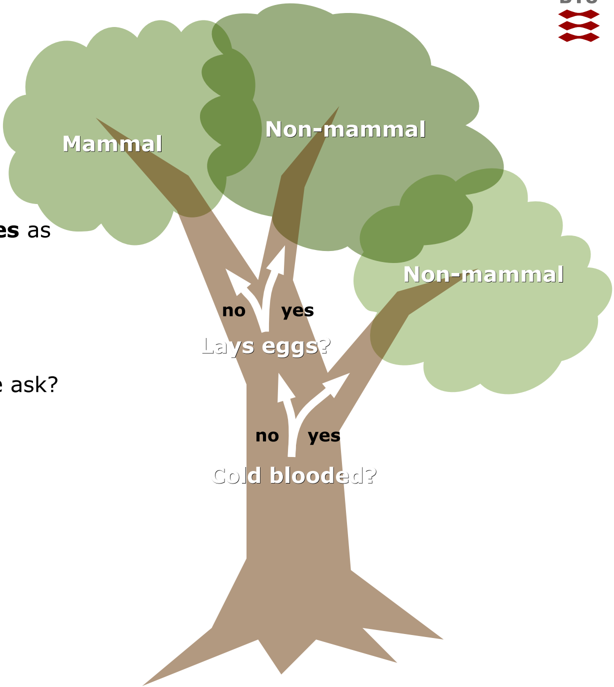
Q1. Is it an Animal? Yes.
Q2. Can you hold it? No.
Q3. Does it live in groups (gregarious)? Yes.
Q4. Are there many different sorts of it? No.
Q5. Can it jump? Yes.
Q6. Does it eat seeds? No.
Q7. Is it white? Sometimes.
Q8. Is it black and white? No.
Q9. Does it have paws? Yes.
Q10. Can you see it in a zoo? Yes.
Q11. Does it roar? Yes.
Q12. Is it worth a lot of money? Yes.
Q13. Does it have spots? Yes.
Q14. Is it multicoloured? Yes.
Q15. Can you make money by selling it? Yes.
Q16. Does it live in the jungle? Yes.
Q17. I guessed that it was a leopard? Wrong.
Q18. Does it like to play? Yes.
Q19. I guessed that it was a cheetah? Wrong.
Q20. I am guessing that it is a siberian tiger? Correct.

Decision trees

- Ask a series of questions until a conclusion is reached

- **Example**: Classify **vertebrates** as
  - **Mammal** or
  - **Non-mammal**

- **Learning task**
  - Which questions should we ask?



Hunts algorithm

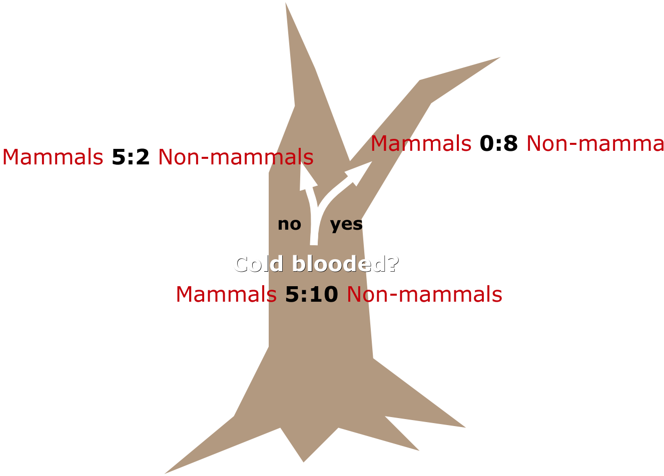
- Assign all data objects to the root

Hunts algorithm

- Select an attribute test condition
  - Find a good question to ask

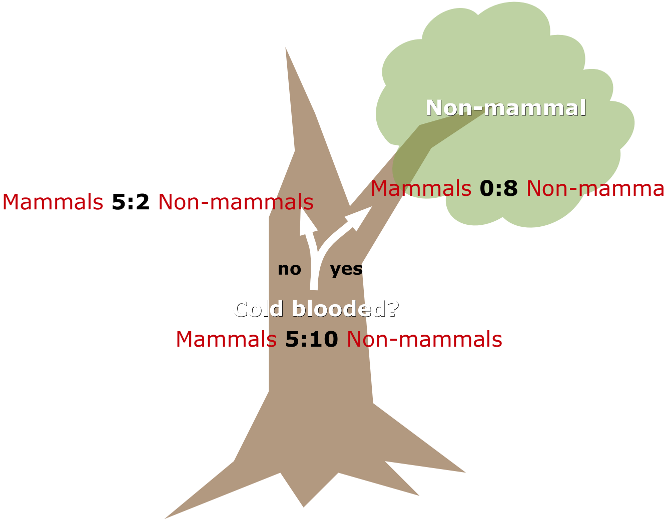
Hunt's Algorithm

- Partition the data objects into subsets according to the test condition



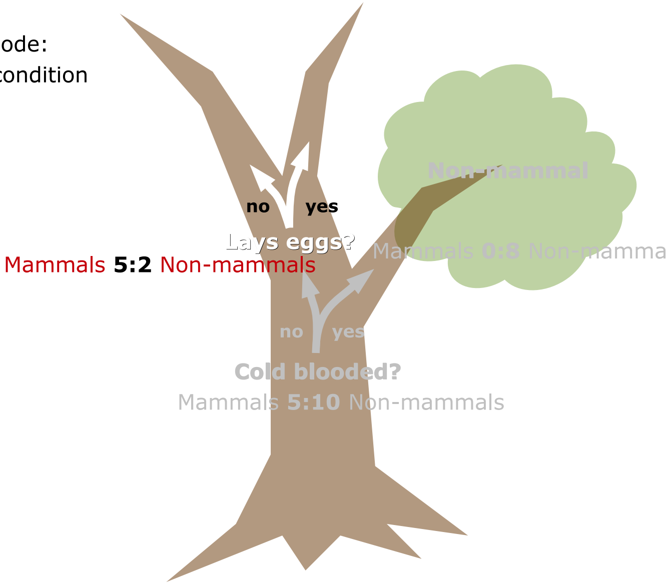
Hunts algorithm

- If all data objects belong to the same class
  - Create a leaf node



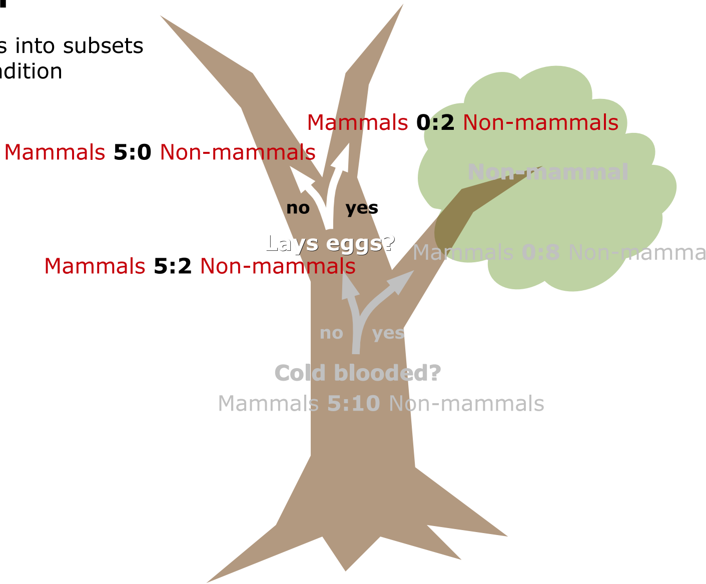
Hunts algorithm

- Repeat for each non-leave node:
  - Select an attribute test condition



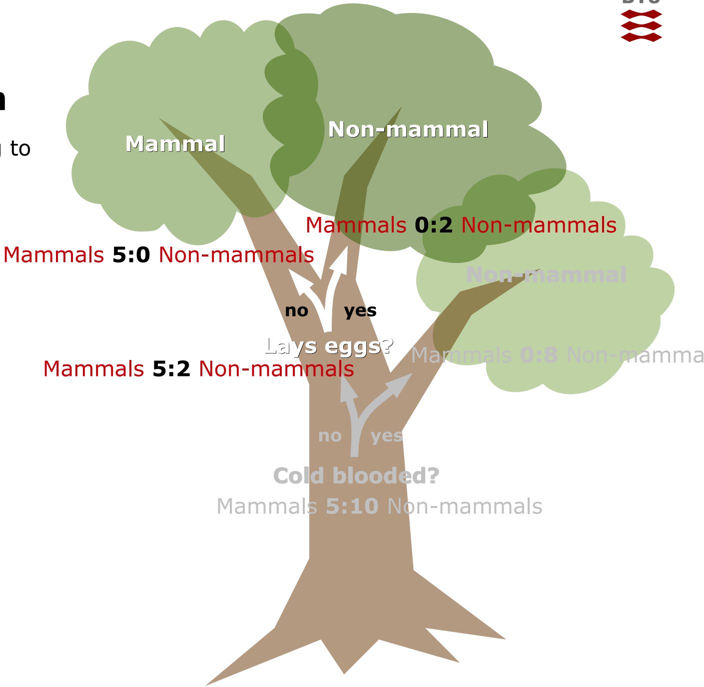
Hunts algorithm

- Partition the data objects into subsets according to the test condition



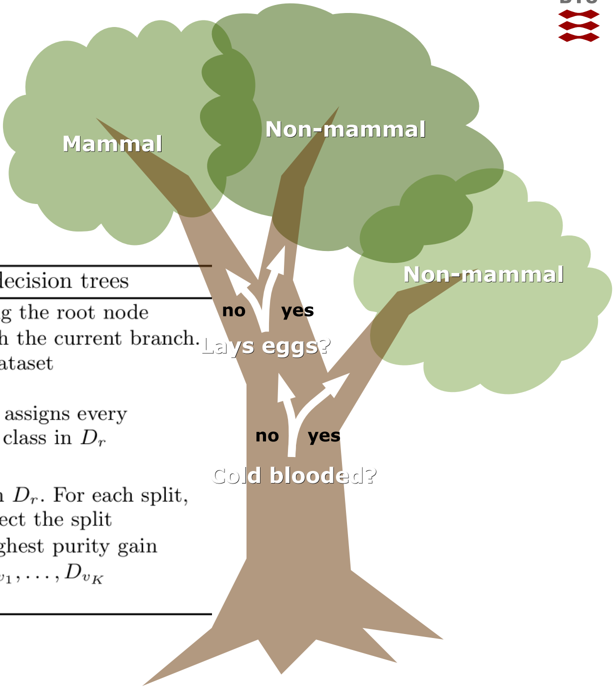
Hunts algorithm

- If all data objects belong to the same class
  - Create a leaf node



Hunts algorithm

- But how do we find the **best question** at each step?

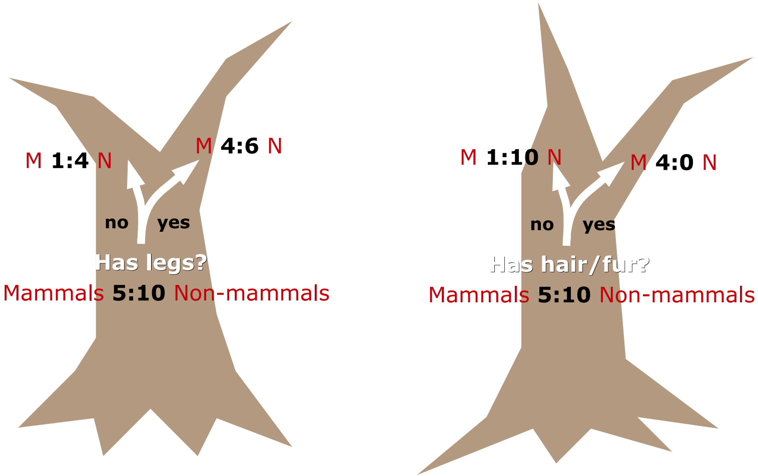


Algorithm 2: Hunt's algorithm for decision trees

Require: Initial tree T only containing the root node
Require: D_r : Dataset associated with the current branch. Initially just the full dataset

if The **stop criterion** is met **then**
  Add a leaf node to the tree which assigns every observation to the most prevalent class in D_r
else
  Try a number of different splits on D_r. For each split, compute the **purity gain** and select the split $D_r = \{D_{v_1}, \ldots, D_{v_K}\}$ with the highest purity gain
  Recursively call the method on $D_{v_1}, \ldots, D_{v_K}$
end if

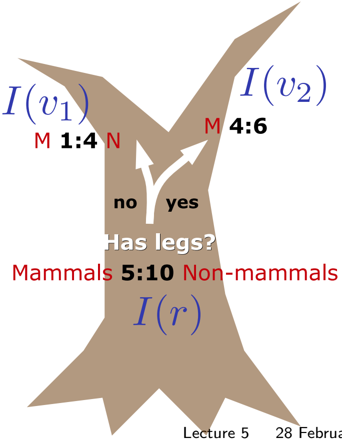
Which split is best?



Which split is best?

- Create a measure Δ (**the purity gain**) of how good a split is
- A binary split creates 3 partitions: the root r and the right/left branches v_1, v_2.
- For each partition, we compute $I(r), I(v_1), I(v_2)$ (the **impurity**)
- Purity gain is the **weighted reduction in impurity**:

$$\Delta = I(r) - \sum_{k=1}^{K=2} \frac{N(v_k)}{N(r)} I(v_k)$$



$N(r) = 15,$
$N(v_1) = 5,$
$N(v_2) = 10$

Which split is best?

- Create a measure Δ (**the purity gain**) of how good a split is
- A binary split creates 3 partitions: the root r and the right/left branches v_1, v_2.
- For each partition, we compute $I(r), I(v_1), I(v_2)$ (the **impurity**) of each partition
- Purity gain is the **weighted reduction in impurity**:

$$\Delta = I(r) - \sum_{k=1}^{K=2} \frac{N(v_k)}{N(r)} I(v_k)$$

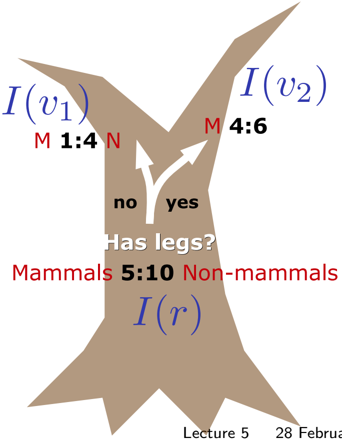
The impurity measure $I()$ can be one of the following

$$\text{Entropy}(v) = -\sum_{c=1}^{C} p(c|v) \log_2 p(c|v),$$

$$\text{Gini}(v) = 1 - \sum_{c=1}^{C} p(c|v)^2,$$

$$\text{ClassError}(v) = 1 - \max_c p(c|v).$$

$$p(c|v) = \frac{\{\text{Nr. in class } c \text{ in branch } v\}}{N(v)}$$



$N(r) = 15,$
$N(v_1) = 5,$
$N(v_2) = 10$

Quiz 1, Impurity gain

If we use the Gini index as impurity measure I, what is the purity gain Δ for the split indicated by the tree?

$$\Delta = I(r) - \sum_{k=1}^{K=2} \frac{N(v_k)}{N(r)} I(v_k)$$

A. ≈ 0.0177

B. ≈ 0.104

C. ≈ 0.129

D. ≈ 0.2

E. Don't know.

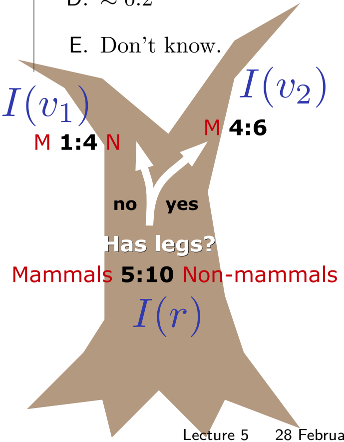
The impurity measure $I()$ can be one of the following

$$\text{Entropy}(v) = -\sum_{c=1}^{C} p(c|v)\log_2 p(c|v),$$

$$\text{Gini}(v) = 1 - \sum_{c=1}^{C} p(c|v)^2,$$

$$\text{ClassError}(v) = 1 - \max_c p(c|v).$$

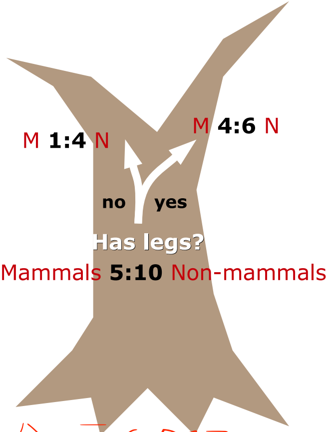
$$p(c|v) = \frac{\{\text{Nr. in class } c \text{ in branch } v\}}{N(v)}$$



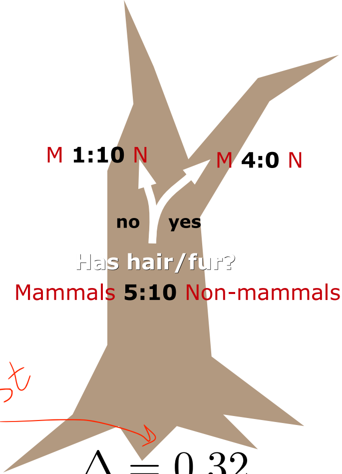
$N(r) = 15,$
$N(v_1) = 5,$
$N(v_2) = 10$

Selecting the best split

- Consider a large number of possible splits
- Compute a measure of impurity after the proposed split
  - For each new branch of the tree
  - Compute weighted average impurity
- Choose split that reduces impurity most



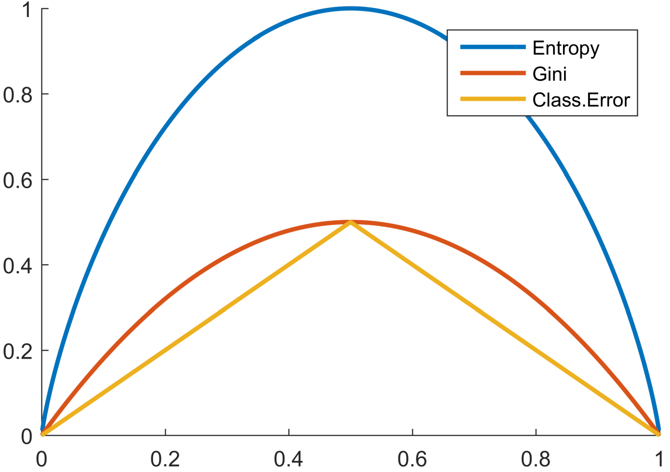
$\Delta = 0.017$



Best

$$\Delta = 0.32$$

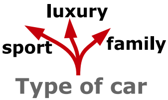
For a two class problem

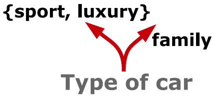


Which splits to consider

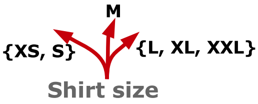
- Binary

- Nominal

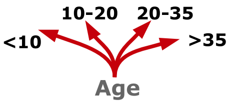


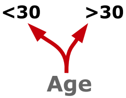


- Ordinal

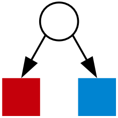


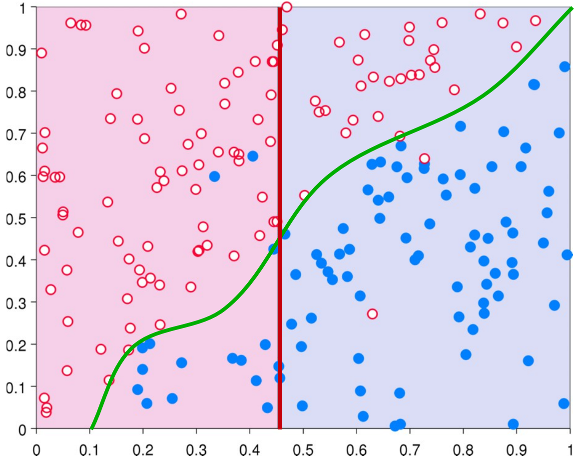
- Continuous



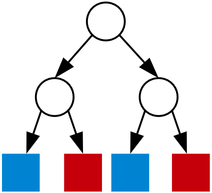


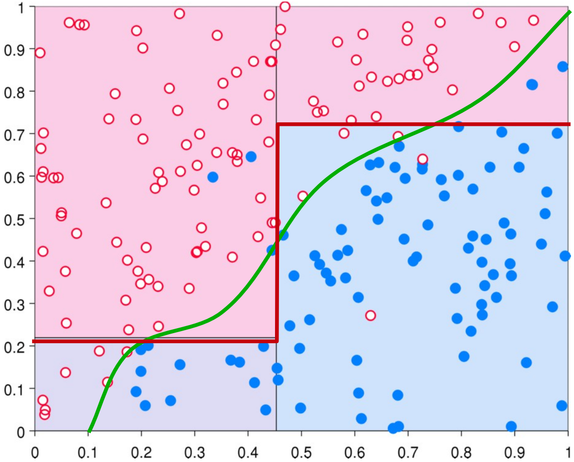
Classification Trees



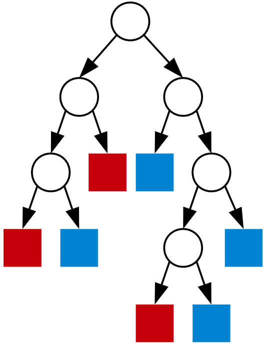


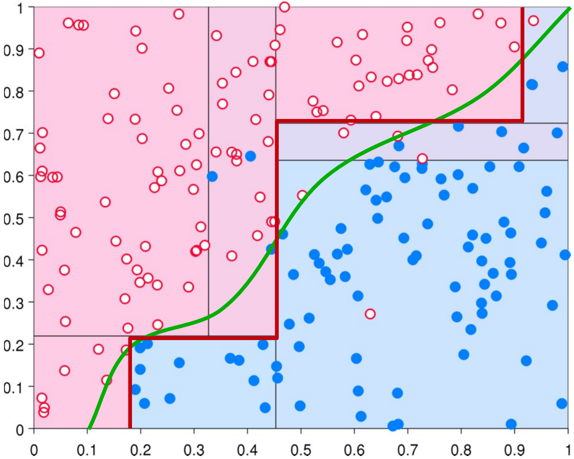
Classification trees





Classification trees



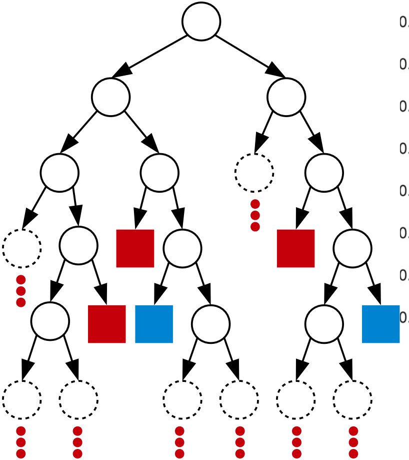


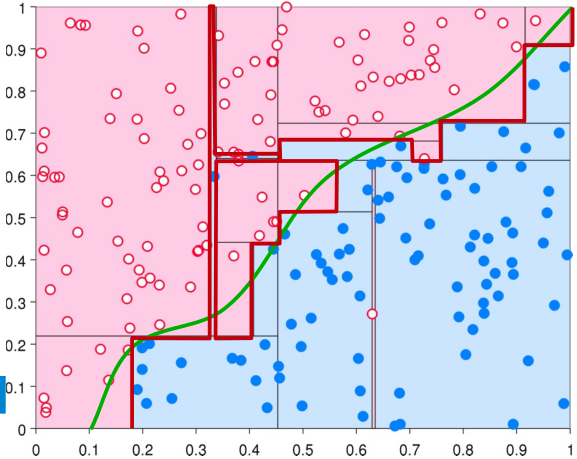
Classification trees

<u>Common stopping criteria:</u>

All records have the same class label

The number of observations have fallen below some minimum treshold





Regression trees

Algorithm 4: Hunt's algorithm for regression trees

Require: Initial tree T only containing the root node
Require: D_r : Dataset associated with the current branch. Initially just the full dataset
if The **stop criterion** is met **then**
Add a leaf node to the tree which assigns every observation the mean value of the nodes in D_r: $y(r) = \frac{1}{N(r)} \sum_{i \in r} y_i$
else
Try a number of different splits on D_r. For each split, compute the **purity gain** using the sum-of-squares impurity measure and select the split $D_r = \{D_{v_1}, \ldots, D_{v_K}\}$ with the highest purity gain
Recursively call the method on $D_{v_1}, \ldots, D_{v_K}$
end if

Use mean square error as purity gain

$$I(v) = \frac{1}{N(v)} \sum_{i \in v} (y_i - \hat{y}_v)^2, \quad \hat{y}_v = \frac{1}{N(v)} \sum_{i \in v} y_i$$

Confusion matrix

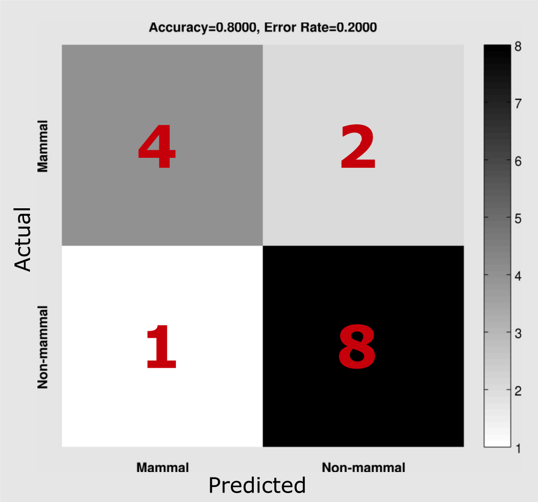
- Visualization of actual versus predicted class labels

- **Accuracy**
  (Number of correctly predicted observations divided by the total number of observations)

$$\frac{4+8}{4+2+1+8} = 80\%$$

- **Error rate**
  (Number of in-correctly predicted observations divided by the total number of observations)

$$\frac{2+1}{4+2+1+8} = 20\%$$



Evaluating a regression model

Compute average loss per observation:

$$E = \frac{1}{N} \sum_{i=1}^{N} L(y_i, f(\boldsymbol{x}_i))$$

Where we either use L_1 or L_2 (Euclidean) loss

$$L_1(y_i, \hat{y}_i) = |y_i - \hat{y}_i|, \quad L_2(y_i, \hat{y}_i) = (y_i - \hat{y}_i)^2$$

(Compare these to p-norms)

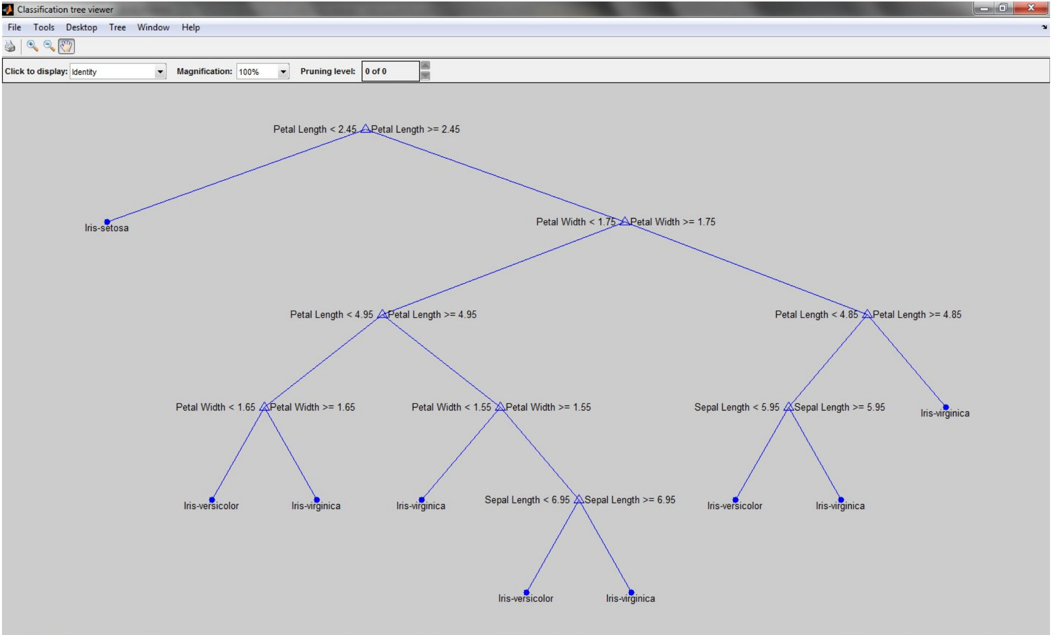
The iris data set

- **Three flowers**
  - 50 instances of each class, 150 in total

- **Attributes**
  - Sepal (outermost leaves)
    - length in cm
    - width in cm
  - Petal (innermost leaves)
    - length in cm
    - width in cm
  - Class of flower
    - Iris Setosa
    - Iris Versicolour
    - Iris Virginica

Fower ID	Attribute			
	Sepal Length	Sepal Width	Petal Length	Petal Width
1	5.1	3.5	1.4	0.2
2	4.9	3.0	1.4	0.2
3	4.7	3.2	1.3	0.2
4	4.6	3.1	1.5	0.2
.	.	.	.	.
.	.	.	.	.
150	5.9	3.0	5.1	1.8

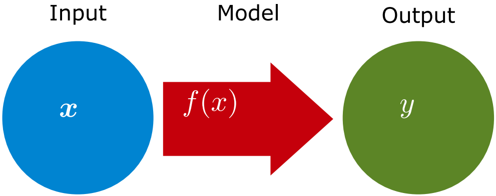
$X^{\text{Observation x Attribute}}$



What would the following iris flower be classsified as?

Sepal Length	Sepal Width	Petal Length	Petal Width
4.0	3.5	3.0	2.0

Supervised learning



- **Mapping between domains**
  - Classification: Discrete (nominal) output
  - Regression: Continuous output

Supervised learning

- **Data**
  - Inputs and outputs $\{\boldsymbol{x}_n, y_n\}_{n=1}^{N}$
- **Model**
  - Function that maps inputs to outputs
$$f(\boldsymbol{x})$$
- **Cost function**
  - Dissimilarity measure between data and model
$$d\big(y, f(\boldsymbol{x})\big)$$

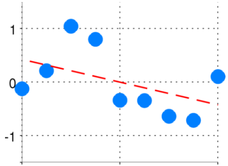
Regression

- **Definition:** Learning a function that maps a data object to a continuous-valued output

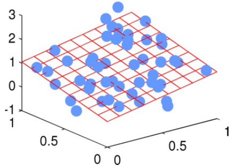
- **Why Regression?**
  - Descriptive modeling
    - Explain / understand the relation between attributes and continous-valued output
  - Predictive modeling
    - Predict the output value of a new data object

Linear regression

- 1-dimensional inputs
  $f(x) = w_0 + w_1 x$



- 2-dimensional inputs
  $f(\boldsymbol{x}) = w_0 + w_1 x_1 + w_2 x_2$



- K-dimensional inputs
  $f(\boldsymbol{x}) = w_0 + w_1 x_1 + w_2 x_2 + \cdots + w_K x_K$

Linear regression

- K-dimensional inputs

$$f(\boldsymbol{x}) = w_0 + w_1 x_1 + w_2 x_2 + \cdots + w_K x_K$$

- Non-linearly transformed inputs

$$f(x) = w_0 + w_1 x + w_2 x^2 + \cdots + w_K x^K$$

$$f(x) = w_0 + w_1 sin(x) + w_2 cos(x)$$

Linear regression

- K-dimensional inputs

$$f(\boldsymbol{x}) = w_0 + w_1 x_1 + w_2 x_2 + \cdots + w_K x_K$$

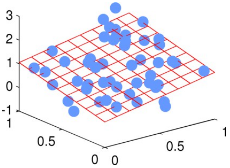
- Non-linearly transformed inputs

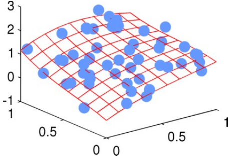
$$f(x) = w_0 + w_1 x + w_2 x^2 + \cdots + w_K x^K$$

$$f(x) = w_0 + w_1 sin(x) + w_2 cos(x)$$

- **Example**

$$f(\boldsymbol{x}) = w_0 + w_1 x_1 + w_2 x_2$$





$$\begin{aligned} f(\boldsymbol{x}) = {} & w_0 + w_1 x_1 + w_2 x_2 \\ & + w_3 x_1^2 + w_4 x_2^2 + w_5 x_1 x_2 \\ & + w_6 x_1^3 + w_7 x_1^2 x_2 + w_8 x_1 x_2^2 + w_9 x_2^3 \end{aligned}$$

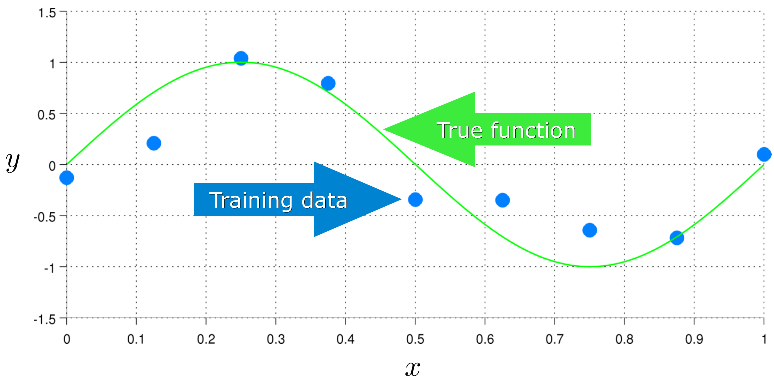
Vector notation

- The linear model can be written compactly using vector notation

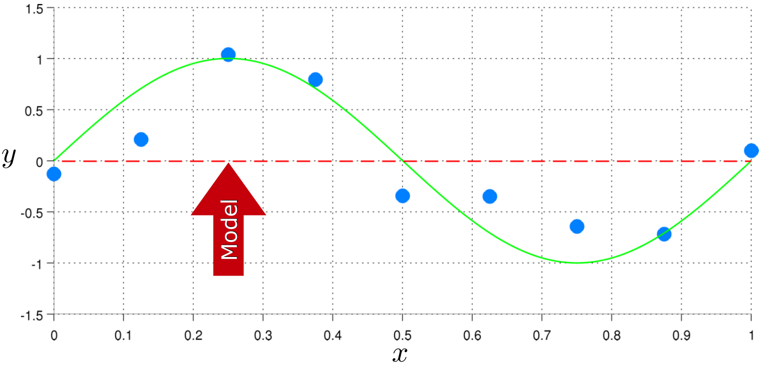
$$f(\boldsymbol{x}) = w_0 + w_1 x_1 + w_2 x_2 + \cdots + w_K x_K$$
$$= \sum_{k=0}^{K} w_k x_k = \boxed{\boldsymbol{x}^\top \boldsymbol{w}}$$

  - where $x_0 = 1$

Linear regresssion



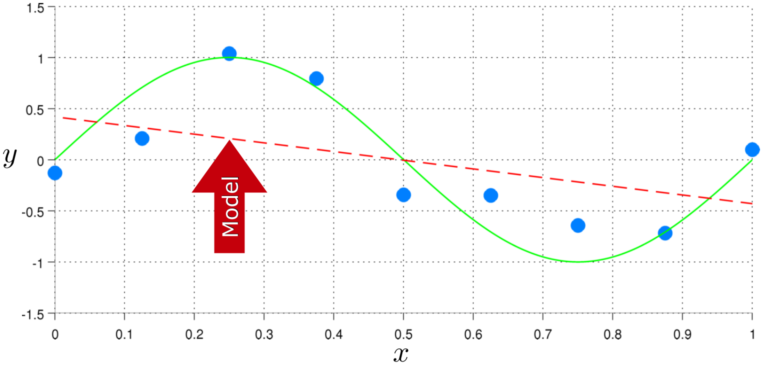
Linear regresssion



Model

$f(x) = w_0$

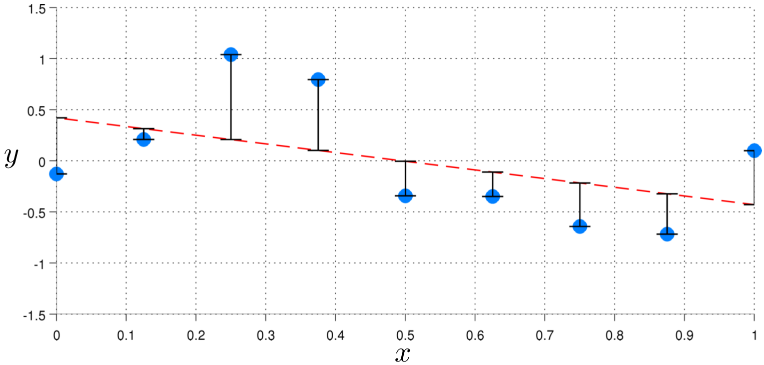
Linear regresssion



Model

$$f(x) = w_0 + w_1 x$$

Residual error



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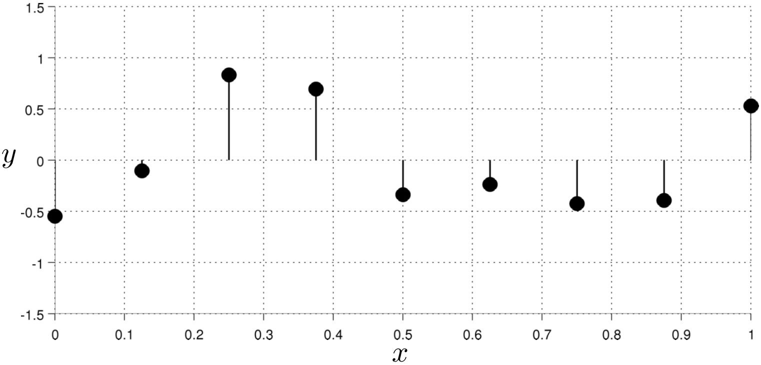
Residual error



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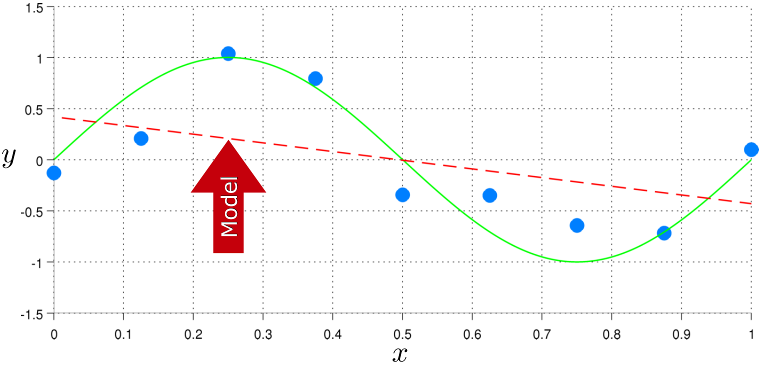
Linear regression



Model

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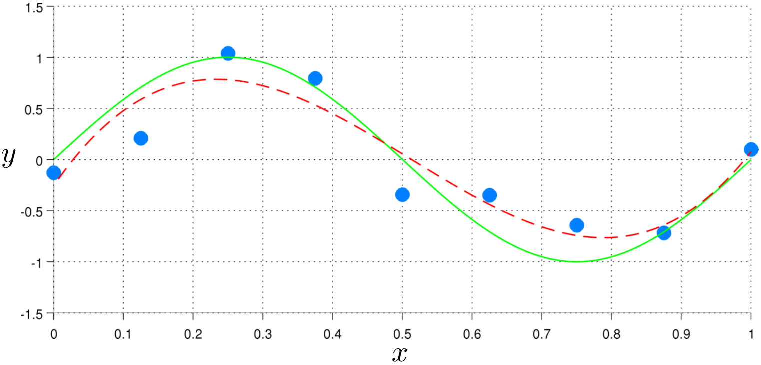
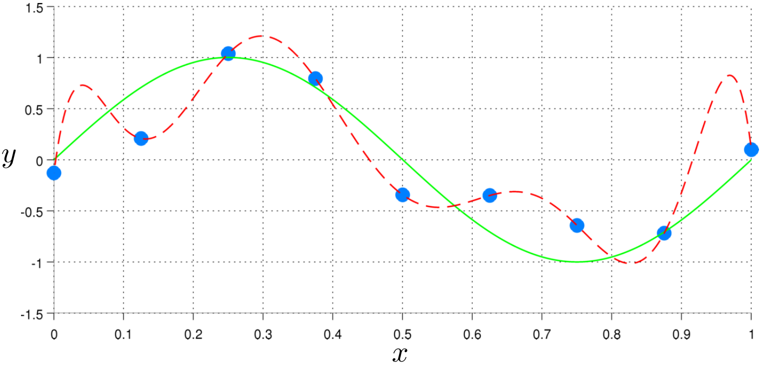
Linear regression



Model

$$f(x) = w_0 + w_1 x + w_2 x^2 + w_3 x^3$$

Linear regression

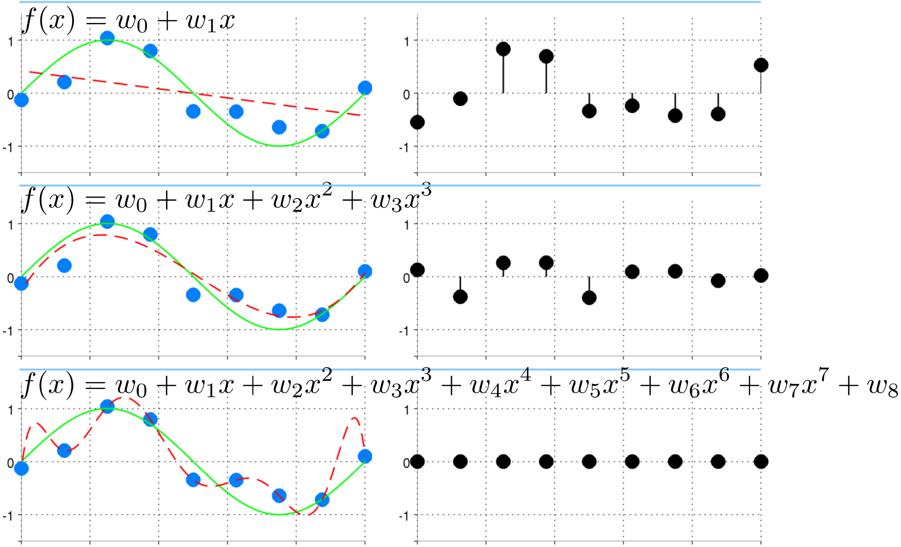


Model

$$f(x) = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + w_4 x^4 + w_5 x^5 + w_6 x^6 + w_7 x^7 + w_8 x^8$$

Model order

- Which model order
  - Gives the best fit?
  - Do you think is most "correct"?



Learning

Setup: We got some data $(\boldsymbol{y}, \boldsymbol{X})$. We have a way to define a function

$$f(\boldsymbol{x}; \boldsymbol{w}) = \tilde{\boldsymbol{x}}^T \boldsymbol{w} = w_0 + \sum_{k=1}^{M} x_k w_k$$

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- **Question:** How do we learn the correct $\boldsymbol{w}$?

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- **Question:** How do we learn the correct $\boldsymbol{w}$?
- Answer: For each observation, assume:

$$y_i = f(\boldsymbol{x}_i; \boldsymbol{w}) + \varepsilon_i$$

where ε_i is a normally distributed noise term $\mathcal{N}(\varepsilon_i | \mu = 0, \sigma^2)$. Recall:

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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- This means that

$$p(y_i|\boldsymbol{x}_i, \boldsymbol{w}) =$$

Recall from last time: Maximum A Posteriori (MAP) learning

- Consider some data $\boldsymbol{X} = \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ and $\boldsymbol{y} = y_1, \ldots, y_N$
- Suppose $\boldsymbol{x}_i$ relates to y_i by some parameters $\boldsymbol{w}$
- **Assume**

$$p(\boldsymbol{y}|\boldsymbol{X}, \boldsymbol{w}) = \prod_{i=1}^{N} p(y_i|\boldsymbol{x}_i, \boldsymbol{w}), \quad p(\boldsymbol{w}|\boldsymbol{X}) = p(\boldsymbol{w})$$

- Then

$$p(\boldsymbol{w}|\boldsymbol{X}, \boldsymbol{y}) = \frac{p(\boldsymbol{y}|\boldsymbol{X}, \boldsymbol{w})p(\boldsymbol{w}|\boldsymbol{X})}{p(\boldsymbol{y}|\boldsymbol{X})} = \frac{\prod_{i=1}^{N} p(\boldsymbol{y}_i|\boldsymbol{w}, \boldsymbol{x}_i)p(\boldsymbol{w})}{p(\boldsymbol{X}|\boldsymbol{y})}$$

- And maximizing: $\boldsymbol{w}^* = \arg\max_{\boldsymbol{w}} p(\boldsymbol{w}|\boldsymbol{X}, \boldsymbol{y})$ is equivalent to

$$\textit{Minimize:} \quad \boldsymbol{w}^* = \arg\min_{\boldsymbol{w}} E(\boldsymbol{w})$$

$$E(\boldsymbol{w}) = \frac{1}{N}\left[-\sum_{i=1}^{N} \log p(y_i|\boldsymbol{x}_i, \boldsymbol{w}) - \log p(\boldsymbol{w})\right]$$

Back to the linear model

$$p(y_i|\boldsymbol{x}_i, \boldsymbol{w}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i - \tilde{\boldsymbol{x}}_i^\top \boldsymbol{w})^2}{2\sigma^2}}$$

Optimal $\boldsymbol{w}^* = \arg\max_{\boldsymbol{w}} p(\boldsymbol{w}|\boldsymbol{X}, \boldsymbol{y})$ found as $\boldsymbol{w}^* = \arg\min_{\boldsymbol{w}} E(\boldsymbol{w})$

$$E(\boldsymbol{w}) = \frac{1}{N}\left[-\sum_{i=1}^{N} \log p(y_i|\boldsymbol{x}_i, \boldsymbol{w}) - \log p(\boldsymbol{w})\right]$$

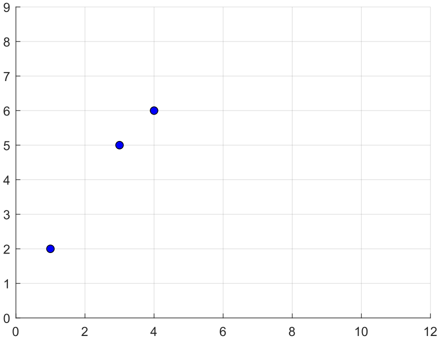
Quiz 2, The linear model

Suppose you observe three points:

$$(x, y) = (1, 2), (3, 5), (4, 6)$$

Knowing what you have learned so far, you first bring these points to the standard format:

$$\boldsymbol{X} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$



You wish to train a linear model of the form $y = ax + b$ on this dataset. What is $\boldsymbol{w} = \begin{bmatrix} b \\ a \end{bmatrix}$ Then, compute the prediction of the model at $x = 5$? (the prediction is given as: $y = \tilde{\boldsymbol{x}}^\top \boldsymbol{w}^*$)

A. 6.5

B. 7

C. 7.5

D. 8

E. Don't know.

$$\tilde{X} = \begin{bmatrix} 1 & \\ 1 & X \\ \vdots & \\ 1 & \end{bmatrix}$$

Recall $\boldsymbol{w}^* = (\tilde{\boldsymbol{X}}^\top \tilde{\boldsymbol{X}})^{-1} \tilde{\boldsymbol{X}}^\top \boldsymbol{y}$

Logistic regression

$f(x) = x^\top w$

- Assume we are given $(\boldsymbol{X}, \boldsymbol{y})$, but assume y is *binary*: $y_i = 0, 1$
- An idea is to use the Bernoulli distribution

$$p(y_i|\boldsymbol{x}_i, \boldsymbol{w}) = \text{Bernouilli}(y_i|\theta_i) = \theta_i^{y_i}(1-\theta_i)^{1-y_i}$$

  Where θ_i depends on $\boldsymbol{w}$ and $\boldsymbol{x}_i$.

- **Problem:** θ_i must belong to the unit interval, but $f(\boldsymbol{x}_i, \boldsymbol{w}) = \tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}$ won't
- **Solution:** Assume

$$\theta_i = \sigma(f(\boldsymbol{x}, \boldsymbol{w})), \quad \text{where } \sigma(z) = \frac{1}{1+e^{-z}} \text{ is the } \textbf{logistic sigmoid}$$

  Then

$$-\log p(y_i|\boldsymbol{x}_i, \boldsymbol{w})) =$$

Recall from 10 minutes ago: Maximum A Posteriori (MAP) learning

- Consider some data $\boldsymbol{X} = \boldsymbol{x}_1, \ldots, x_N$ and $\boldsymbol{y} = y_1, \ldots, y_N$
- **Assume**

$$p(\boldsymbol{y}|\boldsymbol{X}, \boldsymbol{w}) = \prod_{i=1}^{N} p(y_i|\boldsymbol{x}_i, \boldsymbol{w}), \quad p(\boldsymbol{w}|\boldsymbol{X}) = p(\boldsymbol{w})$$

- Then

$$p(\boldsymbol{w}|\boldsymbol{X}, \boldsymbol{y}) = \frac{\prod_{i=1}^{N} p(\boldsymbol{y}_i|\boldsymbol{w}, \boldsymbol{x}_i)p(\boldsymbol{w})}{p(\boldsymbol{X}|\boldsymbol{y})}$$

- Maximizing: $\boldsymbol{w}^* = \arg\max_{\boldsymbol{w}} p(\boldsymbol{w}|\boldsymbol{X}, \boldsymbol{y})$ is equivalent to $\boldsymbol{w}^* = \arg\min_{\boldsymbol{w}} E(\boldsymbol{w})$

$$E(\boldsymbol{w}) = \frac{1}{N}\left[-\sum_{i=1}^{N} \log p(y_i|\boldsymbol{x}_i, \boldsymbol{w}) - \log p(\boldsymbol{w})\right]$$

- By assuming a constant (flat prior) we can ignore we obtain (Máximum Likelihood learning)

$$E(\boldsymbol{w}) = \frac{1}{N}\sum_{i=1}^{N} \left[-y_i \log(\theta_i) - (1 - y_i) \log(1 - \theta_i)\right], \quad \theta_i = \sigma(\tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}) = \frac{1}{1 + e^{-\tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}}}$$

Quiz 3, Logistic regression

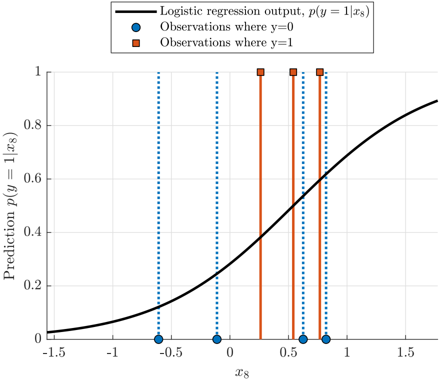


Figure 1: Output of a logistic regression classifier trained on 7 observations from the dataset.

Consider the Avila Bible dataset. We are particularly interested in predicting whether a bible copy was written by copyist 1, and we therefore wish to train a logistic regression classifier to distinguish between copyist one vs. copyist two and three.

To simplify the setup further, we select just 7 observations and train a logistic regression classifier using only the feature x_8 as input (as usual, we apply a simple feature transformation to the inputs to add a constant feature in the first coordinate to handle the intercept term). To be consistent with the lecture notes, we label the output as $y = 0$ (corresponding to copyist one) and $y = 1$ (corresponding to copyist two and three).

In Figure 1 is shown the predicted output probability an observation belongs to the positive class, $p(y = 1|x_8)$. What are the weights?

A. $\begin{bmatrix} -0.93 \\ 1.72 \end{bmatrix}$

B. $\begin{bmatrix} -2.82 \\ 0.0 \end{bmatrix}$

C. $\begin{bmatrix} 1.36 \\ 0.4 \end{bmatrix}$

D. $\begin{bmatrix} -0.65 \\ 0.0 \end{bmatrix}$

E. Don't know.

General linear model

$$\text{Linear regr.: } E(\boldsymbol{w}) = \frac{1}{N}\sum_{i=1}^{N} \|y_i - \tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}\|^2$$

$$\text{Logistic regr.: } E(\boldsymbol{w}) = \frac{-1}{N}\sum_{i=1}^{N} \left[y_i \log(\theta_i) + (1-y_i)\log(1-\theta_i)\right], \;\; \theta_i = \sigma(\tilde{\boldsymbol{x}}_i^\top \boldsymbol{w})$$

$$\text{GLM } E(\boldsymbol{w}) = \frac{1}{N}\sum_{i=1}^{N} d(y_i, g(\tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}))$$

General linear model

$$\text{Linear regr.: } E(\boldsymbol{w}) = \frac{1}{N}\sum_{i=1}^{N} \|y_i - \tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}\|^2$$

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$$\text{GLM } E(\boldsymbol{w}) = \frac{1}{N}\sum_{i=1}^{N} d(y_i, g(\tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}))$$

We call d the cost function and g the link function. In our examples:

$$\text{Lin.reg. : } d(y,z) = \|y-z\|^2, \qquad z = g(\tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}) = \tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}$$

$$\text{Log.reg. : } d(y,z) = -y\log z - (1-y)\log(1-z), \quad z = g(\tilde{\boldsymbol{x}}_i^\top \boldsymbol{w}) = \sigma(\tilde{\boldsymbol{x}}_i^\top \boldsymbol{w})$$

Resources

http://www2.imm.dtu.dk Our interactive regression demo

(http://www2.imm.dtu.dk/courses/02450/DemoComplexityRegression.html)