02610 Optimization and Data Fitting

Homework assignment 1

Hand in on DTU Learn before 6 October 10pm.

1 Report on the exercises for week 5 (60%)

2 Convexity (by hand, 10%)

1. (5%) Consider the function $f(x) = \frac{1}{x^2}$ with the domain $\{x \in \mathbb{R} | x \neq 0\}$. Is $f(x)$ convex? Why?

2. (5%) If $f(x)$ with $x \in \mathbb{R}$ is convex, is $g(x) = \exp(f(x))$ a convex function? Why?

3 Exact line search and steepest descent method (by hand, 20%)

Consider the function

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} \tag{1}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}$$

with $\gamma \geq 1$. That is, the function $f(\boldsymbol{x})$ also can be written as

$$f(x_1, x_2) = \frac{\gamma}{2}x_1^2 + \frac{1}{2}x_2^2.$$

1. (5%) Compute the gradient $\nabla f(\boldsymbol{x})$, and write down the steepest descent iteration formula.

2. (10%) Perform one iteration of the steepest descent method with exact line search, starting at $\boldsymbol{x}_0 = [1, \gamma]^T$. Include your derivation in your answers.

3. (5%) We notice the fact (you don't need prove it) that with the starting point $\boldsymbol{x}_0 = [1, \gamma]^T$ the iterates obtained from the steepest descent method with exact line search in fact follow

$$\boldsymbol{x}_k = \left(\frac{\gamma - 1}{\gamma + 1}\right)^k \begin{bmatrix} (-1)^k \\ \gamma \end{bmatrix}.$$

Prove that this sequence $\{\boldsymbol{x}_k\}$ converges Q-linearly to $[0, 0]^T$.

4 Newton method (by hand, 10%)

Consider a univariate function $f(x) = x^s$ with $s \geq 2$ and $x \in \mathbb{R}_{++}$, i.e. $x > 0$. Prove that, for any starting point $x_0 \in \mathbb{R}_{++}$, Newton's method is well-defined and converges linearly to zero. Compute the convergence factor.

5 Inverse of an increasing convex function (Bonus, 10%)

Suppose $f : \mathbb{R} \to \mathbb{R}$ is increasing and convex on its domain. Let g denote its inverse, i.e., $g(f(x)) = x$. Suppose both f and g are twice differentiable. What can we say about convexity or concavity of g?